

Research Article

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Refinements of quantum Hermite-Hadamard-type inequalities

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Abstract: In this paper, we first obtain two new quantum Hermite-Hadamard-type inequalities for newly defined quantum integral. Then we establish several refinements of quantum Hermite-Hadamard inequalities.

Keywords: Hermite-Hadamard inequality, q -integral, quantum calculus, convex function

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1 Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [1], [2, p. 137]) is one of the most well-established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f: I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $\omega_1, \omega_2 \in I$ with $\omega_1 < \omega_2$, then

$$f\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} f(x) dx \leq \frac{f(\omega_1) + f(\omega_2)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave. We note that the Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years, and a remarkable variety of refinements and generalizations have been studied.

On the other hand, quantum calculus, sometimes called calculus without limits, is equivalent to traditional infinitesimal calculus without the notion of limits. In the field of q -analysis, many studies have recently been carried out. It has applications in numerous areas of mathematics such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and in fields of other sciences such as mechanics, theory of relativity, and quantum theory [3–7]. Apparently, Euler invented this important branch of mathematics when he used the q parameter in Newton's work on infinite series. Later, the q -calculus was first given by Jackson [3]. In 1908–1909, the general form of the q -integral and q -difference operator was defined by Jackson [6]. In 1969, for the first time Agarwal [8] defined the q -fractional derivative. In 1966–1967, Al-Salam [9] introduced a q -analog of the q -fractional integral and

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q -Riemann-Liouville fractional. In 2004, Rajkovic gave a definition of the Riemann-type q -integral, which was generalized to the Jackson q -integral. In 2013, Tariboon introduced the ${}_{\omega_1}D_q$ -difference operator [10].

In recent years, because of the importance of convexity in numerous fields of applied and pure mathematics, it has been significantly investigated. The theory of convexity and inequalities are strongly connected to each other, therefore, various inequalities can be established in the literature which are proved for convex, generalized convex, and differentiable convex functions of single and double variables, see, for example, [10–29].

The general structure of this paper consists of five main sections including Introduction. In Section 2, we give some necessary important notations for concept q -calculus and we also mention some related works in the literature. In Section 3 we present some new Hermite-Hadamard-type inequalities for q^{ω_2} integrals. Some refinements of quantum Hermite-Hadamard-type inequalities are proved in Section 4. We also examine the relation between our results and inequalities presented in the earlier works. Finally, in Section 5, some conclusions and further directions of research are discussed. We note that the opinion and technique of this work may inspire new research in this area.

2 Preliminaries of q -calculus and some inequalities

In this section, we present some required definitions and related inequalities about q -calculus.

We have to give the following notation which will be used many times in the following sections (see [7]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Definition 1. [30] For a function $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$, the q_{ω_1} -derivative of F at $\kappa \in [\omega_1, \omega_2]$ is characterized by the expression:

$${}_{\omega_1}D_q F(\kappa) = \frac{F(\kappa) - F(q\kappa + (1-q)\omega_1)}{(1-q)(\kappa - \omega_1)}, \quad \kappa \neq \omega_1. \quad (2)$$

If $\kappa = \omega_1$, we define ${}_{\omega_1}D_q F(\omega_1) = \lim_{\kappa \rightarrow \omega_1} {}_{\omega_1}D_q F(\kappa)$ if it exists and it is finite.

Definition 2. [18] For a function $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$, the q^{ω_2} -derivative of F at $\kappa \in [\omega_1, \omega_2]$ is characterized by the expression:

$${}^{\omega_2}D_q F(\kappa) = \frac{F(q\kappa + (1-q)\omega_2) - F(\kappa)}{(1-q)(\omega_2 - \kappa)}, \quad \kappa \neq \omega_2.$$

If $\kappa = \omega_2$, we define ${}^{\omega_2}D_q F(\omega_2) = \lim_{\kappa \rightarrow \omega_2} {}^{\omega_2}D_q F(\kappa)$ if it exists and it is finite.

Definition 3. [30] Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a function. Then, the q_{ω_1} -definite integral on $[\omega_1, \omega_2]$ is defined as

$$\int_{\omega_1}^{\omega_2} F(\kappa) {}_{\omega_1}d_q \kappa = (1-q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n F(q^n \omega_2 + (1-q^n)\omega_1) = (\omega_2 - \omega_1) \int_0^1 F((1-t)\omega_1 + t\omega_2) d_q t.$$

In [10], Alp et al. proved the following q_{ω_1} -Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 1. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex differentiable function on $[\omega_1, \omega_2]$ and $0 < q < 1$. Then q -Hermite-Hadamard inequalities are as follows:

$$F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\kappa) {}_{\omega_1}d_q \kappa \leq \frac{qF(\omega_1) + F(\omega_2)}{[2]_q}. \quad (3)$$

On the other hand, Bermudo et al. gave the following new definition and were able to prove the corresponding Hermite-Hadamard-type inequalities:

Definition 4. [18] Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a function. Then, the q^{ω_2} -definite integral on $[\omega_1, \omega_2]$ is defined as

$$\int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x = (1-q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n F(q^n \omega_1 + (1-q^n)\omega_2) = (\omega_2 - \omega_1) \int_0^1 F(t\omega_1 + (1-t)\omega_2) d_q t.$$

Theorem 2. [31] We have the following identities for q^{ω_2} -integrals:

- (i) ${}^{\omega_2}D_q \left(\int_x^{\omega_2} F(t)^{\omega_2} d_q t \right) = -F(x).$
- (ii) $\int_x^{\omega_2} {}^{\omega_2}D_q F(t)^{\omega_2} d_q t = F(\omega_2) - F(x).$
- (iii) $\int_x^{\omega_2} [F(t) + G(t)]^{\omega_2} d_q t = \int_x^{\omega_2} F(t)^{\omega_2} d_q t + \int_x^{\omega_2} G(t)^{\omega_2} d_q t.$
- (iv) $\int_{\omega_1}^{\omega_2} G(x)^{\omega_2} {}^{\omega_2}D_q F(x)^{\omega_2} d_q x = [F(x)G(x)]_{\omega_1}^{\omega_2} - \int_{\omega_1}^{\omega_2} F(qx + (1-q)\omega_2)^{\omega_2} {}^{\omega_2}D_q G(x)^{\omega_2} d_q x.$

Theorem 3. [18] Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex function on $[\omega_1, \omega_2]$ and $0 < q < 1$. Then, q -Hermite-Hadamard inequalities are as follows:

$$F\left(\frac{\omega_1 + q\omega_2}{[2]_q}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x \leq \frac{F(\omega_1) + qF(\omega_2)}{[2]_q}. \quad (4)$$

3 New Hermite-Hadamard-type inequalities for q^{ω_2} -integrals

In this section, we prove two new quantum Hermite-Hadamard inequalities for q^{ω_2} -integrals.

Theorem 4. If $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ is a convex differentiable function on $[\omega_1, \omega_2]$ and $0 < q < 1$, then we have

$$F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) - \frac{(1-q)(\omega_2 - \omega_1)}{[2]_q} F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x \leq \frac{F(\omega_1) + qF(\omega_2)}{[2]_q}. \quad (5)$$

Proof. We can write the equation of the tangent line for the function F at the point $\left(\frac{q\omega_1 + \omega_2}{[2]_q}, F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right)\right)$ as follows:

$$h(x) = F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) + F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \left(x - \frac{q\omega_1 + \omega_2}{[2]_q}\right)$$

for all $x \in [\omega_1, \omega_2]$. Since F is a convex function, it is clear that $h(x) \leq F(x)$ for all $x \in [\omega_1, \omega_2]$. Thus, we have

$$\int_{\omega_1}^{\omega_2} h(x)^{\omega_2} d_q x \leq \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x.$$

By Definition 4, we get

$$\begin{aligned}
 \int_{\omega_1}^{\omega_2} h(\chi) {}^{\omega_2}d_q\chi &= \int_{\omega_1}^{\omega_2} \left[F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) + F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \left(\chi - \frac{q\omega_1 + \omega_2}{[2]_q}\right) \right] {}^{\omega_2}d_q\chi \\
 &= (\omega_2 - \omega_1) F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) + F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \int_{\omega_1}^{\omega_2} \left(\chi - \frac{q\omega_1 + \omega_2}{[2]_q}\right) {}^{\omega_2}d_q\chi \\
 &= (\omega_2 - \omega_1) F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \\
 &\quad + F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \left[(1-q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n (q^n \omega_1 + (1-q^n)\omega_2) - (\omega_2 - \omega_1) \frac{q\omega_1 + \omega_2}{[2]_q} \right] \\
 &= (\omega_2 - \omega_1) F\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) - \frac{(1-q)(\omega_2 - \omega_1)^2}{[2]_q} F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right),
 \end{aligned}$$

which gives the first inequality in (5). The second inequality is the same as in Theorem 3. \square

Remark 1. If we take the limit $q \rightarrow 1^-$ in Theorem 4, then the inequalities (5) reduce to (1).

Theorem 5. If $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ is a convex differentiable function on $[\omega_1, \omega_2]$ and $0 < q < 1$, then we have

$$F\left(\frac{\omega_1 + \omega_2}{2}\right) - \frac{(1-q)(\omega_2 - \omega_1)}{2[2]_q} F'\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi) {}^{\omega_2}d_q\chi \leq \frac{F(\omega_1) + qF(\omega_2)}{[2]_q}. \quad (6)$$

Proof. Similar way as in Theorem 4, we can write tangent line for the function F at the point $\left(\frac{\omega_1 + \omega_2}{2}, F\left(\frac{\omega_1 + \omega_2}{2}\right)\right)$ as follows:

$$k(\chi) = F\left(\frac{\omega_1 + \omega_2}{2}\right) + F'\left(\frac{\omega_1 + \omega_2}{2}\right) \left(\chi - \frac{\omega_1 + \omega_2}{2}\right)$$

for all $\chi \in [\omega_1, \omega_2]$. Since F is a convex function, we get

$$\int_{\omega_1}^{\omega_2} k(\chi) {}^{\omega_2}d_q\chi \leq \int_{\omega_1}^{\omega_2} F(\chi) {}^{\omega_2}d_q\chi.$$

By Definition 4, we get

$$\begin{aligned}
 \int_{\omega_1}^{\omega_2} k(\chi) {}^{\omega_2}d_q\chi &= \int_{\omega_1}^{\omega_2} \left[F\left(\frac{\omega_1 + \omega_2}{2}\right) + F'\left(\frac{\omega_1 + \omega_2}{2}\right) \left(\chi - \frac{\omega_1 + \omega_2}{2}\right) \right] {}^{\omega_2}d_q\chi \\
 &= (\omega_2 - \omega_1) F\left(\frac{\omega_1 + \omega_2}{2}\right) + F'\left(\frac{\omega_1 + \omega_2}{2}\right) \int_{\omega_1}^{\omega_2} \left(\chi - \frac{\omega_1 + \omega_2}{2}\right) {}^{\omega_2}d_q\chi \\
 &= (\omega_2 - \omega_1) F\left(\frac{\omega_1 + \omega_2}{2}\right) \\
 &\quad + F'\left(\frac{\omega_1 + \omega_2}{2}\right) \left[(1-q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n (q^n \omega_1 + (1-q^n)\omega_2) - (\omega_2 - \omega_1) \frac{\omega_1 + \omega_2}{2} \right] \\
 &= (\omega_2 - \omega_1) F\left(\frac{\omega_1 + \omega_2}{2}\right) - \frac{(1-q)(\omega_2 - \omega_1)^2}{2[2]_q} F'\left(\frac{\omega_1 + \omega_2}{2}\right).
 \end{aligned}$$

This gives the first inequality in (6). The second inequality is the same as in Theorem 3. \square

Remark 2. If we take the limit $q \rightarrow 1^-$ in Theorem 5, then the inequalities (6) reduce to (1).

Lemma 1. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex continuous function on $[\omega_1, \omega_2]$ and $0 < q < 1$. Then we have

$$F\left(\frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} (t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y.$$

Proof. Lemma 1 follows directly from Definition 3 and quantum Jensen's inequality. \square

4 Main results

In this section, we present the refinements of quantum Hermite-Hadamard inequalities for q^{ω_2} -integrals.

Theorem 6. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex continuous function on $[\omega_1, \omega_2]$ and $0 < q < 1$. Then we have

$$F\left(\frac{\omega_1 + q\omega_2}{[2]_q}\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi \leq \frac{F(\omega_1) + qF(\omega_2)}{[2]_q} \quad (7)$$

for all $t \in [0, 1]$.

Proof. Since F is convex on $[\omega_1, \omega_2]$, it follows that

$$F(t\chi + (1-t)y) \leq tF(\chi) + (1-t)F(y) \quad (8)$$

for all $\chi, y \in [\omega_1, \omega_2]$ and $t \in [0, 1]$. Taking double q^{ω_2} -integration on both sides of (8) on $[\omega_1, \omega_2] \times [\omega_1, \omega_2]$, we obtain

$$\begin{aligned} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y &\leq \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} [tF(\chi) + (1-t)F(y)]^{\omega_2} d_q \chi^{\omega_2} d_q y \\ &= t \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi^{\omega_2} d_q y + (1-t) \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(y)^{\omega_2} d_q \chi^{\omega_2} d_q y \\ &= t(\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi + (1-t)(\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(y)^{\omega_2} d_q y \\ &= (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi, \end{aligned}$$

which proves the second part of (7). On the other hand, by Lemma 1, we have

$$F\left(\frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} (t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y \quad (9)$$

and we also get

$$\frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} (t\chi + (1-t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y = \frac{\omega_1 + q\omega_2}{[2]_q}. \quad (10)$$

By (9) and (10), we have the first part of (7). \square

Remark 3. If we take the limit $q \rightarrow 1^-$ in Theorem 6, then the inequalities (7) reduce to the following inequalities:

$$F\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) d\kappa dy \leq \frac{1}{(\omega_2 - \omega_1)} \int_{\omega_1}^{\omega_2} F(\kappa) d\kappa \leq \frac{F(\omega_1) + F(\omega_2)}{2},$$

which are given by Dragomir in [32].

Corollary 1. Under assumptions of Theorem 6 with $t = \frac{1}{2}$, we have the inequalities

$$F\left(\frac{q\omega_2 + \omega_1}{[2]_q}\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{\kappa + y}{2}\right) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\kappa) {}^{\omega_2}d_q\kappa \leq \frac{qF(\omega_2) + F(\omega_1)}{[2]_q}. \quad (11)$$

Remark 4. If we take the limit $q \rightarrow 1^-$ in Corollary 1, then (11) reduces to the inequalities

$$F\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{\kappa + y}{2}\right) d\kappa dy \leq \frac{1}{(\omega_2 - \omega_1)} \int_{\omega_1}^{\omega_2} F(\kappa) d\kappa \leq \frac{F(\omega_1) + F(\omega_2)}{2},$$

which are given in [33].

Theorem 7. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex continuous function on $[\omega_1, \omega_2]$ and $0 < q < 1$. Then we have

$$\begin{aligned} \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{q\kappa + y}{[2]_q}\right) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy &\leq \frac{1}{(\omega_2 - \omega_1)^2} \int_0^1 \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy {}^{\omega_2}d_qt \\ &\leq \frac{1}{(\omega_2 - \omega_1)} \int_{\omega_1}^{\omega_2} F(\kappa) {}^{\omega_2}d_q\kappa. \end{aligned} \quad (12)$$

Proof. Let us consider the mapping $\phi : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\phi(t) = \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy.$$

For all $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, by convexity of F , we have

$$\begin{aligned} \phi(\alpha t_1 + \beta t_2) &= \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F((\alpha t_1 + \beta t_2)\kappa + (1 - (\alpha t_1 + \beta t_2))y) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy \\ &= \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(\alpha(t_1\kappa + (1-t_1)y) + \beta(t_2\kappa + (1-t_2)y)) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy \\ &\leq \frac{\alpha}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t_1\kappa + (1-t_1)y) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy \\ &\quad + \frac{\beta}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t_2\kappa + (1-t_2)y) {}^{\omega_2}d_q\kappa {}^{\omega_2}d_qy = \alpha\phi(t_1) + \beta\phi(t_2), \end{aligned}$$

which shows that ϕ is convex on $[0, 1]$. By applying Theorem 3 for the convex function ϕ on $[0, 1]$, we have the inequalities

$$\phi\left(\frac{q}{[2]_q}\right) \leq \int_0^1 \phi(t) {}^1d_qt \leq \frac{\phi(0) + q\phi(1)}{[2]_q}.$$

That is,

$$\begin{aligned} \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{q\kappa + y}{[2]_q}\right) \omega_2 d_q \kappa \omega_2 d_q y &\leq \frac{1}{(\omega_2 - \omega_1)^2} \int_0^1 \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) \omega_2 d_q \kappa \omega_2 d_q y^1 d_q t \\ &\leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\kappa) \omega_2 d_q \kappa. \end{aligned}$$

This completes the proof. \square

Remark 5. If we take the limit $q \rightarrow 1^-$ in Theorem 7, then the inequalities (12) reduce to the inequalities

$$\begin{aligned} \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{\kappa + y}{2}\right) d\kappa dy &\leq \frac{1}{(\omega_2 - \omega_1)^2} \int_0^1 \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) d\kappa dy dt \\ &\leq \frac{1}{(\omega_2 - \omega_1)} \int_{\omega_1}^{\omega_2} F(\kappa) d\kappa, \end{aligned}$$

which are given in [33].

Theorem 8. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a q -differentiable convex continuous function and $0 < q < 1$. Then the inequalities

$$\begin{aligned} 0 &\leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\kappa) \omega_2 d_q \kappa - \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) \omega_2 d_q \kappa \omega_2 d_q y \\ &\leq t \left[\frac{qF(\omega_1) + F(\omega_2)}{[2]_q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(q\kappa + (1-q)\omega_2) \omega_2 d_q \kappa \right] \end{aligned} \quad (13)$$

are valid for all $t \in [0, 1]$.

Proof. Since F is convex on $[\omega_1, \omega_2]$, it follows that

$$F(t\kappa + (1-t)y) \leq tF(\kappa) + (1-t)F(y) \quad (14)$$

for all $\kappa, y \in [\omega_1, \omega_2]$ and $t \in [0, 1]$. Taking double q^{ω_2} -integration on both sides of (14) on $[\omega_1, \omega_2] \times [\omega_1, \omega_2]$, we obtain

$$\begin{aligned} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) \omega_2 d_q \kappa \omega_2 d_q y &\leq \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} [tF(\kappa) + (1-t)F(y)] \omega_2 d_q \kappa \omega_2 d_q y \\ &= (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\kappa) \omega_2 d_q \kappa, \end{aligned}$$

which gives the first part of (13). On the other hand, since F is q -differentiable convex on $[\omega_1, \omega_2]$ and $F' \geq {}^{\omega_2}D_q F$, we have

$$F(t\kappa + (1-t)y) - F(y) \geq t(\kappa - y) {}^{\omega_2}D_q F(y)$$

for all $\kappa, y \in [\omega_1, \omega_2]$ and $t \in [0, 1]$. Taking double q^{ω_2} -integration on both sides of the above inequality on $[\omega_1, \omega_2] \times [\omega_1, \omega_2]$, we obtain

$$\int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\kappa + (1-t)y) \omega_2 d_q \kappa \omega_2 d_q y - (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\kappa) \omega_2 d_q \kappa \geq t \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} (\kappa - y) {}^{\omega_2}D_q F(y) \omega_2 d_q \kappa \omega_2 d_q y. \quad (15)$$

By (iv) of Theorem 2, we have

$$\begin{aligned}
 & \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} (\chi - y) F^{\omega_2} D_q F(y)^{\omega_2} d_q \chi^{\omega_2} d_q y \\
 &= (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} \left(\frac{a + qb}{[2]_q} - y \right)^{\omega_2} D_q F(y)^{\omega_2} d_q y \\
 &= (\omega_2 - \omega_1) \left[\left(\frac{a + qb}{[2]_q} - y \right) F(y) \right]_{\omega_1}^{\omega_2} - \int_{\omega_1}^{\omega_2} F(qy + (1 - q)b)^{\omega_2} D_q \left(\frac{a + qb}{1 + q} - y \right)^{\omega_2} d_q y \\
 &= (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi - (\omega_2 - \omega_1)^2 \frac{qF(\omega_1) + F(\omega_2)}{[2]_q}.
 \end{aligned}$$

It follows from (15) that

$$\begin{aligned}
 & (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi - \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1 - t)y)^{\omega_2} d_q \chi^{\omega_2} d_q y \\
 & \leq t \left[(\omega_2 - \omega_1)^2 \frac{qF(\omega_1) + F(\omega_2)}{[2]_q} - (\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} F(q\chi + (q - 1)\omega_2)^{\omega_2} d_q \chi \right],
 \end{aligned}$$

for all $t \in [0, 1]$, which is the second part of (13). \square

Remark 6. If we take the limit $q \rightarrow 1^-$ in Theorem 8, then the inequalities (13) reduce to the inequalities

$$\begin{aligned}
 0 & \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi) d\chi - \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F(t\chi + (1 - t)y) dx dy \\
 & \leq t \left[\frac{F(\omega_1) + F(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi) d\chi \right],
 \end{aligned}$$

which can be seen in [33,34].

Corollary 2. Under assumptions of Theorem 8 with $t = \frac{1}{2}$, we have the inequalities

$$\begin{aligned}
 0 & \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi)^{\omega_2} d_q \chi - \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{\chi + y}{2}\right)^{\omega_2} d_q \chi^{\omega_2} d_q y \\
 & \leq \frac{1}{2} \left[\frac{qF(\omega_1) + F(\omega_2)}{[2]_q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(q\chi + (1 - q)\omega_2)^{\omega_2} d_q \chi \right].
 \end{aligned} \tag{16}$$

Remark 7. If we take the limit $q \rightarrow 1^-$ in Corollary 2, then the inequalities (16) reduce to the inequalities

$$0 \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi) d\chi - \frac{1}{(\omega_2 - \omega_1)^2} \int_{\omega_1}^{\omega_2} \int_{\omega_1}^{\omega_2} F\left(\frac{\chi + y}{2}\right) dx dy \leq \frac{1}{2} \left[\frac{F(\omega_1) + F(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(\chi) d\chi \right],$$

which are given in [33].

Theorem 9. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a q -differentiable convex continuous function which is defined at the point $\frac{\omega_1 + q\omega_2}{[2]_q} \in (\omega_1, \omega_2)$ and $0 < q < 1$. Then the following inequalities hold:

$$\begin{aligned} 0 &\leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F\left(tx + (1-t)\frac{q\omega_2 + \omega_1}{[2]_q}\right)^{\omega_2} d_q x \\ &\leq (1-t) \left[\frac{qF(\omega_1) + F(\omega_2)}{[2]_q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(qx + (1-q)\omega_2)^{\omega_2} d_q x \right]. \end{aligned} \quad (17)$$

Proof. Since F is convex on $[\omega_1, \omega_2]$, by Theorem 3, it follows that

$$\begin{aligned} \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x &= \frac{t}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x + \frac{1-t}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x \\ &\geq \frac{t}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x + (1-t)F\left(\frac{\omega_1 + q\omega_2}{[2]_q}\right) \\ &\geq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F\left(tx + (1-t)\frac{\omega_1 + q\omega_2}{[2]_q}\right)^{\omega_2} d_q x \end{aligned}$$

for all $t \in [0, 1]$, which becomes the first part of (17). On the other hand, since F is q -differentiable convex on $[\omega_1, \omega_2]$, we have

$$F\left(tx + (1-t)\frac{\omega_1 + q\omega_2}{[2]_q}\right) - F(x) \geq (1-t)\left(\frac{\omega_1 + q\omega_2}{[2]_q} - x\right)^{\omega_2} D_q F(x).$$

Taking the q^{ω_2} -integration on the above inequality on $[\omega_1, \omega_2]$, we obtain

$$\begin{aligned} \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F\left(tx + (1-t)\frac{\omega_1 + q\omega_2}{[2]_q}\right)^{\omega_2} d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x \\ \geq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} (1-t)\left(\frac{\omega_1 + q\omega_2}{[2]_q} - x\right)^{\omega_2} D_q F(x)^{\omega_2} d_q x. \end{aligned} \quad (18)$$

We also have

$$\int_{\omega_1}^{\omega_2} \left(\frac{\omega_1 + q\omega_2}{[2]_q} - x\right)^{\omega_2} D_q F(x)^{\omega_2} d_q x = \int_{\omega_1}^{\omega_2} F(qx + (1-q)\omega_2)^{\omega_2} d_q x - (\omega_2 - \omega_1) \frac{qF(\omega_1) + F(\omega_2)}{[2]_q}. \quad (19)$$

From (18) and (19), we get the second part of (17). \square

Theorem 10. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a q -differentiable convex continuous function which is defined at the point $\frac{q\omega_1 + \omega_2}{[2]_q} \in (\omega_1, \omega_2)$ and $0 < q < 1$. Then the inequalities

$$\begin{aligned} (1-t) \frac{(1-q)(\omega_2 - \omega_1)}{[2]_q} F'\left(\frac{q\omega_1 + \omega_2}{[2]_q}\right) \\ \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x)^{\omega_2} d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F\left(tx + (1-t)\frac{q\omega_1 + \omega_2}{[2]_q}\right)^{\omega_2} d_q x \\ \leq (1-t) \left[\frac{qF(\omega_2) + F(\omega_1)}{[2]_q} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(qx + (1-q)\omega_2)^{\omega_2} d_q x \right] \end{aligned} \quad (20)$$

are valid for all $t \in [0, 1]$.

Proof. The proof of this theorem follows a similar procedure to that in Theorem 9 by using Theorem 4. \square

Theorem 11. Let $F : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a q -differentiable convex continuous function which is defined at the point $\frac{\omega_1 + \omega_2}{2} \in (\omega_1, \omega_2)$ and $0 < q < 1$. Then the inequalities

$$\begin{aligned} & (1-t) \frac{(1-q)(\omega_2 - \omega_1)}{2([2]_q)} F' \left(\frac{\omega_1 + \omega_2}{2} \right) \\ & \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x) \omega_2 d_q x - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F \left(t x + (1-t) \frac{\omega_1 + \omega_2}{2} \right) \omega_2 d_q x \\ & \leq (1-t) \left[\frac{F(\omega_1) + F(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(qx + (1-q)\omega_2) \omega_2 d_q x \right] \end{aligned} \quad (21)$$

are valid for all $t \in [0, 1]$.

Proof. The proof of this theorem follows a similar procedure to that in Theorem 9 by using Theorem 5. \square

Remark 8. If we take the limit $q \rightarrow 1^-$, then the inequalities (17), (20) and (21) reduce to

$$\begin{aligned} 0 & \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x) dx - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F \left(t x + (1-t) \frac{\omega_1 + \omega_2}{2} \right) dx \\ & \leq (1-t) \left[\frac{F(\omega_1) + F(\omega_2)}{2} - \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} F(x) dx \right], \end{aligned}$$

which are given in [33].

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