Reflected wave atypical phase change at a boundary.

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ABSTRACT

According to Fresnel formulae, at normal incidence on an abrupt interface, the reflected wave has a phase difference of zero or π, if the second medium has a lower or larger refractive index than the first. However, what happens if the refractive indices of two media are the same at the interface but the derivative of the refractive index varies abruptly? Since the two media are not homogeneous because the refractive index derivative is finite, the problem cannot be tackled with the Fresnel formalism. In order to deal with this problem the amplitude and phase representation of plane electromagnetic waves is used. An invariant is obtained that permits the decoupling of the amplitude and phase equations, both of which, are nonlinear. The amplitude equation is then solved numerically. No approximations are made regarding how slow or fast refractive index varies compared to the wavelength. Interpretation of the amplitude equation solutions reveal that surfaces where any of the derivatives of the refractive index profile is discontinuous, do enhance reflection. At normal incidence, the reflected wave thus generated will have a phase difference that may be a multiple of $\frac{\pi}{2}$, apparently contradicting the Fresnel equations.

Keywords: electromagnetic propagation, stratified media, reflectivity, discontinuity, rugate filters, atmospheric disturbances, optical coherence tomography

1. INTRODUCTION

1.1 Amplitude equation

Important works trying to describe propagation of electromagnetic waves through stratified media have already been published. Somehow these works end up using approximate expressions, most of them being for either a slowly or strongly varying refractive index, compared to the wavelength.[1–4] This approximations allow analytical expressions to be written for the fields and the reflectivity. Note that this early works were published before commercial computers were widely used by scientists and regular population, numerical analysis was not as easily performed as it is today. In the present work, no approximations of this kind are intended, just those inherent to the use of numerical methods. For that purpose a differential equation for the electric field amplitude will be convenient.[5,6] It is a nonlinear ordinary differential equation of the Ermakov–Milne–Pinney type.

Starting with Maxwell equations, assuming then an isotropic, transparent, dielectric medium with a linear response and no free charges, but letting the electric permittivity and magnetic permeability vary, yields the following second order equation for the electric field:

$$\nabla\left(\mathbf{E}\cdot\nabla\ln\varepsilon\right)+\nabla\ln\mu\times\left(\nabla\times\mathbf{E}\right)=\mu\varepsilon\frac{\partial^{\mathbf{2}}\mathbf{E}}{\partial t^{\mathbf{2}}}-\nabla^{\mathbf{2}}\mathbf{E} \tag{1}$$

Where ϵ and μ are the electric permittivity and magnetic permeability respectively. Let now z be the direction of stratification, meaning thatϵ and μ will depend only on z. Consider also a monochromatic time behavior, being ω its frequency, and assume a TE polarization, then equation 1 leads to:[7]

$$\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}+\omega^2\epsilon\mu E_x=\frac{\mathrm{d}(\ln\mu)}{\mathrm{d}z}\frac{\partial E_x}{\partial z} \tag{2}$$

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Where we choose x as the direction of the electric field. Particularly for normal incidence and non magnetic media, this is $\mu = \mu_0$, the former equation may be written in a very simple way:

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 n^2 E_x = 0 \tag{3}$$

Where $k_0^2 = \omega^2 \mu_0 \epsilon_0$ and $n = \sqrt{\frac{\epsilon}{\epsilon_0}}$, being ϵ_0 the electric permittivity of vacuum and μ_0 its magnetic permeability. Equation 3 looks like a one dimensional Helmholtz equation except that the refractive index is not necessarily a constant, $n = n(z)$. Now, consider a complex E_x, namely $E_x = Ae^{iq}$, where the amplitude A and phase q may depend on z. We are interested in non absorbing media, so we will assume the refractive index to be a real quantity. Substitution in equation 3 and later separation of real and imaginary parts render:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} - A\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 = -k_0^2 n^2 A \tag{4}$$

$$2\left(\frac{\mathrm{d}A}{\mathrm{d}z}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right) + A\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 0 \tag{5}$$

Equation 5 can be readily integrated to obtain an invariant quantity given by:

$$Q = A^2 \frac{\mathrm{d}q}{\mathrm{d}z} \tag{6}$$

A nonlinear ordinary differential equation for the amplitude is obtained upon substitution of this result in equation 7:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} - \frac{Q^2}{A^3} = -k_0^2 n^2 A \tag{7}$$

The former is an Ermakov–Milne–Pinney type equation. In order to work with a dimensionless amplitude function let us introduce$A_d = A\sqrt{\frac{\mathrm{k}_0}{Q}}$, then equation 10 can be rewritten:

$$\frac{1}{k_0^2}\frac{\mathrm{d}^2 A_d}{\mathrm{d}z^2} - \frac{1}{A_d^3} = -n^2 A_d \tag{8}$$

This is the ordinary differential equation for the electric field amplitude. It seems folly to turn a linear differential equation into a non-linear ODE, but equation 8 poses no challenge to be solved numerically if A is real and n is bounded. Also, initial conditions are easily imposed having a clear physical meaning and finally, interpretation of the solutions is straight forward.

1.2 Interpretation of the solutions

A constant n is related to an homogeneous medium, in that case the solutions of equation 8 are known and must be of the following form:[5,6,8]

$$A_d = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos\left(2k_0 nz + \beta\right)} \tag{9}$$

Where A_d is the field amplitude produced by the superposition of two counter propagating waves, with individual constant amplitudes $A_1$and A_2. Also β is the phase difference between both waves at $z = 0$. There is a restriction for this amplitudes $A_1$and A_2 if equation 8 is to be satisfied:

$$\left(A_1^2 - A_2^2\right)^2 = \frac{1}{n^2} \tag{10}$$

This homogeneous media solution $A_d(z)$ oscillates periodically if neither, A_1nor A_2, is zero. Maxima $A_{\max}$ and minima $A_{\min}$ occur when the incoming and outgoing waves are in or out of phase respectively. These extrema can also be related to the ratio $r = \frac{A_2}{A_1}$:[5,6]

$$r = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \tag{11}$$

The local extrema of the oscillations show where the incident and reflected waves are in or out of phase. Wherever there is a minimum, the electric vector fields of these waves point in opposite directions; if there is a maximum, it means the fields point in the same direction. Let Op be the optical path $Op = \int_0^z n\,dz$. If maxima occur at $Op = \frac{1}{4} \pm \frac{m}{2}$ and minima at $Op = \pm\frac{m}{2}$, for $m = 0, 1, 2, 3...$, the electric field waves must have phase difference of $\delta = \pi$ between them at $z = 0$, the discontinuity site. If maxima and minima positions are interchanged, waves must be in phase at $z = 0$. For the former interpretation to be valid, the refractive index $n(z)$ must be sufficiently constant along the interval where the local extrema are found. If there is only one wave propagating, A_1or A_2 is zero and $A_d$is constant, particularly:

$$A_d^2 = \frac{1}{n} \tag{12}$$

To model a single interface, $n(z)$ must be a monotonic continuous function that evolves from n_1 to n_2. Far from the interface n should be practically constant. To evaluate the reflectivity, the convenient initial condition is a single transmitted wave through the second medium, so the incident light is assumed to come only from the first medium side. This will mean that the solution in the second medium, far from the interface, is almost constant $A_d = \sqrt{\frac{1}{n_2}}$. Under this condition, the oscillations in the first medium, far from the interface too, will reveal the reflectivity: $R = r^2$. Given the indices n_1 and n_2, we expect that reflectivity will depend on the interface abruptness. The former property can be characterized by the distance D through which the index varies from $n_1 + \frac{1}{20}\Delta n$ to $n_2 - \frac{1}{20}\Delta n$, so that parameter D can be thought as the interface thickness, corresponding to 90 % of the index change.

In earlier papers[5,6] studies of the interface reflectivity, given different profiles with varying thicknesses, have been done. All profiles were continuous, but some were piecewise defined and their derivatives were discontinuous. For "hard" interfaces, meaning $D \approx \frac{\lambda}{100}$, reflectivity was close to the Fresnel result $R = \left(\frac{n_2-n_1}{n_2+n_1}\right)^2$, regardless of the profile type. For "softer" interfaces, $D \approx \frac{\lambda}{2}$, the reflectivity of almost all $n(z)$ profiles fell to less than 6 % of the former result. When D extended to about a wavelength and more, the reflectivity for analytic profiles continued to drop monotonically, but for the piecewise defined it oscillated. It is important to say that every piecewise defined profile had two attachment points $z_1$and z_2 in order to keep a certain interface symmetry. Oscillations in these cases resembled quite well thin film interference, being its thickness $z_2 - z_1$. Evidently discontinuities in the profile derivatives were causing reflection but it was not clear, only with the interference data, which type of phase change was the reflected wave undergoing at the boundaries $z_1$and z_2. We could only infer the relative change between the two reflections. Our task now is to evaluate such phase change, based on interpretations of the Amplitude equation solutions.

label	$n(z)$	$n(z)$
step+	$\begin{cases} n_a & z < 0 \\ n_g & z \geq 0 \end{cases}$	increasing
step-	$\begin{cases} n_g & z < 0 \\ n_a & z \geq 0 \end{cases}$	decreasing

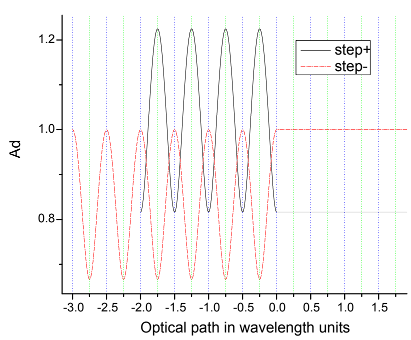

Figure 1. Two discontinuous functions and their corresponding amplitude equation solutions.

label	$n(z)$	$\frac{dn}{dz}$ at the junction
lin&tanh+	$\begin{cases} n_a & z < 0 \\ n_a + (n_g - n_a)\tanh\left(\frac{a_1}{D}z\right) & z \geq 0 \end{cases}$	increasing
tanh&lin+	$\begin{cases} n_g + (n_g - n_a)\tanh\left(\frac{a_1}{D}z\right) & z < 0 \\ n_g & z \geq 0 \end{cases}$	decreasing

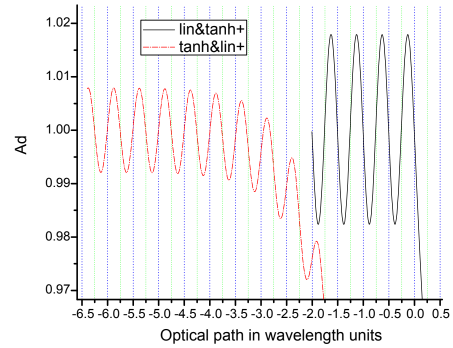

Figure 2. Two continuous functions with discontinuous first derivatives and their corresponding amplitude equation solutions.

label	$n(z)$	$\frac{d^2n}{dz^2}$ at the junction
lin&sech+	$\begin{cases} n_a & z < 0 \\ n_g - (n_g - n_a)\mathrm{sech}^2\left(\frac{a_2}{D}z\right) & z \geq 0 \end{cases}$	increasing
sech&lin-	$\begin{cases} n_g - (n_g - n_a)\mathrm{sech}^2\left(\frac{a_2}{D}z\right) & z < 0 \\ n_a & z \geq 0 \end{cases}$	decreasing

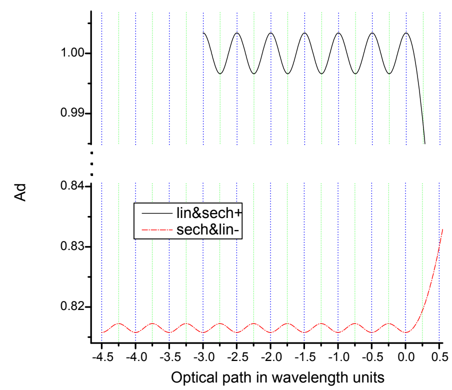

Figure 3. Two continuous functions with discontinuous second derivatives and their corresponding amplitude equation solutions.

label	$n(z)$	$\frac{d^3n}{dz^3}$ at the junction
lin&cubexp+	$\begin{cases} n_a & z < 0 \\ n_g - (n_g - n_a)\exp\left(-\frac{a_3}{D}z\right)^3 & z \geq 0 \end{cases}$	increasing
cubexp&lin+	$\begin{cases} n_a + (n_g - n_a)\exp\left(\frac{a_3}{D}z\right)^3 & z < 0 \\ n_g & z \geq 0 \end{cases}$	decreasing

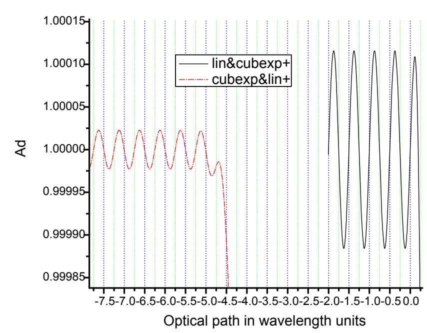

Figure 4. Two continuous functions with discontinuous third derivatives and their corresponding amplitude equation solutions.

2. REFRACTIVE INDEX PROFILES

In order to see how the reflectivity of a single junction profile behaves, piecewise refractive index profiles can be built with only one attachment point. To rule out reflections from the rest of the interface, the D parameter must be grater than one. The junction, always at $z = 0$, can exhibit different types of discontinuities, the function $n(z)$ may be discontinuous itself, or be continuous but not its first derivative and/or the rest. The former figures show different $n(z)$ functions and their corresponding amplitude equation 8solution plotted against the optical path in wavelength units. Figure 1 shows two $n(z)$ functions with a discontinuity at $z = 0$, let $n_a = 1$ and $n_g = 1.5$. Figure 2 shows another couple of profiles, this time $n(z)$ is continuous for both but $\frac{dn}{dz}$ is not, in one case the first derivative increases at the junction and in the other case it decreases. The functions $n(z)$ in figure

3 are continuous as well as $\frac{dn}{dz}$, but $\frac{d^2n}{dz^2}$ is discontinuous at the junction. Profiles $n(z)$ at figure 4 are continuous as well as $\frac{dn}{dz}$ and $\frac{d^2n}{dz^2}$, but not $\frac{d^3n}{dz^3}$.

3. PHASE RELATION BETWEEN INCIDENT AND REFLECTED WAVES

The amplitude equation solution for "step+", with an increasing $n(z)$ at the junction, has its maxima around $Op = -\frac{1}{4} - \frac{m}{2}$ and minima at $Op = -\frac{m}{2}$, the electric field waves must have a phase difference of $\delta = \pi$ between them at the junction. For "step-", with a decreasing $n(z)$ at the junction, maxima and minima positions are interchanged, waves must be in phase at $z = 0$. The former results are also reached by the well known Fresnel formulae.

The amplitude equation solution for "lin&tanh+", with an increasing $\frac{dn}{dz}$ at the junction, has its maxima around $Op = -\frac{1}{8} - \frac{m}{2}$ and minima at $Op = -\frac{3}{8} - \frac{m}{2}$, the electric field waves must have a phase difference of $\delta = \frac{\pi}{2}$ between them at the junction. For "tanh&lin+", with a decreasing $\frac{dn}{dz}$ at the junction, maxima and minima positions are interchanged, the electric field waves must have a phase difference of $\delta = \frac{3\pi}{2}$ between them at the junction. The former results can not be reached by the Fresnel formalism and are very surprising. The following results can not be reached by Fresnel formalism either and certainly are surprising too.

The amplitude equation solution for "lin&sech+", with an increasing $\frac{d^2n}{dz^2}$ at the junction, has its maxima around $Op = -\frac{m}{2}$ and minima at $Op = \frac{1}{4} - \frac{m}{2}$, waves must be in phase at $z = 0$. For "lin&sech-", with a decreasing $\frac{d^2n}{dz^2}$ at the junction, maxima and minima positions are interchanged, the electric field waves must have a phase difference of $\delta = \pi$ between them at the junction.

The amplitude equation solution for "lin&cubexp+", with an increasing $\frac{d^3n}{dz^3}$ at the junction, has its maxima around $Op = -\frac{3}{8} - \frac{m}{2}$ and minima at $Op = -\frac{1}{8} - \frac{m}{2}$, the electric field waves must have a phase difference of $\delta = \frac{3\pi}{2}$ between them at the junction. For "cubexp&lin+", with a decreasing $\frac{d^3n}{dz^3}$ at the junction, maxima and minima positions are interchanged, the electric field waves must have a phase difference of $\delta = \frac{\pi}{2}$ between them at the junction.

4. CONCLUSIONS

The selected profiles and their amplitude equation solutions show a certain order, related to the phase difference between the incident and reflected waves at the junction. Table 1 displays a generalization of that behavior. Other profiles have been tested and they all hold to the rule. Reflectivity of the interface, computed with equation 11, diminishes about an order of magnitude as the lowest order of the discontinuous derivatives escalates. That is why the relevant discontinuity is the one related to the lowest order discontinuous derivative. The former results may be applied to rugate filter design, interpretation of Doppler radar measurements in the clear air atmosphere, optical coherence tomography and other issues related with wave propagation in stratified media.

Table 1. conclusions.

Lowest order discontinuous derivative	Phase difference δ between incident and reflected waves at the junction	
	for increasing lowest order discontinuous derivative	for decreasing lowest order discontinuous derivative
$n(z)$ is discontinuous	π	0
$\frac{dn}{dz}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$
$\frac{d^2n}{dz^2}$	0	π
$\frac{d^3n}{dz^3}$	$\frac{3\pi}{2}$	$\frac{\pi}{2}$

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