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ABSTRACT

Exact equations are derived which relate the (complex) index of refraction and the thickness of a single, absorbing or transparent film on an absorbing or transparent substrate to the changes sustained by polarized light reflected by the film-covered surface. Both film and substrate are assumed to be linear, homogeneous, and isotropic media.

INTRODUCTION

A practical method for determining the optical properties of materials is based on changes in the state of polarization which occur when light is reflected from surfaces or transmitted through layers (films). This report gives a theoretical analysis of the optical properties of an absorbing film on a metal base, with the emphasis on a self-contained treatment with uniform definitions and conventions throughout. Some of the theoretical results on bare metal surfaces, needed here, will be summarized briefly below. Reference may be made to a companion report[3] for detailed derivations of relationships between the complex index of refraction of a bare metal surface and the changes in polarization upon reflection from it.

When polarized light is reflected from a film-covered metal surface, the change in the state of polarization depends upon the refractive indices of metal surface and film and on the thickness of the film. The polarization state is characterized by the ratio of electric field amplitudes parallel and normal to the plane of incidence and by the phase difference between these two components.

Let the subscripts p and s be chosen to denote components parallel and normal to the plane of incidence respectively. Then the electric field components are E_p and E_s before reflection and E_p'' and E_s'' after reflection. The ratios of s and p amplitudes of incident and reflected waves are defined as

$$\tan \psi_i \equiv \frac{|E_s|}{|E_p|} \tag{1}$$

$$\tan\psi_r \equiv \frac{|E''_s|}{|E''_p|} \tag{2}$$

so that, on reflection, the amplitude ratio is changed by the factor

$$\tan\psi \equiv \frac{\tan\psi_r}{\tan\psi_i} \tag{3}$$

Figure 1 illustrates the coordinate system to be used. Figure 2 shows both electric $\vec{E}$ and magnetic $\vec{H}$ field vectors for the cases of electric field parallel to plane of incidence (a) and normal to the plane of incidence (b). The vector $\vec{S}$ is the Poynting vector which gives the direction of energy propagation. The x-z plane is the plane of incidence. The x-y plane defines the surface at which reflection and refraction occur.

The reflection coefficients, in general complex, are defined by

$$r_p \equiv \frac{E''_p}{E_p} = \frac{|E''_p| \quad e^{-i\varepsilon''_p}}{|E_p| \quad e^{-i\varepsilon_p}} = \frac{|E''_p|}{|E_p|}\, e^{-i(\varepsilon''_p - \varepsilon_p)} \tag{4}$$

$$r_s \equiv \frac{E''_s}{E_s} = \frac{|E''_s| \quad e^{-i\varepsilon''_s}}{|E_s| \quad e^{-i\varepsilon_s}} = \frac{|E''_s|}{|E_s|}\, e^{-i(\varepsilon''_s - \varepsilon_s)} \tag{5}$$

where the epsilons are the phases of the various components with respect to an arbitrary time origin. If we define the absolute phase changes by

$$\delta_p \equiv \varepsilon''_p - \varepsilon_p, \qquad \delta_s \equiv \varepsilon''_s - \varepsilon_s \tag{6}$$

the ratio of the reflection coefficients is

$$\rho \equiv \frac{r_s}{r_p} = \left(\frac{|E''_s|}{|E_s|} \Big/ \frac{|E''_p|}{|E_p|} \right) \frac{e^{-i\delta_s}}{e^{-i\delta_p}} \tag{7}$$

$$= \left(\frac{|E''_s|}{|E''_p|} \Big/ \frac{|E_s|}{|E_p|} \right) e^{i(\delta_p - \delta_s)}$$

Substituting into this equation the ratios

$$|E_s| \;/\; |E_p| \text{ and } |E''_s| \;/\; |E''_p|$$

from Eqs. (1) and (2) leads to the result

$$\rho = \frac{\tan\psi_r}{\tan\psi_i} \; e^{i(\delta_p - \delta_s)} \tag{8}$$

Since $\delta_p - \delta_s$ is the relative phase difference Δ imposed between the p and s components on reflection,

$$\Delta \equiv \delta_p - \delta_s \tag{9}$$

$$n(1 - \kappa^2) = n'(1 - \kappa'^2) \tag{13a}$$

$$n^2\kappa = n'^2\kappa' \cos\phi_r \tag{13b}$$

where ϕ_r is the real angle of refraction (Fig. 3).

In contrast to n and κ, the parameters n' and κ' have more easily recognizable physical significance, as n' is the ratio

$$n' \equiv c/v \tag{14}$$

of the phase velocity c of light in vacuum to the (real) phase velocity v in the medium. The significance of κ' is seen from the relation

$$\kappa' \equiv \frac{\alpha c}{\omega n'}, \tag{15}$$

where α is the absorption coefficient. The amplitude of the electric field diminishes by a factor 1/e after traveling a distance

$$d = \frac{1}{\alpha} = \frac{c}{\omega\kappa' n'} = \frac{\lambda_o}{2\pi\kappa' n'}$$

where λ_o is the vacuum wavelength. The quantity d is known as the field penetration depth.*

---

* This result follows from the fact that the fields decay as $e^{-\alpha z}$ where $\frac{\vec{z}}{z}$ is the normal unit vector to the surface. Often, half of this value, $d = \lambda_o/4\pi\kappa' n'$, is given for the penetration depth. The distance d is then defined as the depth at which the intensity is reduced to 1/e of its original value. Since the intensity is proportional to the modulus squared of the field, $I \propto e^{-2\alpha z}$, the intensity diminishes by a factor 1/e after the wave has penetrated to a depth $z = 1/2\alpha$, or one half the field penetration depth.

If light is incident on the plane surface of an absorbing medium from a transparent medium ($\sigma = 0$) of refractive index n_o at an angle ϕ from the normal to the boundary, then the angle between planes of equal phase and equal amplitude in the absorbing medium is ϕ_r (Fig. 3) and is found from Snell's Law

$$n_o \sin\phi = n' \sin\phi_r \tag{16}$$

The (real) angle ϕ_r is known as the angle of refraction.

The ultimate equations describing the reflection are put in their simplest form when written in terms of the complex angle of refraction ϕ' defined by

$$\sin\phi' \equiv \frac{n_o \sin\phi}{n_c} = \frac{n' \sin\phi_r}{n_c} \tag{17}$$

Another relation equivalent to Eq. (17) is

$$n_c \cos\phi' = n'(\cos\phi_r + i\kappa') \tag{18}$$

In the chapters to follow, all electric and magnetic fields will be assumed to be of the form

$$\vec{A} = \vec{A}_o \, e^{i(\vec{k}_c \cdot \vec{r} - \omega t)} \tag{19}$$

where $\vec{A}$ stands for the electric field $\vec{E}$ or the magnetic field $\vec{H}$. The phase is contained in the complex amplitude $\vec{A}_o$. The complex wave vector $\vec{k}_c$ is, in general, written as

$$\vec{k}_c = \vec{k} + i\alpha\hat{z} = \frac{\omega n'}{c}\left[\pm\hat{x}\sin\phi_r \pm \hat{z}(\cos\phi_r + i\kappa')\right] \qquad (20)$$

or, with Eq. (17) and (18), and the definition

$$k_o \equiv \frac{\omega}{c} = \frac{2\pi}{\lambda_o}$$

$$\vec{k}_c = k_o n_c (\pm\hat{x}\sin\phi' \pm \hat{z}\cos\phi') \qquad (20a)$$

In Eqs. (20) and (20a), the $\pm$ signs are chosen depending on whether the real wave vector $\vec{k}$, $|\vec{k}| \equiv \omega/v = \frac{\omega n'}{c}$, has positive or negative x and/or z components, respectively. Eqs. (1) through (20a) are developed in Chapter I of the companion report[3].

Figure 4a shows how a ray of light incident on a film-covered surface is partially transmitted and partially reflected at the first film surface (taken to be the plane z = -L), then at the film-substrate surface (the plane z = 0), and finally again at the film-air surface. In the discussion that follows the x-z plane is the plane of incidence and all surfaces are parallel to the yz plane. This convention is illustrated by Figs. 1 and 2.

It will be assumed that the system of reflected and refracted waves shown in Fig. 4a can be replaced by the system shown in Fig. 4b, where $\vec{E}''$ is a wave equivalent to all the reflected waves leaving the film, $\vec{E}'$ and $\vec{E}''$ are the equivalent of all waves in the film, and $\vec{E}_m$ is the equivalent of all waves in the metal which is taken to be of infinite extent in the +z direction.[7,8]

verified if the resulting $\vec{E}_o$ and $\vec{H}_o$ satisfy Eqs. (21-24) above.* For the wave reflected at the film-metal surface $\vec{k}_c'''$ can be written

$$\vec{k}_c''' = \frac{\omega n'}{c} [\hat{x} \sin\phi_r - \hat{z}(\cos\phi_r + i\kappa')] \quad (26)$$

Substituting Eqs. (25) and (26) into the transversality equation

$$\vec{k}_c''' \cdot \vec{E}_o''' = 0 \quad (21)$$

one has

$$\varepsilon_1''' E_p''' \sin\phi_r - \varepsilon_3''' E_p''' (\cos\phi_r + i\kappa') = 0$$

$$\varepsilon_1''' \sin\phi_r = \varepsilon_3''' (\cos\phi_r + i\kappa')$$

A solution which satisfies this equation and which leads to expressions for $\vec{E}_o'''$ and $\vec{H}_o'''$ that satisfy Maxwell's equations is

$$\varepsilon_1''' = (\cos\phi_r + i\kappa')$$

$$\varepsilon_3''' = \sin\phi_r$$

Therefore, a possible solution for $\vec{E}_o'''$ is

---

* A vector field is uniquely specified when its curl and divergence are known. Since Maxwell's equations give the curl and divergence of $\vec{E}$ and $\vec{H}$, there can be only one solution satisfying Eqs. (21-24).

$$\frac{c}{\omega}\vec{k}_c''' \times \vec{H}_o''' = n'^2\left[\hat{x}\sin\phi_r - \hat{z}(\cos\phi_r + i\kappa')\right] \times$$

$$\left[\hat{x}E_s'''(\cos\phi_r + i\kappa') - \hat{y}E_p'''(1-\kappa'^2 + 2i\kappa'\cos\phi_r)\right.$$

$$\left. + \hat{z}E_s'''\sin\phi_r\right]$$

$$= -n'^2\left[\hat{z}\sin\phi_r E_p'''(1-\kappa'^2 + 2i\kappa'\cos\phi_r) + \hat{y}\sin^2\phi_r E_s''' + \hat{y}E_s'''(\cos\phi_r + i\kappa')^2\right.$$

$$\left. + \hat{x}E_p'''(1-\kappa'^2 + 2i\kappa'\cos\phi_r)(\cos\phi_r + i\kappa')\right]$$

$$= -\left[\hat{z}\sin\phi_r E_p''' + \hat{y}E_s''' + \hat{x}E_p'''(\cos\phi_r + i\kappa')\right]n'^2(1-\kappa'^2 + 2i\kappa'\cos\phi_r)$$

$$= -\vec{E}_o'''\, n_c^2$$

where the last step follows from Eqs. (11), (12), (13a) and (13b). Therefore Eqs. (27) and (28) are the required solutions.

Now that $\vec{E}_o'''$ and $\vec{H}_o'''$ have been found and have been shown to satisfy Maxwell's Equations, the ten electric and magnetic vectors can be summarized.

Writing each electric field vector in the form

$$\vec{E} = \vec{E}_o\, e^{-i(\omega t - \vec{k}_c \cdot \vec{r})}$$

there results

$$\vec{E} = (E_p\cos\phi\hat{x} + E_s\hat{y} - E_p\sin\phi\hat{z})e^{-\frac{2\pi i}{\lambda_o}[ct - n_o(x\sin\phi + z\cos\phi)]} \qquad (29)$$

It will be convenient to rewrite these equations in a simpler form by using the complex angle of refraction. The relations needed to make the simplification are the following:

$$n_c \sin \phi' = n_o \sin\phi = n' \sin \phi_r \quad (17)$$

$$n_c \cos \phi' = n' (\cos \phi_r + i\kappa') \quad (18)$$

$$n_{cm} \sin \phi_m' = n_o \sin \phi = n_{cm}' \sin \phi_{rm} \quad (17)$$

$$n_{cm} \cos \phi_m' = n_{cm}' (\cos \phi_{rm} + i\kappa_m') \quad (18)$$

First Eq. (17) is used to replace $n' \sin \phi_r$ in the expressions for $\vec{E}'$, $\vec{E}'''$, $\vec{H}'$, and $\vec{H}'''$ and $n_m' \sin \phi_{rm}$ in the expressions for $\vec{E}_m$ and $\vec{H}_m$ by $n_o \sin \phi$. Then all ten fields will contain the same term $ct-n_o x\sin \phi$ in the brackets of the exponent. This common term would cancel in the calculations that follow, so it will be ignored in the interest of simplicity. With these changes, and with the definition

$$k_o \equiv 2\pi/\lambda_o$$

the ten field vectors can be written as follows (see Fig. 4b)

$$\vec{E} = (E_p \cos\phi\ \hat{x} + E_s\hat{y} - E_p \sin\phi\ \hat{z})\ e^{ik_o z n_o \cos\phi} \quad (29a)$$

$$\vec{E}'' = (E_p'' \cos\phi\ \hat{x} + E_s''\hat{y} + E_p'' \sin\phi\ \hat{z})\ e^{-ik_o z n_o \cos\phi} \quad (30a)$$

$$\vec{E}' = \left(\frac{n_c}{n'} E_p' \cos\phi'\ \hat{x} + E_s'\hat{y} - E_p' \frac{n_c}{n'} \sin\phi'\ \hat{z}\right) e^{ik_o z n_c \cos\phi'} \quad (31a)$$

B. Evaluation of Boundary Conditions

When there are no surface charges or surface currents the boundary conditions require that the tangential components of $\vec{H}$ and $\vec{E}$ and the normal components of $n_c^2\vec{E}$ and $\mu\vec{H}$ be continuous across each interface. Thus, at the air-film interface z = -L, and with $\mu = 1$ everywhere,

$$n_o^2 \, (\vec{E} + \vec{E}'') \cdot \hat{z} = n_c^2(\vec{E}' + \vec{E}''') \cdot \hat{z} \tag{39}$$

$$(\vec{H} + \vec{H}'') \cdot \hat{z} = (\vec{H}' + \vec{H}''') \cdot \hat{z} \tag{40}$$

$$(\vec{E} + \vec{E}'') \times \hat{z} = (\vec{E}' + \vec{E}''') \times \hat{z} \tag{41}$$

$$(\vec{H} + \vec{H}'') \times \hat{z} = (\vec{H}' + \vec{H}''') \times \hat{z} \tag{42}$$

When Eqs. (29a) through (32a) are evaluated at z = -L and substituted into Eq. (39), there results

$$n_o \left(-E_p e^{-i\tau_o} + E_p'' \, e^{i\tau_o} \right) n_o \sin\phi = \frac{n_c^2}{n'} \left(-E_p' \, e^{-i\tau} + E_p''' \, e^{i\tau} \right) n_c \, \sin\phi'$$

where

$$\tau_o \equiv k_o L n_o \cos\phi = \frac{2\pi}{\lambda_o} L n_o \cos\phi \tag{43}$$

$$\tau \equiv k_o L n_c \cos\phi' = \frac{2\pi}{\lambda_o} L n_c \cos\phi' \tag{44}$$

This result can be rewritten since

$$n_o \sin\phi = n_c \sin\phi' \tag{17}$$

Thus

$$n_o \left(E_p'' \, e^{i\tau_o} - E_p \, e^{-i\tau_o} \right) = \frac{n_c^2}{n'} \left(E_p''' e^{i\tau} - E_p' \, e^{-i\tau} \right) \tag{45}$$

$$n_c^2 \, (\vec{E}' + \vec{E}''') \cdot \hat{z} = n_{cm}^2 \vec{E}_m \cdot \hat{z} \tag{49}$$

$$(\vec{H}' + \vec{H}''') \cdot \hat{z} = \vec{H}_m \cdot \hat{z} \tag{50}$$

$$(\vec{E}' + \vec{E}''') \times \hat{z} = \vec{E}_m \times \hat{z} \tag{51}$$

$$(\vec{H}' + \vec{H}''') \times \hat{z} = \vec{H}_m \times \hat{z} \tag{52}$$

When Eqs. (31a) through (33a) are evaluated at z = 0 and substituted into Eq. (49), there results

$$\frac{n_c^2}{n'} \, (E_p''' - E_p') \, n_c \sin \phi' = - \frac{n_{cm}^2}{n_m'} n_{cm} \sin \phi_m' \, E_{pm}$$

or, since

$$n_c \sin \phi' = n_{cm} \sin \phi_m' \tag{17}$$

$$\frac{n_c^2}{n'} \, (E_p' - E_p''') = \frac{n_{cm}^2}{n_m'} E_{pm} \tag{53}$$

When Eqs. (31a) through (33a) are evaluated at z = 0 and substituted into Eq. (51), there results

$$- \frac{n_c}{n'} E_p' \cos \phi' \, \hat{y} + E_s' \, \hat{x} - \frac{n_c}{n'} E_p''' \cos \phi' \, \hat{y} + E_s''' \, \hat{x} =$$

$$- \frac{n_{cm}}{n_m'} E_{pm} \cos \phi_m' \, \hat{y} + E_{sm} \, \hat{x}$$

This equation implies two others, namely,

$$\frac{n_c}{n'} \, (E_p' + E_p'') \cos \phi' = \frac{n_{cm}}{n_m'} \, E_{pm} \cos \phi_m' \tag{54}$$

$$E_s' + E_s''' = E_{sm} \tag{55}$$

$$\tau_o \equiv k_o L\, n_o \cos\phi = \frac{2\pi L}{\lambda_o} n_o \cos\phi \tag{43}$$

$$\tau \equiv k_o L\, n_c \cos\phi' = \frac{2\pi L}{\lambda_o} n_c \cos\phi' \tag{44}$$

At z = 0 (film-metal interface)

$$\frac{n_c^2}{n'}\left(E_p''' - E_p'\right) = -\frac{n_{cm}^2}{n_m'} E_{pm} \tag{53}$$

$$\frac{n_c}{n'}\left(E_p''' + E_p'\right)\cos\phi' = \frac{n_{cm}}{n_m'} E_{pm}\cos\phi_m' \tag{54}$$

$$E_s' + E_s''' = E_{sm} \tag{55}$$

$$n_c\left(E_s' - E_s'''\right)\cos\phi' = n_{cm} E_{sm}\cos\phi_m' \tag{56}$$

C. Derivation of Reflection Coefficients

For the calculations of the next chapter it will be necessary to know the reflection coefficients for the film-substrate interface. These are defined by

$$r_{2s} \equiv \frac{E_s'''}{E_s'} \qquad r_{2p} \equiv \frac{E_p'''}{E_p'} \tag{57a,b}$$

where the electric field amplitudes E are to be evaluated at z = 0. The pertinent equations are Eqs. (53) through (56) derived in the previous section and summarized on the preceding page.

Divide Eq. (53) by Eq. (54) and multiply by $\frac{\cos\phi'}{n_{cm}}$ to get or

$$(1 - r_{2s})\ n_c \cos\phi' = (1 + r_{2s})\ n_{cm} \cos\phi'_m \tag{60}$$

$$n_c \cos\phi' - r_{2s} n_c \cos\phi' = n_{cm} \cos\phi'_m + n_{cm}\ r_{2s} \cos\phi'_m$$

Thus

$$r_{2s} = \frac{n_c \cos\phi' - n_{cm} \cos\phi'_m}{n_c \cos\phi' + n_{cm} \cos\phi'_m} \tag{61}$$

Note that

$$\frac{1 + r_{2s}}{1 - r_{2s}} = \frac{n_c \cos\phi'}{n_{cm} \cos\phi'_m} \tag{60a}$$

Equations (59) and (61) are the Fresnel coefficients for reflection at the plane boundary between two linear, homogeneous, isotropic, absorbing media.

D. Summary - Fresnel Equations for Two Absorbing Media

The Fresnel Equations for an interface between two absorbing media are as follows:

$$r_p \equiv \frac{E_{p\ ref}}{E_{p\ inc}} = \frac{n_c \cos\phi'_m - n_{cm} \cos\phi'}{n_c \cos\phi'_m + n_{cm} \cos\phi'} \tag{59}$$

$$r_s \equiv \frac{E_{s\ ref}}{E_{s\ inc}} = \frac{n_c \cos\phi' - n_{cm} \cos\phi'_m}{n_c \cos\phi' + n_{cm} \cos\phi'_m} \tag{61}$$

In Eqs. (59) and (61), n_c and n_{cm} are the complex indices of refraction of the incident and base medium respectively. They are the complex square roots of the quantities

CHAPTER II. REFLECTION FROM A FILM-COVERED SURFACE

In this chapter an exact, general equation is derived which describes the reflection of polarized light from an absorbing surface covered by a single, absorbing film. As in the previous two chapters it will be assumed that the media are linear, isotropic, and homogeneous.

For the derivation of the equation it will be necessary to know the reflection coefficients for the incident medium-film interface. These are found from Eqs. (59) and (61) of the last chapter by replacing the complex refractive index n_c and the complex angle of refraction ϕ' by n_o and ϕ respectively (where n_o is the real refractive index of the incident medium and ϕ is the real angle of incidence) and by replacing n_{cm} and ϕ'_m by n_c and ϕ': Thus the reflection coefficients at the incident medium-film interface can be written as

$$r_{1s} = \frac{n_o \cos\phi - n_c \cos\phi'}{n_o \cos\phi + n_c \cos\phi'} \tag{62}$$

$$r_{1p} = \frac{n_o \cos\phi' - n_c \cos\phi}{n_o \cos\phi' + n_c \cos\phi} \tag{63}$$

The analogs of Eqs. (58a) and (60a) are

$$\frac{1 + r_{1p}}{1 - r_{1p}} = \frac{n_o \cos\phi'}{n_c \cos\phi} \tag{64}$$

$$\frac{1 + r_{1s}}{1 - r_{1s}} = \frac{n_o \cos\phi}{n_c \cos\phi'} \tag{65}$$

The equations summarized at the end of Chapter I, Section B will yield the ultimate result. To begin with, the eight equations (45) - (48) and (53) - (56) are condensed into two equations (68) and (71) in the following way.

where the last step is made by the use of Eq. (58a). Division of Eq. (55) by Eq. (56) gives

$$\frac{E_s' + E_s'''}{E_s' - E_s'''} = \frac{n_c \cos\phi'}{n_{cm}\cos\phi_m'} = \frac{1 + r_{2s}}{1 - r_{2s}} \tag{70}$$

where the last step is made by the use of Eq. (60a). Equations (69) and (70) can be summarized as

$$\frac{E_\nu' + E_\nu'''}{E_\nu' - E_\nu'''} = \frac{1 + r_{2\nu}}{1 - r_{2\nu}} \tag{71}$$

where ν stands for p or s.

Next, Eqs. (68) and (71) are solved for the ratio E_ν''/E_ν which is the desired reflection coefficient of the film-covered surface. Divide numerator and denominator of the right-hand side by E_ν' to get

$$\frac{e^{-2i\tau_o} + E_\nu''/E_\nu}{e^{-2i\tau_o} - E_\nu''/E_\nu} = \frac{1 + r_{1\nu}}{1 - r_{1\nu}}\;\frac{1 + r_{2\nu}\, e^{2i\tau}}{1 - r_{2\nu}\, e^{2i\tau}} \tag{73}$$

Define the complex reflection coefficient of the film covered surface by

$$r_\nu \equiv \frac{E_\nu''}{E_\nu} \tag{74}$$

and set the right hand side of Eq. (73) equal to the variable ξ for convenience. Then

$$\frac{e^{-2i\tau_o} + r_\nu}{e^{-2i\tau_o} - r_\nu} = \xi$$

$$e^{-2i\tau_o} + r_\nu = \xi\left(e^{-2i\tau_o} - r_\nu\right)$$

complex optical distance, and $2\tau_o$ is an overall phase angle resulting from the particular origin chosen for the calculation.

This result (Eq. (75)) was originally obtained by Drude,[6] but in his equation the exponents are negative while here they are positive. This difference is due to a different representation of the electromagnetic wave in the complex notation which was here

$$\vec{E} = \vec{E}_o \, e^{-i(\omega t - \vec{k} \cdot \vec{r})}$$

The spatial dependence is in this case

$$\vec{E}_s = \vec{E}_o \, e^{i\vec{k} \cdot \vec{r}}$$

Drude, on the other hand, chose to represent the wave by

$$\vec{E} = \vec{E}_o \, e^{i(\omega t - \vec{k} \cdot \vec{r})}$$

for which the spatial dependence is

$$\vec{E}_s = \vec{E}_o \, e^{-i\vec{k} \cdot \vec{r}}$$

Therefore, the two derivations yield opposite signs in the exponents of Eqs. (29a) through (38a).

In our calculations the different sign has carried all the way through to Eq. (75) where we have a positive exponent and Drude has a negative one.*

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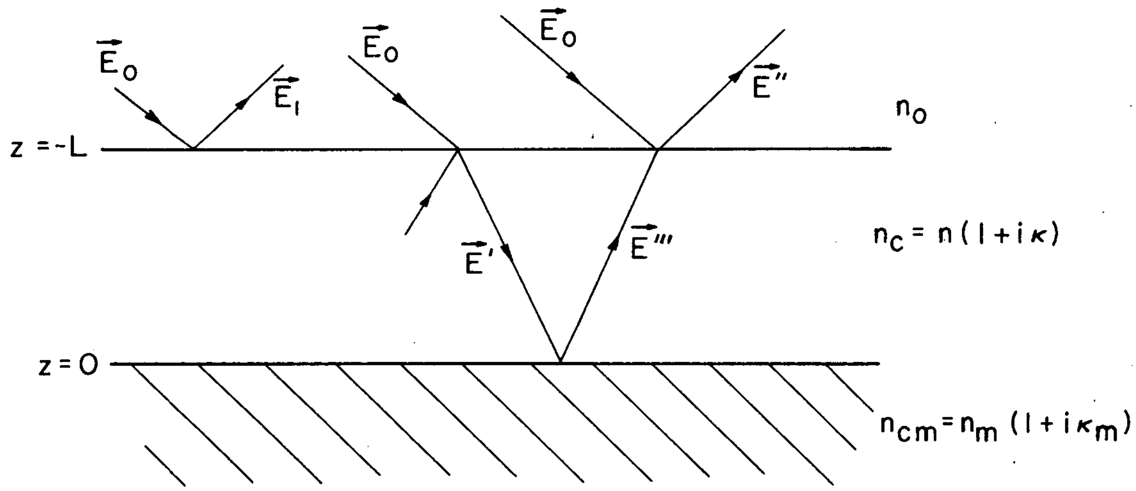
* Others using Drude's convention are Winterbotton,[7] McCrackin,[15] and Leberknight and Lustman.[4]

Fig. 5 Illustration of Meaning of Reflection Coefficients r_{1s}, r_{1p}, r_{2s}, r_{2p}, r_s, and r_p. The electric field vectors $\vec{E}_o$, $\vec{E}'$, $\vec{E}''$, and $\vec{E}'''$ are defined in Fig. 4b and by Eqs. (29a), (30a), (31a), and (32a) (the $\vec{E}_o$ of this figure is the $\vec{E}$ of Eq. (29a)). The electric field vector $\vec{E}_1$ results from reflection at the first interface only and corresponds to reflection from an infinitely thick film. $\vec{E}''$ is the resultant of all reflections at both interfaces as indicated in Fig. 4b. The various waves can be written as sums of component waves parallel (p) and normal (s) to the plane of incidence.



$$\vec{E}_o = (\vec{E}_o \cdot \hat{p})\hat{p} + (\vec{E}_o \cdot \hat{s})\hat{s}$$

$$\vec{E}_1 = r_{1p}(\vec{E}_o \cdot \hat{p})\hat{p} + r_{1s}(\vec{E}_o \cdot \hat{s})\hat{s}$$

$$\vec{E}' = (\vec{E}' \cdot \hat{p})\hat{p} + (\vec{E}' \cdot \hat{s})\hat{s}$$

$$\vec{E}''' = r_{2p}(\vec{E}' \cdot \hat{p})\hat{p} + r_{2s}(\vec{E}'_s \cdot \hat{s})\hat{s}$$

$$\vec{E}'' = r_p(\vec{E}_o \cdot \hat{p})\hat{p} + r_s(\vec{E}_o \cdot \hat{s})\hat{s}$$

The unit vectors $\hat{p}$ and $\hat{s}$ are parallel and normal to the plane of incidence, respectively.

CHAPTER III. COMPUTER SOLUTIONS OF FILM EQUATIONS

The basic equation describing the reflection of polarized light from a film covered metal surface, as derived in the previous section, is

$$r_\nu \equiv \frac{E_\nu \text{ reflected}}{E_\nu \text{ incident}} = \frac{r_{1\nu} + r_{2\nu}\, e^{iD}}{1 + r_{1\nu} r_{2\nu} e^{iD}}\, e^{iD_o} \qquad (75)$$

where r_1, r_2, D and D_o are given in the preceding summary. The complex quantity r_ν is the reflection coefficient of the film-metal combination, giving both the phase and amplitude changes of the reflected light.

By analogy with Eqs. (7) and (10) we write

$$\rho \equiv r_s/r_p = \tan\psi\, e^{i\Delta} \qquad (77)$$

where $\tan\psi$ is the amplitude diminution and Δ the relative phase change caused by the reflection.*

In practice one is primarily interested in finding the thickness L and refractive index $n_c = n + in\kappa$ of a film on a substrate of known properties from measured values of ψ and Δ. Although L and n_c are related to ψ and Δ through Eq. (77), and although Eq. (77) can be solved explicitly for L as a function of n_c, ϕ and Δ (as in Section B below), it is not possible to solve Eq. (77) explicitly for n_c. Two methods will now be presented for obtaining n_c and L with the use of a high speed computer.

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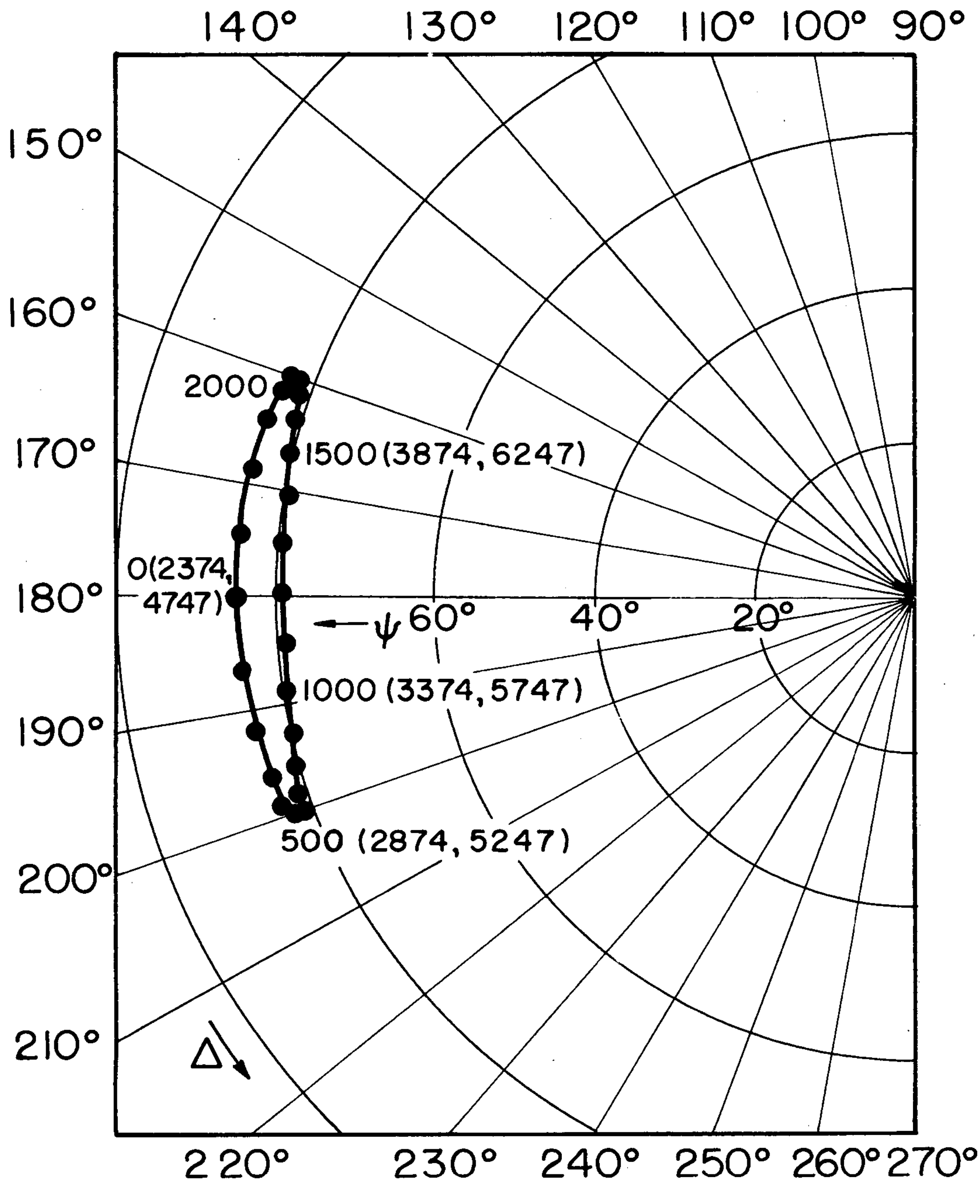
* McCracken[15] and other authors use the definition $\rho = \tan\psi\, e^{i\Delta} = r_p/r_s$. Owing to this and other differences in convention the ψ given here will be the complement of that found in much of the literature, while the Δ given here is essentially the same as that in the literature (i.e., $\Delta = \delta_p - \delta_s$, Eq. (9)). In practice one solves the exact Eq. (77) for $\tan\Delta$ and not Δ. When only the tangent of an angle is known (and not its sine or or cosine) the quadrant in which the angle is to be placed is ambiguous. Thus, further differences arise in the literature.



XBL 671-299

Fig. 6 Transparent film on Transparent Substrate (CaF_2 on glass). Values of ψ and Δ calculated for increasing film thickness for wavelength 5461 Å, angle of incidence 60°, base constants $n_s = 1.5190$, $\kappa_s = 0$ and film constants $n = 1.4339$, $\kappa = 0$. The curve is labeled for various thicknesses and closes on itself at a thickness of 2373.6 Å (optical thickness).

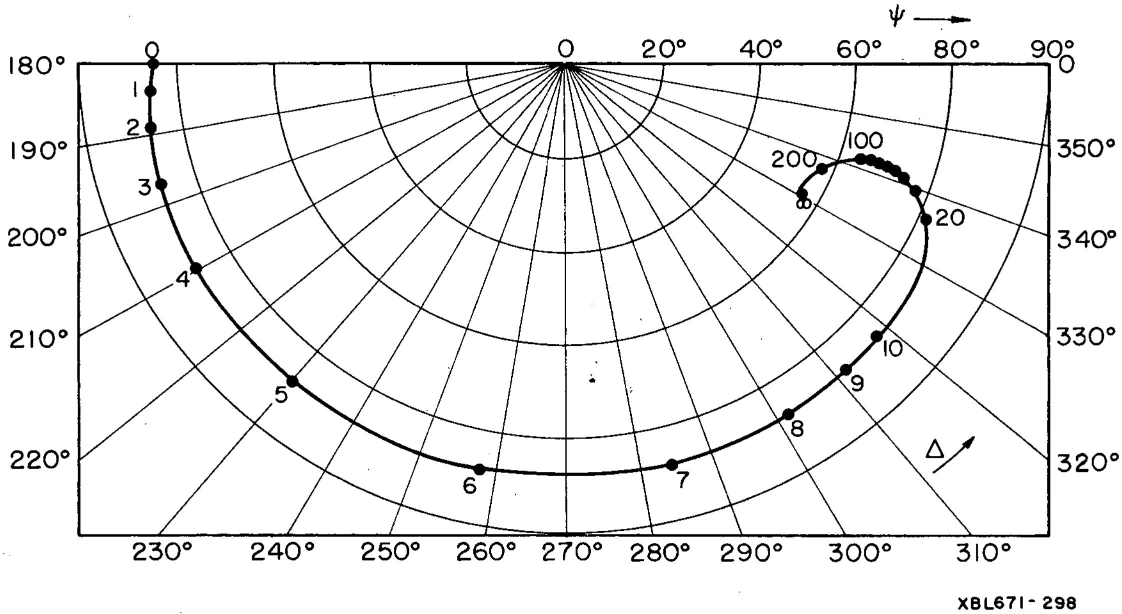


Fig. 8 Absorbing Film on Transparent Base (Chromium on Glass). Values of ψ and Δ calculated for increasing film thickness for wavelength 5461 Å, angle of incidence 60°, base constants $n_s = 1.519$, $\kappa_s = 0.0$, and film constants $n = 2.96$, $n\kappa = 3.45$. The curve does not close on itself but spirals from ψ and Δ corresponding to the bare glass to the values corresponding to bare chromium.

B. Method of McCracken et al.[15,16]

A method similar to the one described above, but more efficient has been devised by McCracken et al. By this method the film thickness is calculated directly from measurements of ψ and Δ and from assumed values of n and $n\kappa$. With this method one need not assume values of film thickness, but only of n and $n\kappa$. Thus, in the example of Section A there would be 16 times fewer combinations to try.

The equations from which the film thickness can be deduced will now be devloped. The starting point is Eq. (77). Substituting from Eq. (75) into Eq. (77), one has

$$\rho = \frac{r_s}{r_p} = \frac{\dfrac{r_{1s} + r_{2s}\, e^{iD}}{1 + r_{1s}\, r_{2s}\, e^{iD}}}{\dfrac{r_{1p} + r_{2p}\, e^{iD}}{1 + r_{1p} r_{2p} e^{iD}}} = \tan\psi\, e^{i\Delta} \tag{78}$$

The value of ρ can be found from the measured values of ψ and Δ, and is thus assumed to be known. Eq. (78) is reduced to the standard form for a quadratic equation in the variable e^{iD} as follows. Write

$$\rho = \frac{\left(r_{1s} + r_{2s}\, e^{iD}\right)\left(1 + r_{1p}\, r_{2p}\, e^{iD}\right)}{\left(1 + r_{1s}\, r_{2s}\, e^{iD}\right)\left(r_{1p} + r_{2p}\, e^{iD}\right)}$$

$$\rho = \frac{r_{1s} + (r_{1s}\, r_{1p}\, r_{2p} + r_{2s})\, e^{iD} + r_{1p}\, r_{2p}\, r_{2s}\, e^{i2D}}{r_{1p} + (r_{1p}\, r_{1s}\, r_{2s} + r_{2p})\, e^{iD} + r_{1s}\, r_{2p}\, r_{2s}\, e^{i2D}}$$

or, multiplying both sides by the denominator of the right side and collecting terms,

Taking the complex logarithm of both sides of Eqs. (82) and (83)*
and letting E stand for either E_1 or E_2 and D for D_1 or D_2

$$iD = \text{Log}\,|E| + i\,\tan^{-1}\frac{\text{Im}\,E}{\text{Re}\,E} \tag{84}$$

where

$$|E| = \sqrt{E \cdot E^*}$$

and Im E and Re E are the imaginary and real parts of E, respectively.
Thus, with the original definition of D

$$D = \frac{4\pi}{\lambda_o}\, L\, n_c \cos\phi' \tag{76}$$

Equation (84) becomes

$$\frac{4\pi i L}{\lambda_o}\, n_c \cos\phi' = \text{Log}\,|E| + i\,\tan^{-1}\frac{\text{Im}\,E}{\text{Re}\,E}$$

or

$$\frac{4\pi L\, n_c \cos\phi'}{\lambda_o} = \tan^{-1}\frac{\text{Im}\,E}{\text{Re}\,E} - i\,\text{Log}\,|E|$$

Solving for L gives

$$L = \frac{\lambda_o}{4\pi\, n_c \cos\phi'}\left[\tan^{-1}\frac{\text{Im}\,E}{\text{Re}\,E} - i\,\text{Log}\,|E|\right] \tag{85}$$

Since there are two solutions for E, there will be two solutions for L. The film thickness L must be a real quantity, so the right-hand

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* Ref. 18, p. 55.

APPENDIX I

Single Film Computer Program "FILM"

The program "FILM" finds the thickness and complex refractive index of a single, absorbing film on an absorbing substrate. It does so by systematically combining all prescribed values of film thickness L and refractive index $n(1 + i\kappa)$ and calculating the relative phase change Δ and amplitude diminution $\tan\psi$ for each combination. Whenever a particular combination of L, n, and $n\kappa$ yields agreement with the experimentally determined quantities Δ and ψ within a specified error ϵ_Δ and ϵ_ψ this combination appears in the output as a solution.

The equations evaluated by the program are the following:

$$\cos \phi'_s = \sqrt{1 - \frac{n_o^2 \sin^2\phi}{n_{cs}^2}} \qquad \text{from Eq. (17)}$$

$$\cos \phi' = \sqrt{1 - \frac{n_o^2 \sin^2\phi}{n_c^2}} \qquad \text{from Eq. (17)}$$

$$r_{1s} = \frac{n_o\cos\phi - n_c\cos\phi'}{n_o\cos\phi + n_c\cos\phi'} \tag{62}$$

$$r_{1p} = \frac{n_o\cos\phi' - n_c\cos\phi}{n_o\cos\phi' + n_c\cos\phi} \tag{63}$$

$$r_{2s} = \frac{n_c\cos\phi' - n_{cs}\cos\phi'_s}{n_c\cos\phi' + n_{cs}\cos\phi'_s} \tag{61}$$

Sequence and Format of Data Cards for Program "FILM"

Card	Col. 1	Col. 10	Col. 20	Col. 30	Col. 40	Col. 50
1	Title and comments (up to 80 columns)					
2	Ranges of n, nκ, and L (up to 80 columns)					
3	n_o	λ_o	n_s	$n_s\kappa_s$		
4	n (initial)	δn (increment)	n (final)	nκ (initial)	δnκ (increment)	nκ (final)
5	L (initial)	δL (increment)	L (final)			
6	ϕ	ψ	Δ	ϵ_ψ	ϵ_Δ	

These six cards constitute a set. Any number of sets may follow. Two blank cards must follow card 6 of the last set of data, and a card with 0.0 punched in the first field (columns 1 - 9) must follow these blank cards. Cards 1 and 2 of each set may contain any comments (or none at all) desired by the user. Their contents appear printed verbatim at the head of the output. The two cards serve to conveniently identify the output data. All numbers entered on cards 3 - 6 must contain a decimal point, and may be located anywhere in the nine columns beginning with the one indicated.

Variables -- real

Name	Symbol	Description
DELM	Δ (measured)	relative phase change (degrees)
DELC	Δ (calculated)	relative phase change (degrees)
DTN2(1)	δn	iteration increment of film index
DTN2(2)	$\delta n\kappa$	iteration increment of film $n\kappa$
DT	δT	iteration increment of film thickness
EPSIM	$\epsilon\psi$	experimental error in ψ (degrees)
EDELM	$\epsilon\Delta$	experimental error in Δ (degrees)
PHI1	ϕ	angle of incidence (degrees)
PSIM	ψ (measured)	arctangent of amplitude diminution (degrees)
PSIC	ψ (calculated)	arctangent of amplitude diminution (degrees)
TNFI	n_o	refractive index of incident medium
TN3(1)	n_s	n of substrate
TN3(2)	$n_s\kappa_s$	$n\kappa$ of substrate
TNI	n_i	lower limit of iteration span for n of film
TNKFI	$(n\kappa)_i$	lower limit of iteration span for $n\kappa$ of film
TN2(1)	n	n of film (result)
TN2(2)	$n\kappa$	$n\kappa$ of film (result)
TN2M(1)	n_m	upper limit of iteration span for n of film
TN2M(2)	$(n\kappa)_m$	upper limit of iteration span for $n\kappa$ of film
TI	L_i	lower limit of iteration span for film thickness

Variables -- Complex

Name	Symbol	Description
CPHI2	$\cos \phi'$	complex cosine of complex angle of refraction in film
CHPI3	$\cos \phi'_s$	complex cosine of complex angle of refraction in substrate
PL	D	complex optical path length
R1S	r_{1s}	Fresnel reflection coefficient at air-film interface for polarization normal to plane of incidence
R1P	r_{1p}	Fresnel reflection coefficient at air-film interface for polarization parallel to plane of incidence
R2S	r_{2s}	Fresnel reflection coefficient at film-metal interface (normal polarization)
R2P	r_{2p}	Fresnel reflection coefficient at film-metal interface (parallel polarization)
RS	r_s	reflection coefficient at film-metal combination (normal polarization)
RP	r_p	reflection coefficient of film-substrate combination (parallel polarization)
RHO	$\rho = r_s/r_p = \tan\psi \; e^{i\Delta}$	
A, B, C, D		temporary variables

A flow sheet and reproduction of the program appear on the next two pages. The output for the program "FILM" is almost the same as the output for the program "SFILM" for which a sample output is given at the end of Appendix II. In the output from "FILM", however, the line beginning with "RANGES" might (if desired by user) be amended to include information concerning the range of film thickness considered along with the iteration increment. Also the line regarding imaginary part of film thickness does not appear.

"FILM" program, Fortran II version

```
      FORTRAN II PROGRAM FILM1(INPUT,OUTPUT,TAPE2=INPUT,TAPE3=OUTPUT)
C     THIS PROGRAM CALCULATES THE THICKNESS AND COMPLEX REFRACTIVE
C     INDEX OF A SINGLE ABSORBING FILM ON AN ABSORBING SUBSTRATE
      DIMENSION A(2), B(2), C(2), D(2), CPHI2(2), CPHI3(2), TN2(2),
     CDTN2(2), TN2M(2), TN3(2), R1S(2), R1P(2), R2S(2), R2P(2), PL(2),
     C RS(2), RP(2), RHO(2), TITLE(8), RANGE (8)
      COR = 0.017453
    1 READ (2,6) TITLE, RANGE
    2 READ (2,7) TN1, WAVE, TN3(1), TN3(2)
      IF (TN1) 3000, 3000, 3
    3 READ (2,8) TNFI, DTN2(1), TN2M(1), TNKFI, DTN2(2), TN2M(2)
    4 READ (2, 9) TI, DT, TM
    5 READ (2,10) PHI1, PSIM, DELM, EPSIM, EDELM
    6 FORMAT (8A10/8A10)
    7 FORMAT (F5.0, 3F10.0)
    8 FORMAT (F9.0, 5F10.0)
    9 FORMAT (F5.0, 2F10.0)
   10 FORMAT (F5.0, 4F10.0)
   11 FORMAT (1H0,/6HPHI = ,F5.2,10X,4HN = ,F7.4, 10X, 13HWAVELENGTH = ,
     C F5.0, 11H  ANGSTROMS//33H REFRACTIVE INDEX OF SUBSTRATE = , F7.4,
     C 2X, 4H+  I, F7.4)
   12 FORMAT (1H0, 27HREFRACTIVE INDEX OF FILM = , F7.4, 2X,
     C 4H+  I, F7.4//18H FILM THICKNESS = , F7.2, 10H ANGSTROMS,
     C //8H PSIC = , F10.5, 10X, 7HDELC = , F10.5,
     C //8H PSIM = , F10.5, 10X, 7HDELM = , F10.5)
   13 FORMAT (1H1, 8A10//8A10)
   14 FORMAT (1H0, 7HPSIM = , F10.5,10X,  7HDELM = , F10.5//32H NO SOLUT
     CION WITHIN GIVEN LIMITS)
      WRITE (3,13) TITLE, RANGE
      PHI1 = COR*PHI1
      M = 1
      TN2(1) = TNFI
   20 TN2(2) = TNKFI
   30 T = TI
C     CALCULATION OF COSINE OF ANGLE OF REFRACTION IN FILM
  100 A(1) = -TN1**2*SINF(PHI1)**2
      A(1) = -TN1**2*SINF(PHI1)**2
      A(2) = 0.0
      B(1) = TN2(1)**2 - TN2(2)**2
      B(2) = 2.0*TN2(1)*TN2(2)
      CALL CD(C, A, B)
      C(1) = 1.0 + C(1)
      CALL CSR(CPHI2, C)
C     CALCULATION OF COSINE OF ANGLE OF REFRACTION IN SUBSTRATE
      B(1) = TN3(1)**2 - TN3(2)**2
      B(2) = 2.0*TN3(1)*TN3(2)
      CALL CD(C, A, B)
      C(1) = 1.0 + C(1)
      CALL CSR(CPHI3, C)
C     CALCULATION OF FRESNEL COEFFICIENT R1S
      CALL CM(C, TN2, CPHI2)
      A(1) = -C(1) + TN1*COSF(PHI1)
      A(2) = -C(2)
      B(1) = C(1) + TN1*COSF(PHI1)
      B(2) = C(2)
      CALL CD(R1S, A, B)
C     CALCULATION OF FRESNEL COEFFICIENT R1P
      A(1) = TN1
      A(2) = 0.0
      CALL CM(C, A, CPHI2)
      A(1) = COSF(PHI1)



      CALL CM(D, A, TN2)
      A(1) = C(1) - D(1)
      A(2) = C(2) - D(2)
      B(1) = C(1) + D(1)
      B(2) = C(2) + D(2)
      CALL CD(R1P, A, B)
C     CALCULATION OF FRESNEL COEFFICIENT R2S
      CALL CM(C, TN2, CPHI2)
      CALL CM(D, TN3, CPHI3)
      A(1) = C(1) - D(1)
      A(2) = C(2) - D(2)
      B(1) = C(1) + D(1)
      B(2) = C(2) + D(2)
      CALL CD(R2S, A, B)
C     CALCULATION OF FRESNEL COEFFICIENT R2P
      CALL CM(C, TN2, CPHI3)
      CALL CM(D, TN3, CPHI2)
      A(1) = C(1) - D(1)
      A(2) = C(2) - D(2)
      B(1) = C(1) + D(1)
      B(2) = C(2) + D(2)
      CALL CD(R2P, A, B)
C     CALCULATION OF PATH LENGTH PL
      CALL CM(C, TN2, CPHI2)
      PL(1) = 4.0*3.1415927*T*C(1)/WAVE
      PL(2) = 4.0*3.1415927*T*C(2)/WAVE
C     CALCULATION OF THE NORMAL REFLECTION COEFFICIENT RS
      CALL CE(C, PL)
      CALL CM(D, C, R2S)
      CALL CM(C, R1S, D)
      A(1) = R1S(1) + D(1)
      A(2) = R1S(2) + D(2)
      B(1) = 1.0 + C(1)
      B(2) = C(2)
      CALL CD(RS, A, B)
C     CALCULATION OF PARALLEL REFLECTION COEFFICIENT RP
      CALL CE(C, PL)
      CALL CM(D, C, R2P)
      CALL CM(C, R1P, D)
      A(1) = R1P(1) + D(1)
      A(2) = R1P(2) + D(2)
      B(1) = 1.0 + C(1)
      B(2) = C(2)
      CALL CD(RP, A, B)
C     CALCULATION OF RHO
      CALL CD(RHO, RS, RP)
C     CALCULATION OF PSI (DEGREES)
      TPSI = SQRTF(RHO(1)**2 + RHO(2)**2)
      PSIC = ATANF(TPSI)/COR
C     CALCULATION OF DELTA (DEGREES)
      CALL ATN1(DELC, RHO(2), RHO(1))
      DELC = DELC/COR
      IF (EPSIM - ABSF(PSIC - PSIM)) 400, 200, 200
  200 IF (EDELM - ABSF(DELC - DELM)) 400, 300, 300
  300 PHI1 = PHI1/COR
      WRITE (3, 11) PHI1, TN1, WAVE, TN3(1), TN3(2)
      WRITE (3,12) TN2(1), TN2(2), T, PSIC, DELC, PSIM, DELM
      M = 2
  400 IF (TM - (T + DT)) 600, 600, 500
  500 T = T + DT
      GO TO 100
  600 IF (TN2M(2) - (TN2(2) + DTN2(2))) 800, 800, 700
  700 TN2(2) = TN2(2) + DTN2(2)
      GO TO 30
  800 IF (TN2M(1) - (TN2(1) + DTN2(1))) 1000, 1000, 900
  900 TN2(1) = TN2(1) + DTN2(1)
      GO TO 20
 1000 GO TO (2000, 1) M
 2000 WRITE (3, 11) PHI1, TN1, WAVE, TN3(1), TN3(2)
      WRITE (3,14) PSIM, DELM
      GO TO 1
 3000 CONTINUE
      END


      SUBROUTINE CM(C, A, B)
C     COMPLEX MULTIPLICATION SUBROUTINE  (C = AB WHERE C, A, B COMPLEX)
      DIMENSION C(2), A(2), B(2)
      C(1) = A(1)*B(1) - A(2)*B(2)
      C(2) = A(2)*B(1) + A(1)*B(2)
      RETURN
      END


      SUBROUTINE CD(C, A, B)
C     COMPLEX DIVISION SUBROUTINE  (C = A/B WHERE C, A, B COMPLEX)
      DIMENSION C(2), A(2), B(2)
      E = B(1)**2 + B(2)**2
      C(1) = (A(1)*B(1) + A(2)*B(2))/E
      C(2) = (A(2)*B(1) - A(1)*B(2))/E
      RETURN
      END


      SUBROUTINE CE(C, B)
C     COMPLEX EXPONENTIAL SUBROUTINE (C = EXP(IB) WHERE C, B COMPLEX)
      DIMENSION C(2), B(2)
      C(1) = EXPF(-B(2))*COSF(B(1))
      C(2) = EXPF(-B(2))*SINF(B(1))
      RETURN
      END


      SUBROUTINE CSR(C, A)
C     COMPLEX SQUARE ROOT SUBROUTINE (C =SQ. RT. OF A WHERE C,A COMPLEX)
      DIMENSION C(2), A(2)
      IF (A(2)) 4, 1, 4
    1 IF (A(1)) 2, 2, 3
    2 C(1) = 0.0
      C(2) = SQRTF(ABSF(A(1)))
      GO TO 5
    3 C(1) = SQRTF(A(1))
      C(2) = 0.0
      GO TO 5
    4 E = 0.5*SQRTF(A(1)*A(1) + A(2)*A(2))
      C(1) = SQRTF(0.5*A(1) + E)
      C(2) = SQRTF(-0.5*A(1) + E)
    5 RETURN
      END


      SUBROUTINE ATN1(X, Y, Z)
C     THE FOLLOWING SUBROUTINE GIVES THE PROPER QUADRANT OF THE ANGLE
C     X WHEN TANX = Y/Z AND Y, Z ARE GIVEN
      IF (Z)11,6,11
   11 R=Y/Z
      IF(Z)2,6,1
    1 X = ATANF(R)
      GO TO 10
    2 IF(Y)3,5,4
    3 X=ATAN (R)-3.14159
      GO TO 10
    4 X=ATAN (R)+3.14159
      GO TO 10
    5 X = 0.0
      GO TO 10
    6 X=1.57079
   10 RETURN
      END
```

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Variables -- real

Name	Symbol	Description
DELM	Δ (measured)	relative phase change (degrees)
DELC	Δ (calculated)	relative phase change (degrees)
DTN1	δn	iteration increment of refractive index n
DTNK	$\delta(n\kappa)$	iteration increment of $n\kappa$
DT	δL	iteration increment of film thickness L
EPSIM	ϵ_ψ	experimental error in ψ (degrees)
EDELM	ϵ_Δ	experimental error in Δ (degrees)
PHI1	ϕ	angle of incidence (degrees)
PHI	ϕ	angle of incidence (radians)
PSIM	ψ (measured)	relative amplitude change (degrees)
PSIC	ψ (calculated)	relative amplitude change (degrees)
TN1	n_o	refractive index of incident medium
TNS	n_s(substrate)	refractive index of substrate
TNKS	$n_s\kappa_s$(substrate)	$n\kappa$ of substrate
TNI	n_i	lower limit of iteration span for film index n
TN	n	n index of film
TNM	n_m	upper limit of iteration span for film index n
TNKI	$(n\kappa)_i$	lower limit of iteration span for $n\kappa$ for film
TNK	$n\kappa$ (film)	$n\kappa$ of film
TNKM	$(n\kappa)_m$	upper limit of iteration span for $n\kappa$ for film
TI	L_i	lower limit of iteration span for film thickness L
T	L	film thickness
TM	L_m	upper limit of iteration span for film thickness
WL	λ_o	vacuum wave length

Chippewa FORTRAN Functions

Name	Evaluates	Converts
CSQRT(C)	$\sqrt{C}$	complex to complex
CEXP(C)	e^C	complex to complex
CABS(C)	$\|C\|$	complex to real
AIMAG (C)	finds the imaginary part of C	complex to real
REAL(C)	finds the real part of C	complex to real
COS(X)	cosx	real to real
SIN(X)	sinx	real to real
ATAN(X)	$\tan^{-1}x$	real to real
ATAN2(X,Y)	$\tan^{-1}(x/y)$	real to real
ABS(X)	$\|x\|$	real to real
CMPLX(A,B)	constructs A + iB from A,B	real to complex

Where C is complex, A,B,X, Y are real.

"FILM" program, Fortran IV version

```
      PROGRAM FILM2(INPUT, OUTPUT, TAPE2=INPUT, TAPE3=OUTPUT)
C     THIS PROGRAM CALCULATES THE THICKNESS AND COMPLEX REFRACTIVE
C     INDEX OF A SINGLE ABSORBING FILM ON AN ABSORBING SUBSTRATE
      COMPLEX TN2, TN3, CPHI2, CPHI3, R1S, R1P, R2S, R2P, D, RS, RP, RHO
      DIMENSION TITLE (8), RANGE (8)
    1 READ (2,2) TITLE, RANGE
    2 FORMAT (8A10/8A10)
    3 WRITE (3,4) TITLE, RANGE
    4 FORMAT (1H1, 8A10//8A10)
    5 READ (2,9) TN1, WL, TNS, TNKS
      IF (TN1) 3000, 3000, 6
    6 READ (2,10) TNI, DTN, TNM, TNKI, DTNK, TNKM
    7 READ (2,11) TI, DT, TM
    8 READ (2,12) PHI1, PSIM, DELM, EPSIM, EDELM
    9 FORMAT (F5.0, 3F10.0)
   10 FORMAT (F5.0, 5F10.0)
   11 FORMAT (F5.0, 2F10.0)
   12 FORMAT (F5.0, 4F10.0)
   13 FORMAT (1H0,/6HPHI = ,F5.2,10X,4HN = ,F7.4, 10X, 13HWAVELENGTH = ,
     C F5.0, 11H  ANGSTROMS//33H REFRACTIVE INDEX OF SUBSTRATE = , F7.4,
     C 2X, 4H+  I, F7.4)
   14 FORMAT (1H0, 27HREFRACTIVE INDEX OF FILM = , F7.4, 2X,
     C 4H+  I, F7.4//18H FILM THICKNESS = , F7.2, 10H ANGSTROMS,
     C //8H PSIC = , F10.5, 10X, 7HDELC = , F10.5,
     C //8H PSIM = , F10.5, 10X, 7HDELM = , F10.5)
   15 FORMAT (1H0, 7HPSIM = , F10.5,10X,  7HDELM = , F10.5//32H NO SOLUT
     CION WITHIN GIVEN LIMITS)
      M = 1
      PHI = 0.017453*PHI1
      CP = COS(PHI)
      SP = SIN(PHI)
      TN3 = CMPLX(TNS, TNKS)
      CPHI3 = CSQRT(1.0 - TN1**2*SP**2/(TN3**2))
      TN = TNI
   20 TNK = TNKI
   30 T = TI
  100 TN2 = CMPLX(TN, TNK)
      CPHI2 = CSQRT(1.0 - TN1**2*SP**2/(TN2**2))
      R1S = (TN1*CP - TN2*CPHI2)/(TN1*CP + TN2*CPHI2)
      R1P = (TN1*CPHI2 - TN2*CP)/(TN1*CPHI2 + TN2*CP)
      R2S = (TN2*CPHI2 - TN3*CPHI3)/(TN2*CPHI2 + TN3*CPHI3)
      R2P = (TN2*CPHI3 - TN3*CPHI2)/(TN2*CPHI3 + TN3*CPHI2)
      D = (0.0,1.0)*(4.0*3.1415927*T/WL)*TN2*CPHI2
      RS = (R1S + R2S*CEXP(D))/(1.0 + R1S*R2S*CEXP(D))
      RP = (R1P + R2P*CEXP(D))/(1.0 + R1P*R2P*CEXP(D))
      RHO = RS/RP
      PSIC = ATAN(CABS(RHO))/0.017453
      DELC = ATAN2(AIMAG(RHO), REAL(RHO))/0.017453
      IF (EPSIM - ABS(PSIC - PSIM)) 400, 200, 200
  200 IF (EDELM - ABS(DELC - DELM)) 400, 300, 300
  300 WRITE (3,13) PHI1, TN1, WL, TNS, TNKS
      WRITE (3,14) TN, TNK, T, PSIC, DELC, PSIM, DELM
      M = 2
  400 IF (TM - (T + DT)) 600, 600, 500
  500 T = T + DT
      GO TO 100
  600 IF (TNKM - (TNK + DTNK)) 800, 800, 700
  700 TNK = TNK + DTNK
      GO TO 30
  800 IF (TNM - (TN + DTN)) 1000, 1000, 900
  900 TN = TN + DTN


      GO TO 20
 1000 GO TO (2000, 1) M
 2000 WRITE (3,13) PHI1, TN1, WL, TNS, TNKS
      WRITE (3,15) PSIM, DELM
      GO TO 1
 3000 CONTINUE
      END
```

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$$r_{2s} = \frac{n_c \cos\phi' - n_{cs} \cos\phi'_s}{n_c \cos\phi' + n_{cs} \cos\phi'_s} \qquad (61)$$

$$r_{2p} = \frac{n_c \cos\phi'_s - n_{cs} \cos\phi'}{n_c \cos\phi'_s + n_{cs} \cos\phi'} \qquad (59)$$

$$\rho_{meas} = \tan \psi_{meas} \, e^{i\Delta_{meas}} \qquad (77)$$

$$A = (\rho r_{1s} - r_{1p}) \, r_{2s} r_{2p} \qquad (79),(80)$$

$$B = \rho(r_{1p} r_{1s} r_{2s} + r_{2p}) - (r_{1s} r_{1p} r_{2p} + r_{2s}) \qquad (79),(80)$$

$$C = \rho r_{1p} - r_{1s} \qquad (79),(80)$$

$$X = [B^2 - 4AC]^{1/2} \qquad \text{see (82)}$$

$$E_{1,2} = (-B \pm X)/(2A) \qquad \text{see (82)}$$

$$L_{1,2} = \frac{\lambda_o}{4\pi n_c \cos\phi'} \left[\tan^{-1}\left(\frac{\mathrm{Im}E_{1,2}}{\mathrm{Re}E_{1,2}} \right) - i \,\mathrm{Log}\, |E_{1,2}| \right] \qquad (85)$$

L = real part of L_1 or L_2, whichever has smaller imaginary part

$$D = \frac{4\pi i L}{\lambda_o} \, n_c \cos\phi' \qquad (76)$$

$$r_s = \frac{r_{1s} + r_{2s} e^D}{1 + r_{1s} r_{2s} e^D} \qquad (75)$$

Sequence and Format of Data Cards for Program "SFILM"

Card	Col. 1	Col. 10	Col. 20	Col. 30	Col. 40	Col. 50
1	Title and comments (up to 80 columns)					
2	Ranges of n and $n\kappa$ (up to 80 columns)					
3	n_o	λ_o	n_s	$n_s \kappa_s$		
4	n (initial)	δn (increment)	n (final)	$n\kappa$ (initial)	$\delta n\kappa$ (increment)	$n\kappa$ (final)
5	ϕ	ψ	Δ	ϵ_ψ	ϵ_Δ	

These five cards constitute a set. Any number of sets may follow. Two blank cards (these are dummy title cards and need not necessarily be blank) must follow card 5 of the last set of data, and a card with 0.0 punched in the first field (columns 1 - 9) must follow these blank cards. (When the computer reads this card it sets $n_o = 0.0$. It thus transfers to the end of the program.) All numbers punched on cards 3 through 5 must contain a decimal point and may be located anywhere in the field of columns 1 - 9, 10 - 19, etc.

Variables -- Complex

Name	Symbol	Description
A	A	intermediate variable
B	B	intermediate variable
C	C	intermediate variable
CPHI2	$\cos\phi'$	cosine of angle of ref. in film
CPHI3	$\cos\phi'_s$	cosine of angle of ref. in substrate
D	D	complex optical path length
E1, E2	e^D	
I	$\sqrt{-1} = i$	imaginary unit
N2	$n + in\kappa$	refractive index of film
N3	$n_s + in_s\kappa_s$	refractive index of substrate
R1S	r_{1s}	Fresnel reflection coefficient at air-film interface for polarization normal to plane of incidence
R1P	r_{1p}	Fresnel reflection coefficient at air-film interface for polarization parallel to plane of incidence
R2S	r_{2s}	Fresnel reflection coefficient at film-substrate interface for polarization normal to plane of incidence
R2P	r_{2p}	Fresnel reflection coefficient at film-substrate interface for polarization parallel to plane of incidence
RS	r_s	reflection coefficient of film-substrate combination for polarization normal to plane of incidence
RP	r_p	reflection coefficient of film-substrate combination for polarization parallel to plane of incidence
RHO	$\rho = r_s/r_p = \tan\psi\ e^{i\Delta}$ (calculated)	
RHOM	$\rho = r_s/r_p = \tan\psi\ e^{i\Delta}$ (measured)	
T1, T2	L_1, L_2	calculated complex film thicknesses

A reproduction of the program SFILM appears on the next page, followed by a sample output. The flow diagram is the same as that of the program "FILM" with two exceptions. Since film thickness is calculated, not guessed, the

"SFILM" Program

```
      PROGRAM SFILM (INPUT, OUTPUT, TAPE2=INPUT, TAPE3=OUTPUT)
C     THIS PROGRAM CALCULATES THE THICKNESS AND COMPLEX REFRACTIVE
C     INDEX OF A SINGLE ABSORBING FILM ON AN ABSORBING SUBSTRATE
      COMPLEX CPHI2, CPHI3, N2, N3, R1S, R1P, R2S, R2P, RS, RP, RHO
      COMPLEX A, B, C, D, E1, E2, I, RHOM, T1, T2, X
      REAL N1, NS, NKS, NI, NM, NKI, NKM, N, NK
      DIMENSION TITLE(8), RANGE(8)
      CDR = 0.017453
      I = CMPLX(0.0, 1.0)
    1 READ (2,2) TITLE, RANGE
    2 FORMAT (8A10/8A10)
    3 WRITE (3,4) TITLE, RANGE
    4 FORMAT (1H1, 8A10//8A10)
    5 READ (2,8) N1, WL, NS, NKS
      IF (N1) 1000, 1000, 6
    6 READ (2,9) NI, DN, NM, NKI, DNK, NKM
    7 READ (2, 10) PHI, PSIM, DELM, EPSIM, EDELM
    8 FORMAT (F9.0, 3F10.0)
    9 FORMAT (F9.0, 5F10.0)
   10 FORMAT (F9.0, 4F10.0)
   11 FORMAT (1H0,/6HPHI = ,F5.2,10X,4HN = ,F7.4, 10X, 13HWAVELENGTH = ,
     C F5.0, 11H  ANGSTROMS//33H REFRACTIVE INDEX OF SUBSTRATE = , F7.4,
     C 2X, 4H+  I, F7.4)
   12 FORMAT (1H0, 27HREFRACTIVE INDEX OF FILM = , F7.4, 2X,
     C 4H+  I, F7.4//18H FILM THICKNESS = , F7.2, 10H ANGSTROMS//
     C 36H IMAGINARY PART OF FILM THICKNESS = , F10.5, 10H ANGSTROMS,
     C //8H PSIC = , F10.5, 10X, 7HDELC = , F10.5,
     C //8H PSIM = , F10.5, 10X, 7HDELM = , F10.5)
   13 FORMAT (1H0, 7HPSIM = , F10.5,10X,  7HDELM = , F10.5//32H NO SOLUT
     CION WITHIN GIVEN LIMITS)
      M = 1
      N3 = CMPLX(NS, NKS)
      RHOM = TAN(PSIM*CDR)*CEXP(I*DELM*CDR)
      SP = SIN(PHI*CDR)
      CP = COS(PHI*CDR)
      CPHI3 = CSQRT(1.0 - N1**2*SP**2/(N3**2))
      P = WL/(4.0*3.1415927)
      N = NI
   20 NK = NKI
  100 N2 = CMPLX(N, NK)
      CPHI2 = CSQRT(1.0 - N1**2*SP**2/(N2**2))
      R1S = (N1*CP - N2*CPHI2)/(N1*CP + N2*CPHI2)
      R1P = (N1*CPHI2 - N2*CP)/(N1*CPHI2 + N2*CP)
      R2S = (N2*CPHI2 - N3*CPHI3)/(N2*CPHI2 + N3*CPHI3)
      R2P = (N2*CPHI3 - N3*CPHI2)/(N2*CPHI3 + N3*CPHI2)
      A = (RHOM*R1S - R1P)*R2S*R2P
      B = RHOM*(R1P*R1S*R2S + R2P) - (R1S*R1P*R2P + R2S)
      C = RHOM*R1P - R1S
      X = CSQRT(B**2 - 4.0*A*C)
      E1 = (-B + X)/(2.0*A)
      E2 = (-B - X)/(2.0*A)
      T1 = P*(ATAN2(AIMAG(E1), REAL(E1)) - I*ALOG(CABS(E1)))/(N2*CPHI2)
      T2 = P*(ATAN2(AIMAG(E2), REAL(E2)) - I*ALOG(CABS(E2)))/(N2*CPHI2)
      IF (ABS(AIMAG(T1)) - ABS(AIMAG(T2))) 110, 110, 120
  110 T = REAL (T1)
      ERROR = AIMAG (T1)
      GO TO 130
  120 T = REAL(T2)
      ERROR = AIMAG(T2)
  130 D = (4.0*3.1415927*T/WL)*I*N2*CPHI2
      RS = (R1S + R2S*CEXP(D))/(1.0 + R1S*R2S*CEXP(D))


      RP = (R1P + R2P*CEXP(D))/(1.0 + R1P*R2P*CEXP(D))
      RHO = RS/RP
      PSIC = ATAN(CABS(RHO))/CDR
      DELC = ATAN2(AIMAG(RHO), REAL(RHO))/CDR
      IF (EPSIM - ABS(PSIC - PSIM))400, 200, 200
  200 IF (EDELM - ABS(DELC - DELM))400, 300, 300
  300 WRITE (3,11) PHI, N1, WL, NS, NKS
      WRITE (3,12) N, NK, T, ERROR, PSIC, DELC, PSIM, DELM
      M = 2
  400 IF (NKM - (NK + DNK)) 600, 500, 500
  500 NK = NK + DNK
      GO TO 100
  600 IF (NM - (N + DN)) 800, 700, 700
  700 N = N + DN
      GO TO 20
  800 GO TO (900, 1) M
  900 WRITE (3,11) PHI, N1, WL, NS, NKS
      WRITE (3,13) PSIM, DELM
      GO TO 1
 1000 CONTINUE
      END
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