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REFLECTIONS AND PROSPECTIVES

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ABSTRACT. Intellectual challenges and opportunities for mathematics are greater than ever. The role of mathematics in society continues to grow; with this growth comes new opportunities and some growing pains; each will be analyzed here.

1. INTRODUCTION

"It was the best of times, it was the worst of times, . . ." (Dickens)

Progress for the past decade or so has been extraordinary. The solution of Fermat's Last Theorem [11] and of the Poincaré Conjecture [1] have resolved two of the most outstanding challenges to mathematics. For both cases, deep and advanced theories and whole subfields of mathematics came into play and were developed further as part of the solutions. And still the future is wide open. Six of the original seven problems from the Clay Foundation challenge remain open (`http://www.claymath.org/millenium`). The 23 DARPA challenge problems (`http://.arsmathematica.net/archives/2007/12/26/605/`) are open. Entire new branches of mathematics have been developed, including financial mathematics and the connection between geometry and string theory, proposed to solve the problems of quantized gravity. New solutions of the Einstein equations, inspired by shock wave theory, suggest a cosmology model which fits accelerating expansion of the universe possibly eliminating assumptions of "dark matter" [9]. Interdisciplinary mathematics is in vogue, and with it an array of novel problems, as well as a deeper examination of many traditional ones. Among the newer entries to this

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list, we mention mathematics and medicine, biology, geology, sociology, economics, linguistics, genetics, and the list goes on.

As a professional discipline, mathematics is also healthy. The number of Ph.D. degrees awarded annually has been stable for a decade. Attendance at the Joint Meetings continues to rise. Journals are as crowded with papers as ever, and continue to expand. The numbers of graduates choosing industrial, laboratory, and other nonacademic employment continues to increase, as it should for a subject with increasing ties to the larger world. Private donations to mathematics are at unprecedented levels, with five major centers, institutes, or major funding for existing institutes established in recent years, and perhaps more that I am not aware of. However, in view of current financial problems, problem spots are appearing in this picture. They first emerged in employment and will no doubt expand from there.

Running through all of this is a sea change in the relation of mathematics to society. Not to interpret too narrowly, this is the mathematical version of a sea change in the relation of science and technology to society.

In simple terms, mathematics works. It is effective. It is essential. It is practical. Its force cannot be avoided, and the future belongs to societies that embrace its power. Its force is derived from its essential role within science, and from the role of science in technology. Wigner's observations concerning "The Unreasonable Effectiveness of Mathematics" [10] are truer today than when they were first written in 1960.

With the benefits of increased recognition and importance come increased obligations and responsibilities, and inexorably, an increase in external influences over the profession. Our interests as mathematicians (as well as those of our society) are best served by being engaged in this process.

The positive aspects of increased respect for mathematics within society include new problems, arising from new instances of the use of mathematics, new resources to address these problems, new colleagues in other disciplines to share this work with, and an enhanced sense of relevance and of coupling to important forces of history. We see many signals that a vigorous coupling to these forces is a magnet to attract young talent into our profession. For example, some of the larger mathematics programs, in terms of the undergraduate degrees as a fraction of the total undergraduate population, are found at UCLA and Stony Brook, where a wide range of interdisciplinary and career-relevant tracks are offered. Negative aspects of these changes include pressures to organize our activities along externally determined directions. Here we need to value the diversity of our activities, and adopt a "big tent" philosophy, whereby within a large range of valuable professional activities, different mathematicians emphasize different directions and even different goals, all of value. With this point of view, the increased expectations which we face will be shared among those with a desire to meet them, and are thus far less onerous.

2. The big tent for mathematics

"such a cooperation and harmony would be the very end and success" (Thoreau)

It would be difficult to recall the number of times someone has said to me something to the effect that, "It is unfortunate that in some decades past, we failed to keep the doors open for some particular subject, which has now established itself as an independent entity." History shows that such divisions are not necessary. For example, the American Physics Society represents all aspects of physics, and physics has not been similarly fractured. It is of course the case that we will not reverse the course of history, nor will we cause water to flow upstream. But a simple examination of the past may help us with decisions that lie in the future; the future is certainly full of wide open issues of this type. I believe the mistakes were made in following a narrow and misplaced sense of equity, which, in not being future oriented, failed to perceive the strength and potential of evolving and emerging subfields of mathematics.

Today we face similar choices, in responding to areas such as scientific computing, biomathematics, quantitative finance, and the mathematics of data-driven science. As the expectations of professional responsibility continue to bear on our profession, we find an increasing flow of some of our members to professional activities and concerns which go beyond a traditional research focus, such as innovation and excellence in teaching, employment in research laboratories, research in interdisciplinary science, and government service. A positive response to all of these trends comprises what I call the "big tent" philosophy: value all of the many and varied forms of professional activities, distinctions and accomplishments.

How does the big tent play out in specific terms? The appointments of N. Weiner and J. Nash at MIT and of J. Tukey at Princeton speak to a success for this point of view from a previous generation. The existence today of a number of universities with strength in both pure and applied mathematics and faculty who are comfortable in moving from one to the other provides a current benchmark of success. Partly due to the success of these efforts, and partly due to the success of analogous interdisciplinary work in other areas of science and technology, the standards for this type of integration within and across disciplines continue to increase. Accordingly, it is the opportunities and responsibilities of today and tomorrow that we address here.

We have already accomplished the principle of cordial relations among the multiple mathematics organizations and among the points of view they represent. A next step, and one that I hope is taken, is to engage in joint activities, where there is added value in doing so. The Joint Meetings, which have a long and positive history, are a clear example. Recently, special sessions sponsored by SIAM have been added to these meetings. As the AMS no longer holds a summer meeting, SIAM, with its summer meeting, could consider its own version of this cooperation. The regional meetings of the AMS are unique, in the sense that secondary SIAM meetings are national, and restricted to specialized subject matter. Regional meetings are an opportunity for graduate students to attend at a reasonable cost. These meetings could have a broader intellectual focus, including some aspects of applied mathematics as well as traditional research areas. I believe there has been a recent trend in this direction, which I hope will continue.

In terms of teaching, our natural liaison is with MAA. Many mathematicians belong to both organizations, and can simply wear one hat or the other depending on the activity being pursued. A clear instance of the added value of explicit cooperation between the societies is the issue of education within research universities, as this issue requires a greater participation by research faculty than has been traditional. The Task Force on the First Year Undergraduate Experience in Research Universities (see §3) is an example of such a working group. Another example is the teaching of K–12 teachers, insofar as much of this takes place in research universities. There is a sizable effort by AMS members in this direction.

3. The importance of teaching well

"And gladly would he learn, and gladly teach" (Chaucer)

The vast preponderance of the funding for mathematics comes from the universities and colleges. The support from NSF and other federal agencies is far smaller. Studies using focus groups have concluded that the mathematics faculty believes that its courses are well taught, while the students, parents, deans and faculty colleagues in "client departments" do not. While we can hope that these reports are out of date, surely there is much progress still to be made. If public spiritedness and a sense of shared responsibility is not sufficient to focus attention on the topic of this section, I hope that simple self-interest will do the trick: I have yet to meet a good teacher who does not love his/her teaching. And I have yet to meet a poor one who derives satisfaction from it. My estimate, from my two decades experience as a department chair, is that the added time needed for good teaching is not over 5%, and the added satisfaction is immense.

This being the case, as President of the AMS, I convened a task force to focus on the teaching of first year college/university students. I charged the task force to look for actions that were (a) likely to be effective in making a substantial improvement in the quality of the teaching and (b) were not so intrusive and time consuming as to be damaging to the other professional activities of the faculty. The task force focused on three topical areas:

- Use of technology, specifically machine-graded homework systems.
- Departmental leadership, from the chair and the senior professors in support of the goal of good teaching.
- Training of TAs in methods of good teaching.

The report, written by the task force chair and deputy chair, Jim Lewis and Alan Tucker, will be published in the Notices of the AMS [4] and develops these themes. The NSF has funded, and the AMS will conduct, a follow-on study of the technology issue specifically. Its focus will be what might be called a user manual or frequently asked questions (with answers) regarding the use of computer technology for grading of homework. It will address such questions as required computer and personnel support to install and maintain such a system, reasons why it was adopted or not at responding institutions, and whether or why it was continued or not. Attitudes of the faculty before and after adoption will also be sought.

4. The research agenda, looking forward

"you shall seek all day ere you find them" (Shakespeare)

Within the broad divide between pure and applied mathematics, there is no need to decide which has influenced the other, as there are ample examples of influence passing in both directions. There are ample examples of the added value which accrues as subjects are passed back and forth, often acquiring powerful new theoretical foundations and completely novel applications, unrelated to the original, after making a round trip. Fourier analysis, group theory, probability and partial differential equations provide only a few of the many examples of this phenomena.

The traditional rationale within applied mathematics has been to solve or provide insight for equations which describe some part of the physical world. Much of pure mathematics is an abstraction of this theme, and much, even where the connection is lost in the mists of time, had its origins there. String theory and its connection to geometry has greatly enriched mathematics. We hope that through its connections to the quantization of gravity, it does as much for physics. Both turbulence and the quantum mechanics of many bodies are challenges to physics that will continue to engage mathematicians as well. Even where this point of view shows signs of maturity, I believe it has decades of life left in it.

But there is a paradigm shift afoot. Mathematics is used to describe data, without the benefit of an interpolating equation or physical principle. Consider the problem of protein folding. One can integrate the classical equations of molecular dynamics, based on a potential energy between atoms derived from more fundamental equations of quantum mechanics. This is an active line of investigation. But the same problem can be attacked from a knowledge- (as opposed to theory-) based set of principles. Many biomolecules have a measured geometry (folding pattern), from X-ray crystallography or from NMR. So there is a statistical library of experimental "solutions" to these equations. These experimentally based "solutions" can be used to populate probability models, which can generate sample folding patterns of previously unfolded biomolecules. This approach is called a knowledge-based method. In annual competitions organized by biomodelers, with molecules having unknown folding patterns as a test problem, the knowledge-based methods usually win except for proteins very dissimilar to all others in the library of known solutions. Similarly, in the decoding of the human genome, competing teams placed different emphasis on mathematics and computation vs. laboratory experiment. In the end, the official version was that the contest was a draw, but all observers know that the mathematical and computational ideas were extremely powerful. The mathematics used here was not based on physical laws. It was based on pattern recognition, and similarity of overlapping fragments, with statistical tests for the assembly of the fragments into a reliable whole.

This shift is not limited to the biological sciences. It is said that there are as many data points measured for the atmosphere as there are computational grid cells. For voice recognition, there is no such thing as a law of physics to convey or interpret from the sound waves to the meaning. Still, mathematical ideas and computer programs have a degree of success with this important problem. Hidden Markov models provide the mathematical framework for this area. There is no law of physics to guide the recognition of finger prints, nor to interpret an image and decide whether there is a face in it. Computer recognition of handwriting and even of digits remains a challenging problem. Even for digits, and even for postal zip codes, where the accuracy requirements are not as high as they would be in bank deposit slips, human beings are still in the lead. This advantage seems to be only a matter of time. The leading chess champion is a computer. The mathematical models of finance, although based on familiar mathematics (stochastic integrals and differential equations), arrive at their formulations without use of physical principles.

These new areas for mathematics encompass new areas of knowledge. The social sciences are beginning to participate in this mathematization of thought and understanding. Here we include certainly linguistics and economics, where the role of mathematics is well established. The dynamics of voting preferences has been described through a Markov process. In the aggregate behavior of large groups of individuals, collective patterns emerge. Agent-based models have been used to study automobile traffic patterns and to plan for disruptions caused by repairs. Networks of associations and correlations between purchase patterns have an obvious commercial interest to merchandisers. Out of such utilitarian concerns will emerge general principles, including mathematical ones. A typical and generic problem is to describe a manifold and its inherent and possibly low-dimensional geometry, when it is presented through noisy data embedded in a high-dimensional space.

If we have had four centuries of physically based and motivated mathematics, it does not seem a stretch of the imagination to assume that we will have one or more centuries of mathematics based on the organization of data and the intelligence to be derived from it, perhaps to be named the mathematics of knowledge and intelligence.

Mathematics and pure mathematicians have a long tradition of exploring the issues of data, intelligence, noise and meaning. The classical works of Kolmogorov and of Shannon illustrate this point. The future is bright for an expansion of this type of inquiry.

5. What makes us unique

We point out metrics in which mathematics excels, relative to other sciences.

"Love truth, but pardon error" (Voltaire)

5.1. **The nature of truth, and how it is established.** There is an absolute nature to truth in mathematics, which is unmatched in any other branch of knowledge. A theorem, once proven, requires independent checking but not repetition or independent derivation to be accepted as correct. In contrast, experiments in physics must be repeated to be regarded as confirmed. In medicine, the number of independent confirmations required is much higher. Truth in mathematics is totally dependent on pure thought, with no component of data to be added. This is unique. Associated with truth in mathematics is an absolute certainty in its validity.

Why does this matter, and why does it go beyond a cultural oddity of our profession? The answer is that mathematics is deeply embedded in the reasoning used within many branches of knowledge. That reasoning often involves conjectures, assumptions, intuition. But whatever aspect has been reduced to mathematics has an absolute validity. As other subjects search for truth, the mathematical components embedded in their search are like the boulders in the stream, providing a solid footing on which to cross from one side to the other.

As science acquires an increasingly computational basis, the search for truth becomes systematized. Verification of a computation asserts that the computation solves the problem as it has been formulated mathematically, as an equation to be solved. Validation means that this mathematical formulation is an accurate representation of the intended domain of application, that is, a correct description of the problem as it occurs in the real world. Uncertainty quantification refers to establishing limits or probability distributions for possible errors in the computation, both those arising from the numerical approximations (verification) and from the mathematical approximations to the true problem (validation). Intuitively, it refers to error bars associated with a computation, including all possible types of errors. Continuing with this new set of responsibilities, we are asked to estimate quantified margins of uncertainty. These are the simulation equivalent of an engineering safety margin, and the final test that a design is safe (enough) to build and use. These margins serve to estimate and compare uncertainty in design and the design point to the distance (and its own uncertainty) to some unsafe or unknown region of design space.

Herein lies a new (or newly growing and popular) branch of mathematics: the study of the various types of errors in various types of approximations and their propagation, from input to output within a computation.

There is a darker side to the search for truth, one of systematic error and fraud.

We start with a softer version of this issue which is worthy of our professional consideration: systematic miscommunication of truth across management levels. Examples of truth perceived by the scientists in the trenches and ignored by managers have occasionally blown up into public scandals. The famous case of the O-rings involved in a shuttle disaster is an example.

Less well publicized is the opposite possibility, in which excuses and misinformation are passed up the line. To give a sense of the possibilities of this phenomena within a mathematical context, consider the mathematics of risk management. Probabilities for rare events can be inferred from the probabilities of common ones, but only if there is a valid law relating the two. The normal distribution, and its claimed justification in terms of the law of large numbers, is an example. Data will exist only for common events, and extrapolation of this data to predict rare events is a theoretical enterprise, not itself supported by the data. Data concerning correlation between events is normally missing, but an assumption of independence could falsely predict as rare an actually common event.

To make this point explicitly, I have the possibly unjustified impression that "100-year storms" occur approximately every 25 years. Even if there were 100 years of storm data, the weather patterns may have changed, and certainly the capability of the ground to absorb and hold back a flood surge has changed, so it would seem that the categorization of a 100- or 500-year storm has no basis in data and certainly has no basis in the laws of probability applied to the differential equations which govern the weather.

This is a personal impression concerning a broad aspect of the application or misapplication of mathematical ideas to areas of societal concern. Estimates of risk and uncertainty are an essential output from the modeling and numerical simulation of today, and will be increasingly so in the future. The mathematical risk-assessement models used by the now defunct hedge fund Long Term Capital Management misestimated risk for essentially the reasons given above, leading to a well-publicized failure. Structural changes in the application of risk models cloud the relevance of historical data. Outliers seldom are accompanied by sufficient data for validation of risk models. Possible correlations between different data streams are even more difficult to assess, and historical data is potentially scarce, missing, or lacking relevance.

In this context, we ask, "What are the professional responsibilities of a mathematical scientist to communicate the basis of estimates of risk to management, and what are the chances that these responsibilities can feasibly be discharged?" To make these questions less abstract and more easily understood, I will translate them to the context of engineering and the design and safety of bridges. The famous failure of the Tacoma bridge was due to wind induced vibrations in the superstructure. This was a type of failure never previously encountered in the history of bridge building, and never considered in the design calculations. Post-failure, it is (as far as I understand) routinely considered in the design of new bridges. Likewise, the failure of the I-95 bridge in Connecticut was traced to a lax program of inspections. So we can find problems in the technology and problems in the regulatory regimen. To the extent that these problems exist in the mathematics of risk management and have not been adequately communicated to their users, their elimination or amelioration should be a professional responsibilty of the mathematics community. Certainly the communication of the nature and limitations of risk models, and of their combined validation in mathematics and in data, are professional responsbilities, as is a program to extend and improve the scientific and mathematical basis for this technology.

A related issue with overtones of miscommunication between disciplines rather than between management levels illustrates a theme often expressed among mathematicians. Verification of numerical solutions is a step in their scientific acceptability, and a formal requirement of many engineering oriented inquiries. Compressible fluid flow, solved numerically, often is solved at the level of the Euler equations, that is without use of the regularizations such as viscosity and mass diffusion which are part of physics, but which may be small enough that there is a temptation to set them to zero. However, these equations are known to have nonunique solutions [3, 7, 8], so any assertion that a particular numerical approximation is correct runs into a big problem: If the true answer is not unique, what does correctness of the approximation mean? How would you verify the mathematical correctness of a numerical method to calculate the value of 0/0? This question and its import have yet to reach the world of practical simulations, and so we only offer our own proposal for its resolution. From the point of view of physics, the nonuniqueness originates in the ratio of the regularizing terms, as the coefficients all go to zero. Thus, the Schmidt number (ratio of viscosity to mass diffusion) and the Prandtl number (ratio of viscosity to heat conducivity) are dimensionless quantities that must be set within the numerical solution process. One hopes that once this is done, verification of the numerical process can resume [5, 6].

We should also mention more direct ethical challenges to our community. Misuse of privileged information (as in the use of unpublished papers or unfunded research proposals, available through a review or decision process) does occur. Multiple simultaneous submissions of the identical paper to different journals can occur. Journals may minipulate citations to enhance ratings, and so may authors. Use of numerical ratings as a substitute for scientific judgment and exercise of mathematical taste is an administrative shortcut which encourages such behavior, and where not supported by the independent use of judgment and taste, should be seen as dangerous and potentially unethical in its own right. Plagarizing of passages in papers or even of entire manuscripts does occur. Professional societies have an obligation to encourage adherance to ethical behavior, to state clearly the boundaries of unethical behavior, and to the extent practical, to enforce these boundaries.

"The universe is written in the language of mathematics" (Galileo)

5.2. **Universality and re-usability.** Knowledge tends to pass from descriptive to qualitative to quantitative. At the end of this transition, numerical and mathematical aspects emerge. Perhaps it is this progression which accounts for the universality of mathematics as a language for systematic knowledge in science, technology, and even beyond these boundaries.

Perhaps it is the exceptional simplicity and beauty of mathematical thought and its extraordinary ability to organize knowledge that is the driving force for the universality and re-usability of mathematics.

Perhaps it is the amazing increase in the power of computation that is the driving force. It can be said that following Newton, we expect laws of physics to be expressed in mathematical terms. But following the development of the computer, we expect the solution also to come from mathematics, from computation. Von Neumann observed a half century ago the virtual stagnation of science broadly at the border between the linear and the nonlinear, and advocated the use of computers to attack nonlinear problems in science. This plan has succeeded admirably, to the extent that nonlinearity is an inconvenience but not a fundamental obstacle. One decade ago, we observed a similar stagnation across broad stretches of science, and at a point of difficulty that the computer does not (yet) address, multiscale science [2]. While the science community has picked up the challenge of this point of view, it is safe to say that the nub of the problem largely remains.

Perhaps it is the diversity of applications of computation, i.e., their importance in the solution of so many different problems, and the need to reduce knowledge to mathematical terms accessible to computation.

Perhaps it is the social forces which, observing striking successes of mathematical ideas in unexpected areas (such as the decoding of the genome or the assessment of risk in financial markets), have gone on to seek similar benefits more broadly.

For whatever reasons, there is no doubt that science, and more generally the pursuit of knowledge, is increasingly interdisciplinary. And mathematics is one of the few disciplines afforded interdisciplinary opportunities with virtually all of the others. To some extent this trait is shared with physics (at least within the "physical sciences", defined to be those for which this universality relative to physics is true) and computer science. But the universality of mathematics embraces also the social sciences, where the laws of physics are not in play.

In an interdisciplinary and knowledge-driven society, mathematics plays a privileged and nearly unique role. It is the mortar that holds the bricks together. It is the grand design and the architect.

Universality of mathematics is a fact, and one we should be proud of.

6. Conclusions

We see a broadening of the intellectual and professional opportunities and responsibilities for mathematicians. These trends are also occuring across all of science. The response can be at the level of the professional societies, which can work to deepen their interactions, not only within the mathematical sciences, but also with other scientific societies. At a deeper level, the choices to be made will come from individual mathematicians. Here, of course, the individual choices will be varied, and we argue for respect and support for this diversity of responses. In such a manner, we hope to preserve the best of the present while welcoming the best of the new.

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