Refinements of Hermite-Hadamard Type Inequalities Involving Fractional Integrals*

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Abstract

In this paper, Hermite-Hadamard type inequalities involving Hadamard fractional integrals via convex functions are studied. An important integral identity and new Hermite-Hadamard type integral inequalities involving Hadamard fractional integrals are also presented. Some applications to special means of real numbers are given.

1 Introduction

Let $f : [a,b] \subset R \to R$ be a convex function, i.e., $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. The classical Hermite-Hadamard type inequality provides a lower and an upper estimations for the integral average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. More precisely, if $f : [a,b] \to R$ is a convex function, then it is integrable in sense of Riemann and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

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As we all known, the inequality (1) was firstly discovered by Hermite in 1881 in the journal Mathesis (see Mitrinović and Lacković [1]). However, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see Pečarić et al. [2]).

For more recent results which generalize, improve, and extend the inequalities presented above, one can see Abramovich et al. [3], Cal et al. [4], Avci et al. [5], Ödemir et al. [6, 7], Dragomir [8, 9], Sarikaya et al. [10, 11], Xiao et al. [12], Bessenyei [13], Tseng et al. [14], Niculescu [15] and references therein.

It is remarkable that Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.1. *(see Theorem 2, [11]) Let $f : [a,b] \to R$ be a positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(b) + {}_{RL}J^{\alpha}_{b^-}f(a)] \le \frac{f(a)+f(b)}{2}.$$

We remark that the symbol ${}_{RL}J^{\alpha}_{a^+}f$ and ${}_{RL}J^{\alpha}_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ are defined by

$$({}_{RL}J^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ (0 \le a < x \le b),$$

and

$$({}_{RL}J^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ (0 \le a \le x < b),$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function.

Theorem 1.2. *(see Lemma 2, [11]) Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $0 \le a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(b) +{}_{RL} J^{\alpha}_{b^-}f(a)] = \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f'(ta+(1-t)b)dt.$$

Theorem 1.3. *(see Theorem 3, [11]) Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following equality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(b) +{}_{RL} J^{\alpha}_{b^-}f(a)]\right| \le \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)|f'(a)+f'(b)|.$$

Remark 1.4. *In fact, $|f'|$ is convex on $[a,b]$ can be changed to f' is convex on $[a,b]$.*

Motivated by the interesting works in [11], Wang et al. [16] established another two fundamental integral identities including the second order derivatives of a given function and proved some Hermite-Hadamard type inequalities involving left-sided and right-sided Riemann-Liouville fractional integrals for m-convex and (s, m)-convex functions respectively.

Recently, Hermite-Hadamard inequality involving Riemann-Liouville fractional integrals have been paid more and more attentions, however, there are few work on the Hadamard fractional integrals, even if it has been reported many years ago. Thus, it is natural to offer to study Hermite-Hadamard type inequalities involving Hadamard fractional integrals.

In the following, we recall some necessary definitions and mathematical preliminaries of left-sided and right-sided Hadamard fractional calculus theory which are used further in this paper. For more recent development on fractional calculus, one can see the monographs of Baleanu et al. [17], Diethelm [18], Kilbas et al. [19], Lakshmikantham et al. [20], Miller and Ross [21], Michalski [22], Podlubny [23] and Tarasov [24].

Definition 1.5. *The left-sided and right-sided Hadamard fractional integrals of order $\alpha \in R^+$ of function $f(x)$ are defined by*

$$({}_HJ^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0 < a < x \le b),$$

and

$$({}_HJ^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0 < a \le x < b),$$

where $\Gamma(\cdot)$ is the Gamma function.

The following two inequalities are also needed.

Lemma 1.6. *(see [25]) For $0 < \sigma \le 1$ and $0 < a < b$, we have*

$$|a^{\sigma} - b^{\sigma}| \le (b-a)^{\sigma}.$$

Lemma 1.7. *(see [26]) For all $\lambda, v, w > 0$, then for any $t > 0$, we have*

$$t^{1-v}\int_0^t (t-s)^{v-1}s^{\lambda-1}e^{-ws}ds \le \max\left\{1, 2^{1-v}\right\}\Gamma(\lambda)\left(1+\frac{\lambda}{v}\right)w^{-\lambda}.$$

In the present paper, we establish some new Hermite-Hadamard's inequalities involving left-sided and right-sided Hadamard fractional integrals and some other integral inequalities using the identity obtained for Hadamard fractional integrals.

2 Hermite-Hadamard's inequalities for Hadamard fractional integrals

Hermite-Hadamard's inequalities for Hadamard fractional integrals can be represented as follows.

Theorem 2.1. *Let $f : [a,b] \to R$ be a positive function with $0 < a < b$ and $f \in L[a,b]$. If f is a nondecreasing and convex function on $[a,b]$, then the following inequality for fractional integrals hold*

$$f(\sqrt{ab}) \leq \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} \left[{}_HJ^{\alpha}_{a+}f(b) \ + \ {}_HJ^{\alpha}_{b-}f(a) \right] \leq f(b). \tag{2}$$

Proof. Since f is a nondecreasing and convex function on $[a,b]$, we have for $x, y \in [a,b]$ with $\lambda = \frac{1}{2}$,

$$f(\sqrt{xy}) \leq f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{3}$$

Set $x = e^{\ln b - t(\ln b - \ln a)}$ and $y = e^{\ln a + t(\ln b - \ln a)}$ for $0 < t < 1$, then

$$2f(\sqrt{ab}) = 2f(\sqrt{e^{\ln b + \ln a}}) \leq f(e^{\ln b - t(\ln b - \ln a)}) + f(e^{\ln a + t(\ln b - \ln a)}). \tag{4}$$

Multiplying both sides of (4) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
\frac{2}{\alpha} f(\sqrt{ab}) &= \frac{2}{\alpha} f(\sqrt{e^{\ln b + \ln a}}) \\
&\leq \int_0^1 t^{\alpha-1} f(e^{\ln b - t(\ln b - \ln a)}) dt + \int_0^1 t^{\alpha-1} f(e^{\ln a + t(\ln b - \ln a)}) dt \\
&= \frac{1}{\ln a - \ln b} \int_b^a \left(\frac{\ln b - \ln u}{\ln b - \ln a}\right)^{\alpha-1} f(u) \frac{du}{u} \\
&\quad + \frac{1}{\ln b - \ln a} \int_a^b \left(\frac{\ln v - \ln a}{\ln b - \ln a}\right)^{\alpha-1} f(v) \frac{dv}{v} \\
&= \frac{\Gamma(\alpha)}{(\ln b - \ln a)^{\alpha}} [{}_HJ^{\alpha}_{a+}f(b) \ + \ {}_HJ^{\alpha}_{b-}f(a)],
\end{aligned}$$

which implies that

$$f(\sqrt{ab}) \ \leq \ \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}} [{}_HJ^{\alpha}_{a+}f(b) \ + \ {}_HJ^{\alpha}_{b-}f(a)].$$

On the other hand, note that f is nondecreasing, we have

$$f(e^{\ln b - t(\ln b - \ln a)}) + f(e^{\ln a + t(\ln b - \ln a)}) \leq 2f(e^{\ln b}) = 2f(b). \tag{5}$$

Then multiplying both sides of (5) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\alpha-1} f(e^{\ln b - t(\ln b - \ln a)})dt + \int_0^1 t^{\alpha-1} f(e^{\ln a + t(\ln b - \ln a)})dt \\ &= \frac{\Gamma(\alpha)}{(\ln b - \ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b) \ + \ {}_HJ^\alpha_{b-}f(a)] \\ &\leq 2f(e^{\ln b})\int_0^1 t^{\alpha-1}dt \\ &= \frac{2}{\alpha}f(b), \end{aligned}$$

which yields that

$$\frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b) \ + \ {}_HJ^\alpha_{b-}f(a)] \ \leq \ 2f(b).$$

The proof is completed. ■

Example 2.2. *Let $a = 1, b = e, \alpha = 2, f(x) = x^2$. Then all the assumptions in Theorem 2.1 are satisfied. Clearly,*

$$\begin{aligned} {}_HJ^2_{1+}f(e) &= \int_1^e (1-\ln t)tdt = \int_0^1 se^{2(1-s)}ds = \frac{1}{4}e^2 - \frac{3}{4}, \\ {}_HJ^2_{e-}f(1) &= \int_1^e t\ln tdt = \int_0^1 se^{2s}ds = \frac{1}{4}e^2 + \frac{1}{4}. \end{aligned}$$

Thus,

$$(2) \iff e < \frac{\Gamma(3)}{2(\ln e - \ln 1)^2}\left[{}_HJ^2_{1+}f(e) \ + \ {}_HJ^2_{e-}f(1)\right] = \frac{1}{2}e^2 - \frac{1}{2} < e^2.$$

3 Hermite-Hadamard type inequalities for Hadamard fractional integrals

We first establish the following important lemma.

Lemma 3.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b) + {}_HJ^\alpha_{b-}f(a)] \\ &= \frac{\ln b - \ln a}{2}\int_0^1 [(1-t)^\alpha - t^\alpha]e^{t\ln a + (1-t)\ln b}f'(e^{t\ln a+(1-t)\ln b})dt. \qquad (6) \end{aligned}$$

Proof. Denote

$$\begin{aligned} I &= \int_0^1 [(1-t)^\alpha - t^\alpha]e^{\ln b - t(\ln b - \ln a)}f'(e^{\ln b - t(\ln b - \ln a)})dt \qquad (7) \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} e^{\ln b - t(\ln b - \ln a)} f'(e^{\ln b - t(\ln b - \ln a)})dt, \\ I_2 &= -\int_0^1 t^{\alpha} e^{\ln b - t(\ln b - \ln a)} f'(e^{\ln b - t(\ln b - \ln a)})dt. \end{aligned}$$

Integrating the term I_1 with t over $[0,1]$,

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} e^{\ln b - t(\ln b - \ln a)} f'(e^{\ln b - t(\ln b - \ln a)})dt \qquad (8) \\ &= (1-t)^{\alpha} \frac{f(e^{\ln b - t(\ln b - \ln a)})}{\ln a - \ln b}\Big|_0^1 - \frac{\alpha}{\ln b - \ln a}\int_0^1 (1-t)^{\alpha-1} f(e^{\ln b - t(\ln b - \ln a)})dt \\ &= \frac{f(b)}{\ln b - \ln a} + \frac{\alpha}{(\ln a - \ln b)^2}\int_b^a \left(\frac{\ln u - \ln a}{\ln b - \ln a}\right)^{\alpha-1} f(u)\frac{du}{u} \\ &= \frac{f(b)}{\ln b - \ln a} - \frac{\alpha}{(\ln b - \ln a)^{\alpha+1}}\int_a^b (\ln u - \ln a)^{\alpha-1} f(u)\frac{du}{u} \\ &= \frac{f(b)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}\ {}_HJ^{\alpha}_{b-}f(a). \end{aligned}$$

Similarly, we get

$$\begin{aligned} I_2 &= -\int_0^1 t^{\alpha} e^{\ln b - t(\ln b - \ln a)} f'(e^{\ln b - t(\ln b - \ln a)})dt \qquad (9) \\ &= t^{\alpha} \frac{f(e^{\ln b - t(\ln b - \ln a)})}{\ln b - \ln a}\Big|_0^1 - \frac{\alpha}{\ln b - \ln a}\int_0^1 t^{\alpha-1} f(e^{\ln b - t(\ln b - \ln a)})dt \\ &= \frac{f(a)}{\ln b - \ln a} - \frac{\alpha}{(\ln b - \ln a)^{\alpha+1}}\int_a^b (\ln b - \ln u)^{\alpha-1} f(u)\frac{du}{u} \\ &= \frac{f(a)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}\ {}_HJ^{\alpha}_{a+}f(b). \end{aligned}$$

Submitting (8) and (9) into (7), it follows that

$$I = \frac{f(a)+f(b)}{\ln b - \ln a} - \frac{\Gamma(\alpha+1)}{(\ln b - \ln a)^{\alpha+1}}[\ {}_HJ^{\alpha}_{a+}f(b) + \ {}_HJ^{\alpha}_{b-}f(a)]. \qquad (10)$$

Thus, by multiplying both sides by $\frac{\ln b - \ln a}{2}$ in (10), we have conclusion (6) immediately. ■

Example 3.2. *Let $a = 1$, $b = e$, $\alpha = 2$, $f(x) = x^2$. Then all the assumptions in Lemma 3.1 are satisfied. Clearly,*

$$\begin{aligned}
\text{the left-sided term of (6)} &\iff \frac{1+e^2}{2} - \frac{1}{2}e^2 + \frac{1}{2} \;=\; 1, \\
\text{the right-sided term of (6)} &\iff \int_0^1 (1-2t)e^{2(1-t)}dt \\
&= \int_0^1 e^{2(1-t)}dt - 2\int_0^1 te^{2(1-t)}dt \\
&= \frac{1}{2}(e^2-1) - 2\left(\frac{1}{4}e^2 - \frac{3}{4}\right) \\
&= 1.
\end{aligned}$$

Using the above lemma, we can obtain the following explicit estimate for some $\alpha \in (0,1]$.

Theorem 3.3. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $0 < a < b$. If $\alpha \in (0,1]$, $f' \in L[a,b]$ and is nondecreasing, then the following equality for fractional integrals holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b) +{}_H J^\alpha_{b-}f(a)]\right| \\
\leq\; & \frac{b(\ln b - \ln a)}{2}\left[\frac{\alpha+2}{\alpha+1}\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)}\right]|f'(b)|. \qquad (11)
\end{aligned}$$

Proof. Using Lemma 3.1 and the nondecreasing property of f', we find

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b) +{}_H J^\alpha_{b-}f(a)]\right| \\
\leq\; & \frac{\ln b - \ln a}{2}\int_0^1 |(1-t)^\alpha - t^\alpha|\, e^{\ln b - t(\ln b - \ln a)}|f'(b)|dt \\
=\; & \frac{b(\ln b - \ln a)|f'(b)|}{2}\int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]e^{-t(\ln b - \ln a)}dt \\
& + \frac{b(\ln b - \ln a)|f'(b)|}{2}\int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]e^{-t(\ln b - \ln a)}dt \\
=\; & \frac{b(\ln b - \ln a)|f'(b)|}{2}(K_1 + K_2), \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]e^{-t(\ln b - \ln a)}dt, \\
K_2 &= \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]e^{-t(\ln b - \ln a)}dt.
\end{aligned}$$

Calculating K_1 we have

$$
\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] e^{-t(\ln b - \ln a)} dt \\
&\leq \int_0^{\frac{1}{2}} (1-2t)^\alpha e^{-t(\ln b - \ln a)} dt \\
&= \frac{1}{2} \int_0^1 (1-s)^{(\alpha+1)-1} e^{-\frac{\ln b - \ln a}{2} s} ds \\
&\leq \max\{1, 2^{-\alpha}\} \left(1 + \frac{1}{\alpha+1}\right) \left(\frac{\ln b - \ln a}{2}\right)^{-1} \\
&\leq \frac{\alpha+2}{\alpha+1} \left(\frac{\ln b - \ln a}{2}\right)^{-1}, \qquad (13)
\end{aligned}
$$

where Lemma 1.6 and Lemma 1.7 is used.

Calculating K_2 we have

$$
\begin{aligned}
K_2 &= \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] e^{-t(\ln b - \ln a)} dt \\
&\leq \int_{\frac{1}{2}}^1 (2t-1)^\alpha e^{-t(\ln b - \ln a)} dt \\
&= \frac{1}{2} \int_1^2 (s-1)^\alpha e^{-\frac{\ln b - \ln a}{2} s} ds \\
&= \frac{1}{2} e^{-(\ln b - \ln a)} \int_0^1 (1-\tau)^\alpha e^{\frac{\ln b - \ln a}{2} \tau} d\tau \\
&\leq \frac{1}{2} e^{-\frac{\ln b - \ln a}{2}} \int_0^1 (1-\tau)^\alpha d\tau \\
&= \frac{\sqrt{\frac{a}{b}}}{2(\alpha+1)}, \qquad (14)
\end{aligned}
$$

where Lemma 1.6 is used.

Thus if we use (13) and (14) in (16), we obtain the inequality of (11). This completes the proof. ■

In general, for $\alpha \in R^+$, one can obtain the following result.

Theorem 3.4. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $0 < a < b$. If $f' \in L[a, b]$ and is nondecreasing, then the following equality for fractional integrals holds*

$$
\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J^\alpha_{a+} f(b) + {}_H J^\alpha_{b-} f(a)] \right| \\
\leq\; & \frac{b(\ln b - \ln a)}{\alpha+1} \left(1 - \frac{1}{2^\alpha}\right) |f'(b)|. \qquad (15)
\end{aligned}
$$

Proof. Using Lemma 3.1 and the nondecreasing property of f', one can obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[{}_HJ^{\alpha}_{a+}f(b)+{}_HJ^{\alpha}_{b-}f(a)]\right|\\
\leq\ &\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\,e^{\ln b-t(\ln b-\ln a)}|f'(b)|dt\\
\leq\ &\frac{b(\ln b-\ln a)|f'(b)|}{2}\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]dt\\
&+\frac{b(\ln b-\ln a)|f'(b)|}{2}\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]dt\\
=\ &\frac{b(\ln b-\ln a)|f'(b)|}{2}\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\\
=\ &\frac{b(\ln b-\ln a)}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)|f'(b)|, \qquad (16)
\end{aligned}$$

where $e^{\ln b-t(\ln b-\ln a)}\leq e^{\ln b}=b$ is used. The proof is completed. ■

Remark 3.5. *Theorem 3.3 and Theorem 3.4 give an upper bound for the approximation of the integral average* $\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[{}_HJ^{\alpha}_{a+}f(b)+{}_HJ^{\alpha}_{b-}f(a)]$. *There exist some integral functions that can not be expressed by elementary functions. So Theorem 3.3 and Theorem 3.4 are useful to deal with such integral functions. For example, set* $a=1$, $b=e$, $f(x)=x^2$, $\alpha=\frac{1}{2}$, *then the left-sided hand of (16) reduces to*

$$\begin{aligned}
&\left|\frac{1+e^2}{2}-\frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}}\left(\int_0^1e^{2(1-s^2)}ds+\int_0^1e^{2s^2}ds\right)\right|\\
=\ &\left|\frac{1+e^2}{2}-\frac{1}{2}\left(\frac{e^2}{\sqrt{2}}\int_0^{\sqrt{2}}e^{-s^2}ds+\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}}e^{s^2}ds\right)\right|. \qquad (17)
\end{aligned}$$

It is obvious that the term $\int_0^{\sqrt{2}}e^{-s^2}ds$ *can not solved directly due to* $\int e^{-s^2}ds$ *can not be expressed by elementary functions. But, applying Theorem 3.3 we can give an upper bound* $\frac{10}{3}e^2+\frac{\sqrt{e^3}}{3}$ *for (17).*

4 Applications to some special means

Consider the following special means (see Pearce and Pečarić [27]) for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$ as follows:

$$\begin{aligned}
H(\alpha,\beta) &= \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\ \alpha,\ \beta\in R\setminus\{0\},\\
A(\alpha,\beta) &= \frac{\alpha+\beta}{2},\ \alpha,\beta\in R,\\
L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\ |\alpha|\neq|\beta|,\ \alpha\beta\neq 0,\\
L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\ n\in Z\setminus\{-1,0\},\ \alpha,\beta\in R,\ \alpha\neq\beta.
\end{aligned}$$

Now, using the theory results in Section 3, we give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in R^+$, $a < b$. Then*

$$|A(a,b) - L(a,b)| \leq \frac{b(\ln b - \ln a)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right], \tag{18}$$

and

$$|A(a,b) - L(a,b)| \leq \frac{b(\ln b - \ln a)}{4}. \tag{19}$$

Proof. Applying Theorem 3.3 and Theorem 3.4 respectively, for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. ■

Proposition 4.2. *Let $a, b \in R^+$, $a < b$ and $n \in Z$, $|n| \geq 2$. Then*

$$\left|A(a^{n+1}, b^{n+1}) - \frac{b-a}{\ln b - \ln a} L_n^n(a,b)\right| \leq \frac{(n+1)b^{n+1}(\ln b - \ln a)}{8}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right], \tag{20}$$

and

$$\left|A(a^{n+1}, b^{n+1}) - \frac{b-a}{\ln b - \ln a} L_n^n(a,b)\right| \leq \frac{(n+1)b^{n+1}(\ln b - \ln a)}{4}. \tag{21}$$

Proof. Applying Theorem 3.3 and Theorem 3.4 respectively, for $f(x) = x^{n+1}$ and $\alpha = 1$, $x \in R$, $n \in Z$, $|n| \geq 2$, one can obtain the results immediately. ■

Proposition 4.3. *Let $a, b \in R^+ (a < b)$, $a^{-1} > b^{-1}$. For $n \in Z$, $|n| \geq 2$, we have*
(i)

$$|H^{-1}(b,a) - L(b^{-1}, a^{-1})| \leq \frac{(\ln b - \ln a)}{8a}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right],$$

(ii)

$$|H^{-1}(b,a) - L(b^{-1}, a^{-1})| \leq \frac{(\ln b - \ln a)}{4a},$$

(iii)

$$\left|H^{-1}(a^{n+1}, b^{n+1}) - \frac{a^{-1} - b^{-1}}{\ln b - \ln a} L_n^n(b^{-1}, a^{-1})\right| \leq \frac{(n+1)(\ln b - \ln a)}{8a^{n+1}}\left[6\left(\frac{\ln b - \ln a}{2}\right)^{-1} + \sqrt{\frac{a}{b}}\right],$$

(iv)

$$\left|H^{-1}(a^{n+1}, b^{n+1}) - \frac{a^{-1} - b^{-1}}{\ln b - \ln a} L_n^n(b^{-1}, a^{-1})\right| \leq \frac{(n+1)(\ln b - \ln a)}{4a^{n+1}}.$$

Proof. Making the substitutions $a \to b^{-1}, b \to a^{-1}$ in the inequalities (18)–(21), one can obtain desired inequalities respectively. ■

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