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Reflection Symmetry of the Eigenvalues in the Wick-Cutkosky Model

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The operator previously introduced for the Wick-Cutkosky eigenvalue problem is expressed in terms of the generators in the positive discrete series of unitary representations of $SU(1,1)$. A reflection symmetry of the eigenvalues follows from the representation thus obtained.

§ 1. Introduction

In the Bethe-Salpeter formalism,[1] the Wick-Cutkosky model[2] provides a rather tractable and yet nontrivial subject for the bound state problem in quantum field theory. The eigenvalue problem associated with the model has thus been investigated extensively. The main results and references are summarized in Nakanishi's review article.[3] The perturbation method with respect to the bound state energy has usually been employed in the calculation of the eigenvalues at nonvanishing energy. As a new perturbation scheme global in the energy variable, an expansion method with respect to the inverse of the principal quantum number was introduced in the previous paper.[4),*)] The asymptotic expansion obtained in (I) seems to display a square-root type threshold singularity and suggests a kind of reflection symmetry.

In the present note the reflection symmetry of the eigenvalues is proved nonperturbatively. In the next section, a summary of (I) is presented together with the next higher order perturbation calculation. The Wick-Cutkosky operator introduced in (I) is expressed in § 3, in terms of the generators in the positive discrete series of unitary representations of the group $SU(1,1)$.[5] The reflection symmetry is derived with the aid of this representation and a certain property of the Lie algebra of $SU(1,1)$. The final section is devoted to discussion.

§ 2. Summary of the previous results

The eigenvalue problem for the coupling constant, at a given bound state energy squared $4\sigma=s(<4)$ in the rest frame, in the Wick-Cutkosky model[2] was reduced in (I) to the eigenvalue problem for the operator

*) This is referred to as (I) hereafter.

$$\mathrm{L}_n(\sigma) = \sqrt{(n+\mathrm{N})(n+\mathrm{N}+1)}\left[1-\sigma g(\mathrm{N}, n) + \sigma\left(\frac{h(\mathrm{N}, n)}{\sqrt{\mathrm{N}}}a^\dagger\right)^2 + \sigma\left(a\frac{h(\mathrm{N}, n)}{\sqrt{\mathrm{N}}}\right)^2\right]\sqrt{(n+\mathrm{N})(n+\mathrm{N}+1)}, \qquad (2\cdot 1)$$

defined on the representation space of the commutation relation $[a, a^\dagger]=1$, N being the number operator $\mathrm{N}=a^\dagger a$. The functions g and h are given by

$$g(\kappa, n) = \frac{2[(n+\kappa)(n+\kappa+1)+n^2-1]}{(2n+2\kappa-1)(2n+2\kappa+3)},$$

$$h(\kappa, n) = \left(\frac{\kappa(2n+\kappa)}{(2n+2\kappa-1)(2n+2\kappa+1)}\right)^{1/2}.$$

The operator $\mathrm{L}_n(\sigma)$ is expanded for large principal quantum number n as

$$\mathrm{L}_n(\sigma) = n^2(1-\sigma) + n\mathrm{H}_0 + \mathrm{H}_1 + \frac{1}{n}\mathrm{H}_2 + \frac{1}{n^2}\mathrm{H}_3 + \cdots$$

with

$$\mathrm{H}_0 = 2\sqrt{1-\sigma}\left(\mathrm{M}+\frac{1}{2}\right),$$

$$\mathrm{H}_1 = \frac{1}{8}\{(2-\sigma)(2\mathrm{M}+1)^2 - (2+\sigma) - \sigma[b^{\dagger 2}(2\mathrm{M}+3) + (2\mathrm{M}+3)b^2]\},$$

$$\mathrm{H}_2 = \frac{\sigma}{2\sqrt{1-\sigma}}\left(\frac{1}{8}\right)^2[(48+\sigma)(2\mathrm{M}+1) - (50-\sigma)(b^{\dagger 2}+b^2)],$$

$$\mathrm{H}_3 = \frac{2\sigma}{(2\sqrt{1-\sigma})^2}\left(\frac{1}{8}\right)^3\{-3(192-46\sigma+\sigma^2)[(2\mathrm{M}+1)^2+1] + 4(146-2\sigma+\sigma^2)[b^{\dagger 2}(2\mathrm{M}+3) + (2\mathrm{M}+3)b^2] - 2(146-\sigma)\sigma(b^{\dagger 4}+b^4)\}.$$

The new creation and annihilation operators $b^\dagger$ and b are the unitary transforms of $a^\dagger$ and a;

$$b^\dagger = \alpha a^\dagger + \beta a, \quad b = \beta a^\dagger + \alpha a$$

with

$$\alpha = \tfrac{1}{2}[(1-\sigma)^{-1/4} + (1-\sigma)^{1/4}], \quad \beta = \tfrac{1}{2}[(1-\sigma)^{-1/4} - (1-\sigma)^{1/4}]$$

and $\mathrm{M}=b^\dagger b$ is the new number operator.

By taking H_0 as an unperturbed Hamiltonian, the usual perturbation method yields the asymptotic expansion for the eigenvalue $\lambda(s; n, \kappa)$ (denoted as $\lambda_{\kappa n}(s)$ in (I)):

$$\lambda(s; n, \kappa) = n^2(1-\sigma) + 2n\sqrt{1-\sigma}\left(\kappa+\frac{1}{2}\right) + \frac{1}{8}[(2-\sigma)(2\kappa+1)^2 - (2+\sigma)]$$

$$+\frac{2\sigma}{2n\sqrt{1-\sigma}}\left(\frac{1}{8}\right)^2[-\sigma(2\kappa+1)^3+3(8-\sigma)(2\kappa+1)]$$

$$+\frac{2\sigma}{(2n\sqrt{1-\sigma})^2}\left(\frac{1}{8}\right)^3[5\sigma(2-\sigma)(2\kappa+1)^4$$

$$-2(144-34\sigma+17\sigma^2)(2\kappa+1)^2-9(32-2\sigma+\sigma^2)]+\cdots.$$

This suggests a reflection symmetry of the eigenvalues

$$\lambda(s;-n,-\kappa-1)=\lambda(s;n,\kappa). \tag{2·2}$$

The equality is to be understood as a brief expression that each term in the $1/n$ expansion or in the power series expansion with respect to s is invariant, as a rational function of n and κ, under the replacement n and κ by $-n$ and $-\kappa-1$. This will be discussed in § 4.

§ 3. Proof of reflection symmetry

We shall derive another representation for $L_n(\sigma)$. Instead of the creation and annihilation operators, the operator is expressed in terms of the generators in the positive discrete series of unitary representations of the group $SU(1,1)$.

The representation space of $D_j^+(j=1/2, 1, 3/2, \cdots)$ in the positive discrete series of unitary representations of $SU(1,1)$[5)] is spanned by the orthonormal basis vectors $|j,m)$, $m=j, j+1, \cdots$. The generators $J_\pm$ and J_3 are represented as

$$J_+|j,m)=\sqrt{(m+1-j)(m+j)}\,|j,m+1),$$

$$J_-|j,m)=\sqrt{(m-j)(m-1+j)}\,|j,m-1),$$

$$J_3|j,m)\ =m|j,m).$$

The Casimir invariant $\mathbf{J}^2:=J_3^2-(J_+J_-+J_-J_+)/2$ is the identity operator multiplied by $j(j-1)$.

Under the replacement of $m-j$ and j by κ and $n+1/2$, the identification $|j,m)$ with $|n\kappa\rangle$ in (I) leads to

$$J_+|n\kappa\rangle=\sqrt{(\kappa+1)(2n+\kappa+1)}\,|n\kappa+1\rangle,$$

$$J_-|n\kappa\rangle=\sqrt{\kappa(2n+\kappa)}\,|n\kappa-1\rangle,$$

$$J_3|n\kappa\rangle\ =(n+\kappa+\tfrac{1}{2})\,|n\kappa\rangle,$$

so that the generators $J_\pm$ and J_3 are represented by $a^\dagger$ and a as

$$J_+=\sqrt{2n+N}a^\dagger,$$

$$J_-=a\sqrt{2n+N},$$

$$J_3\ =n+N+\tfrac{1}{2}. \tag{3·1}$$

As is easily verified, the operator $L_n(\sigma)$ in (2·1) is expressed in terms of $\mathbf{J}$ as follows:

$$L_n(\sigma)=\sqrt{J_3^2-\frac{1}{4}}\left[1-\frac{\sigma}{2}\left(1+\frac{\mathbf{J}^2}{J_3^2-1}\right)+\frac{\sigma}{4}\left(\frac{1}{\sqrt{J_3}}J_+\frac{1}{\sqrt{J_3}}\right)^2\right.$$
$$\left.+\frac{\sigma}{4}\left(\frac{1}{\sqrt{J_3}}J_-\frac{1}{\sqrt{J_3}}\right)^2\right]\sqrt{J_3^2-\frac{1}{4}}\,. \tag{3·2}$$

It is to be noted here that (3·1) is the analogue of the Holstein-Primakoff representation for the generators of $SO(3)$, and the expansion method employed in (I) is a version of the Holstein-Primakoff treatment of spin waves.[6]

It is well known that there exists an (outer) automorphism $\mathbf{J}\to\mathbf{J}'$,

$$J_+\to J_+'=J_-\,,\quad J_-\to J_-'=J_+\,,\quad J_3\to J_3'=-J_3\,, \tag{3·3}$$

in the Lie algebra of $SU(1,1)$, that is, the commutation relation remains unchanged by the replacement (3·3). The reflection symmetry follows from the invariance of $L_n(\sigma)$ in (3·2) under the automorphism.*) (The representation of $\mathbf{J}'$ in the space of D_j^+ is equivalent to D_j^-, the negative discrete series of unitary representations.) Let $L_n(\sigma)$ be diagonalized by a unitary transformation $V=V(\sigma;\mathbf{J})$. Since both $V^{-1}L_n(\sigma)V$ and J_3 are of diagonal form, they commute with each other. The irreducibility of D_j^+ implies that $V^{-1}L_n(\sigma)V$ is a function of $\mathbf{J}^2$ and J_3:

$$V^{-1}L_n(\sigma)V=F(\sigma;\mathbf{J}^2,J_3)\,. \tag{3·4}$$

Here V is assumed to be an identity operator at $\sigma=0$ and vary continuously with σ. Because of the invariance $L_n(\sigma)'=L_n(\sigma)$, application of the automorphism to (3·4) yields

$$V'^{-1}L_n(\sigma)V'=F(\sigma;\mathbf{J}'^2,J_3')=F(\sigma;\mathbf{J}^2,-J_3)\,.$$

There exist thus two operators, V and V' $(=V(\sigma;\mathbf{J}'))$, which diagonalize $L_n(\sigma)$. Since they coincide at $\sigma=0$ and vary continuously with σ and since all the eigenvalues of $L_n(\sigma)$ are shown to be multiplicity-free,[2] the eigenvector $V|j,m)$ and $V'|j,m)$ must be equal up to a possible phase factor. It follows therefore that the eigenvalue of $V|j,m)$ and that of $V'|j,m)$ coincide with each other:

$$F(\sigma;j(j-1),m)=F(\sigma;j(j-1),-m)\,.$$

From the equalities $j=n+1/2$ and $m=n+\kappa+1/2$, the reflection symmetry

$$\lambda(s;n,\kappa)=F\left(\sigma;n^2-\frac{1}{4},n+\kappa+\frac{1}{2}\right)=F\left(\sigma;n^2-\frac{1}{4},-n-\kappa-\frac{1}{2}\right)$$
$$=\lambda(s;-n,-\kappa-1)$$

follows immediately.

*) In both branches $\sqrt{-J_3}=\pm i\sqrt{J_3}$, the automorphism leads to $(1/\sqrt{J_3})J_\pm(1/\sqrt{J_3})\to-(1/\sqrt{J_3})J_\mp(1/\sqrt{J_3})$.

§ 4. Discussion

The representation (3·2) can be used to develop a modified scheme for the perturbation with respect to σ. The operator $\mathrm{L}_n(\sigma)$ is written as

$$\mathrm{L}_n(\sigma)=\mathrm{h}_0+\frac{\sigma}{4}\mathrm{h}_1\,,$$

where h_0 and h_1 are defined by

$$\mathrm{h}_0:=\left(\mathrm{J}_3^2-\frac{1}{4}\right)\left[1-\frac{\sigma}{2}\left(1+\frac{j(j-1)}{\mathrm{J}_3^2-1}\right)\right],$$

$$\mathrm{h}_1:=\sqrt{\mathrm{J}_3^2-\frac{1}{4}}\left[\left(\frac{1}{\sqrt{\mathrm{J}_3}}\mathrm{J}_+\frac{1}{\sqrt{\mathrm{J}_3}}\right)^2+\left(\frac{1}{\sqrt{\mathrm{J}_3}}\mathrm{J}_-\frac{1}{\sqrt{\mathrm{J}_3}}\right)^2\right]\sqrt{\mathrm{J}_3^2-\frac{1}{4}}\,.$$

The unperturbed Hamiltonian h_0 is of a diagonal form

$$\mathrm{h}_0|j,m)=E(j,m)|j,m)$$

with

$$E(j,m):=\left(m^2-\frac{1}{4}\right)\left[1-\frac{\sigma}{2}\left(1+\frac{j(j-1)}{m^2-1}\right)\right]$$

and the nonvanishing matrix elements of h_1 are

$$(j,m|\mathrm{h}_1|j,m-2)=f_-(j,m)\,,\qquad (j,m|\mathrm{h}_1|j,m+2)=f_+(j,m)$$

with

$$f_-(j,m):=\left[\left(m^2-\frac{1}{4}\right)\left(1-\frac{2j(j-1)}{m(m-2)}+\frac{[j(j-1)]^2}{m(m-1)^2(m-2)}\right)\left((m-2)^2-\frac{1}{4}\right)\right]^{1/2},$$

$$f_+(j,m):=\left[\left(m^2-\frac{1}{4}\right)\left(1-\frac{2j(j-1)}{m(m+2)}+\frac{[j(j-1)]^2}{m(m+1)^2(m+2)}\right)\left((m+2)^2-\frac{1}{4}\right)\right]^{1/2}.$$

Since $E(j,m)$, $f_-(j,m)^2$ and $f_+(j,m)^2$ are rational functions of $j(j-1)$ and m, the equalities $E(j,-m)=E(j,m)$ and $f_+(j,-m)=f_-(j,m)$ ensure that, at each step of the perturbation, the approximate eigenvalue is a rational function of $j(j-1)$ and m^2. This is nothing but the reflection symmetry proved in the preceding section. It is meaningless, however, to say that the eigenvalue is a function of $j(j-1)$ and m^2, unless the analyticity property is known with respect to these variables. It is highly desirable to clarify the analyticity of the eigenvalues in connection with the investigation of Regge trajectories in the Wick-Cutkosky model.[7]

The modified perturbation method does not constitute, unfortunately, a Padé approximant scheme. The advantage of the modified method over the usual one consists in that a part of the multiple poles at $m^2=1,\ 4,\ 9,\ \cdots$ appearing in the usual perturbation can be absorbed as moving poles in the complex m^2 plane.

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