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Reflection, Transmission and Diffraction of *SH*-Waves in Linear Viscoelastic Solids

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Summary

The classical problem of a line source, emitting an axisymmetric time-harmonic disturbance, and situated at a finite distance from a plane interface separating two different viscoelastic solids, is considered. Asymptotic expressions for the reflected, transmitted and diffracted waves are obtained. The emphasis of the work is directed towards the study of the spatial amplitude variations, induced by the presence of dissipation.

1. Introduction

Studies of seismic records indicate an amplitude decay, varying with epicentral distance, which is not accounted for by geometric spreading, loss at sharp boundaries etc. The effect, which is small is usually attributed to scattering and anelasticity. Some success has been obtained in the interpretation of the data, on the assumption that certain regions of the Earth, particularly the upper mantle, are linear viscoelastic. However, little theoretical work has been done on the propagation of seismic waves through such a medium. The work presented in this paper is an attempt to provide a better understanding to such problems and in particular, to provide a global picture of the amplitude decay associated with the dissipation in linear viscoelastic solids. To this end, all the essential features are exhibited in the simplified problem of a line source situated at a finite distance from a plane interface separating two linear viscoelastic solids which are in a state of anti-plane strain. The source is assumed to be time-harmonic. This has the advantage that no explicit model of viscoelasticity need be chosen—the dissipation can be simply represented by a small imaginary part in the wave numbers. It transpires that important differences occur when the dissipation in one medium is greater than or less than the dissipation in the other. The problem is solved using a well-known technique. The formal solution is obtained in integral form and after suitable deformation of the contours, asymptotic expressions are derived for the principal phases. The method employed here essentially parallels that of Brekhovskikh (1960).

2. Integral representation of the solution

The geometry of the problem is depicted in Fig. 1. M and M' are two linear viscoelastic solids in welded contact along the plane $z = 0$. M occupies the region $-\infty < x, y < \infty, z \leqslant 0$ and M' the region $-\infty < x, y < \infty, z \geqslant 0$. The line source S is situated along $x = 0$, $z = -h$ and is assumed to emit an axisymmetric disturbance of harmonic time variation $\exp(-i\omega t)$. It is further assumed that the wavelength λ, of the disturbance in M is less than the corresponding wavelength λ', in M'. As is well known, this condition assures the existence of the 'head wave' or diffracted wave.

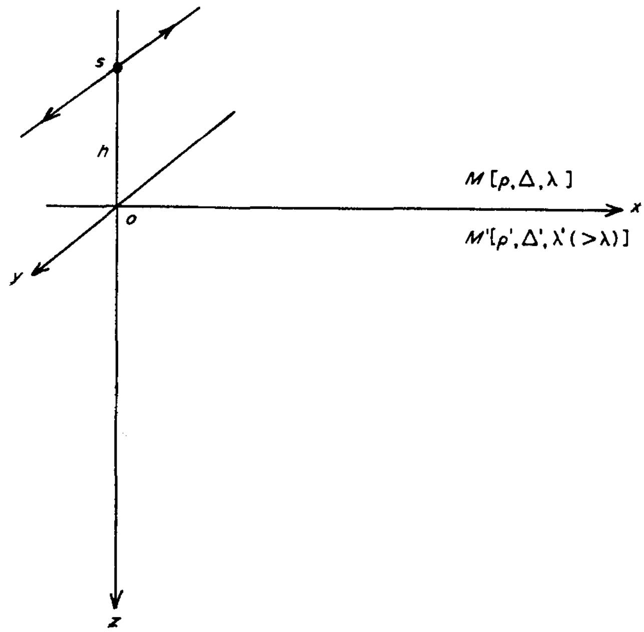

FIG. 1. The geometry of the problem.

For the *SH*-mode of propagation the displacement vector has the form $(0, V(x, z) \exp(-i\omega t), 0)$ in M, where

$$\nabla^2 V + k^2 V = 0, z \leqslant 0. \tag{2.1}$$

k is the complex wave number and can be written in the form

$$k = \kappa(1+i\delta), \delta \ll 1 \tag{2.2}$$

where $\kappa = 2\pi/\lambda$ and δ is a small parameter which determines the extent of dissipation. In terms of the more familiar 'loss parameters' Q (the quality factor) and Δ (the logarithmic decrement),

$$\delta = \frac{\mathscr{I}m\, k}{\mathscr{R}e\, k} = \tfrac{1}{2}Q^{-1} = \Delta/2\pi. \tag{2.3}$$

Similarly, for the disturbance in M',

$$\nabla^2 V' + k'^2 V' = 0, z \geqslant 0 \tag{2.4}$$

where

$$k' = \kappa'(1+i\delta'), \delta' \ll 1 \tag{2.5}$$

and $\kappa > \kappa'$.

Across the boundary the displacements and normal stresses are continuous. Thus if ρ and ρ' are the densities, the boundary conditions on $z = 0$ are

$$\left.\begin{aligned} V &= V' \\ \rho/k^2 \frac{\partial V}{\partial z} &= \rho'/k'^2 \frac{\partial V'}{\partial z} \end{aligned}\right\} \tag{2.6}$$

In the absence of the plane boundary, the line source radiates a field which can be written as

$$V_0 = H_0^{(1)}\left(k\sqrt{(x^2+(z+h)^2)}\right) = H_0^{(1)}(kR) \tag{2.7}$$

where $H_0^{(1)}$ is the Hankel function of zero order and of the first kind. For distances which are large compared with the wavelength

$$V_0 \sim \left(\frac{2}{\pi\kappa R}\right)^{\frac{1}{2}} \exp(-\kappa\delta R) \exp i(\kappa R - \delta/2 - \pi/4). \tag{2.8}$$

Thus the line source emits a disturbance with cylindrical wave fronts and which asymptotically displays an exponential decay of decrement $\Delta = 2\pi\delta$. The amplitude is constant on the wave front.

We seek the solution to our problem in the form,

$$V = V_0 + \frac{1}{\pi}\int_{-\infty}^{\infty} v^{-1} A(\zeta) \exp i\{\zeta x + (h-z)v\}\, d\zeta,\ z \leqslant 0 \tag{2.9}$$

$$V' = \frac{1}{\pi}\int_{-\infty}^{\infty} v^{-1} A'(\zeta) \exp i\{\zeta x + v'z + vh\}\, d\zeta,\ z \geqslant 0 \tag{2.10}$$

where the integrals represent the perturbation on the main field V_0 due to the presence of the boundary. A and A' are coefficients chosen to satisfy the boundary conditions and

$$\left.\begin{aligned} v &= \sqrt{(k^2-\zeta^2)},\ \mathscr{I}m\, v \geqslant 0 \\ v' &= \sqrt{(k'^2-\zeta^2)},\ \mathscr{I}m\, v' \geqslant 0. \end{aligned}\right\} \tag{2.11}$$

Noticing that V_0 can be expressed in integral form as

$$V_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp i\{\zeta x + (h+z)v\} \frac{d\zeta}{v},\ z+h \geqslant 0 \tag{2.12}$$

the boundary conditions (2.6) are satisfied with

$$A(\zeta) = \frac{(\rho/\rho')k'^2 v - k^2 v'}{(\rho/\rho')k'^2 v + k^2 v'} \tag{2.13}$$

and

$$A'(\zeta) = \frac{2(\rho/\rho')k'^2 v}{(\rho/\rho')k'^2 v + k^2 v'}. \tag{2.14}$$

The expressions (2.9) and (2.10) are the formal solutions of our problem. The integrals, or their equivalents, have been obtained previously by many authors, for the case of either zero dissipation or perhaps when only one of the media is dissipative. At this point, we could obtain asymptotic expressions for the integrals as done by Lapwood (1949) or Gerjuoy (1953). However, we find it more suitable to first make a conformal transformation from the ζ-plane, thereby reducing the integrals to a form similar to that obtained by Brekhovskikh (1960).

3. The conformal transformation $\zeta = k \cos \gamma$

Under the conformal mapping $\zeta = k \cos \gamma$, the integrals (2.9) and (2.10) become

$$V = V_0 + \frac{1}{\pi} \int_{\mathscr{C}} A(\gamma) \exp\left(ikr \cos(\theta-\gamma)\right) d\gamma \tag{3.1}$$

and

$$V' = \frac{1}{\pi} \int_{\mathscr{C}} A'(\gamma) \exp i\{kr' \cos\phi' \cos\gamma + \nu' r' \sin\phi' + kh \sin\gamma\} d\gamma \tag{3.2}$$

respectively, where for $z \leqslant 0$, $x = r \cos\theta$, $z = h - r \sin\theta$ and for $z \geqslant 0$, $x = r' \cos\phi'$, $z = r' \sin\phi'$.

Further,

$$\nu \to k \sin\gamma \tag{3.3a}$$

and

$$\nu' \to \sqrt{(k'^2 - k^2 \cos^2\gamma)}, \; \mathscr{I}m\, \nu' \geqslant 0. \tag{3.3b}$$

The transformation takes the real axis of the ζ-plane into the path $\mathscr{C}$ of the γ-plane given by $\mathscr{I}m\, k \cos\gamma = 0$ or with $\gamma = \alpha + i\beta$ by

$$\sin\alpha \sinh\beta - \delta \cos\alpha \cosh\beta = 0. \tag{3.4}$$

The path starts at $\delta + i\infty$, passes through $(\pi/2, 0)$ at an angle $-\delta$ with the α-axis, and ends at $\pi - \delta - i\infty$, as shown in Fig. 2. When M is perfectly elastic, i.e. $\delta = 0$, the path reduces to the familiar Weyl contour.

The ζ-plane is a 4-fold Riemann surface (corresponding to the four combinations of sign of $\mathscr{I}m\, \nu$ and $\mathscr{I}m\, \nu'$) joined along the branch cuts defined by $\mathscr{I}m\, \nu = 0$ and $\mathscr{I}m\, \nu' = 0$. The transformation 'unfolds' the branches through $\zeta = \pm k$ and divides the γ-plane into a 2-fold Riemann surface. The top-sheet ($\mathscr{I}m\, \nu' \geqslant 0$), consists of alternate curvilinear strips for which $\mathscr{I}m\, \nu \gtrless 0$, corresponding to sheets I and III of the ζ-plane. The path $\mathscr{C}$ is taken in the strip for which $\mathscr{I}m\, \nu \geqslant 0$, $\mathscr{I}m\, \nu' \geqslant 0$. Similarly, the lower-sheet of the γ-plane ($\mathscr{I}m\, \nu' \leqslant 0$) consists of alternate curvilinear strips corresponding to the sheets II ($\mathscr{I}m\, \nu \geqslant 0$) and IV ($\mathscr{I}m\, \nu \leqslant 0$) of the ζ-plane.

The two sheets of the γ-plane are joined along the branch cuts defined by

$$\mathscr{I}m \sqrt{(k'^2 - k^2 \cos^2\gamma)} = 0 \tag{3.5}$$

and the corresponding branch points are situated at γ_0 and $\pi - \gamma_0$ where

$$\cos\gamma_0 = k'/k. \tag{3.6a}$$

Writing $\gamma_0 = \alpha_0 + i\beta_0$, we find to the first order of δ and δ' that

$$\alpha_0 = \cos^{-1}(\kappa'/\kappa), \; \beta_0 = \frac{\delta - \delta'}{\sqrt{(\kappa^2/\kappa'^2 - 1)}}. \tag{3.6b}$$

Thus the branch point at $\gamma = \gamma_0$ lies above or below the real axis according as $\delta \gtrless \delta'$ and lies on it only when $\delta = \delta'$. Further, since $\mathscr{I}m\, k \cos\gamma_0 = \kappa'\delta' \geqslant 0$ the branch point γ_0 lies below the path of integration $\mathscr{C}$. Also, since

$$\mathscr{I}m\, k \sin\gamma_0 \sim \frac{\kappa^2 \delta - \kappa'^2 \delta'}{\sqrt{(\kappa^2 - \kappa'^2)}}$$

to first order, γ_0 lies in the strip corresponding to sheet I or III of the ζ-plane according as $\kappa^2 \delta \gtrless \kappa'^2 \delta'$.

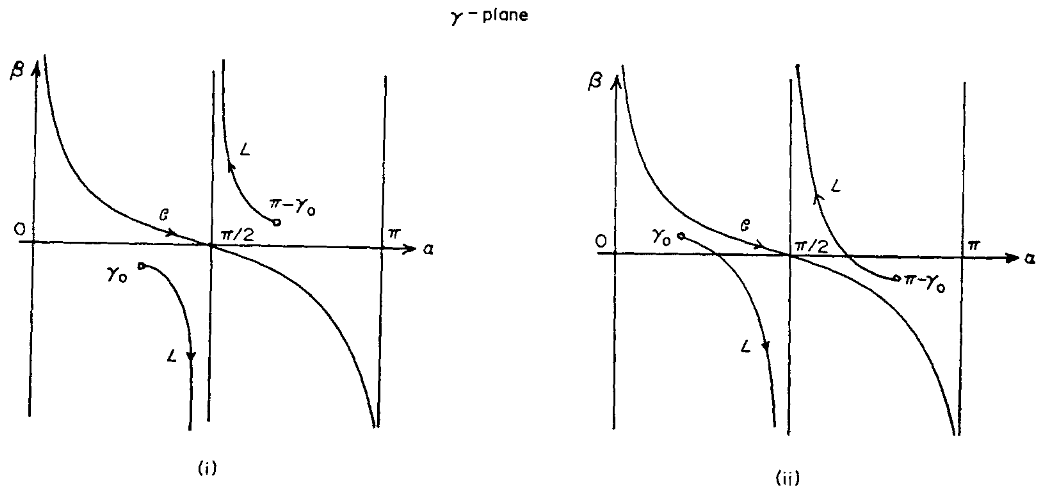

FIG. 2. The γ-plane indicating the path of the integration $\mathscr{C}$, the branch cuts L, and the branch points γ_0, $\pi-\gamma_0$ for the cases (i) $\delta<\delta'$, and (ii) $\delta>\delta'$.

For any point on the branch cut given by (3.5), $k'^2-k^2\cos^2\gamma = X^2$ where X^2 is real and positive. For large $|X|$, $k\cos\gamma \to \pm iX$ or $\mathscr{R}e\, k\cos\gamma \to 0$. Thus the branch cut through γ_0 tends to $\pi/2-\delta-i\infty$ and the branch cut through $\pi-\gamma_0$ tends to $\pi/2+\delta+i\infty$. The situation is depicted in Fig. 2.

The poles of the integrand must also be investigated. To simplify the discussion, we shall assume that $\rho \approx \rho'$. This condition is generally satisfied in geophysical applications and is unlikely to detract anything from the qualitative results. The poles are then given by the roots of

$$k'^2\, v+k^2\, v = 0 \tag{3.7a}$$

or

$$k'^2 \sin\gamma+k\sqrt{(k'^2-k^2\cos^2\gamma)} = 0. \tag{3.7b}$$

There are two roots of (3.7b) in $0<\gamma<\pi$ of the form $\gamma=\gamma_1$, $\pi-\gamma_1$ where $\tan\gamma_1 = k/k'$. With $\gamma_1 = \alpha_1+i\beta_1$, we find to first order

$$\alpha_1 = \tan^{-1}(\kappa/\kappa'),\ \beta_1 = \frac{\kappa/\kappa'(\delta-\delta')}{\kappa^2/\kappa'^2+1}. \tag{3.8}$$

However, since (3.9) was obtained by squaring, these roots need not lie on the top-sheet of the γ-plane. They do so when $\mathscr{I}m\,(k'^2/k)\sin\gamma_1 < 0$ or when

$$\delta/\delta' > 2+\kappa'^2/\kappa^2. \tag{3.9}$$

It is easy to deduce that $\alpha_1 > \alpha_0$, $|\beta_1| < |\beta_0|$ and that γ_1 lies on the same side of the real axis as γ_0. Note, that for $\delta<\delta'$, the pole always lies on the lower-sheet. The position of the poles relative to the branch cuts and path of integration is shown in Fig. 3.

4. The reflected wave

The integral in (3.1) can be written in the form

$$W = \frac{1}{\pi}\int_{\mathscr{C}} A(\gamma)\exp\left(\kappa r f(\gamma)\right)d\gamma \tag{4.1}$$

where

$$f(\gamma) = i\,(l + i\delta)\cos(\theta - \gamma). \qquad (4.2)$$

The asymptotic expansion of (4.1), for $\kappa r \gg 1$, can be obtained by the method of steepest descent. The saddle point γ_s is given by $f'(\gamma_s) = 0$ or $\gamma_s = \theta$, and lies on the real axis. The path of steepest descent is given by $\mathscr{I}m\, f(\gamma) = \mathscr{I}m\, f(\gamma_s)$ and leads to the contour, $S(\theta)$ expressed explicitly by

$$\cos(\theta - \alpha)\cosh\beta - \delta\sin(\theta - \alpha)\sinh\beta = 1. \qquad (4.3)$$

The path starts at $-\pi/2 + \theta + \delta + i\infty$, passes through the saddle point at an angle $\varepsilon = -(\pi/4 + \delta/2)$ with the real axis and ends at $\pi/2 + \theta - \delta - i\infty$. When $\delta = 0$, $S(\theta)$ reduces to the contour obtained by Brekhovskikh (1960). The path of integration $\mathscr{C}$ is distorted into the path of steepest descent $S(\theta)$, and any singularities which are crossed in the process must be taken into account. It transpires that $S(\theta)$ cannot be contained to the top-sheet alone. However, this is permissible provided the path begins and ends on the top-sheet. The distortion into the path of steepest descent is shown in Fig. 3, from which we can determine the following:

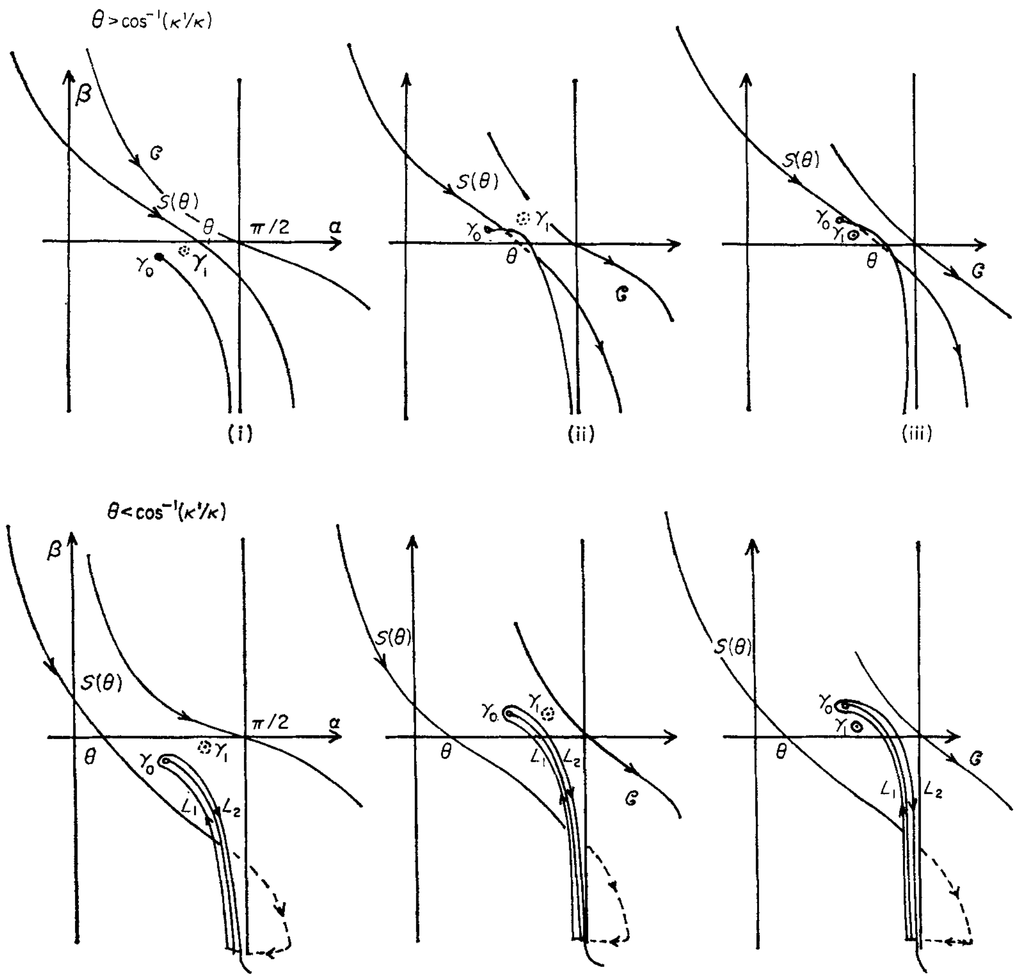

FIG. 3. The path of steepest descent $S(\theta)$ and pole γ_1 for the cases $\theta \lessgtr \cos^{-1}(\kappa'/\kappa)$ and (i) $\delta < \delta'$, (ii) $1 < \delta/\delta' < 2 + \kappa'^2/\kappa^2$, and (iii) $\delta/\delta' > 2 + \kappa'^2/\kappa^2$. Full lines represent positions on the top-sheet, dashed lines for the lower-sheet.

(i) No singularities are crossed when $\theta > \cos^{-1}(\kappa'/\kappa)$. Note however, that for $\delta > \delta'$ and $\theta < \cos^{-1}((\kappa'/\kappa)\sqrt{(\delta'/\delta)})$, the saddle point must be taken on the lower sheet of the γ-plane.

(ii) For $\theta < \cos^{-1}(\kappa'/\kappa)$, the branch point γ_0 is traversed so that a corresponding branch line contribution must be included.

(iii) For $\theta < \cos^{-1}(\kappa'/\kappa)$ and $\delta/\delta' > 2+\kappa'^2/\kappa^2$ the pole γ_1 is traversed so that its contribution must also be included.

Thus (4.1) can be expressed as

$$W = W_s + W_B + W_p \tag{4.4}$$

where the three terms correspond to the saddle point, branch line and pole contributions respectively. We now, in turn, investigate these contributions to the field in M.

In the regions for which $A(\gamma)$ is slowly varying compared with $\kappa r f(\gamma)$, the saddle point contribution gives the asymptotic result (Jeffreys & Jeffreys 1966)

$$W_s \sim \frac{1}{\pi} A(\gamma_s) \left(\frac{2\pi}{\kappa r |f''(\gamma_s)|}\right)^{\frac{1}{2}} \exp\left(\kappa r f(\gamma_s) + i\varepsilon\right)$$

$$= \left(\frac{2}{\pi\kappa r}\right)^{\frac{1}{2}} A(\theta) \exp(-\kappa\delta r) \exp i(\kappa r - \delta/2 - \pi/4). \tag{4.5}$$

This is the geometrically reflected wave with angle of reflection θ and reflection coefficient $A(\theta)$. The wave fronts are cylindrical and appear to come from an image source at $z = h$. The amplitude variation, like the radiated or direct wave, is constant on the wave front. The reflection coefficient, which is analysed in more detail in Section 8, is complex, indicating that there is a phase shift relative to the direct wave.

5. The head wave

The branch line contribution to (4.4) gives the diffracted field or head wave. The integration is taken along the two borders L_1 and L_2 of the branch cut $\mathscr{I}m\, \nu' = 0$. The integrands on L_1 and L_2 differ only in the sign of $\mathscr{R}e\, \nu'$. On L_1, $A(\gamma) = A^-(\gamma)$ with $\mathscr{R}e\, \nu' < 0$ and on L_2, $A(\gamma) = A^+(\gamma)$ with $\mathscr{R}e\, \nu' > 0$. Thus combining the two integrations into one along the path $L = L_2 = -L_1$ we obtain

$$W_B = \frac{1}{\pi} \int_L B(\gamma) \exp\left(ikr\cos(\theta-\gamma)\right) d\gamma,\ \theta < \alpha_0 \tag{5.1}$$

where

$$B(\gamma) = A^+(\gamma) - A^-(\gamma)$$

$$= \frac{-4(\rho/\rho')(k'^2/k^2)\nu\nu'}{(\rho^2/\rho'^2)(k'/k)^4\nu^2 - \nu'^2},\ \mathscr{R}e\, \nu' > 0. \tag{5.2}$$

For $\kappa r \gg 1$, the asymptotic form of W_B is obtained by deforming the path L in such a way that it goes from the branch point $\gamma_0 = \alpha_0 + i\beta_0$, along the path Γ, on which the exponential in the integrand decreases most rapidly. The major contribution to the integral will then come from the initial part only of Γ. With $f(\gamma)$ given by (4.2), Γ is the contour given by,

$$\mathscr{I}m f(\gamma) = \mathscr{I}m f(\gamma_0),\ \mathscr{R}e f(\gamma) < 0. \tag{5.3}$$

The contour is shown in Fig. 4. Disregarding, for the present, any possible poles or other singularities which may be crossed, we can deform L into Γ to obtain

$$W_B = \frac{1}{\pi} \exp\left(i\,\mathscr{I}m f(\gamma_0)\right) \int_\Gamma B(\gamma) \exp\left(\kappa r\,\mathscr{R}e f(\gamma)\right) d\gamma. \tag{5.4}$$

Now in the vicinity of the branch point

$$\gamma \approx \gamma_0 + \eta \exp(i\theta_0),\ |\eta/\gamma_0| \ll 1 \tag{5.5}$$

$$B(\gamma)\, d\gamma \approx F(\gamma_0)\, \eta^{\frac{1}{2}}\, d\eta \tag{5.6}$$

where

$$F(\gamma_0) = -4(\rho'/\rho)(k^2/k'^2)\sqrt{(2 \cot \gamma_0)} \exp(i^{\frac{1}{2}}\theta_0) \tag{5.7}$$

and

$$f(\gamma) \approx f(\gamma_0) + \eta \exp(i\theta_0) f'(\gamma_0). \tag{5.8}$$

To satisfy (5.3) we choose

$$\theta_0 = \pi - \arg f'(\gamma_0) \tag{5.9}$$

whence, on Γ,

$$\mathscr{R}e f(\gamma) = \mathscr{R}e f(\gamma_0) - \eta |f'(\gamma_0)|. \tag{5.10}$$

Thus, we have approximately

$$W_B \sim \frac{1}{\pi} F(\gamma_0) \exp\left(\kappa r f(\gamma_0)\right) \int_0^\infty \eta^{\frac{1}{2}} \exp\left(-\kappa r |f'(\gamma_0)| \eta\right) d\eta \tag{5.11}$$

$$= \frac{1}{2\sqrt{\pi}} \left(\kappa r |f'(\gamma_0)|\right)^{-\frac{3}{2}} F(\gamma_0) \exp\left(\kappa r f(\gamma_0)\right) \tag{5.12}$$

$$= -\frac{2}{\sqrt{\pi}} \frac{\rho' k^2}{\rho k'^2} \sqrt{(2 \cot \gamma_0)} \exp(i^{\frac{1}{2}}\theta_0) \frac{\exp\left(ikr \cos(\theta - \gamma_0)\right)}{\{\kappa r \sin(\alpha_0 - \theta)\}^{\frac{3}{2}}} \tag{5.13}$$

where

$$\theta_0 \approx \pi/2 - \delta - \beta_0 \cot(\alpha_0 - \theta). \tag{5.14}$$

The physical interpretation of the head wave is obtained from its complex phase

$$\Phi(\gamma) = kr \cos(\theta - \gamma_0). \tag{5.15}$$

The wave fronts are given by the curves $\mathscr{R}e\,\Phi = $ const. and the dissipation induces an amplitude variation which is constant on the curves $\mathscr{I}m\,\Phi = $ const.

The results of Appendix I determine the familiar ray path of the head wave and the loss of amplitude along it. A purely geometrical interpretation for this amplitude decay is given in Appendix II.

From (3.6) and (5.15), the wave fronts of the head wave are given by the planes

$$x - z \tan \alpha_0 = \text{const.},\ z \leqslant 0 \tag{5.16}$$

and the lines of constant amplitude (induced by the dissipation) by

$$x - \left(\frac{\sqrt{(\kappa^2/\kappa'^2 - 1)}}{\kappa^2 \delta/\kappa'^2 \delta' - 1}\right) z = \text{const.},\ z \leqslant 0. \tag{5.17}$$

It is thus evident, that in general, the amplitude is not constant on the wave front, as it would be in the absence of any dissipation. The following special cases illustrate the point:

(i) $\delta = 0$. When M (the medium containing the source) is free of dissipation, the lines of constant amplitude, induced by the dissipation in M', are perpendicular to the wave fronts. The head wave is then a plane inhomogeneous wave propagating in a perfectly elastic medium.

(ii) $\kappa^2 \delta = \kappa'^2 \delta'$. The lines of constant amplitude are normal to the interface $z = 0$.

(iii) $\delta = \delta'$. When the attenuation per unit wavelength is the same for both media, the amplitude is so adjusted to be constant on the wave front.

(iv) $\delta' = 0$. When M' is free of dissipation, the lines of constant amplitude, induced by the dissipation in M, are parallel to the interface $z = 0$.

We can further show that the induced amplitude on the wave fronts of the head wave, increases exponentially from the interface for $\delta < \delta'$ and decreases exponentially for $\delta > \delta'$.

6. The pole contribution

There is no pole contribution in the perfectly elastic case, so it is not too surprising that the same net result is obtained in the presence of small dissipation.

As was discussed earlier there is a pole contribution to W of equation (4.4) when $\theta < \cos^{-1}(\kappa'/\kappa)$ and $\delta/\delta' > 2+\kappa'^2/\kappa^2$. This contribution is readily obtained to give

$$Wp = \frac{4ik/k'}{k^4/k'^4-1} \exp\left(ikr\cos(\theta-\gamma_1)\right). \tag{6.1}$$

We now investigate the possibility of a pole contribution from the branch line integral evaluated in Section 5. Here, the branch line L was distorted into a new path Γ and poles of the integrand may have been crossed in the process. The poles of the integrand are those of $B(\gamma)$, given by (5.2), with $\mathscr{Re}\, v' > 0$. The path L can be distorted into Γ without crossing the curve $\mathscr{Re}\, v' = 0$ (see Fig. 4.) so all that remains is to investigate if $B(\gamma) = A^+(\gamma)-A^-(\gamma)$ has a pole between L and Γ for which $\mathscr{Re}\, v' > 0$.

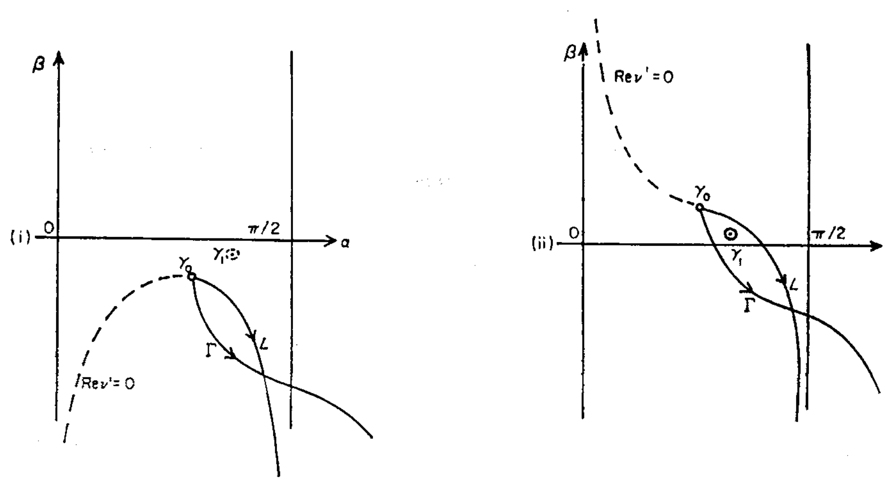

FIG. 4. Branch line L and steepest descent path Γ for the head wave. The two cases are (i) $\delta/\delta' < \kappa'^2/\kappa^2 < 1$, and (ii) $\delta/\delta' > 2+\kappa'^2/\kappa^2$.

We have already seen that $A(\gamma)$ has a pole lying between L and Γ when $\delta/\delta' > 2+\kappa'^2/\kappa^2$ and since

$$\mathscr{R}e\,(k'^2/k)\sin\gamma_1 \approx \frac{\kappa'^2}{\sqrt{(\kappa^2+\kappa'^2)}} > 0,$$

$B(\gamma)$ does have the required pole at $\gamma = \gamma_1$, corresponding to a zero of the denominator of $A^-(\gamma)$, and at this point $A^+(\gamma) = 0$. It is now a simple matter to show that the contribution from this pole is equal in magnitude to (6.1), but of opposite sign. The net pole contribution is now seen to be zero, in accordance with our previous remarks.

7. The transmitted wave

The disturbance in the lower medium M' is given by the integral (3.2) which contains the complex phase

$$\Phi(\gamma) = k\cos\gamma r'\cos\phi' + \sqrt{(k'^2-k^2\cos^2\gamma)}\,r'\sin\phi' + kh\sin\gamma. \tag{7.1}$$

The saddle point γ_s, satisfies $\Phi'(\gamma_s) = 0$, giving

$$-k\sin\gamma_s r'\cos\phi' + \frac{k^2\sin\gamma_s\cos\gamma_s}{\sqrt{(k'^2-k^2\cos^2\gamma_s)}}\,r'\sin\phi' + kh\cos\gamma_s = 0 \tag{7.2}$$

or

$$\frac{r'\sin(\gamma_s'-\phi')}{\sin\gamma_s'} = h\cot\gamma_s \tag{7.3}$$

where

$$k'\cos\gamma_s' = k\cos\gamma_s. \tag{7.4}$$

(7.4) is of course just the mathematical expression (in complex form) of Snell's Law of Refraction.

The solution of (7.3) and (7.4) for γ_s and γ_s' can be obtained for $\theta > \cos^{-1}(\kappa'/\kappa)$ in the form

$$\gamma_s = \theta + i\beta, \; |\beta/\theta| \ll 1 \tag{7.5}$$

and

$$\gamma_s' = \theta' + i\beta', \; |\beta'/\theta'| \ll 1. \tag{7.6}$$

Thus to the first order of δ and δ', we find that θ and θ' correspond geometrically to the angles of incidence and refraction respectively (see Appendix I) and

$$\beta = \frac{(\delta-\delta')\,P'\kappa\sin\theta\cos\theta}{P\kappa'\sin^2\theta' + P'\kappa\sin^2\theta} \tag{7.7}$$

$$\beta' = \frac{-(\delta-\delta')\,P\kappa'\sin\theta'\cos\theta'}{P\kappa'\sin^2\theta' + P'\kappa\sin^2\theta} \tag{7.8}$$

where $P = (h/\sin\theta)$ and $P' = (r'\sin\phi'/\sin\theta')$ are the ray paths of the transmitted wave in M and M' respectively. Thus the saddle point for the transmitted wave (unlike that for the reflected wave) does not in general lie on the real axis, but above it for $\delta > \delta'$ and below it for $\delta < \delta'$. The position of the saddle point also depends on the length of the ray path and from (7.7), $|\beta|$ has the maximum value

$$|\beta_{\max}| = \frac{|\delta-\delta'|}{\tan\theta}. \tag{7.9}$$

But for $\theta > \cos^{-1}(\kappa'/\kappa)$,

$$|\beta_{\max}| < \frac{|\delta-\delta'|}{\sqrt{(\kappa^2/\kappa'^2-1)}} = |\beta_0|,$$

so the saddle point will always lie closer to the real axis than does the branch point. The path of steepest descent through the saddle point is the curve given by

$$\mathcal{R}e\,\Phi(\gamma) = \mathcal{R}e\,\Phi(\gamma_s). \tag{7.10}$$

This starts at $-\psi_+ +\delta+i\infty$ and ends at $\psi_- -\delta-i\infty$ where

$$\psi_\pm = \tan^{-1}\left(\frac{r'\cos\phi'}{r'\sin\phi'\pm h}\right). \tag{7.11}$$

In the vicinity of the saddle point, let us set $\gamma = \gamma_s+\zeta$ where $|\zeta/\gamma_s| \ll 1$. Then, to the second order in ζ,

$$\Phi(\gamma) \approx \Phi(\gamma_s)+\tfrac{1}{2}\zeta^2\,\Phi''(\gamma_s) \tag{7.12}$$

since $\Phi'(\gamma_s) \equiv 0$. Thus the path of steepest descent in the neighbourhood of the saddle point, approximates to the curve

$$\mathcal{R}e\,\zeta^2\,\Phi''(\gamma_s) = 0. \tag{7.13}$$

Now to the first order in δ, δ' we have

$$\Phi''(\gamma_s) \approx \Phi''(\theta)+i\beta\Phi'''(\theta) \tag{7.14}$$

where β is given by (7.7). (7.12) thus represents a pair of straight lines through the saddle point and inclined to the real axis at angles of $-\varepsilon'/2\pm\pi/4$, $(\varepsilon' = (\Phi'''/\Phi'')\,\beta)$ respectively. We chose the lower sign in this expression since the upper corresponds to the path of steepest ascent. Thus, assuming $A'(\gamma)$ is slowly varying compared with the phase $\Phi(\gamma)$, we obtain, for $\theta > \cos^{-1}(\kappa'/\kappa)$, the saddle point approximation for the transmitted wave as

$$V' \sim \left(\frac{2}{\pi|\Phi''(\gamma_s)|}\right)^{\frac{1}{2}} A'(\gamma_s)\exp\left(-(\kappa\delta P+\kappa'\delta' P')\right)\exp i(\kappa P+\kappa' P'-\varepsilon'/2-\pi/4). \tag{7.15}$$

The wave fronts are given by $\kappa P+\kappa' P' =$ const. and the lines of constant amplitude induced by the dissipation by $\kappa\delta P+\kappa'\delta' P' =$ const. When $\delta' = 0$, the latter reduces to the family of curves $P =$ const. and these are the straight line rays in M' which are orthogonal to the wave fronts. When $\delta = \delta'$ the amplitude is constant on the wave front. Fig. 5 displays, for various values of δ/δ', the lines of constant amplitude in relation to the wave fronts. It is evident that the amplitude on the wave front decreases with depth for $\delta > \delta'$ and increases for $\delta < \delta'$.

To calculate the field in M' for $\theta < \cos^{-1}(\kappa'/\kappa)$ we notice that the term $v'r'\sin\phi'$ in (7.1) is slowly varying compared to the other terms and can hence be taken outside the integration. The saddle point is then given by $\Phi'(\gamma_s) = 0$ where

$$\Phi(\gamma) = k\cos\gamma r'\cos\phi'+kh\sin\gamma. \tag{7.16}$$

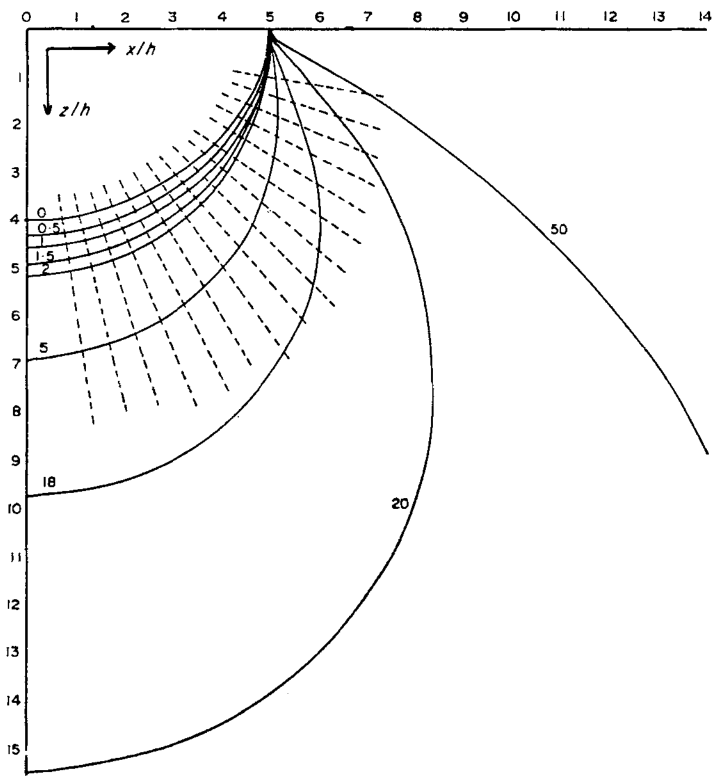

FIG. 5. The solid lines are the curves of constant amplitude for the transmitted wave, i.e. $\kappa\delta P + \kappa'\delta' P' =$ const. κ'/κ is taken as 0·7 and the numbers indicated represent the ratio δ/δ'.
The dashed lines are the ray paths, to which the solids curves approach as $\delta/\delta' \to \infty$.
The solid curve labelled 1 is also a wave front.

This gives $\gamma_s = \theta$ on the real axis where

$$\theta = \tan^{-1}\left(\frac{h}{r'\cos\phi'}\right), \; r'\cos\phi' > \frac{h}{\surd(\kappa^2/\kappa'^2 - 1)}. \tag{7.17}$$

At the saddle point, the phase takes the value

$$\Phi(\theta) = k\surd(h^2 + r'^2\cos^2\phi') = kP \tag{7.18}$$

where P is the ray path in M from the source to the interface subject to the restriction in (7.17). Proceeding as in the previous Sections, we find that for $\theta < \cos^{-1}(\kappa'/\kappa)$

$$V' \sim \left(\frac{2}{\pi\kappa P}\right)^{\frac{1}{2}} A'(\theta)\exp\left(iv'(\theta)\,r'\sin\phi'\right)\exp\left(-\kappa\delta P\right)\exp i(\kappa P - \delta/2 - \pi/4). \tag{7.19}$$

Now the term $\exp\left(iv'(\theta)\,r'\sin\phi'\right)$ is equal, to first order, to

$$\exp\left(-\surd(\kappa^2\cos^2\theta - \kappa'^2)\,r'\sin\phi'\right)\exp i\left(\frac{\kappa'^2\delta' - \kappa^2\delta\cos^2\theta}{\surd(\kappa^2\cos^2\theta - \kappa'^2)}\,r'\sin\phi'\right). \tag{7.20}$$

Thus the disturbance in M' for $\theta < \cos^{-1}(\kappa'/\kappa)$ decays rapidly with depth. It does not satisfy Fermat's Principle so that no true ray path can be associated with it. However, the phase factor in (7.20) indicates that there is a small energy fluctuation normal to the interface—this does not occur in the perfectly elastic case when $\delta = \delta' = 0$.

A similar discussion as in Section 6 shows that there is no pole contribution to V'. There is however, a branch line contribution when $\theta < \cos^{-1}(\kappa'/\kappa)$ which gives a ray path corresponding, once again, to the head wave.

8. The reflection and transmission coefficients

Both the reflection and transmission coefficients are affected by the presence of dissipation. For $\theta > \cos^{-1}(\kappa'/\kappa)$, we have $\kappa\cos\theta = \kappa'\cos\theta'$ and setting

$$\sigma = \frac{\rho'\,\kappa\,\sin\theta'}{\rho\kappa'\,\sin\theta},$$

we find to first order,

$$A(\theta) = \frac{1-\sigma}{1+\sigma} + 2i\,\frac{\sigma(\delta-\delta')(\cot^2\theta'-1)}{(1+\sigma)^2} \tag{8.1}$$

and

$$A'(\theta+i\beta) = \frac{2}{1+\sigma} + 2i\,\frac{\sigma(\delta-\delta')(\cot^2\theta-1)}{(1+\sigma)^2} \tag{8.2}$$

where β is assumed here to be given by (7.9).

For $\theta < \cos^{-1}(\kappa'/\kappa)$ we can define an angle ψ' so that $\kappa\cos\theta = \kappa'\cosh\psi'$ and setting

$$\tau = \frac{\rho'\,\kappa\,\sinh\psi'}{\rho\kappa'\,\sin\theta},$$

we find to first order,

$$A(\theta) = \frac{1-i\tau}{1+i\tau} + \frac{2\tau(\delta-\delta')(1+\coth^2\psi')}{(1+i\tau)^2} \tag{8.3}$$

and

$$A'(\vartheta) = \frac{2}{1+i\tau} - \frac{2\tau(\delta-\delta')(1+\coth^2\psi')}{(1+i\tau)^2}. \tag{8.4}$$

Thus for $\theta > \cos^{-1}(\kappa'/\kappa)$ the dissipation introduces a first order phase shift in both reflected and transmitted waves, the direction of which depends on whether $\delta > \delta'$ or $\delta < \delta'$. For $\theta < \cos^{-1}(\kappa'/\kappa)$ the dissipation introduces a first order change in amplitude. It is interesting to observes that for $\delta > \delta'$, $|A(\theta)| > 1$ and this appears to violate energy principles. However, for $\theta < \cos^{-1}(\kappa'/\kappa)$ the 'transmitted' wave contains the term (7.20) and for $\delta > \delta'$ this indicates a small energy flux across the interface from M' to M.

9. Discussion

We have presented the asymptotic expressions for the direct, reflected and diffracted waves in M and the transmitted and inhomogeneous waves in M'. It is important to mention that these expressions are not valid in the neighbourhood of the 'critical' angle $\theta = \cos^{-1}(\kappa'/\kappa)$. The concept of criticality for the case of small dissippation is somewhat obscure. For example, differences in its interpretation were obtained by Lockett (1962) and Cooper (1967), in their work on plane waves in linear viscoelastic solids. In fact, there is not a sharp transition from sub-critical reflection to super-critical reflection, but a rapid and continuous one—and this is also true for the case of plane waves. This interpretation is confirmed by M. Schoenberg (1970, private communication).

Distinct differences in the amplitude distributions for the head wave and transmitted wave occur if the Q for one medium is greater or less than the Q for the other. When Q is the same for both media, the waves display properties similar to the case of perfect elasticity. For example, the amplitude on the wave fronts is constant, as it is when there is no dissipation, and there is no change, to first order, in the reflection and transmission coefficients.

Suppose our interface represents the Earth's crust. Then it is evident from the discussion of the amplitude of the head wave, that at least theoretically, it is possible to determine a mean Q for the upper mantle by observations on the Earth's surface alone.

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Appendix I

In this Appendix we shall determine by simple arguments, the ray paths and the variation of amplitude along them, when a small amount of dissipation is present. In particular, the results are applicable to both the head wave and the transmitted wave.

In the absence of any dissipation, the geometric ray paths for both these waves are obtained from a phase function $\Phi(\kappa, \kappa'; S, R)$ which depends on the (real) wave numbers of the two media (M and M') and on the relative positions of the source and receiver. For a fixed source and receiver, the phase can be written in general form, as

$$\Phi(\kappa, \kappa') = \kappa P + \kappa' P' \qquad \text{(AI.1)}$$

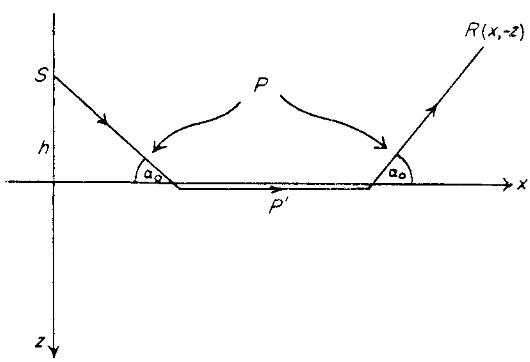
where P and P' are the ray paths in M and M' respectively.

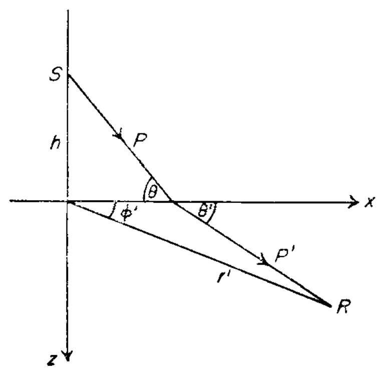
Head wave

$$\Phi = \kappa r \cos(\theta-\alpha_0)$$
$$= \kappa' x+\sqrt{(\kappa^2-\kappa'^2)}(h-z), z \leqslant 0$$
$$\alpha_0 = \cos^{-1}(\kappa'/\kappa)$$

Transmitted wave

$$\Phi = \kappa' r' \cos(\theta'-\phi')+\kappa h \sin\theta$$
$$\kappa \cos\theta = \kappa' \cos\theta'$$
$$\frac{r' \sin(\theta'-\phi')}{\sin\theta'} = h \cot\theta$$





$$P = \frac{h-z}{\sin\alpha_0} \qquad\qquad P = \frac{h}{\sin\theta}$$

$$P' = x-\frac{h-z}{\tan\alpha_0} \qquad\qquad P' = \frac{r' \sin\phi'}{\sin\theta'}$$

It is clear from the previous diagram that for fixed h, x, z, r' and ϕ' the ray paths depend on κ and κ' only through their ratio. That is,

$$\Phi(\kappa,\kappa') = \kappa P(\kappa/\kappa')+\kappa' P'(\kappa/\kappa'). \tag{AI.2}$$

Now, both the head wave and the transmitted wave satisfy Fermat's Principle. Mathematically, this means that, for fixed source and receiver, the phase remains stationary with respect to small variations in κ/κ'. Therefore,

$$\kappa\dot{P}+\kappa' \dot{P}' = 0. \tag{AI.3}$$

When a small amount of dissipation is introduced, we replace κ by $\kappa+i\kappa\delta$ and κ' by $\kappa'+i\kappa'\delta'$. Thus, by Taylor's Theorem, we have to the first order in δ and δ',

$$\Phi(\kappa+i\kappa\delta,\ \kappa'+i\kappa'\delta') \approx \Phi(\kappa,\kappa')+i\left(\kappa\delta\frac{\partial\Phi}{\partial\kappa}+\kappa'\delta'\frac{\partial\Phi}{\partial\kappa'}\right). \tag{AI.4}$$

But from (AI.2) and (AI.3),

$$\frac{\partial \Phi}{\partial \kappa} = P+(\kappa/\kappa')\,\dot{P}+\dot{P}' = P \tag{AI.5}$$

$$\frac{\partial \Phi}{\partial \kappa'} = -\frac{\kappa^2}{\kappa'^2}\,\dot{P}+P'-\frac{\kappa}{\kappa'}\,\dot{P}' = P'. \tag{AI.6}$$

Finally then

$$\mathcal{R}e\,\Phi \approx \kappa P+\kappa' P' \tag{AI.7}$$

and

$$\mathcal{I}m\,\Phi \approx \kappa\delta P+\kappa'\delta' P'. \tag{AI.8}$$

The interpretation of this result is obvious. In addition, the amplitude on the wave front $\mathcal{R}e\,\Phi =$ const., varies as $\exp\left(-\kappa(\delta-\delta')\,P\right)$ or $\exp\left(-\kappa'(\delta'-\delta)\,P'\right)$, which is constant only when $\delta = \delta'$.

Appendix II

Here, we provide a purely geometrical description for the variation of amplitude of the head wave. For this purpose it is more convenient to use the 'natural' wave parameters—the wavelength ($\lambda = 2\pi/\kappa$) and the decrement ($\Delta = 2\pi\delta$).

The wave fronts of the head wave can be obtained from a Huygen's construction. Consider two such wave fronts, separated by exactly one wavelength, as shown in Fig. 6.

An observer, who makes measurements on a line parallel to OA (i.e. normal to the wave fronts) would measure the true wavelength λ and the true decrement Δ. Similarly, a hypothetical observer who makes measurements along the line OA' parallel to the interface, would obtain the apparent wavelength λ' and apparent decrement Δ'. Now suppose our observer makes his measurements along some arbitrary line parallel to OA_0 inclined at an angle θ_0 to the interface, obtaining an apparent wavelength λ_0 and apparent decrement Δ_0. Then,

$$\lambda_0 = OA_0 = \lambda \sec(\theta_c-\theta_0) = \frac{\lambda\lambda'}{\lambda \cos\theta_0+\sqrt{(\lambda'^2-\lambda^2)}\sin\theta_0}. \tag{AII.1}$$

Now suppose the amplitudes at A, A_0 and A' are W, W_0 and W' respectively. Assuming an exponential variation of amplitude along the wave front, we can write

$$W_0 = W\exp(-\alpha p),\; W' = W_0\exp(-\alpha q) \tag{AII.2}$$

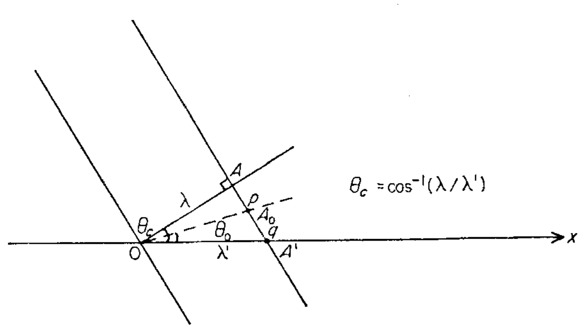


FIG. 6. Geometrical interpretation of the amplitude variation of the head wave. The amplitude at an arbitrary point A_0 can be expressed in terms of the amplitudes at A and A' which lie on the wave front through A_0.

where $p = AA_0$, $q = A_0 A'$ and α may be positive, negative or zero. Eliminating α we obtain

$$W_0^{p+q} = W^q W'^p. \tag{AII.3}$$

Now suppose U_0 is the amplitude at O. Then dividing (AII.3) by U_0^{p+q} and taking logarithms, we obtain

$$(p+q)\,\Delta_0 = q\Delta + p\Delta'. \tag{AII.4}$$

Now from the geometry of Fig. 6. we easily find that

$$\left.\begin{aligned} p = \lambda \tan(\theta_c - \theta_0),\ q &= \frac{\lambda' \sin\theta_0}{\cos(\theta_c-\theta_0)} \\ p+q = \lambda' \sin\theta_c &= \lambda \tan\theta_c. \end{aligned}\right\} \tag{AII.5}$$

Thus, from (AII.4) we get an expression for the apparent decrement on the line OA_0 as

$$\Delta_0 = \frac{\Delta \sin\theta_0 + \Delta' \sin(\theta_c - \theta_0)\cos\theta_c}{\sin\theta_c \cos(\theta_c - \theta_0)}, \tag{AII.6}$$

Note that for $\theta_0 = \theta_c$, $\Delta_0 = \Delta$ and for $\theta_0 = 0$, $\Delta_0 = \Delta'$. In particular, when the observer makes measurements on a line on which the amplitude is constant, his observed decrement is zero. It must be remembered that by 'amplitude' we mean that induced by the dissipation only. Other amplitude variations such as geometric spreading are assumed to have been already taken into account. Thus from (AII.6) when $\Delta_0 = 0$ we find

$$\tan\theta_0 = \frac{\sin\theta_c \cos\theta_c}{\cos^2\theta_c - \Delta/\Delta'} = \frac{\sqrt{(\lambda'^2/\lambda^2 - 1)}}{1 - \lambda'^2\,\Delta/\lambda^2\,\Delta'} \tag{AII.7}$$

and this expression agrees precisely with (5.17).