Reflection and transmission of waves near the localization threshold

van Tiggelen, B.A.; Lagendijk, A.; Wiersma, D.S.

DOI
10.1103/PhysRevLett.84.4333

Publication date
2000

Published in
Physical Review Letters

Reflection and Transmission of Waves near the Localization Threshold

B. A. van Tiggelen,[1] A. Lagendijk,[2] and D. S. Wiersma[3]
[1]*CNRS/Laboratoire de Physique et Modélisation des Milieux Condensés, Université Joseph Fourier, Maison des Magistères, B.P. 166 38042 Grenoble Cedex 9, France*
[2]*Van der Waals-Zeeman Laboratory, University of Amsterdam, Valckenierstraat 65-67, 1018 XE Amsterdam, The Netherlands*
[3]*European Laboratory for non-Linear Spectroscopy and Istituto Nazionale per la Fisica della Materia, Largo E. Fermi 2, 50125 Florence, Italy*
(Received 10 January 2000)

A theory is presented for propagation of waves in bounded media near the mobility edge, based on the self-consistent theory for localization. It predicts a spatially inhomogeneous diffusion constant that leads to scale dependence in enhanced backscattering and transmission.

PACS numbers: 42.25.Hz, 78.90.+t

Enhanced backscattering (EB) and localization of waves are two related subjects that have received a lot of attention in recent years [1–5]. They both find their origin in interference effects in multiple scattering of waves. EB with classical waves has elucidated the crucial role of reciprocity [6–8]. For electrons, interest has concentrated on weak localization effects, whose interpretation calls upon the same interference events that are observed directly in EB [9,10]. Recent experiments [4,11] call for a theory capable of describing reflection and transmission around the mobility edge. For open systems, the random-matrix theory [12] and the self-consistent (SC) theory of localization [13–15] have been developed. The first is nonperturbational and can even deal with fluctuations, but is restricted to quasi-1D systems as studied in microwave experiments [3].

The SC theory provides an implicit equation for the dynamic "ac" diffusion constant $D(\Omega)$ of the waves [15],

$$\frac{1}{D(\Omega,\mathbf{r})} = \frac{1}{D_B} + \frac{C_\Omega(\mathbf{r},\mathbf{r})}{\pi v_E \rho(k)}. \tag{1a}$$

The Boltzmann diffusion constant D_B is free from interference. The second term contains the average "return probability" $C(\mathbf{r},\mathbf{r})$ by constructive interference of reciprocal paths at position $\mathbf{r}$, which lowers the diffusion constant. We will ignore the difference between extinction length, scattering and Boltzmann transport mean-free path and represent all by ℓ. With v_E being the transport speed of light and k its wave number, we have (in 3D) the familiar relations $D_B = \frac{1}{3}v_E\ell$ [1], and $\rho(k) \approx k^2/\pi^2 v_E$ for the density of states per unit volume. k, ℓ, and v_E have been calculated near the localization threshold [16].

Without magnetic fields, the reciprocity principle requires the amount of constructive interference $C_\Omega(\mathbf{r},\mathbf{r})$ to be *exactly* equal to the amount of "incoherent" radiation returning at $\mathbf{r}$ [15,17]. C_Ω must thus obey the dynamic diffusion equation,

$$[-i\Omega - \nabla \cdot D(\Omega,\mathbf{r})\nabla]C_\Omega(\mathbf{r},\mathbf{r}') = \frac{4\pi}{\ell}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{1b}$$

The factor $4\pi/\ell$ appears when single scattering is adopted as a source for multiple scattering. In the rest of this paper we consider $\Omega = 0$, which describes stationary diffusion flow.

Equations (1a) and (1b), here formulated for three dimensions, must contain *the same* diffusion constant, and one seeks a "self-consistent" solution. In infinite media, $C(\mathbf{r},\mathbf{r}')$ is translationally invariant, so that the return probability $C(\mathbf{r},\mathbf{r})$ and diffusion constant $D(\mathbf{r})$ do not depend on $\mathbf{r}$. In reciprocal space $C(q) = 4\pi/\ell Dq^2$, so that $C(\mathbf{r},\mathbf{r}) = \sum_q C(q) \sim \mu/D\ell^2$, assuming an upper cutoff $q_{\max} = \mu/\ell$. Hence,

$$D(\Omega = 0) = D_B\left(1 - \frac{\mu}{k^2\ell^2}\right). \tag{2}$$

The mobility edge, defined by $D = 0$, obeys an Ioffe-Regel–type criterion as derived microscopically by John *et al.* [18] and Economou *et al.* [19], and agrees with the numerical studies of the Anderson tight binding model [20,21]. The cutoff removes short wave paths from the return probability and influences the exact location of the mobility edge [19,22]. For $\mu = 1$ the mobility edge is located at $k\ell = 1$.

In finite media, translational symmetry is absent and Eq. (1a) requires that the diffusion constant $D(\Omega,\mathbf{r})$ be dependent on $\mathbf{r}$. This has not to our knowledge been considered before, but is unavoidable if one doesn't wish to give up the basic ingredients of the SC theory of localization: reciprocity and flux conservation. Previous work focused on a homogeneous but "scale-dependent" diffusivity kernel $D(\Omega,\mathbf{r}-\mathbf{r}')$, with Fourier transform $D(\Omega,\mathbf{q})$. Near the mobility edge, $D(\Omega,q) \sim q$ has been suggested [23–25]. The absence of such q dependence in the SC theory is sometimes considered a serious failure, in spite of its agreement with scaling arguments for the dynamic diffusivity $D(\Omega)$ [26] and its qualitative agreement with scaling theory [27], as shown by Wölfle and Vollhardt [15]. At the mobility edge, our local formulation of the SC theory predicts the scale dependence $D(z) \sim 1/z$, which leads to a transmission $T \sim 1/L^2$ of a slab with thickness L, and a rounding of the EB line shape. Both properties were previously interpreted as consequences of a scale-dependent diffusivity $D(q) \sim q$ [25]. Contrary to all other approaches, our local variant of the SC theory deals elegantly and explicitly with boundary conditions.

We consider stationary propagation, in a slab geometry of thickness L, and Fourier transform ($\mathbf{q}_\parallel$) the transverse coordinate. For $0 < z < L$, Eqs. (1a) and (1b) become

$$D(z)^{-1} = D_B^{-1} + \frac{2}{k^2\ell}\int_0^{1/\ell} dq_\parallel q_\parallel C(z,z,q_\parallel), \quad (3a)$$

$$-\partial_z D(z)\partial_z C(z,z',q_\parallel) + D(z)q_\parallel^2 C(z,z',q_\parallel) = \frac{4\pi}{\ell}\delta(z-z'), \quad (3b)$$

$$C(0,z',q_\parallel) - z_e(0)\partial_z C(0,z',q_\parallel) = 0, \quad (3c)$$

$$C(L,z',q_\parallel) + z_e(L)\partial_z C(L,z',q_\parallel) = 0. \quad (3d)$$

The last two equations are the radiative boundary conditions at both sides of the slab, featuring the "extrapolation lengths" $z_e(0/L) \equiv 3z_0 D(0/L)/v_E$ [28]. Berkovits and Kaveh [25] emphasized that flux conservation requires them to contain the diffusion constant D, including interference. In our theory, D is finite at the boundaries so that z_e is always nonzero, even in the localized regime, when D vanishes in the bulk; $z_0 = \frac{2}{3}$ corresponds to no internal reflection, and increases with increasing internal reflection. The value for z_0 in recent localization experiments [4,11] is estimated to be typically 10.

Equation (3b) is recognized as an ordinary, second-order differential equation with a source term. Without the latter, two independent solutions $f_\pm(z)$ exist with constant and nonzero Wronskian $W(q_\parallel) \equiv D(z) \times (f'_+ f_- - f'_- f_+)$. In a "quasi-1D" medium with transverse surface $A < \ell^2$, only the transverse mode $q_\parallel = 0$ contributes to Eq. (3a), with weight $1/A$. Equations (3a)–(3d) have analytical solutions with a similar scale dependence of the transmission as predicted by random matrix theory [29]. For a semi-infinite quasi-1D medium their solution is $D(z) = D(0)\exp(-2z/\xi)$, with $\xi = A\rho(k)v_E\ell$ the same localization length as found in random matrix theory [29]. Furthermore, $1/D(0) = 1/D_B + 2z_0/\xi$.

For the slab geometry, Eqs. (3a)–(3d) have been studied numerically. We first discuss the semi-infinite medium $L = \infty$. In that case $C(z,z',q_\parallel)$ must be bounded at large z, z' so that, with $f_+(z)$ the growing solution, Eq. (3b) is solved for

$$C(z,z',q_\parallel) = \frac{f_+(z_<)f_-(z_>)}{W(q_\parallel)\ell/4\pi} - P(q_\parallel)f_-(z)f_-(z'), \quad (4)$$

where $z_< = \min(z,z')$, $z_> = \max(z,z')$, and $P(q_\parallel)$ is determined by the boundary condition (3c) at $z = 0$. For the critical value $k\ell = 1$ we have compared the numerical solution to the simple algebraic form,

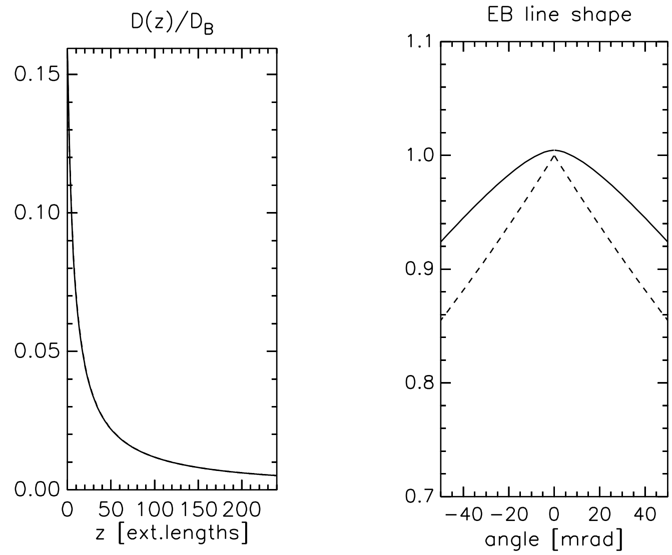

FIG. 1. Solution of the self-consistent equation at the mobility edge of a semi-infinite slab with internal reflection parameter $z_0 = 10$. Left panel: the diffusion constant $D(z)$ as a function of depth (in units of extinction lengths). Right panel: line profile in enhanced backscattering. The dashed line is the conventional cusp for $k\ell \gg 1$ using the same extrapolation length z_e.

$$D(z) = \frac{D(0)}{1 + z/\xi_c}, \quad (5)$$

with two free parameters $D(0)$ and ξ_c. The homogeneous solutions would then be

$$f_+(z) = (z + \xi_c)I_1(q_\parallel[z + \xi_c]); \quad f_-(z) = (z + \xi_c)K_1(q_\parallel[z + \xi_c]), \quad (6)$$

in terms of the modified Bessel functions I_1 and K_1 with Wronskian $W = D(0)\xi_c$ [30]. Equation (4) shows that $C(z,z,q_\parallel)$ rises linearly in z for large z and that Eq. (3a) is asymptotically satisfied. The SC equation for $z = 0$ gives a relation between $D(0)$ and ξ_c. The remaining freedom in ξ_c was chosen to optimize self-consistency below 0.05%. (see Fig. 1 and Table I). Both ξ_c and $D(0)$ depend heavily on the parameter z_0 in the boundary condition.

The line shape $I_c(\theta)$ in EB can be obtained from $C(z,z',q_\parallel)$ using standard methods [31] and is shown in Fig. 1. Insight is provided by the approximate formula

TABLE I. Solution $D(z) = D(0)/(1 + z/\xi_c)$ of the self-consistent equations (3a)–(3d) at the mobility edge $k\ell = 1$ for a semi-infinite slab as a function of the parameter z_0 that controls internal reflection at the boundary. The middle column shows the relation $D(0) \sim 1/z_0$.

z_0	$D(0)/D_B$	ξ_c/ℓ
2/3	0.642	1.5
3	0.336	3
5	0.249	4
7	0.203	6
10	0.159	8
20	0.0968	25

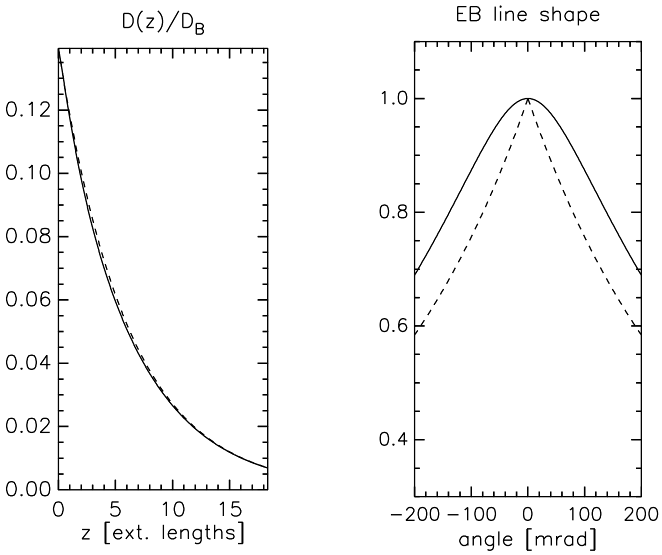

FIG. 2. Approximate solution of the self-consistent equations in the localized regime ($k\ell = 0.96$) of a semi-infinite slab with internal reflection parameter $z_0 = 10$, assuming an exponentially decaying diffusion constant. The dashed line in the left panel shows one iteration of the self-consistent equation. The dashed line in the right panel is the conventional cusp for $k\ell \gg 1$ using the same extrapolation length z_e.

$I_c(\theta) \approx C(z = \ell, z' = \ell, q_\parallel = 2k \sin\theta/2)$, used by Lagendijk *et al.* [28]. The line shape exhibits a logarithmic rounding $I_c(\theta) \sim 1 + z_e(0)\xi_c q_\parallel^2 \log(q_\parallel \xi_c)$ when $q_\parallel \xi \ll 1$, rather than the familiar cusp $I_c(\theta) \sim 1 - z_e|q_\parallel|$ [31]. Berkovits and Kaveh [25] predicted a rounding of the line shape on the basis of the nonlocal diffusion kernel $D(q)$.

The localized regime corresponds to $k\ell < 1$. As in quasi-1D media, we may assert the solution $D(z) = D(0)\exp(-2z/\xi)$, with ξ the localization length. We find $f_\pm(z) = \exp(-\lambda_\pm z)$ with $\lambda_\pm = 1/\xi \pm \sqrt{q_\parallel^2 + 1/\xi^2}$, and Wronskian $W = 2D(0)\sqrt{q_\parallel^2 + 1/\xi^2}$. The SC equation (3a) is satisfied for $z \gg \xi$ if $\xi/\ell = 2(k\ell)^2/[1 - (k\ell)^4]$, whereas the SC equation at $z = 0$ provides $D(0)$. Figure 2 shows the above exponential ansatz for D to be satisfactory for $z_0 = 10$, but for smaller internal reflection we found less agreement. The EB line shape is approximately given by

$$I_c(\theta) \approx \frac{1}{1 - z_e(0)/\xi + z_e(0)\sqrt{q_\parallel^2 + 1/\xi^2}}, \qquad (7)$$

i.e., an analytical rounding for $\theta < 1/k\xi$ (Fig. 2). This EB line shape is reminiscent of an absorbing semi-infinite medium in the delocalized regime, a case that must be excluded experimentally [11].

Our final subject is the length dependence of the total transmission $T(L)$ of a slab with length L. For a point source close to the boundary $z = 0$, the diffusion equation predicts

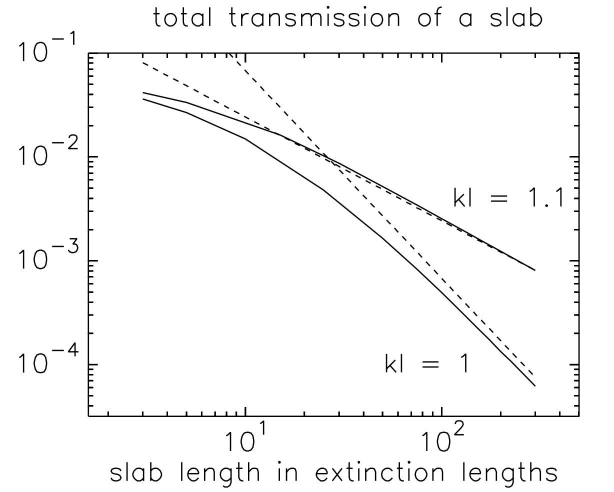

FIG. 3. Numerical solution of the self-consistent equations for a finite slab. The total transmission coefficient is displayed as a function of the slab length L, for the critical value $k\ell = 1$ and in the delocalized regime $k\ell = 1.1$. The dashed lines have slopes -2 and -1. We have adopted an internal reflection parameter $z_0 = 10$.

$$T(L) = z_0\ell\left(2z_0\ell + \int_0^L dz\, \frac{D_B}{D(z)}\right)^{-1}. \qquad (8)$$

The integral is proportional to the optical thickness of the slab. If $D(z)$ is constant, Eq. (8) reduces to the familiar result of radiative transfer with internal reflection [32], with the $1/L$ scaling. For a very long slab we expect the solution $D_\infty(z)$ for a semi-infinite medium to be relevant. More precisely, $D(z) \approx D_\infty(\frac{1}{2}L - |\frac{1}{2}L - z|)$. Equation (8) gives

$$T(L) \to \begin{cases} 4z_0[D_\infty(0)/D_B](\xi_c/\ell) \times (\ell/L)^2, & k\ell = 1, \\ z_0[D_\infty(0)/D_B](\ell/\xi) \times \exp(-L/\xi), & k\ell < 1. \end{cases} \qquad (9)$$

This scale dependence agrees with scaling theory [15,33], but has large and precise prefactors. (2.6 for $k\ell = 1$ and $z_0 = \frac{2}{3}$, and increasing with z_0). In Figure 3 we compare these results to the numerical solutions of Eqs. (3a)–(3d). The $1/L^2$ law predicted at the mobility edge is seen to disappear rapidly in the delocalized regime $k\ell > 1$. It has been reported by Garcia and Genack [2] and Wiersma *et al.* [4].

In summary, we have developed a theory for wave propagation in finite media near the mobility edge, adopting a local diffusion picture. Globally, scale-dependent diffusion is seen to emerge that captures quantitatively known scaling properties in transmission and enhanced backscattering.

We thank David Lacoste for discussions. The work at LPM2C is supported by the GdR 1847 PRIMA of the French CNRS. The work at UVA is part of the Dutch FOM research program and is made possible by financial support of NWO. The work at LENS is supported by the EC Grant No. HPRI-CT1999-00111.

[1] P. Sheng, *Introduction to Wave Scattering, Localization and Mesoscopic Phenomena* (Academic, San Diego, 1995).

[2] N. Garcia and A. Z. Genack, Phys. Rev. Lett. **66**, 1850 (1991).

[3] A. Z. Genack and N. Garcia, Phys. Rev. Lett. **66**, 2064 (1991).

[4] D. S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, Nature (London) **390**, 671 (1997).

[5] For recent reviews, are *Scattering and Localization of Classical Waves,* edited by P. Sheng (World Scientific, Singapore, 1990); B. A. van Tiggelen, in *Diffuse Waves in Complex Media,* edited by J. P. Fouque (Kluwer, Dordrecht, 1999).

[6] F. C. MacKintosh and S. John, Phys. Rev. B **37**, 1884 (1988).

[7] F. Erbacher, R. Lenke, and G. Maret, Europhys. Lett. **21**, 55 (1993).

[8] D. S. Wiersma, M. P. van Albada, B. A. van Tiggelen, and A. Lagendijk, Phys. Rev. Lett. **74**, 4193 (1995).

[9] B. L. Altshuler, A. G. Aronov, and B. Z. Spivak, JETP Lett. **33**, 94 (1981); D. Yu. Sharvin and Yu. V. Sharvin, JETP Lett. **34**, 272 (1981).

[10] *Localization, Interaction and Transport Phenomena,* edited by B. Kramer, G. Bergmann, and Y. Bruynseraede (Springer, Berlin, 1985).

[11] F. J. P. Schuurmans, M. Megens, D. Vanmaekelbergh, and A. Lagendijk, Phys. Rev. Lett. **83**, 2183 (1999).

[12] C. W. J. Beenakker, Phys. Rep. **69**, 731 (1997).

[13] W. Götze, J. Phys. C **12**, 1279 (1979).

[14] D. Vollhardt and P. Wölfle, Phys. Rev. B **22**, 4666 (1980).

[15] P. Wölfle and D. Vollhardt, in *Electronic Phase Transitions* (Elsevier Science, Amsterdam, 1992).

[16] K. Busch and C. M. Soukoulis, Phys. Rev. Lett. **75**, 3442 (1995).

[17] B. A. van Tiggelen, D. S. Wiersma, and A. Lagendijk, Europhys. Lett. **30**, 1 (1995).

[18] S. John, H. Sompolinsky, and M. J. Stephen, Phys. Rev. B **27**, 5592 (1983).

[19] E. N. Economou, C. M. Soukoulis, and A. D. Zdetsis, Phys. Rev. B **30**, 1686 (1984).

[20] A. D. Zdetsis, C. M. Soukoulis, E. N. Economou, and G. S. Grest, Phys. Rev. B **32**, 7811 (1985).

[21] J. Kroha, T. Kopp, and P. Wölfle, Phys. Rev. B **41**, 888 (1990).

[22] P. Sheng and Z. Q. Zhang, Phys. Rev. Lett. **57**, 1879 (1986).

[23] Y. Imry, Y. Gefen, and D. Bergman, Phys. Rev. B **26**, 3436 (1982).

[24] E. Abrahams and P. A. Lee, Phys. Rev. B **33**, 683 (1986).

[25] R. Berkovits and M. Kaveh, Phys. Rev. B **36**, 9322 (1987).

[26] F. J. Wegner, Z. Phys. B **25**, 327 (1976).

[27] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[28] A. Lagendijk, B. Vreeker, and P. de Vries, Phys. Lett. A **136**, 81 (1989).

[29] M. R. Zirnbauer, Phys. Rev. Lett. **69**, 1584 (1992).

[30] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972), Sect. 9.6.1.

[31] E. Akkermans, P. E. Wolf, and R. Maynard, Phys. Rev. Lett. **56**, 1471 (1986); M. B. van der Mark, M. P. van Albada, and A. Lagendijk, Phys. Rev. B **37**, 3575 (1988).

[32] Th. M. Nieuwenhuizen and J. M. Luck, Phys. Rev. E **48**, 569 (1993).

[33] P. W. Anderson, Philos. Mag. B **52**, 505 (1985).