*Geophys. J. R. astr. Soc.* (1973) **32**, 459–478.

# Reflection and Transmission of Rayleigh Waves in a Wedge—II

**K. Viswanathan* and Arabinda Roy†**

(Received 1972 July 3)

*Summary*

The problem of the reflection and transmission of Rayleigh waves in an elastic wedge discussed in an earlier paper for the case of incidence from infinity is now studied in more detail and for the more general case of an incidence from a finite distance from the corner. A detailed application is made of the effect of the critical regions of Rayleigh waves in Lamb's half-space problem when a series of interactions with the wedge faces are considered. These interactions are two-fold, viz. those due to the inwardly progressing waves and those due to the outwardly receding waves. Both lead to contributions given by certain integral equations. While in the latter case the integral equations behave like the Fredholm equations, in the case of the former the behaviour is like Volterra equations of second kind at lower range of wedge angles and like the Fredholm equations at higher range and there is a mixed character in the intermediate values. These approximations lead to dividing the range of the wedge angle, which we take to be from 0° to 180°, into five parts at points depending on the critical angles of Lamb's problem. The solutions in these parts are piece-wise continuous. A brief outline of the corner wave effects is also included. The numerical results show that the present theory can explain well some of the important experimental features of the problem that were only partially achieved by previous theories.

## 1. Introduction

In a previous part of this work Viswanathan, Kuo & Lapwood (1971) showed that the problem of the reflection and transmission of Rayleigh waves in an elastic wedge is significantly influenced by the actual regions in which these waves can exist in the more fundamental problem of a half-space with a source usually known as the Lamb's problem.

In the above work which we refer to as Part I henceforth, we treated the case when the incident field was from infinity. Moreover, the effects of the critical regions for the existence of Rayleigh waves defined in the context of the Lamb's problem were only partially incorporated while dealing with the interactions with the wedge boundaries. In particular such effects were not applied to the waves that travel towards the corner.

The purpose of the present work is to study the more general case when the source of the initial Rayleigh field lies at a finite distance $l$ from the corner. Further, we

*Present address: A-242, G-Type Flats, Kidwai Nagar East, New Delhi–23, India.

†Present address: Department of Applied Mathematics, 92 Acharyaa Prafulla Chandra Road, Calcutta-53, India.

attempt to incorporate the effects of the critical regions referred to above also to all the incoming and outgoing waves. It is shown that this leads to dividing the range of the wedge angle $\theta$ into five parts between 0° and 180° at points depending on the critical angles defined later. The approximations for the reflection and transmission coefficients in these regions will be distinct but piece-wise continuous. These arise from certain integral equations behaving like the Volterra or Fredholm equations as determined by the wedge angle chosen.

Another aspect which we briefly discuss is the effect of the corner. A workable method is indicated for modifying certain routine steps for minimizing the cost of ignoring the corner altogether in the conventional approach.

## 2. The problem

Fig. 1 shows an elastic wedge of angle $\theta \leqslant 180°$ bounded by the stress-free planes $S_1$ and $S_2$. A Rayleigh wave issues from a source at the point $P_0$ on $S_1$ towards the corner 0. It is required to find the reflected and transmitted Rayleigh waves produced on $S_1$ and $S_2$ respectively.

The situation that we consider here is one of two-dimensional plane strain. Further, we deal with time-harmonic solutions and define two elastic potentials $\phi$ and $\psi$ in the usual form

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}, \quad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1} \tag{1}$$

where $(u_1, u_3)$ are the displacements in the $(x_1, x_3)$ system so that $\phi$ and $\psi$ are solutions of the Helmholtz equations

$$\left.\begin{aligned}(\nabla^2+k_\alpha{}^2)\phi &= 0\\ (\nabla^2+k_\beta{}^2)\psi &= 0\end{aligned}\right\}, \quad k_\alpha = \frac{\omega}{\alpha}, \quad k_\beta = \frac{\omega}{\beta} \tag{2}$$

where $\alpha$, $\beta$ are the velocities of the compressional and shear waves. The incident field is now assumed in the form

$$\begin{aligned}\phi \equiv \phi_0 &= \exp\left(-i\omega t + i\omega\frac{(x_1+l)}{\gamma} - \omega\eta_\alpha x_3\right)\\ \psi \equiv \psi_0 &= (\gamma/2i\eta_\beta)\left(\eta_\beta{}^2+\frac{1}{\gamma^2}\right)\exp\left(-i\omega t + i\omega\frac{(x_1+l)}{\gamma} - \omega\eta_\beta x_3\right)\end{aligned} \tag{3}$$

$$(x_1 > -l, \; x_3 \geqslant 0)$$

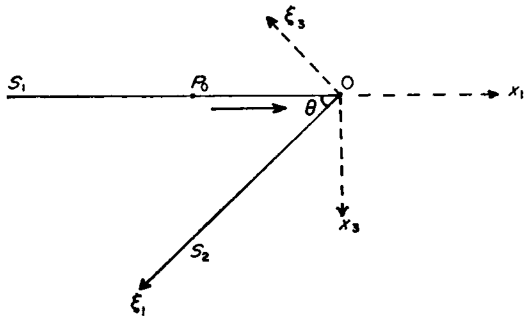


FIG. 1. The geometry of the problem ($0P_0 = l$).

where $\gamma$ denotes the velocity of Rayleigh waves, and

$$\eta_\alpha = \left(\frac{1}{\gamma^2} - \frac{1}{\alpha^2}\right)^{\frac{1}{2}}, \eta_\beta = \left(\frac{1}{\gamma^2} - \frac{1}{\beta^2}\right)^{\frac{1}{2}}.$$

Since the incident field arises from a source at a *finite* distance from the corner, we must invoke the critical regions associated with these pair of potentials. Fig. 2 shows these regions in terms of the critical angles $\theta_\alpha = \cos^{-1}(\gamma/\alpha)$ and $\theta_\beta = \cos^{-1}(\gamma/\beta)$ from which we notice that the functions $\phi_0$ and $\psi_0$ in (3) must be multiplied by the factors

$$\begin{aligned} &H(x_1+l-x_3\cot\theta_\alpha) \quad \text{for} \quad \phi_0; \\ &H(x_1+l-x_3\cot\theta_\beta) \quad \text{for} \quad \psi_0; \end{aligned} \tag{4}$$

where $H(x)$ denotes the step function being equal to zero or unity according as $x \lessgtr 0$. For the limiting case of incidence from infinity, $l \to \infty$, these factors become unity and hence no longer require special mention.

Now we turn to the main problem and let $A_R(l)$ and $A_T(l)$ denote the reflection and transmission coefficients of the Rayleigh waves generated at large distances from the corner on $S_1$ and $S_2$. For convenience we write them in the form

$$\begin{aligned} A_R(l) &= A_R^{(1)}(l)+A_R^{(2)}(l)+A_R^{(3)}(l) \\ A_T(l) &= A_T^{(1)}(l)+A_T^{(2)}(l)+A_T^{(3)}(l) \end{aligned} \tag{5}$$

where the leading terms denote the contributions from the interaction of the incoming waves with the wedge, the middle terms those from outgoing waves on reintersecting the wedge, and the last terms those from the corner effects. These are next discussed.

## 3. The integral equations for $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$

Let us suppose that a first-order approximation be worked out similar to that of Mal & Knopoff (1966) in which we define suitable virtual sources on $S_2$ which give a transmitted Rayleigh wave on $S_2$ of amount $T(\theta, l)$. Also let each virtual source's contribution to the excitation of an additional incoming Rayleigh wave, now on $S_2$, be $\sigma(\xi_1', l)$ where $(\xi_1', 0)$ denotes the virtual source on $S_2$. Further, let the first-order approximation to the reflected Rayleigh wave on $S_1$ be $R_1(\theta, l)$. This is again given by Mal & Knopoff, and will be discussed shortly.

Since the Rayleigh wave on $S_2$ excited towards the corner by any virtual source will further be reflected and transmitted just as the field from $P_0$, it is readily possible

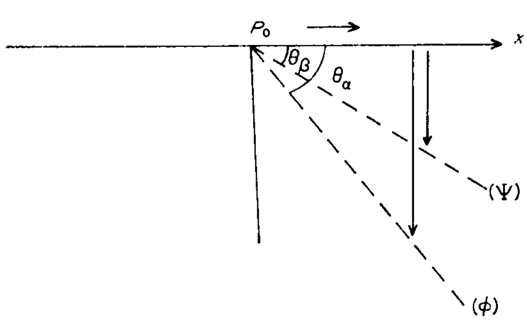


FIG. 2. The critical regions for the Rayleigh potentials in the wave travelling to the right. Vertical arrows show the depths of penetration.

to generalize the above first-order approximation to yield the higher-order formulae for $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$:

$$\begin{aligned} A_R^{(1)}(l) &= R_1(\theta, l) + \int_0^\infty \sigma(\xi_1', l)\, A_T^{(1)}(\xi_1')\, d\xi_1' \\ A_T^{(1)}(l) &= T(\theta, l) + \int_0^\infty \sigma(\xi_1', l)\, A_R^{(1)}(\xi_1')\, d\xi_1'. \end{aligned} \tag{6}$$

These integral equations are easily decoupled by mutual substitution.

Now we outline how exactly the first-order approximations are obtained conventionally.

First we note that the incident field (3) gives rise to the following stresses at any point $(\xi_1', 0)$ on the boundary $S_2$:

$$\left.\begin{aligned} \sigma_{13}^s(\xi_1', 0) &= -\mu\omega^2\{C_1 e^{-i\omega a_0 \xi_1'} H(\xi_\phi - \xi_1') - D_1 e^{-i\omega b_0 \xi_1'} H(\xi_\psi - \xi_1')\} \\ \sigma_{33}^s(\xi_1', 0) &= \mu\omega^2\{C_3 e^{-i\omega a_0 \xi_1'} H(\xi_\phi - \xi_1') - D_3 e^{-i\omega b_0 \xi_1'} H(\xi_\psi - \xi_1')\} \end{aligned}\right\}. \tag{7}$$

Here we have used the appropriate components in the $(\xi_1, \xi_3)$ system of co-ordinates and also omitted the factor $e^{-i\omega(t - l/\gamma)}$ to save it from being repeatedly reproduced. Other symbols have been defined in the Appendix. In deriving (7) we have used the actual portions of $S_2$ which will lie within the regions of the incident field shown in Fig. 2 and thus we have

$$\left.\begin{aligned} &\xi_\phi = k_1 l, \xi_\psi = k_2 l;\ (0 \leqslant \theta \leqslant 180^\circ - \theta_\alpha) \\ &\xi_\phi = \infty, \xi_\psi = k_2 l\ ;\ (180^\circ - \theta_\alpha \leqslant \theta \leqslant 180^\circ - \theta_\beta) \\ &\xi_\phi = \xi_\psi = \infty\qquad ;\ (180^\circ - \theta_\beta \leqslant \theta \leqslant 180^\circ) \end{aligned}\right\} \tag{8}$$

where

$$\begin{aligned} k_1 &= \sin\theta_\alpha \operatorname{cosec}(\theta + \theta_\alpha) \\ k_2 &= \sin\theta_\beta \operatorname{cosec}(\theta + \theta_\beta). \end{aligned} \tag{9}$$

Since $S_2$ is a stress-free boundary, we can imagine every point of it to act as a virtual source with a stress across $S_2$ being equal to the negative of (7). Then as a first approximation, we ignore the presence of $S_1$, and solve a half-space problem for $\mathscr{H}_2 : \xi_3 \geqslant 0, -\infty < \xi_1 < \infty$ in which on $\xi_3 = 0$ for $\xi_1 > 0$ these virtual sources are present while for the remaining part $\xi_1 < 0$ of the boundary we assume a stress-free condition to hold good. This last piece of assumption is the most common to come across in literature and we shall return to its discussion later in Section 5.

Now the solution to the first-order problem for $\mathscr{H}_2$ defined above is straightforward being also the formula (11) of Mal & Knopoff (1966), when we perform the integration over the virtual sources. Using their equation (13) the corresponding parts of the Rayleigh waves that are generated in the two directions $\xi_1 \to \pm\infty$ are readily obtained. Thus it is not difficult to show that the functions $T(\theta, l)$ and $\sigma(\xi_1', l)$ are given by

$$T(\theta, l) = \int_0^\infty \sigma'(\xi_1', l)\, d\xi_1', \tag{10}$$

$$\sigma'(\xi_1', l) = \frac{\omega e^{-i\omega\xi_1'/\gamma}}{F'(1/\delta)} \{C e^{-i\omega a_0 \xi_1'} H(\xi_\phi - \xi_1') + D e^{-i\omega b_0 \xi_1'} H(\xi_\psi - \xi_1')\}, \tag{11}$$

$$\sigma(\xi_1', l) = \frac{\omega\, e^{i\omega\xi_1'/\gamma}}{F'(1/\gamma)} \{A\, e^{-i\omega a_0 \xi_1'} H(\xi_\phi - \xi_1') + B\, e^{-i\omega b_0 \xi_1'} H(\xi_\psi - \xi_1')\}, \quad (12)$$

where the various symbols have been defined in the Appendix. $\sigma(\xi_1', l)$ and $\sigma'(\xi_1', l)$ denote the amounts of incoming and outgoing Rayleigh wave on $S_2$ due to one virtual source.

The procedure for obtaining $R_1(\theta, l)$ is slightly different. Here we start with the elastodynamic representation theorem

$$u_k(\mathbf{x}) = \int_S \{G_{kl}(\mathbf{x}', \mathbf{x})\, \tau_{ij}(u) - u_i(\mathbf{x}')\, \tau_{ij}(\mathbf{G}_k)\}\, n_j\, dS(\mathbf{x}') \quad (13)$$

where $S$ is a closed contour enclosing the field point $\mathbf{x}$, $n_j$ is the outward unit normal to $S$ at $\mathbf{x}'$, and $\mathbf{G}_k(\mathbf{x}'\,\mathbf{x}')$ is the Green's function, which, for a fixed $k$, gives the displacement vector due to a point source at $\mathbf{x}'$ where a force is prescribed in the $k$-th direction. To calculate $R_1(\theta, l)$, the contour $S$ is taken as the sum of $S_1$, $S_2$ and two perpendiculars $S_3$ and $S_4$ to these at large distances from the corner. Also the Green's function is chosen to satisfy stress-free conditions on $S_1$. Then the integral over $S_1$ will vanish identically. Similar is the case of the integral over $S_4$ the perpendicular to $S_2$. The integral over $S_3$, the perpendicular to $S_1$, when subtracted from the left-hand side will balance the incident field part there. Thus the remainder, viz. the reflected Rayleigh wave on $S_1$ will be given by the integral over $S_2$ of which the first term on the right of (13) is zero due to our boundary conditions and the remaining term involves only the unknown displacement of $S_2$. Therefore, in deriving $R_1(\theta, l)$ we deal with certain displacement-sources within the half-space

$$\mathscr{H}_1 : x_3 \geqslant 0, \; -\infty < x_1 < \infty$$

being distributed over the position of $S_2$. For the first approximation, the displacement of $S_2$ is taken to be the sum of those in the primary field (3) and its reflected (body) waves calculated on the plane-wave reflection theory. These steps are summarised by Mal & Knopoff (1966) which give the following expression for $R_1(\theta, l)$ in our case:

$$R_1(\theta, l) = \int_0^\infty \sigma^*(\xi_1', l)\, d\xi_1' \quad (14)$$

where

$$\sigma^*(\xi_1', l) = \omega H(\xi_\phi - \xi_1')\{E_1\, e^{-2i\omega a_0 \xi_1'} + E_2\, e^{-i\omega(a_0+b_0)\xi_1'}\} + \omega H(\xi_\psi - \xi_1')\{E_3\, e^{-i\omega(a_0+b_0)\xi_1'} + E_4\, e^{-2i\omega b_0 \xi_1'}\}, \quad (15)$$

and other symbols have been defined in the Appendix.

This completes the necessary definitions for the integral equations (6) for $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$. Their solution will be discussed in Section 6.

## 4. The integral equations for $A_R^{(2)}(l)$ and $A_T^{(2)}(l)$

Next consider the terms $A_R^{(2)}(l)$ and $A_T^{(2)}(l)$ which arise from a further interaction between the above outgoing reflected and transmitted waves and the faces of the wedge. This is possible at least for adequately small values of $\theta$. Note that this effect will disappear for $\theta > \theta_\alpha$.

Although the solution of (6) contains a sequence of outgoing contributions on the two faces, we pick out only the leading terms here for constructing $A_R^{(2)}(l)$ and $A_T^{(2)}(l)$. The contributions from the remaining terms can be obtained in the same manner but will be relatively small and hence not evaluated here.

We recall that the function $T(\theta, l)$ defined in (10) is an integral of contributions from several outgoing waves from various virtual sources on $S_2$. And for any virtual source, the outgoing wave is restricted by the corresponding critical lines for $\phi$ and $\psi$. By construction it is quickly seen what parts of $S_1$ come under their interaction. This interaction is again a first-order problem like the previous analysis and we can evaluate the Rayleigh wave part of this half-space problem for $\mathscr{H}_1$. Of these, the waves going along $S_1$ towards $x_1 \to -\infty$ give the first approximation to $A_R^{(2)}(l)$. This is seen to be

$$A_R^{(2)}(l) = \int_0^\infty \sigma'(\xi_1', l)\, d\xi_1' \int_0^\infty \sigma''(\xi_1'', \xi_1')\, d\xi_1'' \tag{16}$$

where $\sigma''(\xi_1', l)$ is the counterpart of $\sigma(\xi_1', l)$ defined in (12) and it represents the amount of outgoing Rayleigh wave produced on one boundary by a virtual source on it caused by a unit outgoing Rayleigh wave from an initial source on the other boundary. In (16) we have considered the synthesis over $\xi_1'$ to get the contribution corresponding to $T(\theta, l)$. For favour of future use we give below the expression for $\sigma''(\xi_1', l)$:

$$\sigma''(\xi_1', l) = -\frac{\omega \exp(-i\omega\xi_1'/\gamma)}{F'(1/\gamma)} \{\bar{A} \exp(i\omega\bar{a}_0 \xi_1') H(\xi_1' - \xi_\phi') + \bar{B} \exp(i\omega\bar{b}_0 \xi_1') H(\xi_1' - \xi_\psi')\} \tag{17}$$

where $\bar{x} = \text{conjg}(x)$,

$$\left.\begin{aligned} &\xi_\phi' = k_1' l,\ \xi_\psi' = k_2' l, && (0 \leqslant \theta < \theta_\beta) \\ &\xi_\phi' = k_1' l,\ \xi_\psi' = \infty && (\theta_\beta \leqslant \theta < \theta_\alpha) \\ &\xi_\phi' = \xi_\psi' = \infty && (\theta_\alpha \leqslant \theta) \end{aligned}\right\} \tag{18}$$

and

$$\left.\begin{aligned} k_1' &= \sin\theta_\alpha \operatorname{cosec}(\theta_\alpha - \theta) \\ k_2' &= \sin\theta_\beta \operatorname{cosec}(\theta_\beta - \theta). \end{aligned}\right\} \tag{19}$$

In order to get a similar result for $A_T^{(2)}(l)$ that arises from $R_1(\theta, l)$ we note from (14) that the sources contributing to $R_1(\theta, l)$ are internal sources of displacements in the half-space problem for $\mathscr{H}_1 : x_3 \geqslant 0,\ -\infty < x, < \infty$. Fig. 3 shows the critical lines

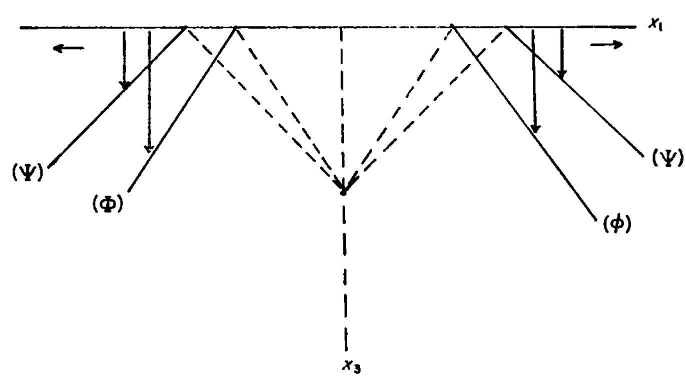


FIG. 3. Critical regions for the Rayleigh waves due to a buried source in a half-space. Only the far sides of the solid lines represent the regions of existence of the Rayleigh waves. These lines are inclined at the critical angles $\theta_\alpha$ and $\theta_\beta$ to the horizontal boundary.

associated with an internal source. The interaction of an outgoing wave on $S_1$, for this geometry, with the second face is again governed by the appropriately modified parts of $S_2$ which can intersect these waves. This involves a modified form for $\sigma''(\xi_1'', \xi_1')$ and the corresponding contribution from $R_1(\theta, l)$ to $A_T^{(2)}(l)$ is given by

$$A_T^{(2)}(l) = \int_0^\infty \sigma^*(\xi_1', l)\, d\xi_1' \int_0^\infty \sigma_I''(\xi_1'', \xi_1')\, d\xi_1'' \tag{20}$$

where we have defined $\sigma_I''(\xi_1', l)$ for internal sources by

$$\sigma_I''(\xi_1', l) = -\frac{\omega \exp(-i\omega\xi_1'/\gamma)}{F'(1/\gamma)} \{\bar{A} \exp(i\omega\bar{a}_0 \xi_1' H(\xi_1' - \xi_\phi'') + \bar{B} \exp(i\omega\bar{b}_0 \xi_1') H(\xi_1' - \xi_\psi'')\} \tag{21}$$

with

$$\left.\begin{aligned} &\xi_\phi'' = k_1'' l,\ \xi_\psi'' = k_2'' l \quad (0 \leqslant \theta < \theta_\beta) \\ &\xi_\phi'' = k_1'' l, \xi_\psi'' = \infty \qquad (\theta_\beta \leqslant \theta < \theta_\alpha) \\ &\xi_\phi'' = \xi_\psi'' = \infty \qquad\quad (\theta_\alpha \leqslant \theta) \end{aligned}\right\} \tag{22}$$

and

$$\left.\begin{aligned} k_1'' &= \sin(\theta + \theta_\alpha) \operatorname{cosec}(\theta_\alpha - \theta) \\ k_2'' &= \sin(\theta + \theta_\beta) \operatorname{cosec}(\theta_\beta - \theta). \end{aligned}\right\} \tag{23}$$

(16) and (20) represent two independent first-order contributions, one for $A_R^{(2)}(l)$ and the other for $A_T^{(2)}(l)$. From each of these there will arise by further sequence of interactions with the wedge-faces an appropriate higher-order contribution for both the functions. From the definition of $\sigma''(\xi_1', l)$ as the counterpart of $\sigma(\xi_1', l)$, the combined generalization becomes

$$\begin{pmatrix} A_R^{(2)}(l) \\ A_T^{(2)}(l) \end{pmatrix} = \int_0^\infty \sigma'(\xi_1', l)\, d\xi_1' \int_0^\infty \sigma''(\xi_1'', \xi_1') \begin{pmatrix} f_R(\xi_1'') \\ f_T(\xi_1'') \end{pmatrix} d\xi_1''$$
$$+ \int_0^\infty \sigma^*(\xi_1', l)\, d\xi_1' \int_0^\infty \sigma_I''(\xi_1'', \xi_1') \begin{pmatrix} f_T(\xi_1'') \\ f_R(\xi_1'') \end{pmatrix} d\xi_1'', \tag{24}$$

where $f_R(l)$ and $f_T(l)$ are determined from the pair of integral equations

$$\left.\begin{aligned} f_R(l) &= 1 + \int_0^\infty \sigma''(\xi_1', l) f_T(\xi_1')\, d\xi_1' \\ f_T(l) &= \int_0^\infty \sigma''(\xi_1', l) f_R(\xi_1')\, d\xi_1' \end{aligned}\right\}. \tag{25}$$

These equations will also be discussed in Section 6.

## 5. Corner-waves and contributions to $A_R^{(3)}(l)$ and $A_T^{(3)}(l)$

It remains to say on the role of the corner waves and to assess how far these contribute to the reflected and transmitted Rayleigh waves.

Obviously we have posed a sequence of auxiliary problems at each stage of our approximation for the two half-spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. In each case $S_1$ or $S_2$ forms one half of the half-space boundary on which we take the stress to be the negative of a contribution from the waves of an earlier approximation but over the remaining half of the boundary we arbitrarily impose a stress-free condition. This conventional approach is capable of modification and one can think of a more suitable choice of conditions on these remaining parts of the boundaries beyond $S_1$ or $S_2$ as the case may be.

Consider for instance the problem for $\mathscr{H}_2$ which gave rise to the first-order transmission coefficient $T(\theta, l)$. There, we assumed that $S_2$ is under stresses being the negative of (7) from the incident field. But over $\xi_3 = 0$, $\xi_1 < 0$ we supposed a stress-free condition. Thus we had introduced a stress-jump in the boundary-values of both the normal and shear stresses at the corner point. Looking at (7) these are seen to be

$$\left.\begin{aligned} [\sigma_{13}]_{0-}^{0+} &= \mu\omega^2 (C_1 - D_1) = \mu\omega^2 (\eta_\alpha^2 - \eta_\beta^2) \sin 2\theta, \quad \text{for shear stress;} \\ [\sigma_{33}]_{0-}^{0+} &= \mu\omega^2 (-C_3 + D_3) \qquad\qquad\qquad , \quad \text{for normal stress.} \\ &= \mu\omega^2 (\eta_\alpha^2 - \eta_\beta^2)(1 - \cos 2\theta) \end{aligned}\right\} \quad (26)$$

The factors depending on the wedge angle $\theta$ in these jumps are shown in Fig. 4. We note that the strongest discontinuity occurs in the normal stress at $\theta = 90°$ and moreover the combined jump values indicate that the whole range from $\theta = 45°$ to 135° may be governed by rather strong corner waves. It is well known that the corner waves will consist of expanding circular fronts which will be made up of pieces of arcs with oppositely oriented stress jumps* that will render the solution ultimately smooth for points removed into the medium from the corner. (The conical fronts connecting the circular fronts are not usually discontinuous anywhere, that is, along their length.)

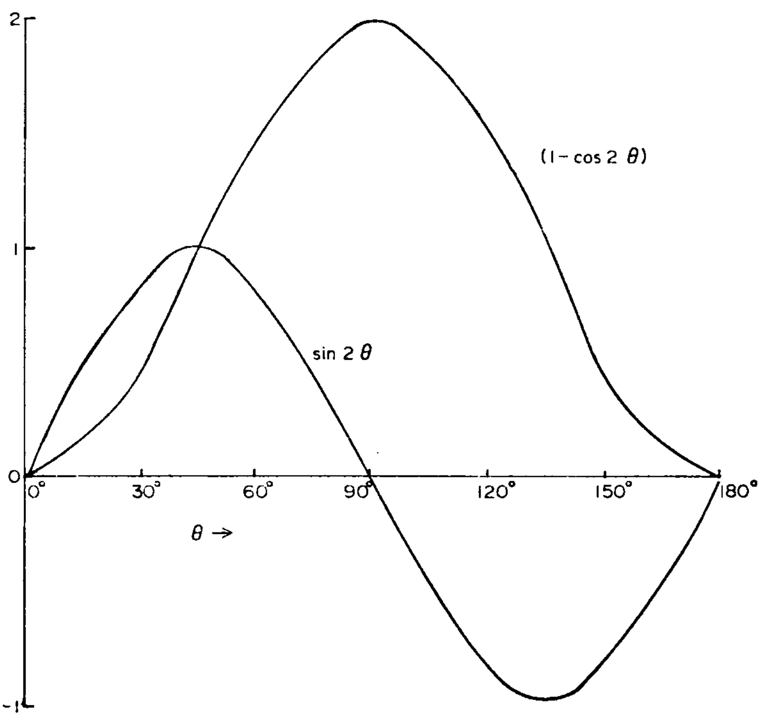


Fig 4. The stress-jump factors in the equation (26).

* At their joints.

The best method of tackling such situations in these auxiliary problems perhaps lies in making appropriate modifications in the choice of the boundary conditions beyond the $S_1$ or $S_2$ parts, which also incidentally possess certain corner effects but considerably minimized. One such choice suggests that at least for wedges of angle $\theta \approx 90°$, the region beyond $S_1$ or $S_2$ be considered as loaded by similar stresses as on the parts $S_1$ or $S_2$ concerned. In fact if $S_1$ or $S_2$ were uniformly loaded, this assumption will remove the corner waves altogether. However, the stresses in (7) are non-uniform and hence we do expect only a reasonable reduction in intensity. Then by symmetry, the additional contribution due to this modification over the already obtained approximation $T(\theta, l)$ for $A_T^{(1)}(l)$ will be the integral

$$Q(\theta, l) = \int_0^\infty \sigma(\xi_1', l)\, d\xi_1' \tag{27}$$

where $\sigma(\xi_1', l)$ was already defined in (12) as the inward going wave when stress-free condition was imposed beyond $S_2$.

Thus a first approximation to $A_T^{(3)}(l)$ may be taken as

$$A_T^{(3)}(l) = Q(\theta, l), \tag{28}$$

apparently good for $\theta \in (45°, 135°)$. There is no such first-order approximation for $A_R^{(3)}(l)$ and hence we take $A_R^{(3)}(l) = 0$.

Higher approximations are not attempted here. Although these can be formulated on the above lines, we do not emphasize that this is an adequate procedure for the corner effects. In fact the principal interest of the present work lies in the results of the preceding two sections.

## 6. Solution of the integral equations

Now we seek to solve the integral equations of Sections 3 and 4 in standard form.

Before this, we observe that the kernels $\sigma(\xi_1', l)$ and $\sigma''(\xi_1', l)$ of the integral equations [defined in (12) and (17)] are such that each is written as a sum of two parts, viz. the compressional and shear parts. Each part is associated with a step function whose argument fixes an upper or lower limit to the variable of integration $\xi_1'$. These limits are given by $\xi_\phi$, $\xi_\psi$, $\xi_\phi'$ and $\xi_\psi'$ which from (8) and (18) fall into five groups depending on $\theta$ as follows:

$$\left.\begin{array}{l}
\text{(i) } 0 \leqslant \theta < \theta_\beta : \xi_\phi = k_1 l, \xi_\psi = k_2 l, \xi_\phi' = k_1' l, \xi_\psi' = k_2' l \\
\text{(ii) } \theta_\beta < \theta < \theta_\alpha : \xi_\phi = k_1 l, \xi_\psi = k_2 l, \xi_\phi' = k_1' l, \xi_\psi' = \infty \\
\text{(iii) } \theta_\alpha < \theta < 180° - \theta_\alpha : \xi_\phi = k_1 l, \xi_\psi = k_2 l, \xi_\phi' = \xi_\psi' = \infty \\
\text{(iv) } 180° - \theta_\alpha < \theta < 180° - \theta_\beta : \xi_\phi = \infty, \xi_\psi = k_2 l, \xi_\phi' = \xi_\psi' = \infty \\
\text{(v) } 180° - \theta_\beta < \theta \leqslant 180° : \xi_\phi = \xi_\psi = \xi_\phi' = \xi_\psi' = \infty.
\end{array}\right\} \tag{29}$$

As each set of values is distinct, the corresponding solutions will be distinct in the five sub-intervals for $\theta$. They will be piece-wise continuous at the points $\theta_\beta$, $\theta_\alpha$, $180° - \theta_\alpha$ and $180° - \theta_\beta$.

Now we discuss the solution to these equations. A closed form solution to (6) and (24) becomes possible only when $\theta > 180° - \theta_\beta$. Obviously, then,

$$A_R^{(2)}(l) = A_T^{(2)}(l) = 0,$$

while $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$ become independent of $l$ and are given by the algebraic equations

$$\left.\begin{aligned} A_R^{(1)} &= R_1(\theta)+Q(\theta)\,A_T^{(1)} \\ A_T^{(1)} &= T(\theta)+Q(\theta)\,A_R^{(1)} \end{aligned}\right\} \quad (\theta > 180^\circ - \theta_\beta) \tag{30}$$

where $R_1(\theta)$, $T(\theta)$ and $Q(\theta)$ are the corresponding expressions for $R_1(\theta, l)$, $T(\theta, l)$ and $Q(\theta, l)$ respectively. Solving the above equations we have

$$\left.\begin{aligned} A_R^{(1)} &= (R_1+QT)/(1-Q^2) \\ A_T^{(1)} &= (T+QR_1)/(1-Q^2) \end{aligned}\right\} \quad (\theta > 180^\circ - \theta_\beta) \tag{31}$$

The solution given here is of interest because it is not very different from the approximations used by Mal & Knopoff (1966), viz.

$$A_R = aR_1 + bT,$$
$$A_T = aT + bR_1,$$

with

$$a = (1+Q^2)/(1-Q^2),$$
$$b = 2Q/(1-Q^2)\,.$$

The difference is due to the fact that they assume that the outgoing waves can interact with the wedge faces right up to $\theta = 180°$. Although this is easily seen to be incorrect, the error is not too great to call it a good approximation for large enough angles of

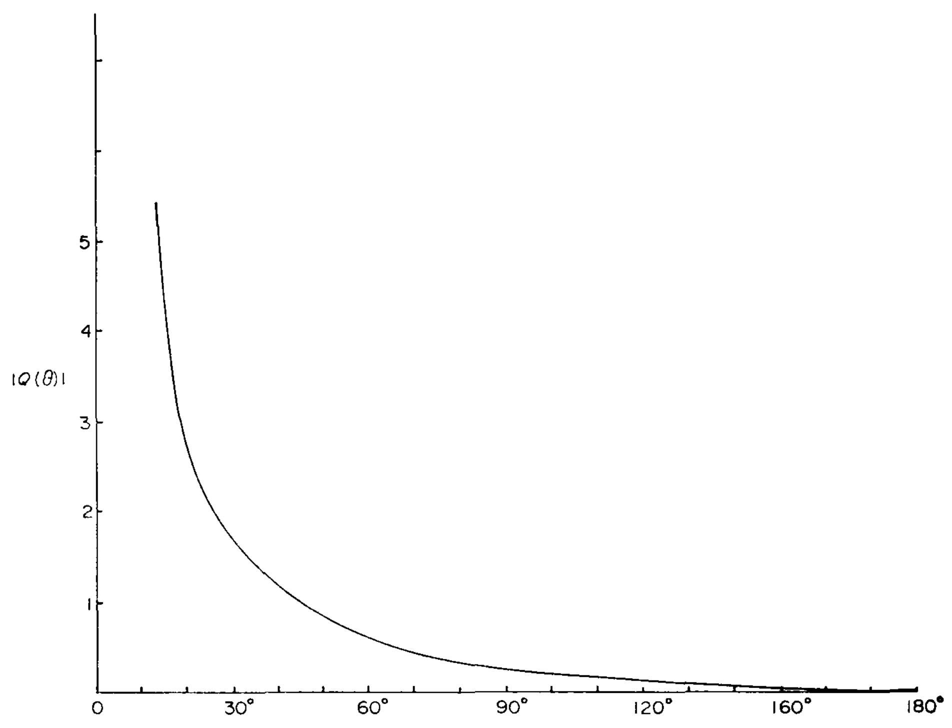


FIG. 5. Amplitude of $Q(\theta)$.

the wedge. However, there is another aspect of their solution which may be more serious. This comes from the fact that they employ their solution also for all values of $\theta$ from 0° to 180°. Now the solution (31), for instance, of the coupled equations (30) has been obtained by the usual elimination process. However the equations (30) represent also a physical process, viz. a sequence of contributions from a number of waves. Hence in a real situation we would normally solve (30) by the method of successive substitution, and in fact, this will be done shortly for the region $\theta < 180°-\theta_\beta$ not covered by that equation. The necessary condition that this second mode of solving (30) will truly converge to the sums in (31) is that $|Q(\theta)| < 1$. The function $Q(\theta)$ which is known to be purely imaginary, has been computed for a Poisson material ($\alpha = \sqrt{3}\,\beta$) and we have shown its magnitude in Fig. 5. We note that $|Q(\theta)| > 1$ for $\theta < 45°$ approximately. Thus, the solution of Mal & Knopoff will suffer from the above dual interpretation for wedge angles less than 45°.

Next we turn to the solution of (6) for $\theta < 180°-\theta_\beta$, and similarly of (24) and (25) for $\theta < \theta_\alpha$ in which range they exist. As remarked earlier, these equations will be solved only by the usual method of successive substitutions; (see Mikhlin 1964). The convergence of these solutions offers an interesting examination and we therefore deal with some of these aspects here. This is facilitated by considering their simplified models. First consider the equations in (6). Although these are easily decoupled for $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$, it is simpler to define new variables $F_\pm(l) = A_R^{(1)}(l) \pm A_T^{(1)}(l)$ which would satisfy integral equations like

$$F_\pm(l) = G_\pm(l) \pm \int_0^\infty \sigma(\xi_1', l)\, F_\pm(\xi_1')\, d\xi_1', \tag{32}$$

where $G_\pm(l)$ represents $R_1(\theta, l) \pm T(\theta, l)$. Making use of (12) these take the form, whenever $\theta < 180°-\theta_\alpha$:

$$\begin{aligned} F_\pm(l) = G_\pm(l) &\pm \frac{\omega A}{F'(1/\gamma)} \int_0^{k_1 l} F_\pm(\xi_1') \exp\left(-i\omega\xi_1'(a_0 - 1/\gamma)\right) d\xi_1' \\ &\pm \frac{\omega B}{F'(1/\gamma)} \int_0^{k_2 l} F_\pm(\xi_1') \exp\left(-i\omega\xi_1'(b_0 - 1/\gamma)\right) d\xi_1', \end{aligned} \tag{33}$$

and for the remaining range, viz. $180°-\theta_\alpha < \theta < 180°-\theta_\beta$ the discussion below will apply identically. The above equations are obviously generalized forms of the Volterra-type integral equations since the variable $l$ occurs in the upper limits of the integrals *but* with the added complexity that the parameters $k_1$ and $k_2$ also occur in these limits. If $k_1$ and $k_2$ were equal to unity, we have precisely the Volterra equation of the second type. But $k_1$ and $k_2$ are functions of $\theta$ and can take a continuous range of values including unity. However, it is possible to examine such general cases by means of the typical equation

$$\phi(x) = g(x) + \lambda^* \int_0^{px} k(x, \mathcal{S})\, \phi(\mathcal{S})\, d\mathcal{S} \tag{34}$$

where $k(x, \mathcal{S}) = 0$ for $\mathcal{S} > px$, and $p$ can take real positive values different from unity. For the Volterra case, i.e. for $p = 1$, the solution by the method of successive substitution is known to converge like an exponential series for all values of $\lambda^*$ and with an infinite radius of convergence for $x$. This is achieved by the possibility of writing the iterated kernels in the form

$$k_m(x, \mathscr{S}) = \int_{\mathscr{S}}^{x} k_1(x, \mathscr{S}')\, k_{m-1}(\mathscr{S}', \mathscr{S})\, H(x-\mathscr{S})\, d\mathscr{S}'$$

$$[k_1(x, \mathscr{S}) = k(x, \mathscr{S})],\ m > 1 \quad (35)$$

from which it is then shown that if $|k(x, \mathscr{S})| < M$ for all $x$ and $\mathscr{S}$, then

$$|k_m(x, \mathscr{S})| < \frac{M^m (x-\mathscr{S})^{m-1}}{(m-1)!} \quad . \quad (36)$$

This criterion ensures the above properties of convergence. However, for $p \neq 1$ (35) gives place to

$$k_m(x, \mathscr{S}) = H(x-\mathscr{S}/p^m) \int_{\mathscr{S}/p^{m-1}}^{px} k_1(x, \mathscr{S}')\, k_{m-1}(\mathscr{S}', \mathscr{S})\, d\mathscr{S}' \quad (37)$$

$$(m > 1),$$

so that if $|k(x, \mathscr{S})| < M$, it could be shown that

$$|k_m(x, \mathscr{S})| < \frac{M^m (xp^{m/2} - \mathscr{S}p^{-m/2})^{m-1}}{(m-1)!}$$

$$< \frac{M^m x^{m-1} p^{\frac{1}{2}m(m-1)}}{(m-1)!} \quad . \quad (38)$$

This is a sufficient condition to show that equation (34) will continue to behave like the Volterra equation (of the second type) when $p < 1$. But when $p > 1$, this approach would not be meaningful since it can be shown that if $p = (1+\eta)^2$ with $\eta > 0$ and $x\lambda^* > 1/(M\eta)$, then

$$\frac{M^m x^{m-1} p^{\frac{1}{2}m(m-1)}}{(m-1)!} > \frac{M^m (x\eta)^{m-1} e^m}{\sqrt{(2\pi)}\, m^{\frac{1}{2}}}$$

$$(m \gg 1),$$

so that the intended comparison series is divergent and no longer useful. For our problem the parameter $p$ which stands for $k_1$ or $k_2$ will exceed unity for values of $\theta$ exceeding $180° - 2\theta_\alpha$ and $180° - 2\theta_\beta$ respectively. But we already know that for the closed-form solution (31) pertaining to $\theta > 180° - \theta_\beta$ the equations (30) do admit a solution by successive substitution and that this solution converges like a geometric series characteristic of a Fredholm-type integral equation.

Thus we conclude that the integral equations (6) will behave like Volterra equations of second type for lower ranges of wedge angles and like the Fredholm equations in the higher ranges. At intermediate values of $\theta$ there will be a mixed behaviour since there are values of $\theta$ when only one of $k_1$ or $k_2$ may exceed unity.

The situation with (24) and (25) is much simpler because in this case we come across equations of the type

$$\phi(x) = g(x) + \lambda^* \int_{qx}^{\infty} k(x, \mathscr{S})\, \phi(\mathscr{S})\, d\mathscr{S}, \quad (39)$$

where $k(x, \mathscr{S}) = 0$ for $\mathscr{S} < qx$ and the parameter $q$ which stands for $k_1'$ and $k_2'$ defined in (19) always remains greater than unity. Hence the integral equations will behave like Fredholm equations and their solutions converge like geometric series.

All the solutions are summarized in the next section.

## 7. Summary of the approximations

The final results for the reflection and transmission coefficients (5) of the main problem are summarized below. The incident field being given by (3) along with (4), $A_R(l)$ and $A_T(l)$ are obtained as

$$\left.\begin{aligned} A_R(l) &= A_R^{(1)}(l) + A_R^{(2)}(l) + A_R^{(3)}(l) \\ A_T(l) &= A_T^{(1)}(l) + A_T^{(2)}(l) + A_T^{(3)}(l) \end{aligned}\right\} \tag{40}$$

where

$$\left.\begin{aligned} A_R^{(1)}(l) &= R_1(\theta, l) + \sum_{n=1}^{\infty} \mathscr{L}^{2n}\{R_1(\theta, l)\} + \sum_{n=0}^{\infty} \mathscr{L}^{2n+1}\{T(\theta, l)\} \\ A_T^{(1)}(l) &= T(\theta, l) + \sum_{n=1}^{\infty} \mathscr{L}^{2n}\{T(\theta, l)\} + \sum_{n=0}^{\infty} \mathscr{L}^{2n+1}\{R_1(\theta, l)\} \\ A_R^{(2)}(l) &= \sum_{n=0}^{\infty} \mathscr{D}\{\mathscr{L}^{*2n}(1)\} + \sum_{n=0}^{\infty} \mathscr{D}^*\{\mathscr{L}^{*2n+1}(1)\} \\ A_T^{(2)}(l) &= \sum_{n=0}^{\infty} \mathscr{D}\{\mathscr{L}^{*2n+1}(1)\} + \sum_{n=0}^{\infty} \mathscr{D}^*\{\mathscr{L}^{*2n}(1)\} \\ A_R^{(3)}(l) &\simeq 0 \\ A_T^{(3)}(l) &\approx \int_0^{\infty} \sigma(\xi_1', l)\, d\xi_1', \ \theta \in (45°, 135°) \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} \mathscr{D}\{\phi(l)\} &= \int_0^{\infty} \sigma'(\xi_1', l)\, d\xi_1' \int_0^{\infty} \sigma''(\xi_1'', \xi_1')\, \phi(\xi_1'')\, d\xi_1'' \\ \mathscr{D}^*\{\phi(l)\} &= \int_0^{\infty} \sigma^*(\xi_1', l)\, d\xi_1' \int_0^{\infty} \sigma_1''(\xi_1'', \xi_1')\, \phi(\xi_1'')\, d\xi_1'' \end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned} \mathscr{L}\{f(\theta, l)\} &= \int_0^{\infty} \sigma(\xi_1', l)\, f(\theta, \xi_1')\, d\xi_1' \\ \mathscr{L}^*\{f(\theta, l)\} &= \int_0^{\infty} \sigma''(\xi_1', l)\, f(\theta, \xi_1')\, d\xi_1' \end{aligned}\right\}. \tag{43}$$

We recall that the corner wave effects represented by the first-order approximations of $A_R^{(3)}(l)$ and $A_T^{(3)}(l)$ are recommended only cautiously for the range $\theta \in (45°, 135°)$.

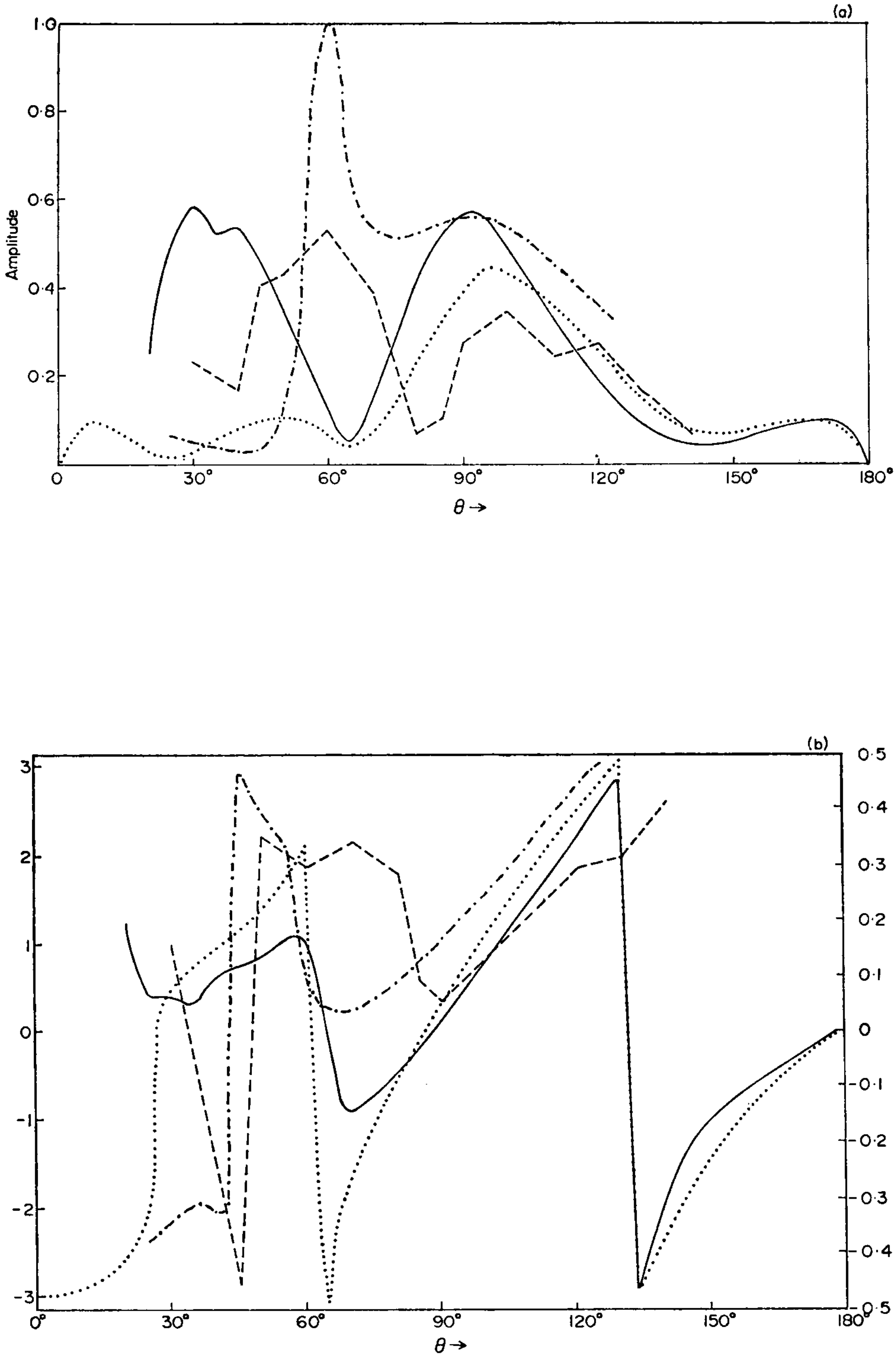


FIG. 6. (a) Amplitude, and (b) Phase of the Reflection coefficient for incidence from infinity ($l \rightarrow \infty$). ————, present theory; - - - - - - -, experimental; — · — · — · — · —, from Viswanathan *et al.* (1971); . . . . . . . . . . , from Mal & Knopoff (1966).

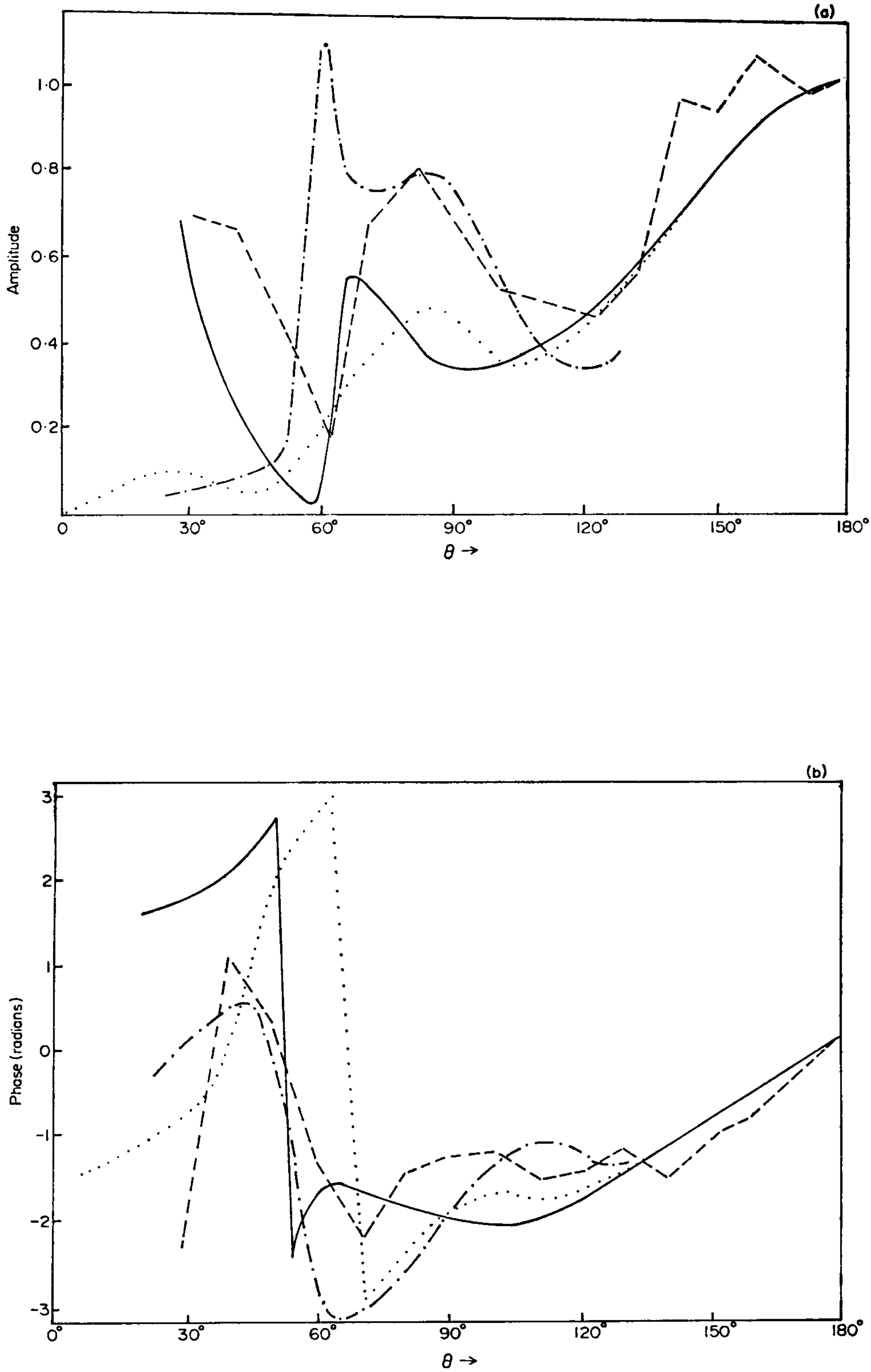


FIG. 7. Same as Fig. 6 for the transmission coefficient ($l \to \infty$).

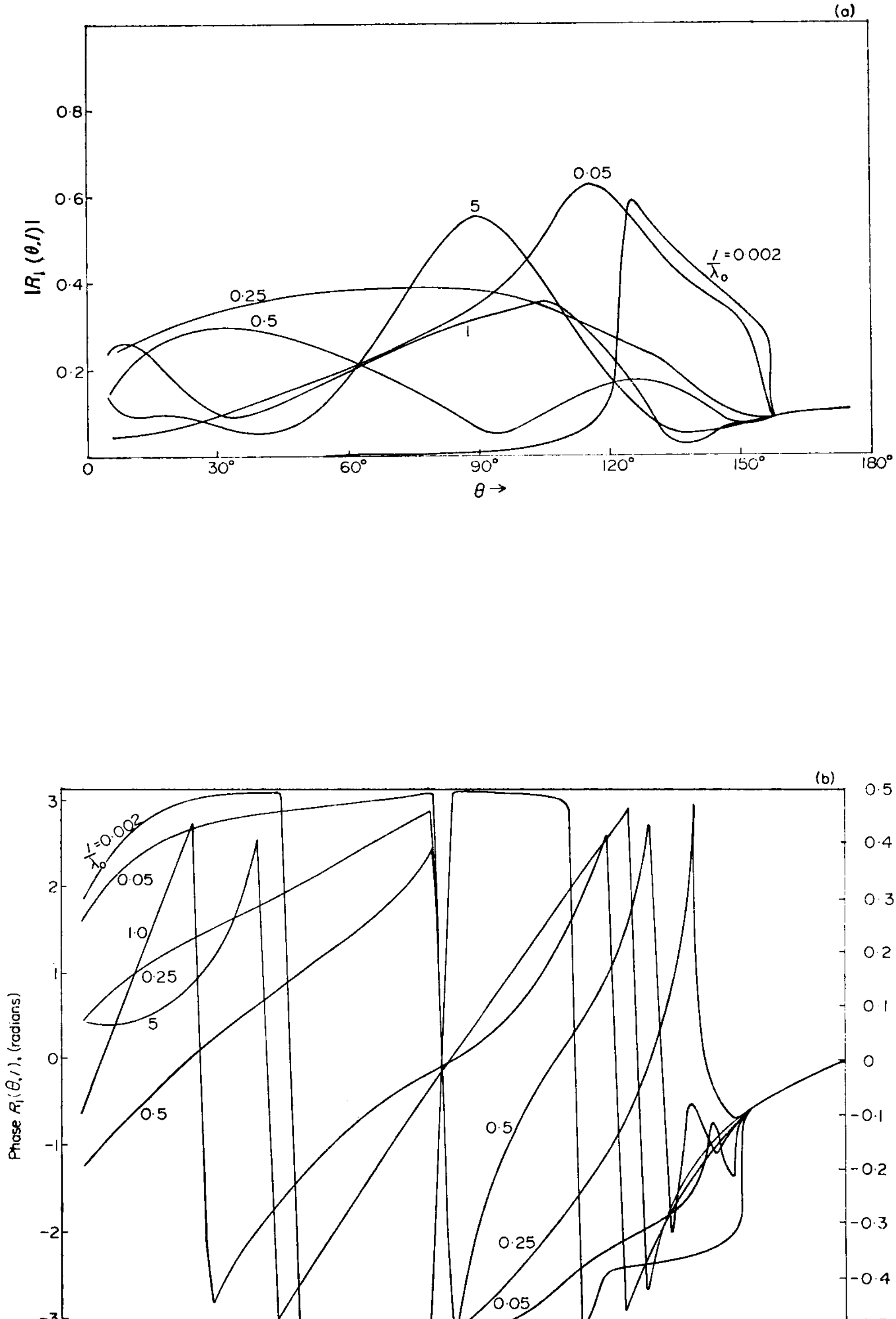


FIG. 8. Computation of the first-order reflection coefficient $R_1(\theta, l)$ for finite $l$. $\lambda_0$ is the wavelength of the incident field.

## 8. Numerical results and concluding remarks

We have made our calculations for a Poisson material for which $\alpha = \sqrt{3}\,\beta$ and the critical angles are given by $\theta_\alpha = 57{\cdot}93°$ and $\theta_\beta = 23{\cdot}15°$. The parameter $k_1$ exceeds unity when $\theta > 180° - 2\theta_\alpha$ and similarly $k_2$ exceeds unity when $\theta > 180° - 2\theta_\beta$.

First we consider the case $l \to \infty$, i.e. when the incident field is from infinity. The amplitude and phase of the reflection coefficient $A_R(l)|_{l\to\infty}$ are shown in Fig. 6. The experimental results of Pilant, Knopoff & Schwab (1964) as well as previous theoretical results are included for comparison. If we recall that the present theory is based on unifying the concept of the critical regions and multiple interactions recommended in the previous theories referred to, the results of the present theory are remarkably good in so far as the hitherto less-matched experimental features are concerned. Quantitative improvements are possible only by more painstaking computations. Some ideas are set forth later.

The corresponding transmission coefficient $A_T(l)|_{l\to\infty}$ of the present theory is also numerically estimated. We note here that in this case there is also a contribution from the corner wave correction term $A_T^{(3)}(l)|_{l\to\infty}$ and its limitations have already been mentioned. We have randomly employed this correction to the range* $60° < \theta < 120°$ which lies within the range of possible importance of the corner effect discussed earlier in this paper. The final results are shown in Fig. 7 along with other theories and the experimental data. Again the qualitative features show some promise but not much can be achieved without invoking also a detailed calculation of the effect of the corner waves.

It is worth noting that the convergence of the series for $A_R^{(1,2)}(l)$ and $A_T^{(1,2)}(l)$ in (41) is very rapid and we found it sufficient to compute four or five terms in each case

Next we consider the case when $l$ is finite. A parameter of interest in this case is the ratio of $l$ to the wavelength $\lambda_0 = 2\pi(\omega/\gamma)^{-1}$ of the incident field. The first-order reflection and transmission coefficients $R_1(\theta, l)$ and $T(\theta, l)$ given by (14) and (10) respectively, have been computed for several values of this ratio in the case of a Poisson material. The results are shown in Figs 8 and 9 from which we gather that a variation of $l/\lambda_0$ is likely to cause appreciable effects only when we consider the range $l/\lambda_0 \leqslant 1$. Presumably this will hold good also in the higher-order approximations for finite $l$. Such computations are not made here since at present it is only of analytical interest.

Finally we mention some ways of improvement. The first is readily seen to be the inclusion of the contributions to $A_R^{(2)}(l)$ and $A_T^{(2)}(l)$ from the higher-order terms of $A_R^{(1)}(l)$ and $A_T^{(1)}(l)$ than only from $R_1(\theta, l)$ and $T(\theta, l)$. The second factor is the accurate treatment of the corner wave effects. As noted already, this effect is rather important in the range $45° < \theta < 135°$ and apparently a first-order correction is inadequate especially for $\theta \approx 90°$. This effect could be treated in many different ways either by solving the auxiliary problems more accurately or by modifying the boundary values of the stresses in these auxiliary problems and so on. Last we also find the possibility of a totally different mode of contribution which has been overlooked in most of the previous studies. This is described easily by noting that in the first-order approximation studies, we normally assume that a reflected Rayleigh wave of amount $R_1(\theta, l)$ on $S_1$, a transmitted Rayleigh wave of amount $T(\theta, l)$ on $S_2$ and certain incoming Rayleigh waves of amount $Q(\theta, l) = \int_0^\infty \sigma(\xi_1', l)\, d\xi_1'$ on $S_2$ (where the integrand gives the contribution from one virtual source only) are generated. However, by symmetry (which may not be so obvious at the first thought), there must also be a similar *first-order incoming* Rayleigh wave excited on $S_1$. This fact becomes obvious when we look at equation (13) and the arguments used in deriving $R_1(\theta, l)$, the Rayleigh wave excited on $S_1$ outwards. In that case the Green's function $\mathbf{G}_k(\mathbf{x}, \mathbf{x}')$

* The 'randomness' refers to the sub-range selected.

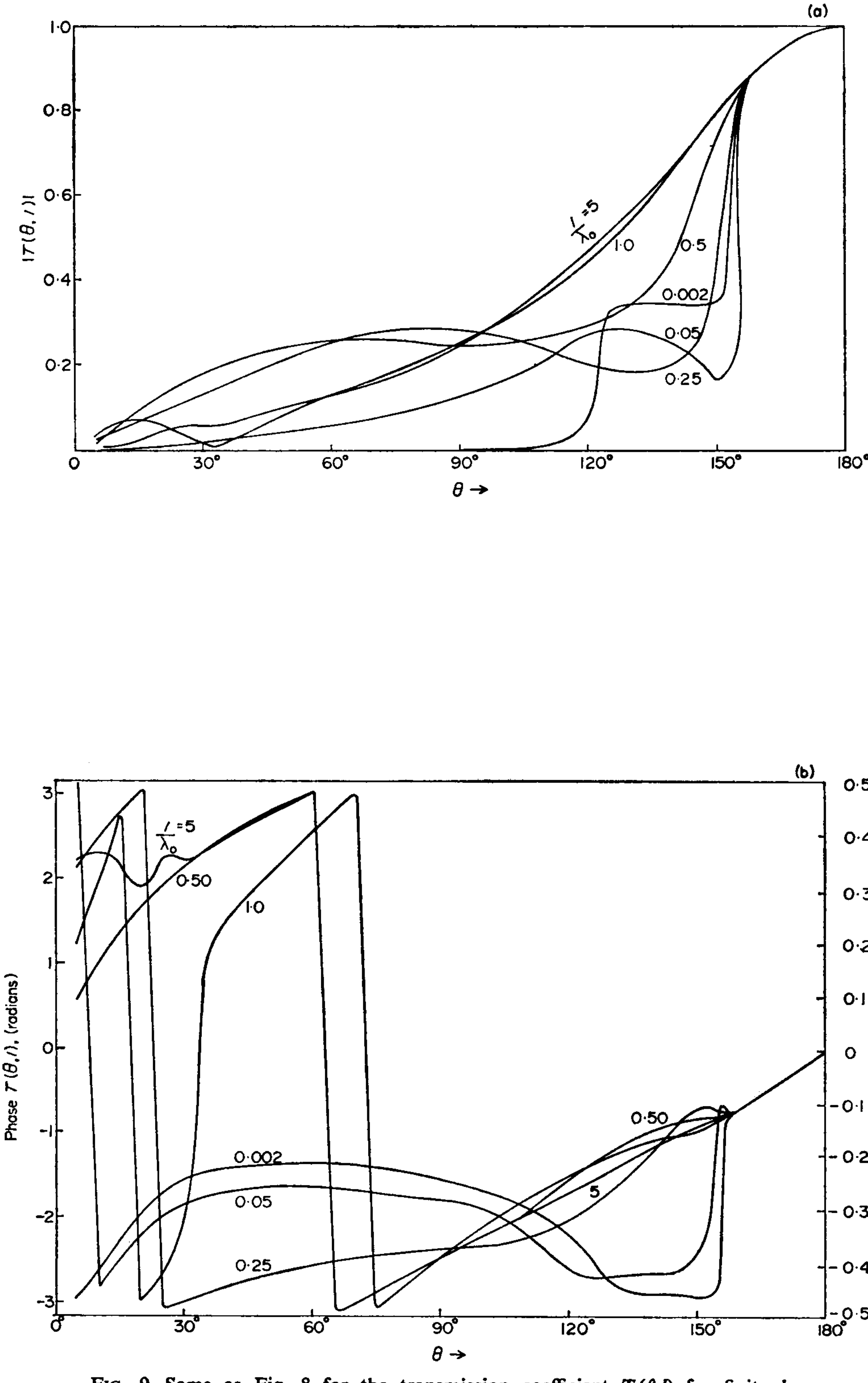


FIG. 9. Same as Fig. 8 for the transmission coefficient $T(\theta,l)$ for finite $l$.

was taken to be appropriate to the half-space $x_3 \geqslant 0$ and for obtaining $R_1(\theta, l)$ we chose the Rayleigh wave term of $\mathbf{G}_k$ as the one which represents the wave travelling towards $x_1 \to -\infty$ from the source at $\mathbf{x}'$. But the Green's function is symmetric about the source and hence it will also contain the corresponding Rayleigh wave excited towards $x_1 \to +\infty$, due to any given virtual source at $\mathbf{x}'$. These incoming Rayleigh waves on $S_1$ constitute an independent first-order phenomenon which have been mentioned before. It is felt that their contribution to the main problem may be important at least for small values of $\theta$.

## Acknowledgments

The authors are glad to thank Dr E. R. Lapwood for several useful discussions. They are also thankful to the Department of Applied Mathematics and Theoretical Physics, University of Cambridge for facilities extended and to the Commonwealth Scholarships Commission (U.K.) for financial support.

*Department of Applied Mathematics and Theoretical Physics*
*University of Cambridge*

## References

Mal, A. K. & Knopoff, L., 1966. Transmission of Rayleigh waves at a corner. *Bull. seism. Soc. Am.*, **56**, 455–466.
Mikhlin, S. G., 1964. *Integral equations*, Pergamon Press Ltd, Oxford.
Pilant, W., Knopoff, L. & Schwab, F., 1964. Transmission and reflection of surface waves at a corner, 3, Rayleigh waves (experimental), *J. geophys. Res.*, **69**, 291–297.
Viswanathan, K., Kuo, J. T., & Lapwood, E. R., 1971. Reflection and transmission of Rayleigh waves in a wedge-I, *Geophys. J. R. astr. Soc.*, **24**, 401–414.

## Appendix

Here we define the various symbols that were employed in the text of the paper. $(\lambda, \mu)$ denote Lamé's constants;

$$\left.\begin{aligned} C_1 &= (\eta_\alpha^2+1/\gamma^2)\sin 2\theta+(2i\eta_\alpha/\gamma)\cos 2\theta \\ D_1 &= (\eta_\beta^2+1/\gamma^2)\sin 2\theta+(2i\eta_\alpha/\gamma)\cos 2\theta \\ C_3 &= (\eta_\alpha^2+1/\gamma^2)\cos 2\theta-(\eta_\alpha^2-\eta_\beta^2)-(2i\eta_\alpha/\gamma)\sin 2\theta \\ D_3 &= (\eta_\beta^2+1/\gamma^2)\cos 2\theta-(2i\eta_\alpha/\gamma)\sin 2\theta \end{aligned}\right\}, \quad (A1)$$

$$\left.\begin{aligned} a_0 &= (1/\gamma)\cos\theta-i\eta_\alpha\sin\theta \\ b_0 &= (1/\gamma)\cos\theta-i\eta_\beta\sin\theta \end{aligned}\right\}, \quad (A2)$$

$$\left.\begin{aligned} A &= i\,[2i\eta_\beta/\gamma\, C_1-(\eta_\beta^2+1/\gamma^2)\, C_3] \\ B &= -i\,[2i\eta_\beta/\gamma\, D_1-(\eta_\beta^2+1/\gamma^2)\, D_3] \\ C &= -i\,[2i\eta_\beta/\gamma\, C_1+(\eta_\beta^2+1/\gamma^2)\, C_3] \\ D &= i\,[2i\eta_\beta/\gamma\, D_1+(\eta_\beta^2+1/\gamma^2)\, D_3] \end{aligned}\right\}, \quad (A3)$$

$$\left.\begin{aligned} F(x) &= (2x^2-1/\beta^2)^2-4x^2\,(1/\alpha^2-x^2)^{\frac{1}{2}}\,(1/\beta^2-x^2)^{\frac{1}{2}} \\ &\quad \mathrm{Re}\,(1/\alpha^2-x^2)^{\frac{1}{2}} \geqslant 0,\ \mathrm{Re}(1/\beta^2-x^2)^{\frac{1}{2}} \geqslant 0 \\ F'(x) &= d/dx\, F(x) \end{aligned}\right\}, \quad (A4)$$

$$\left.\begin{aligned} E_1 &= -(i/P)(A_1 C_1 - A_3 C_3) \\ E_2 &= (i/P)(A_1 D_1 - A_3 D_3) \\ E_3 &= -(i/P)(B_1 C_1 - B_3 C_3) \\ E_4 &= (i/P)(B_1 D_1 - B_3 D_3) \end{aligned}\right\}, \tag{A5}$$

$$\left.\begin{aligned} A_1 &= -(4i\eta_\beta/\gamma)\, a_0\, \alpha_0\, (1/\beta^2 - a_0^{\,2})^{\frac{1}{2}}/F_1(a_0) \\ B_1 &= -(\eta_\beta^{\,2} + 1/\gamma^2)\, \beta_0\, (2b_0^{\,2} - 1/\beta^2)/F_1(b_0) \\ A_3 &= (2i\eta_\beta/\gamma)\, \alpha_0\, (2a_0^{\,2} - 1/\beta^2)/F_1(a_0) \\ B_3 &= -2(\eta_\beta^{\,2} + 1/\gamma^2)\, b_0\, \beta_0\, (1/\alpha^2 - b_0^{\,2})^{\frac{1}{2}}/F_1(b_0) \end{aligned}\right\}, \tag{A6}$$

$$\left.\begin{aligned} \alpha_0 &= 1/\gamma \sin\theta + i\eta_\alpha \cos\theta \\ \beta_0 &= 1/\gamma \sin\theta + i\eta_\beta \cos\theta \end{aligned}\right\}, \tag{A7}$$

$$F_1(x) = (2x^2 - 1/\beta^2)^2 + 4x^2 (1/\alpha^2 - x^2)^{\frac{1}{2}} (1/\beta^2 - x^2)^{\frac{1}{2}}, \tag{A8}$$

$$P = 1/\gamma^2 \left((\eta_\alpha^{\,2} + \eta_\beta^{\,2})/\eta_\alpha \eta_\beta\right) + 2\eta_\alpha \eta_\beta - 2(\eta_\beta^{\,2} + 1/\gamma^2) \tag{A9}$$