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Reflection Symmetries of *q*-Bernoulli Polynomials

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Abstract

A large part of the theory of classical Bernoulli polynomials $B_n(x)$'s follows from their reflection symmetry around $x = 1/2$: $B_n(1-x) = (-1)^n B_n(x)$. This symmetry not only survives quantization but has two equivalent forms, classical and quantum, depending upon whether one reflects around 1/2 the classical x or quantum $[x]_q$.

1 Introduction

The classical Bernoulli polynomials are the difference analogs of the indefinite integrals of the power functions. These polynomials are defined by the generating function

$$\mathcal{B}(x;t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}. \tag{1.1}$$

Since

$$\mathcal{B}(1-x;-t) = \mathcal{B}(x;t) : \tag{1.2}$$

$$\mathcal{B}(1-x;-t) = \frac{-t}{e^{-t}-1} e^{(1-x)(-t)} = \frac{-te^{-t}}{e^{-t}(1-e^t)} e^{xt} =$$
$$= \frac{t}{e^t-1} e^{xt} = \mathcal{B}(x;t),$$

we find that

$$B_n(1-x) = (-1)^n B_n(x), \quad n \in \mathbf{Z}_{\geq 0}. \tag{1.3}$$

The question is: what happens with the reflection symmetry (1.3) in the q-domain, when one considers q-Bernoulli polynomials?

Originally q-Bernoulli numbers and polynomials were introduced by Carlitz in 1948 [1], but they do not seem to be the most natural ones; in particular, they don't appear as the values at nonpositive integers of various Riemann and Hurwitz q-zeta functions (see [4]). The particular version of q-Bernoulli polynomials which is used in the next four Sections, up to a rescaling, can be shown to be equivalent to the one defined by Tsumura from different considerations and by different formulae [5,6].

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2 q-Notations

We shall be working with the following quantization scheme: if x is a classical object, such as a complex number, its q-version $[x]_q$ is defined as

$$[x]_q = \frac{q^x - 1}{q - 1} = \frac{e^{hx} - 1}{e^h - 1} \quad , \tag{2.1}$$

where h is considered as a fixed complex number or a formal parameter:

$$q = e^h, \tag{2.2}$$
$$q \neq 0,\ 1. \tag{2.3}$$

The q-numbers $[x]_q$ satisfy many simple relations, all easily verified, such as:

$$[x + y]_q = [x]_q + q^x [y]_q \ , \tag{2.4}$$
$$[x]_{q^{-1}} = q^{1-x} [x]_q \ , \tag{2.5}$$
$$[-x]_q = -q^{-x} [x]_q \ , \tag{2.6}$$
$$[-x]_{q^{-1}} = -q[x]_q \ , \tag{2.7}$$
$$[1 - x]_q = 1 - [x]_{q^{-1}} \ . \tag{2.8}$$

3 q-Bernoulli polynomials

Let $\hat{\beta}(t)$ be the generating function of the classical Bernoulli *numbers*:

$$\hat{\beta}(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1} \ . \tag{3.1}$$

We shall write $\hat{\beta}(\frac{\partial}{\partial x})$ for the formal power series

$$\hat{\beta}\left(\frac{\partial}{\partial x}\right) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{\partial}{\partial x}\right)^n. \tag{3.2}$$

This formal differential operator acts properly on polynomials in x − where it terminates −

$$\hat{\beta}\left(\frac{\partial}{\partial x}\right)(x^k) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{\partial}{\partial x}\right)^n (x^k) = \sum_{n=0}^{k} \binom{k}{n} B_n x^{k-n} = B_k(x), \tag{3.3}$$

and also on exponents:

$$\hat{\beta}\left(\frac{\partial}{\partial x}\right)(e^{tx}) = \hat{\beta}(t) e^{tx} = \mathcal{B}(x; t). \tag{3.4}$$

Now set:

$$B_n(x|q) = \hat{\beta}\left(\frac{\partial}{\partial x}\right)(([x])^n), \quad n \in \boldsymbol{Z}_{\geq 0}, \tag{3.5}$$

$$\mathcal{B}(x;t|q) = \sum_{n=0}^{\infty} B_n(x|q)\frac{t^n}{n!} = \hat{\beta}\left(\frac{\partial}{\partial x}\right)(e^{t[x]}), \tag{3.6}$$

where $[x]$ stands for $[x]_q$ where there is no risk of confusion.

These are our q-versions of the classical Bernoulli polynomials. Since

$$[x]_q = x + h\binom{x}{2} + O(h^2), \quad h \to 0, \tag{3.7}$$

we see that

$$B_n(x|q) = B_n(x) + \frac{nh}{2}(B_{n+1}(x) - B_n(x)) + O(h^2), \tag{3.8}$$

when $h \to 0$. The first few q-Bernoulli polynomials are:

$$B_0(x|q) = 1, \tag{3.9}$$

$$B_1(x|q) = \frac{h}{q-1}\left([x]_q + \frac{\hat{\beta}(h) - 1}{h}\right), \tag{3.10}$$

$$B_2(x|q) = \frac{2h}{q^2-1}\left(([x]_q)^2 - [x]_q + \frac{q^2 - 1 - 2hq}{2h(q-1)^2}\right); \tag{3.11}$$

the corresponding q-Bernoulli numbers, thus, are:

$$B_0(0|q) = 1, \tag{3.12}$$

$$B_1(0|q) = \frac{h - q + 1}{(q-1)^2}, \tag{3.13}$$

$$B_2(0|q) = \frac{q^2 - 1 - 2hq}{(q^2-1)(q-1)^2}. \tag{3.14}$$

In general,

$$([x]_q)^n = (q-1)^{-n}(q^x - 1)^n = (q-1)^{-n}\sum_{j=0}^{n}\binom{n}{j}q^{jx}(-1)^{n-j}, \tag{3.15}$$

so that

$$B_n(x|q) = (q-1)^{-n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\hat{\beta}(jh)q^{jx}. \tag{3.16}$$

From formula (3.5) we find:

$$B_n(x+1|q) - B_n(x|q) = \frac{\partial}{\partial x}(([x]_q)^n) = \frac{h}{q-1}q^x n([x]_q)^{n-1}. \tag{3.17}$$

This is our particular version of the classical difference equation

$$B_n(x+1) - B_n(x) = nx^{n-1}. \tag{3.18}$$

Another classical relation,

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x), \tag{3.19}$$

is easily seen to be quantized into the form:

$$\frac{\partial B_n(x|q)}{\partial x} = \frac{h}{q-1} nB_{n-1}(x|q) + hnB_n(x|q). \tag{3.20}$$

All these formulae can be profitably viewed from the point of view of the umbral calculus, as the properties of the objects

$$\mathcal{U}\left(\frac{\partial}{\partial x}\right)(e^{t[x]}), \tag{3.21}$$

where

$$\mathcal{U}(t) = \sum_{n=0}^{\infty} u_n \frac{t^n}{n!} \tag{3.22}$$

is an arbitrary formal power series. We won't need such an interpretation for arguments that follow, and it is left as an Exercise to the reader.

4 The expected form of the reflection symmetry

Let's consider the effect of reflection at $x = 1/2$ on the q-Bernoulli polynomials.

We have:

$$\begin{aligned}
&\mathcal{B}(1-x; t|q) = \hat{\beta}\left(-\frac{\partial}{\partial x}\right)(exp(t[1-x]_q)) \quad \text{[by (4.3)]} = \\
&= \hat{\beta}\left(\frac{\partial}{\partial x}\right) exp\left(\frac{\partial}{\partial x}\right)(exp(t[1-x]_q)) = \hat{\beta}\left(\frac{\partial}{\partial x}\right)(exp(t[-x]_q)) \quad \text{[by (2.7)]} = \\
&= \hat{\beta}\left(\frac{\partial}{\partial x}\right)(exp(-tq^{-1}[x]_{q^{-1}})) = \mathcal{B}(x; -tq^{-1}|q^{-1}) : \\
&\qquad \mathcal{B}(1-x; t|q) = \mathcal{B}(x; -tq^{-1}|q^{-1}).
\end{aligned} \tag{4.1}$$

In components, this is:

$$B_n(1-x|q) = (-q)^{-n} B_n(x|q^{-1}), \quad n \in \boldsymbol{Z}_{\geq 0}, \tag{4.2}$$

our first q-analog of the classical reflection formula (1.3).

In the Proof above, we used the classical formula

$$\hat{\beta}(-t) = e^t \hat{\beta}(t). \tag{4.3}$$

It has the following q-analog. Let

$$\hat{\beta}(t|q) = \mathcal{B}(0; t|q) \tag{4.4}$$

be the generating function of the q-Bernoulli numbers. Then

$$\hat{\beta}(-t|q) = e^{t/q} \hat{\beta}(q^{-2}t|q^{-1}). \tag{4.5}$$

In other words, the combination

$$e^{t/2} \hat{\beta}(qt|q) \tag{4.6}$$

is invariant with respect to the involution

$$t \mapsto -t, \quad q \mapsto q^{-1}. \tag{4.7}$$

Formulae (4.1), (4.5), and (3.17) imply that

$$e^t \hat{\beta}(qt|q) - \hat{\beta}(t|q) = \hat{\beta}(h)t, \tag{4.8}$$

which is what becomes of the classical definition $\hat{\beta}(t) = t/(e^t - 1)$ under quantization. We won't need below the last 5 formulae.

5 The unexpected form of the reflection symmetry

To make the argument more transparent, we first separate the variable x from the quantization parameter q.

Set

$$X = [x]_q, \tag{5.1}$$

and define the polynomials $B_n^{ext}(X|q)$ by the rule:

$$B_n(x|q) = B_n^{ext}([x]_q|q). \tag{5.2}$$

For example, formulae (3.9-11) yield:

$$B_0^{ext}(X|q) = 1, \tag{5.3}$$

$$B_1^{ext}(X|q) = \frac{h}{q-1}\left(X + \frac{\hat{\beta}(h) - 1}{q-1}\right), \tag{5.4}$$

$$B_2^{ext}(X|q) = \frac{2h}{q^2-1}\left((X^2 - X) + \frac{q^2 - 1 - 2hq}{(q^2-1)(q-1)^2}\right). \tag{5.5}$$

Thus, while $B_n(x|q)$ is a polynomial of degree $= n$ in q^x, $B_n^{ext}(X|q)$ is a polynomial of degree $= n$ in X; the coefficients in both cases are functions of h and q which are polynomial in h and rational in q.

We are going now to consider reflection at $1/2$ of X, *not* x.

If we suggestively denote by $*$ the result of reflection at $X = 1/2$ on functions of X, with the simultaneous change $q \mapsto q^{-1}$:

$$X^* = 1 - X, \quad q^* = q^{-1}, \tag{5.6}$$

$$f(X|q)^* = f(1 - X|q^{-1}), \tag{5.7}$$

and if we remember that

$$X = [x]_q = (q-1)^{-1}(q^x - 1), \tag{5.8}$$

the $X = 1/2$ reflection (5.6) becomes, in the x-language:

$$\begin{aligned} X^* = [x^*]_{q^{-1}} = \frac{q^{-x^*} - 1}{q^{-1} - 1} = q\frac{1 - q^{-x^*}}{q - 1} = \frac{q - q^{1-x^*}}{q - 1} = \\ = 1 - X = 1 - [x]_q = 1 - \frac{q^x - 1}{q - 1} = \frac{q - q^x}{q - 1} \quad \Leftrightarrow \end{aligned}$$

$$x^* = 1 - x. \tag{5.9}$$

Thus,

$$B_n^{ext}(1 - X|q) = (-q)^{-n} B_n^{ext}(X|q^{-1}), \quad n \in \boldsymbol{Z}_{\geq 0}, \tag{5.10}$$

The nature of the 2^{nd} reflection formula (5.10) is somewhat mysterious. I have stumbled on it by an accident, trying to understand a still bigger mystery: proper quantum analogs of the Faulhaber phenomenon (see [3,2].)

6 Perspectives

The Bernoulli polynomials form just one subclass among the large class of objects connected to various zeta functions and Eisenstein series. It is natural to inquire if the reflection symmetry survives quantization of this class. It is indeed so for the case of 1-dimensional (circular) Eisenstein series (see [7]).

Even more important is the question of what happens with the reflection symmetry of Bernoulli polynomials under quantization schemes *different* from the one given by formula (2.1), such as the arithmetic quantizations

$$[x] = \frac{q^x - q^{-rx}}{q - q^{-r}}, \quad r \in \boldsymbol{Z}_{\geq 0}, \tag{6.1}$$

or even the general two-parameter quantization family

$$[x]_{q,Q} = \frac{q^x - Q^x}{q - Q}, \quad q \neq Q \tag{6.2}$$

and its quasiclassical limit

$$[x]_q^{quas} = \lim_{Q \to q} \frac{q^x - Q^x}{q - Q} = xq^{x-1}. \tag{6.3}$$

An analysis of the derivation of formula (4.1) shows that the reflection symmetry $x \mapsto 1 - x$ survives a given quantization scheme $\{x \mapsto [x]_q\}$ provided there exists an involution

$$I : \boldsymbol{q} \ \mapsto \boldsymbol{q}^* \tag{6.4}$$

in the space of quantization parameters $\boldsymbol{q} = (q_1, \ \ldots, q_N)$ such that the ratio

$$[-x]_{\boldsymbol{q}}/[x]_{\boldsymbol{q}^*} = -\rho(q) \tag{6.5}$$

is x-independent. In such a case, the $\boldsymbol{q}$-Bernoulli polynomials possess the symmetry

$$B_n(1 - x|\boldsymbol{q}) = (-\rho(\boldsymbol{q}))^n B_n(x|\boldsymbol{q}^*), \tag{6.6}$$

where

$$B_n(x|\boldsymbol{q}) = \hat{\beta}\left(\frac{\partial}{\partial x}\right)(([x]_{\boldsymbol{q}})^n), \quad n \in \mathbf{Z}_{\geq 0}. \tag{6.7}$$

For the two-parameter case (6.2),

$$[-x]_{q,Q}/[x]_{q^{-1},Q^{-1}} = -q^{-1}Q^{-1}, \tag{6.8}$$

so that

$$B_n(1 - x|q, Q) = (-qQ)^{-n} B_n(x|q^{-1}, Q^{-1}), \tag{6.9}$$

and we recover formula (4.2) for $Q = 1$.

There is no quantum form of the reflection symmetry for the variable $X = [x]_{q,Q}$; that variable should be considered as not the right one. The correct choice is:

$$X = \frac{1}{2} + \rho(\boldsymbol{q})^{-1/2}[x - \frac{1}{2}]_{\boldsymbol{q}}. \tag{6.10}$$

Indeed, applying the involution $* : x \mapsto 1 - x, \boldsymbol{q} \mapsto \boldsymbol{q}^*$ to formula (6.10), we find:

$$X^* = \frac{1}{2} + \rho(\boldsymbol{q}^*)^{-1/2}[\frac{1}{2} - x]_{\boldsymbol{q}^*}. \tag{6.11}$$

But formula (6.5) tells us that

$$\rho(\boldsymbol{q}^*) = \rho(\boldsymbol{q})^{-1}, \tag{6.12}$$

and therefore,

$$\rho(\boldsymbol{q})^{-1/2}[-y]_{\boldsymbol{q}} + \rho(\boldsymbol{q}^*)^{-1/2}[y]_{\boldsymbol{q}^*} = \rho(\boldsymbol{q})^{-1/2}(-\rho(\boldsymbol{q}))[y]_{\boldsymbol{q}^*} + \\ +\rho(\boldsymbol{q})^{1/2}[y]_{\boldsymbol{q}^*} = 0, \quad y = \frac{1}{2} - x. \tag{6.13}$$

Thus,

$$X + X^* = 1, \tag{6.14}$$

and we recover the quantum form of the reflection symmetry.

For the simplest quantum scheme we have started with, $[x]_q = (q^x - 1)/(q-1)$, the general recipe (6.10) produces a *different* variable X^{new}, than the old variable $X^{old} = [x]_q$:

$$X^{new} - X^{old} = \left(\frac{1}{2} + q^{1/2}\frac{q^{x-1/2}-1}{q-1}\right) - \frac{q^x-1}{q-1} = \frac{1}{2} - \frac{q^{1/2}-1}{q-1} = \\ = \frac{1}{2} - [\frac{1}{2}]_q; \tag{6.15}$$

and the equality

$$[\frac{1}{2}]_q + [\frac{1}{2}]_{q^{-1}} = \frac{q^{1/2}-1}{q-1} + \frac{q^{-1/2}-1}{q^{-1}-1} = \frac{(q^{1/2}-1) - q(q^{-1/2}-1)}{q-1} = 1 \tag{6.16}$$

shows that

$$(X^{new} - X^{old}) + (X^{new} - X^{old})^* = 0. \tag{6.17}$$

I conclude with a remark on other quantization schemes, a fascinating and largely unexplored subject.

One such scheme, a sort of second quantization of the $[x]_q$ - scheme, is:

$${}_1[x]_{h;Q} = \frac{(1\dot{+}h)^x_Q - 1}{h}, \tag{6.18}$$

where

$$(1\dot{+}h)^n_q = \begin{cases} \prod\limits_{n=0}^{n-1} (1 + hQ^j), & n \in \mathbf{Z}_{\geq 1} \\ 1, & n = 0 \end{cases} \tag{6.19}$$

and, in general,

$$(1\dot{+}h)^x_Q = (1\dot{+}h)^\infty_Q/(1\dot{+}hQ^x)^\infty_Q, \quad 0 < |Q| < 1, \tag{6.20a}$$

$$(1\dot{+}h)^x_{Q^{-1}} = (1\dot{+}hQ^{1-x})^x_Q, \quad 0 < |Q| < 1. \tag{6.20b}$$

For $Q = 1$, formula (6.18) becomes:

$$ {}_1[x]_{h;1} = \frac{(1+h)^x - 1}{h} = \frac{q^x - 1}{q - 1} = [x]_q, \quad q = 1 + h, \tag{6.21} $$

the familiar formula (2.1). However, the quantization scheme (6.18) is easily seen to lack the reflection symmetry $x \mapsto 1 - x$, since the formula (6.5) is not satisfied. Nevertheless, since

$$ (1\dot{+}h)^x_Q = 1 + h[x]_Q + h^2 Q \begin{bmatrix} x \\ 2 \end{bmatrix}_Q + O(h^3), \quad h \to 0, \tag{6.22} $$

we have:

$$ {}_1[x]_{h;Q} = [x]_Q + hQ \begin{bmatrix} x \\ 2 \end{bmatrix}_Q + O(h^2), \quad h \to 0, \tag{6.23} $$

and we can set:

$$ < x >_{h;Q} = [x]_Q + hQ^2 \begin{bmatrix} x \\ 2 \end{bmatrix}_Q, \quad h^2 = 0, \tag{6.24} $$

a sort of quasiclassical extension of the quantum scheme $\{x \mapsto [x]_Q\}$. Miraculously, the quantization scheme (6.24) reacquires the reflection symmetry $\{x \mapsto x^* = 1 - x\}$ under the involution

$$ Q^* = Q^{-1}, \quad h^* = -h, \tag{6.25} $$

because

$$ \begin{aligned} -\rho(h;Q) &= \frac{< -x >_{h;Q}}{< x >_{h^*;Q^*}} = \frac{[-x]_Q(1 + hQ^2[-x-1]_Q/[2]_Q)}{[x]_{Q^{-1}}(1 - hQ^{-2}[x-1]_{Q^{-1}}/[2]_{Q^{-1}})} = \\ &= -Q^{-1}(1 + \frac{hQ^2}{1+Q}\frac{Q^{-x-1}-1}{Q-1})(1 + \frac{hQ^{-2}}{1+Q^{-1}}\frac{Q^{-x+1}-1}{Q^{-1}-1}) = -Q^{-1}(1-h). \end{aligned} \tag{6.26} $$

A few comments are in order. First, if we set

$$ q = 1 + h, \quad h^2 = 0, \tag{6.27} $$

in the quantization scheme (6.2), we find:

$$ {}_2[x]_{h;Q} = [x]_Q \left(1 + \frac{h}{Q-1}\left(1 - \frac{x}{[x]_Q}\right)\right), \quad h^2 = 0, \tag{6.28} $$

which is quite different from formula (6.24). Second, one can restore the reflection symmetry to the $(1\dot{+}h)^x_Q$ - quantization scheme by taking $[x]_{h;Q|h^*;Q^*}$ as either

$$ \frac{(1\dot{+}h)^x_Q - 1}{2h} - \frac{(1\dot{+}h^*)^{-x}_{Q^*}}{2h^*} \tag{6.29} $$

or

$$\frac{(1\dot{+}h)_Q^x - (1\dot{+}h^*)_{Q^*}^{-x}}{h - h^*}, \tag{6.30}$$

and choosing the involution $(h;Q) \mapsto (h^*;Q^*)$ at will. Third, at $Q = 1$, formula (6.24) becomes:

$$< x >_h = x + h\binom{x}{2}, \quad h^2 = 0. \tag{6.31}$$

This is the simplest possible deformation, a nilpotent one, of the usual number system, and it suggests a very particular way to calculate the 1-jet of classical mathematics, a sort of an even version of the supersymmetric extension. Fourth, if we apply the perturbation form (6.24) to the quantization scheme $[x]_{q,Q} = (q^x - Q^x)/(q - Q)$:

$$_1 < x >_{h;q,Q} = [x]_{q;Q} + h\begin{bmatrix} x \\ 2 \end{bmatrix}_{q,Q}, \quad h^2 = 0, \tag{6.32}$$

the reflection symmetry criterion (6.5) no longer holds. It *does reappear* when we change formula (6.32) into

$$< x >_{h;q,Q} = [x]_{q,Q}(1 + \frac{h}{([2]_{q,Q})^\alpha}[x]_{q,Q}), \quad h^2 = 0, \tag{6.33}$$

with α being an arbitrary constant, and the involution in the quantum parameter space $\{h; q, Q\}$ being given by the formulae

$$q^* = q^{-1}, \; Q^* = Q^{-1}, \quad h^* = -h/(qQ)^{\alpha+1}. \tag{6.34}$$

These subjects will be dealt with on another occasion.

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