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REFLEXIVE AND ANTISYMMETRIC RELATIONS AND THEIR SYSTEMS

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This paper investigates systems of Y-relations on partial algebras being reflexive and antisymmetric among others. Systems of the so called X-relations, special cases of which are systems of equivalences, tolerances or quasi-orders, have been studied in [5]. In contrast to X-relations, Y-relations do not form lattices but they form complete $\wedge$-semilattices. Therefore, the first part of this paper deals with generalizing compact elements, algebraic lattices, closure operators, etc. for the case of semilattices. In the second part Y-relations are defined, special cases of which are orders or semi-orders. It is proved that the set of all Y-relations on an arbitrary set is an algebraic $\wedge$-semilattice. Similarly, the set of all Y-relations compatible with any partial algebra is also an algebraic $\wedge$-semilattice. The concluding part studies categories of systems of Y-relations, i.e. such categories the objects of which are ordered pairs $(A, \mathscr{Y})$, where A is a set and $\mathscr{Y}$ is a system of Y-relations on A containing id_A and closed under intersections of non-empty subsystems and under unions of bounded directed subsystems; the morphisms of such categories are the mappings of the underlying sets of objects with the property that the coimages of the relations from the appropriate systems of Y-relations are the relations from the corresponding systems of Y-relations. (The first category of the analogous type studied in literature was the category of equivalence systems that was defined by M. Armbrust in [1].) In this paper the basic properties of these categories are shown. (The description of mono- and epimorphisms, injective and projective objects, separators and coseparators and the study of completeness and cocompleteness of categories.) In the paper we use the notions from the universal algebra, see [3], [4] and [6], and those from the category theory, see [2] and [7].

1. ALGEBRAIC SEMILATTICES

Let A, B be sets, φ a partial mapping from A into B. Then Dom φ means the domain of definition of φ.

Definition 1.1. Let $S = (S, \leqq)$ be an ordered set with the smallest element 0 and let λ be a partial mapping from S into S such that

1. $0 \in \operatorname{Dom} \lambda$ and $0\lambda = 0$;
2. $\forall a \in S;\ a \in \operatorname{Dom} \lambda \Rightarrow a \leqq a\lambda$;
3. $\forall a, b \in S;\ a \leqq b,\ b \in \operatorname{Dom} \lambda \Rightarrow a \in \operatorname{Dom} \lambda,\ a\lambda \leqq b\lambda$;
4. $\forall a \in S;\ a \in \operatorname{Dom} \lambda \Rightarrow a\lambda \in \operatorname{Dom} \lambda,\ (a\lambda)\,\lambda = a\lambda$.

Then λ is called a *partial closure operator* in S.

Lemma 1.1. *Let $S = (S, \leqq)$ be a complete $\wedge$-semilattice, λ a partial closure operator in S. Then $S\lambda$ is a closed $\wedge$-subsemilattice of S.*

Proof. $0 \in S\lambda$, hence $S\lambda \neq \emptyset$. Let $a_\alpha \in S\lambda$ $(\alpha \in A)$. If $a = \bigwedge_{\alpha \in A} a_\alpha$, then $a \in \operatorname{Dom} \lambda$ and $a\lambda \leqq a_\alpha\lambda = a_\alpha$ $(\alpha \in A)$, thus $a\lambda \leqq a$. Therefore $a\lambda = a$.

Let us suppose that $S = (S, \leqq)$ is a complete $\wedge$-semilattice. If $S' \subseteq S$, then $\bigvee S'$ means the supremum of S' in S (provided it exists). It is clear that $\bigvee S'$ exists for each non-void upper bounded subset S' of S.

Definition 1.2. Let S be a complete $\wedge$-semilattice. Then an element $x \in S$ is said to be *compact* if the following implication holds:

If $\{y_\alpha;\ \alpha \in A\} \subseteq S$ is an arbitrary system such that $\bigvee_{\alpha \in A} y_\alpha$ exists and $x \leqq \bigvee_{\alpha \in A} y_\alpha$, then there exists a finite subset $B \subseteq A$ for which $x \leqq \bigvee_{\beta \in B} y_\beta$.

Definition 1.3. A complete $\wedge$-semilattice S is called an *algebraic $\wedge$-semilattice* if each element of S is equal to the supremum of a set of compact elements of S.

If S is a complete $\wedge$-semilattice, then we denote the set of its compact elements by S^*.

Definition 1.4. Let S be an algebraic $\wedge$-semilattice. Then a partial closure operator λ in S is called *algebraic* if

$$\forall a \in S^*\ \forall x \in S\,;\ \ (x \in \operatorname{Dom} \lambda\,,\ \ a \leqq x\lambda \Rightarrow \exists x' \in S^*\,;\ \ x' \leqq x\,,\ \ a \leqq x'\lambda)\,.$$

Let us suppose that S is a complete $\wedge$-semilattice, λ a partial closure operator in S. Then we shall denote the supremum in $S\lambda$ (ordered by the induced ordering) by Υ.

Lemma 1.2. *Let S be an algebraic $\wedge$-semilattice, λ a partial algebraic closure operator in S. Then $S\lambda$ is an algebraic $\wedge$-semilattice.*

Proof. By Lemma 1.1, $S\lambda$ is a closed $\wedge$-subsemilattice of S. Let $\{x_\alpha\lambda;\ \alpha \in A\} \subseteq S\lambda$ and let $\Upsilon_{\alpha \in A} x_\alpha\lambda$ exist. Then $\bigvee_{\alpha \in A} x_\alpha\lambda$ exists, too, and we have $\Upsilon_{\alpha \in A} x_\alpha\lambda \geqq \bigvee_{\alpha \in A} x_\alpha\lambda$. But $\Upsilon_{\alpha \in A} x_\alpha\lambda \in S\lambda$, hence $\bigvee_{\alpha \in A} x_\alpha\lambda \in \operatorname{Dom} \lambda$ and $\Upsilon_{\alpha \in A} x_\alpha\lambda \geqq (\bigvee_{\alpha \in A} x_\alpha\lambda)\,\lambda$. Moreover, $x_\alpha \in$

$\in \operatorname{Dom} \lambda$ and $(\bigvee_{\alpha \in A} x_\alpha \lambda) \lambda \geqq x_\alpha \lambda$ for each $\alpha \in A$, therefore $(\bigvee_{\alpha \in A} x_\alpha \lambda) \lambda \geqq \curlyvee_{\alpha \in A} x_\alpha \lambda$. This means

$$(*) \qquad \curlyvee_{\alpha \in A} x_\alpha \lambda = (\bigvee_{\alpha \in A} x_\alpha \lambda) \lambda .$$

Let $c \in S^* \cap \operatorname{Dom} \lambda$, $\{x_\alpha; \alpha \in A\} \subseteq \operatorname{Dom} \lambda$. Suppose that $\curlyvee_{\alpha \in A} x_\alpha \lambda$ exists and $c\lambda \leqq \curlyvee_{\alpha \in A} x_\alpha \lambda$. Then by $(*)$ we have $c \leqq (\bigvee_{\alpha \in A} x_\alpha \lambda) \lambda$. The algebraicity of λ yields the existence of $x' \in S^*$ such that $x' \leqq \bigvee_{\alpha \in A} x_\alpha \lambda$ and $c \leqq x'\lambda$. But then there exists a finite subsystem $\{x_\beta \lambda; \beta \in B\} \subseteq \{x_\alpha \lambda; \alpha \in A\}$ such that $x' \leqq \bigvee_{\beta \in B} x_\beta \lambda$. Now $c \leqq x'\lambda$ implies $c\lambda \leqq x'\lambda \leqq (\bigvee_{\beta \in B} x_\beta \lambda) \lambda = \curlyvee_{\beta \in B} x_\beta \lambda$. Hence $c\lambda \in (S\lambda)^*$.

Now, let $y \in \operatorname{Dom} \lambda$. Then there exists $\{x_\alpha; \alpha \in A\} \subseteq S^*$ such that $y = \bigvee_{\alpha \in A} x_\alpha$. (It is evident that $x_\alpha \in \operatorname{Dom} \lambda$ for each $\alpha \in A$.) We have $y\lambda = (\bigvee_{\alpha \in A} x_\alpha) \lambda = (\bigvee_{\alpha \in A} x_\alpha \lambda) \lambda = \curlyvee_{\alpha \in A} x_\alpha \lambda$, hence $S\lambda$ is an algebraic $\wedge$-semilattice.

Definition 1.5. An algebraic $\wedge$-semilattice S is called *finitely compact* if for each finite subset $\{x_\beta; \beta \in B\} \subseteq S^*$ the supremum $\bigvee_{\beta \in B} x_\beta$ (if it exists) is also in S^*.

Lemma 1.3. *Let S be a finitely compact algebraic $\wedge$-semilattice, λ a partial algebraic closure operator in S, $x \in \operatorname{Dom} \lambda$. Then $x\lambda \in (S\lambda)^*$ if and only if there exists $y \in S^* \cap \operatorname{Dom} \lambda$ such that $x\lambda = y\lambda$.*

Proof. Let $x\lambda \in (S\lambda)^*$. Since S is an algebraic $\wedge$-semilattice, there exists $\{x_\alpha; \alpha \in A\} \subseteq S^*$ such that $x = \bigvee_{\alpha \in A} x_\alpha$. But then $x_\alpha \in \operatorname{Dom} \lambda$ for each $\alpha \in A$. Here $\curlyvee_{\alpha \in A} x_\alpha \lambda$ exists and satisfies $\curlyvee_{\alpha \in A} x_\alpha \lambda = x\lambda$. Indeed, let $z \in S\lambda$ be such that $z \geqq x_\alpha \lambda$ for each $\alpha \in A$. Then $z \geqq \bigvee_{\alpha \in A} x_\alpha = x$ and so $z \geqq x\lambda$. By the assumption we have $x\lambda \in (S\lambda)^*$, hence there exists a finite subset B of A such that $x\lambda = \curlyvee_{\beta \in B} x_\beta \lambda$. But then $x\lambda = (\bigvee_{\beta \in B} x_\beta \lambda) \lambda = (\bigvee_{\beta \in B} x_\beta) \lambda$. Now, the finite compactness implies $\bigvee_{\beta \in B} x_\beta \in S^*$.

Corollary 1.4. *Let S be a finitely compact algebraic $\wedge$-semilattice, λ a partial algebraic closure operator in S. Then the correspondence $a \mapsto a\lambda$ is a mapping of $S^* \cap \operatorname{Dom} \lambda$ onto $(S\lambda)^*$ which satisfies the implication: If $x\lambda, y\lambda \in (S\lambda)^*$ and if $x\lambda \curlyvee y\lambda$ exists, then $x\lambda \curlyvee y\lambda = (x \vee y) \lambda$.*

Proof follows immediately from $(*)$ and Lemma 1.3.

Lemma 1.5. *Let S be a finitely compact algebraic $\wedge$-semilattice, λ a partial* *closure operator in S. Then λ is algebraic if and only if for each upper bounded directed subset $M \subseteq S\lambda$, $\bigvee M \in S\lambda$ holds.*

Proof. Let S be an algebraic $\wedge$-semilattice, λ a partial algebraic closure operator in S. Let M be a bounded directed subset of $S\lambda$. Let us denote $m = \bigvee M$. Suppose that $z \in S^*$ is such that $z \leqq m\lambda$. Then there exists $m' \in S^*$ such that $m' \leqq m$ and $z \leqq m'\lambda$. However, $m' \leqq m = \bigvee M$, thus it must exist a finite subset $M' \subseteq M$ for which $m' \leqq \bigvee M'$. M is directed, therefore $m' \leqq m_1 \in M$. Then $z \leqq m'\lambda \leqq m_1 \leqq m$, and so each compact element which is less than $m\lambda$ is also less than m. This implies $m = m\lambda$, hence $m \in S\lambda$.

Let us suppose that S is finitely compact and that each bounded directed subset $M \subseteq S\lambda$ satisfies $\bigvee M \in S\lambda$. Let $a \in S^*$, $x \in \operatorname{Dom} \lambda$, let $\{x_\alpha;\ \alpha \in A\}$ be the system of all compact elements that are less than x and let $a \leqq x\lambda$. Then $x = \bigvee_{\alpha \in A} x_\alpha$, $x_\alpha \in \operatorname{Dom} \lambda$ for each $\alpha \in A$ and the finite compactness of S implies that the system $\{x_\alpha;\ \alpha \in A\}$ is directed. But then system $\{x_\alpha\lambda;\ \alpha \in A\}$ is directed, too, and it is bounded in $S\lambda$, thus by the assumption $\bigvee_{\alpha \in A} x_\alpha\lambda \in S\lambda$. This implies $\left(\bigvee_{\alpha \in A} x_\alpha\lambda\right)\lambda = \bigvee_{\alpha \in A} x_\alpha\lambda$, i.e. $a \leqq \bigvee_{\alpha \in A} x_\alpha\lambda$. The element a is compact in S, therefore there exists a finite subset B of A such that $a \leqq \bigvee_{\beta \in B} x_\beta\lambda$. But $\bigvee_{\beta \in B} x_\beta\lambda = \Upsilon_{\beta \in B} x_\beta\lambda$, thus by Corollary 1.4, $a \leqq \left(\bigvee_{\beta \in B} x_\beta\right)\lambda$. The finite compactness of S implies $\bigvee_{\beta \in B} x_\beta \in S^*$, and hence λ is algebraic.

2. SEMILATTICES OF REFLEXIVE AND ANTISYMMETRIC RELATIONS COMPATIBLE WITH PARTIAL ALGEBRAS

If A is a set, then $(\mathrm{RAs})_0(A)$ denotes the set of all reflexive and antisymmetric binary relations on A.

Lemma 2.1. *If A is a set, then $(\mathrm{RAs})_0(A)$ ordered by the set-inclusion is an algebraic $\wedge$-semillattice in which the infimum is formed by the intersection. The smallest element is id_A.*

Proof. It is clear that $(\mathrm{RAs})_0(A)$ is a complete $\wedge$-semilattice. Let $\varrho' = \{(x_i, y_i);\ i = 1, \ldots, n\}$ be a finite antisymmetric relation on A and $\varrho = \varrho' \cup \mathrm{id}_A$. Let us suppose that $\{\varrho_\alpha;\ \alpha \in I\}$ is a system in $(\mathrm{RAs})_0(A)$ such that $\sigma = \bigvee_{(\mathrm{RAs})_0(A)} \{\varrho_\alpha;\ \alpha \in I\}$ exists and $\varrho \subseteq \sigma$. Now, $\sigma = \bigcup_{\alpha \in I} \varrho_\alpha$, and for each $(x_i, y_i) \in \varrho'$ there exists ϱ_{α_i} $(\alpha_i \in I)$ such that $(x_i, y_i) \in \varrho_{\alpha_i}$. This implies $\varrho \subseteq \bigcup_{i=1}^{n} \varrho_{\alpha_i} = \bigvee_{i=1}^{n} \varrho_{\alpha_i}$. Hence ϱ is a compact element in $(\mathrm{RAs})_0(A)$. Hereby, it is evident that exactly all compact elements in $(\mathrm{RAs})_0(A)$ are formed in this manner.

But then each element of $(\mathrm{RAs})_0(A)$ is the supremum of compact elements, therefore the $\wedge$ semilattice $(\mathrm{RAs})_0(A)$ is algebraic.

Now, let us consider a system Y of relational quasi-identities of the type $\langle 2 \rangle$ with the signature $\langle A_1^2 \rangle$ which contains the identity of reflexivity and the quasi-identity of antisymmetry, and let us suppose that any other quasi-identity of Y (if it exists) is in the form

$$\forall x_1 \ldots \forall x_n (\mathscr{A}_1 \,\&\, \ldots \,\&\, \mathscr{A}_p \Rightarrow \mathscr{A}) ,$$

where $\mathscr{A}_1, \ldots, \mathscr{A}_p$, $\mathscr{A}$ are primitive formulas and the following conditions are satisfied:

a) for each x_i $(i = 1, \ldots, n)$ there exists at least one of the formulas $\mathscr{A}_1, \ldots, \mathscr{A}_p$ containing x_i,

b) if $p > 1$, then each $\mathscr{A}_k$ $(k = 2, \ldots, p)$ contains at least one of the variables contained in $\mathscr{A}_{k-1}$,

c) $\mathscr{A} = A_1^2(x_r, x_q)$, $r, q \in \{1, \ldots, n\}$.

Definition 2.1. A binary relation on a set A satisfying all quasi-identities of Y is called a *Y-relation* on A.

We denote the set of all Y-relations on A by $Y_0(A)$.

Lemma 2.2. *$Y_0(A)$ ordered by inclusion is a closed subsemilattice of the $\wedge$-semilattice $(\mathrm{RAs})_0(A)$ with the smallest element* id_A.

Proof. Evident.

Let A be a set. We put $\Gamma = \{\varrho \in (\mathrm{RAs})_0(A);\ \exists \sigma \in Y_0(A) : \varrho \subseteq \sigma\}$. If $\varrho \in \Gamma$, then $\bar{\varrho}$ means the intersection of all relations of $Y_0(A)$ containing ϱ.

We define the partial mapping $\lambda : (\mathrm{RAs})_0(A) \to (\mathrm{RAs})_0(A)$ by $\mathrm{Dom}\,\lambda = \Gamma$ and $\varrho\lambda = \bar{\varrho}$ for each $\varrho \in \Gamma$. It is evident that λ is a partial closure operator in $(\mathrm{RAs})_0(A)$ and that $((\mathrm{RAs})_0(A))\,\lambda = Y_0(A)$.

Let us show that λ is algebraic. Evidently, the algebraic $\wedge$-semilattice $(\mathrm{RAs})_0(A)$ is finitely compact. Hence by Lemma 1.5 it suffices to prove that for any bounded directed subsystem $\Sigma \subseteq Y_0(A)$, $\bigvee \Sigma \in Y_0(A)$ holds.

Lemma 2.3. *Let Σ be a bounded directed subsystem in $Y_0(A)$. Then $\bigcup \Sigma \in Y_0(A)$.*

Proof. The antisymmetry of $\bigcup \Sigma$ follows from the boundedness of Σ in $Y_0(A)$. For the other quasi-identities of Y see [5, Proof of Lemma 2].

Therefore we have

Theorem 2.4. *If A is a set, then $Y_0(A)$ is an algebraic $\wedge$-semilattice.*

Proof. Since the suprema in $(\mathrm{RAs})_0(A)$ are formed by the unions, Lemmas 1.5 and 2.3 imply that the partial closure operator λ is algebraic. Thus, by Lemma 1.2, $((\mathrm{RAs})_0(A))\,\lambda = Y_0(A)$ is an algebraic $\wedge$-semilattice.

Let now $\mathfrak{A} = (A, F)$ be a partial algebra, where $A \neq \emptyset$ is the support of $\mathfrak{A}$ and $F \neq \emptyset$ is a set of partial operations on A. If $f \in F$, then n_f denotes the arity of f and $D(f, A)$ denotes the definition domain of f.

Definition 2.2. Let θ be a Y-relation on A. Then θ is called a *$\mathscr{Y}$-relation* on $\mathfrak{A}$ if the following implication holds for each $f \in F$:

If $(a_1, \ldots, a_{n_f})$, $(b_1, \ldots, b_{n_f}) \in D(f, A)$ and if $(a_i, b_i) \in \theta$, $i = 1, \ldots, n_f$, then $(a_1 \ldots a_{n_f} f, b_1 \ldots b_{n_f} f) \in \theta$.

The set of all $\mathscr{Y}$-relations on $\mathfrak{A}$ is denoted by $\mathscr{Y}(\mathfrak{A})$.

Theorem 2.5. *If $\mathfrak{A} = (A, F)$ is a partial algebra, then $\mathscr{Y}(\mathfrak{A})$ ordered by inclusion is a closed $\wedge$-subsemilattice of the complete $\wedge$-semilattice $Y_0(A)$. The smallest element in $\mathscr{Y}(\mathfrak{A})$ is* id_A.

Proof. Evident.

Theorem 2.6. *Let Σ be a bounded directed subsystem in $\mathscr{Y}(\mathfrak{A})$. Then $\bigcup \Sigma \in \mathscr{Y}(\mathfrak{A})$.*

Proof follows from Lemma 2.3.

For a partial algebra $\mathfrak{A} = (A, F)$ let us define the partial mapping $\lambda : Y_0(A) \to$ $\to Y_0(A)$ such that Dom λ is formed by exactly all relations from $Y_0(A)$ each of which is contained in some relation from $\mathscr{Y}(\mathfrak{A})$, and $\varrho\lambda = \bigcap\{\sigma \in \mathscr{Y}(\mathfrak{A});\ \varrho \subseteq \sigma\}$ for each $\varrho \in \mathrm{Dom}\,\lambda$. Obviously λ is a partial closure operator in $Y_0(A)$. Therefore Lemmas 1.2 and 1.5 and Theorem 2.6 imply

Theorem 2.7. *If $\mathfrak{A} = (A, F)$ is a partial algebra, then $\mathscr{Y}(\mathfrak{A})$ is an algebraic $\wedge$-semilattice.*

In particular, let $\mathfrak{A} = (A, F)$ be a partial algebra and $(\mathscr{R}\mathscr{A}\mathscr{o})(\mathfrak{A})$ the complete $\wedge$-semilattice of all compatible reflexive and antisymmetric relations on $\mathfrak{A}$. Let us denote $\Xi = \{\varrho \subseteq A \times A;\ \varrho \subseteq \sigma\}$ for some $\sigma \in (\mathscr{R}\mathscr{A}\mathscr{o})(\mathfrak{A})\}$. If $\varrho \in \Xi$, then Ξ_ϱ means the intersection of all relations of $(\mathscr{R}\mathscr{A}\mathscr{o})(\mathfrak{A})$ containing ϱ. Consider the partial mapping $\lambda : \exp(A \times A) \to \exp(A \times A)$ defined by $\mathrm{Dom}\,\lambda = \Xi$ and $\varrho\lambda = \Xi_\varrho$ for each $\varrho \in \Xi$. Then

1. $\mathrm{Dom}\,\lambda \neq \emptyset$;
2. $\varrho \in \mathrm{Dom}\,\lambda \Rightarrow \varrho \subseteq \varrho\lambda$;
3. $\varrho_1 \subseteq \varrho_2,\ \varrho_2 \in \mathrm{Dom}\,\lambda \Rightarrow \varrho_1 \in \mathrm{Dom}\,\lambda,\ \varrho_1\lambda \subseteq \varrho_2\lambda$;
4. $\varrho \in \mathrm{Dom}\,\lambda \Rightarrow \varrho\lambda \in \mathrm{Dom}\,\lambda,\ (\varrho\lambda)\,\lambda = \varrho\lambda$.

Theorem 2.8. *Let $\varrho_\alpha \subseteq A \times A$ $(\alpha \in I)$ and $\varrho = \bigcup_{\alpha \in I} \varrho_\alpha$. Let $\varrho \in \mathrm{Dom}\,\lambda$. Then $\varrho\lambda =$ $= \bigvee_{\alpha \in I} \varrho_\alpha\lambda$ in $(\mathscr{R}\mathscr{A}\mathscr{o})(\mathfrak{A})$.*

Proof. Since $\varrho_\alpha \subseteq \varrho$, we have $\varrho_\alpha \in \mathrm{Dom}\,\lambda$ and $\varrho_\alpha\lambda \subseteq \varrho\lambda$ for each $\alpha \in I$. Hence

$\bigvee_{\alpha\in I} \varrho_\alpha\lambda$ exists and $\varrho\lambda \supseteq \bigvee_{\alpha\in I} \varrho_\alpha\lambda$. Let $\sigma \in (\mathscr{R}\mathscr{A}\mathfrak{s})(\mathfrak{A})$ be such that $\sigma \supseteq \bigcup_{\alpha\in I} \varrho_\alpha$. Then $\sigma \supseteq \varrho_\alpha$ for each $\alpha \in I$, thus $\sigma \supseteq \bigcup_{\alpha\in I} \varrho_\alpha$. But then $\sigma = \sigma\lambda \supseteq \varrho\lambda$, therefore $\varrho\lambda = \bigvee_{\alpha\in I} \varrho_\alpha\lambda$.

For $\{(a, b)\} \in \operatorname{Dom}\lambda$ we denote $(\{(a, b)\})\lambda$ by $(a, b)\lambda$.

Corollary 2.9. *If* $\varrho \in \operatorname{Dom}\lambda$, *then* $\varrho\lambda = \bigvee_{(a,b)\in\varrho} (a, b)\lambda$. *In particular, if* $\varrho \in (\mathscr{R}\mathscr{A}\mathfrak{s})(\mathfrak{A})$, *then* $\varrho = \bigvee_{(a,b)\in\varrho} (a, b)\lambda$.

Let $\mathfrak{A} = (A, F)$ be a partial algebra. If $a \in A$, then a^0 means the nullary operation on A with the value a. Further, we put $A^0 = \{a^0;\ a \in A\}$, $\mathfrak{A}^0 = (A, F \cup A^0)$. It is clear that the algebraic functions of $\mathfrak{A}$ are exactly the polynomials of $\mathfrak{A}^0$.

Now, let $\varrho \in \operatorname{Dom}\lambda$. We denote $\varrho^F = \{(u, v) \in A \times A;$ there exist an algebraic function $x_1 \dots x_n p$ and $(a_i, b_i) \in \varrho$, $i = 1, \dots, n$, such that $a_1 \dots a_n p$, $b_1 \dots b_n p$ exist and $u = a_1 \dots a_n p$, $v = b_1 \dots b_n p\}$.

Theorem 2.10. *Let* $\mathfrak{A} = (A, F)$ *be a partial algebra,* $\varrho \in \operatorname{Dom}\lambda$. *Then* $\varrho\lambda = \varrho^F$.

Proof. Since $\varrho \subseteq \varrho^F \subseteq \varrho\lambda$ holds, it suffices to prove $\varrho^F \in (\mathscr{R}\mathscr{A}\mathfrak{s})(\mathfrak{A})$. Let $c \in A$, $xp = (c, x)e^{1,2}$, $(a_1, a_2) \in \varrho$. Then $a_1 p = a_2 p = c$, hence ϱ^F is reflexive. Since $\varrho^F \subseteq \varrho\lambda$, ϱ^F is antisymmetric.

Let now $f \in F$, $(a_1, \dots, a_{n_f})$, $(b_1, \dots, b_{n_f}) \in D(f, A)$, $(a_1, b_1), \dots, (a_{n_f}, b_{n_f}) \in \varrho^F$, $c = a_1 \dots a_{n_f} f$, $d = b_1 \dots b_{n_f} f$. Then by the definition of ϱ^F, for an appropriate $m \in N$ there exist m-ary polynomial symbols $p_1, \dots, p_{n_f}$ over $F \cup A^0$ and elements $u_1^{(1)}, \dots, u_m^{(1)}$, $v_1^{(1)}, \dots, v_m^{(1)}, \dots, u_1^{(n_f)}, \dots, u_m^{(n_f)}$, $v_1^{(n_f)}, \dots, v_m^{(n_f)} \in A$ such that $a_1 = u_1^{(1)} \dots u_m^{(1)} p_1$, $b_1 = v_1^{(1)} \dots v_m^{(1)} p_1, \dots, a_{n_f} = u_1^{(n_f)} \dots u_m^{(n_f)} p_{n_f}$, $b_{n_f} = v_1^{(n_f)} \dots v_m^{(n_f)} p_{n_f}$. But $p_1 \dots p_{n_f} f$ is also an m-ary polynomial symbol over $F \cup A^0$ and

$$c = u_1^{(1)} \dots u_m^{(1)} p_1 \dots u_1^{(n_f)} \dots u_m^{(n_f)} p_{n_f} f\,,$$

$$d = v_1^{(1)} \dots v_m^{(1)} p_1 \dots v_1^{(n_f)} \dots v_m^{(n_f)} p_{n_f} f\,,$$

hence ϱ^F is compatible with f.

Therefore $\varrho^F \in (\mathscr{R}\mathscr{A}\mathfrak{s})(\mathfrak{A})$.

3. CATEGORIES OF SYSTEMS OF Y-RELATIONS

In the sequel, Y again means a system of relational quasi-identities of the type $\langle 2 \rangle$ with the properties described before Definition 2.1.

It is known that a system of equivalences on a non-empty set A forms the system of all congruences of some partial algebra with the underlying set A if and only if it is closed under intersections of arbitrary subsystems and under unions of directed subsystems and if it contains id_A. Hence, in [1], the author studied the category the objects of which are sets together with the above described systems of equivalences.

In general, in [5] we investigated categories the objects of which are sets together with systems of X-relations. The concluding part of the paper describes some properties of analogous categories of Y-systems.

Definition 3.1. A *y-system* is any ordered pair $(A, \mathscr{Y})$, where A is a set and $\mathscr{Y} \subseteq \subseteq Y_0(A)$ is a system that is closed with respect to the intersections of non-void subsystems and under the unions of bounded directed subsystems, and that contains id_A.

Definition 3.2. Let $(A, \mathscr{Y})$, $(B, \mathscr{X})$ be y-systems, $\varphi : A \to B$. Then φ is called a *y-morphism* from $(A, \mathscr{Y})$ to $(B, \mathscr{X})$ if $\varrho\varphi^{-1} \in \mathscr{Y}$ for each $\varrho \in \mathscr{X}$.

Definition 3.3. We denote by $\mathbf{Y}$ the category whose class of objects is exactly the class of all y-systems and whose morphisms are precisely the y-morphisms between these y-systems.

Theorem 3.1. *If* $\varphi : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \mathrm{Mor}\, \mathbf{Y}$, *then* φ *is an injection.*

Proof. Let $a_1, a_2 \in A$, $a_1 \neq a_2$, $a_1\varphi = a_2\varphi$. Then (a_1, a_2), $(a_2, a_1) \in \mathrm{id}_B\, \varphi^{-1}$. But $\mathrm{id}_B\, \varphi^{-1} \in \mathscr{Y}$, so it is antisymmetric, a contradiction.

Theorem 3.2. *Let* $\varphi : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \mathrm{Mor}\, \mathbf{Y}$. *Then*

a) φ *is a monomorphism*;

b) φ *is an epimorphism if and only if it is bijective.*

Proof. a) Follows from Theorem 3.1.

b) Let $\varphi : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \mathrm{Mor}\, \mathbf{Y}$, $B \setminus A\varphi \neq \emptyset$. Let $b \in B \setminus A\varphi$, $d \notin B$, $D = B \cup \{d\}$, $\mathscr{U} = \{\mathrm{id}_D\}$. Denote by $\chi_1, \chi_2 : B \to D$ such mappings that $\chi_1 = 1_{B,D}$, $\chi_2 \mid (B \setminus \{b\}) = \chi_1 \mid (B \setminus \{b\})$, $b\chi_2 = d$. Then $\mathrm{id}_D\, \chi_1^{-1} = \mathrm{id}_B \in \mathscr{X}$, $\mathrm{id}_D\, \chi_2^{-1} = \mathrm{id}_{B\setminus\{b\}} \cup \{(b, b)\} = \mathrm{id}_B \in \mathscr{X}$. But this implies that φ is not an epimorphism.

Theorem 3.3. *The category* $\mathbf{Y}$ *is concrete.*

Proof. Let $A = \{a\}$, $\mathscr{Y} = \{(a, a)\}$. Consider an arbitrary $(B, \mathscr{X}) \in \mathrm{Ob}\, \mathbf{Y}$. Let $\varphi : A \to B$ be a mapping. Every relation of $\mathscr{X}$ is reflexive, hence φ is a morphism from $(A, \mathscr{Y})$ to $(B, \mathscr{X})$ and hereby it is the unique extension of the mapping φ. Therefore $(A, \mathscr{Y})$ is a free object over $\mathbf{Y}$, and so $\mathbf{Y}$ is concrete.

Lemma 3.4. *If* $(A_\gamma, \mathscr{Y}_\gamma)$, $\gamma \in \Gamma$, *are objects in* $\mathbf{Y}$ *and if* card $\Gamma > 1$, *then the product of the objects* $(A_\gamma, \mathscr{Y}_\gamma)$, $\gamma \in \Gamma$, *exists if and only if* card $A_\gamma = 1$ *for each* $\gamma \in \Gamma$.

Proof. Let $(A_\gamma, \mathscr{Y}_\gamma) \in \mathrm{Ob}\, \mathbf{Y}$, $\gamma \in \Gamma$, card $\Gamma > 1$ and let there exist $\gamma_1 \in \Gamma$ such that card $A_{\gamma_1} > 1$. Suppose that $(A, \mathscr{Y}) \in \mathrm{Ob}\, \mathbf{Y}$ together with $\varphi_\gamma : (A, \mathscr{Y}) \to (A_\gamma, \mathscr{Y}_\gamma) \in \mathrm{Mor}\, \mathbf{Y}$, $\gamma \in \Gamma$, form the product of $(A_\gamma, \mathscr{Y}_\gamma)$, $\gamma \in \Gamma$. Let $x, y \in A_{\gamma_1}$, $x \neq y$. Consider

$(\{a\}, \{(a, a)\}) \in \mathrm{Ob}\,\mathbf{Y}$ and choose $z_\gamma \in A_\gamma$ for each $\gamma \in \Gamma \setminus \{\gamma_1\}$. Then the mappings $\psi_1^x, \psi_1^y : \{a\} \to A_{\gamma_1}$ and $\psi_\gamma : \{a\} \to A_\gamma$ $(\gamma \in \Gamma \setminus \{\gamma_1\})$ defined by $a\psi_1^x = x$, $a\psi_1^y = y$ and $a\psi_\gamma = z_\gamma$ $(\gamma \in \Gamma \setminus \{\gamma_1\})$ are morphisms of the corresponding objects in $\mathbf{Y}$. Since $(A, \mathscr{Y})$ is the product of the objects $(A_\gamma, \mathscr{Y}_\gamma)$, $\gamma \in \Gamma$, there exist unique morphisms $\psi^x, \psi^y : (\{a\}, \{(a, a)\}) \to (A, \mathscr{Y}) \in \mathrm{Mor}\,\mathbf{Y}$ such that

$$\psi^x \varphi_{\gamma_1} = \psi_1^x\,, \quad \psi^x \varphi_\gamma = \psi_\gamma$$

and

$$\psi^y \varphi_{\gamma_1} = \psi_1^y\,, \quad \psi^y \varphi_\gamma = \psi_\gamma\,.$$

Now, for each $\gamma \in \Gamma \setminus \{\gamma_1\}$ the equality $\psi^x \varphi_\gamma = \psi^y \varphi_\gamma$ holds and since φ_γ is a monomorphism, $\psi^x = \psi^y$. But this means that $\psi_1^x = \psi_1^y$, a contradiction.

Now, we immediately obtain

Theorem 3.5. *The category* $\mathbf{Y}$ *is not complete.*

Now let Y contain either exactly the identity of reflexivity and the quasi-identity of antisymmetry or exactly the identity of reflexivity and the quasi-identities of antisymmetry and transitivity. Then the following theorem holds:

Theorem 3.6. *The category* $\mathbf{Y}$ *is not cocomplete.*

Proof. Consider $(A_1, \mathscr{Y}_1)$, $(A_2, \mathscr{Y}_2) \in \mathrm{Ob}\,\mathbf{Y}$ such that $A_1 = \{a_1, a_2\}$, $\mathscr{Y}_1 = \{\mathrm{id}_{A_1}, \mathrm{id}_{A_1} \cup \{(a_1, a_2)\}\}$, $A_2 = \{b_1, b_2, b_3\}$, $\mathscr{Y}_2 = \{\mathrm{id}_{A_2}, \mathrm{id}_{A_2} \cup \{(b_1, b_2), (b_1, b_3)\}\}$. Let $(C, \mathscr{X})$ together with $\varphi_1 : (A, \mathscr{Y}_1) \to (C, \mathscr{X})$ and $\varphi_2 : (A_2, \mathscr{Y}_2) \to (C, \mathscr{X})$ be the coproduct of $(A_1, \mathscr{Y}_1)$ and $(A_2, \mathscr{Y}_2)$ in $\mathbf{Y}$. Since each morphism in $\mathbf{Y}$ is an injection, $\mathrm{card}\, C \geqq 3$. On the other hand, there exists $(D, \mathscr{U}) \in \mathrm{Ob}\,\mathbf{Y}$ such that $\mathrm{card}\, D = 3$ and that the sets of morphisms from $(A_1, \mathscr{Y}_1)$ to $(D, \mathscr{U})$ and from $(A_2, \mathscr{Y}_2)$ to $(D, \mathscr{U})$ are non-empty. (For example, if $\mathscr{U} = \{\mathrm{id}_D\}$.) Hence $\mathrm{card}\, C = 3$. Moreover, $\chi : A_1 \to A_2$ defined by $a_1\chi = b_1$, $a_2\chi = b_2$ is a morphism from $(A_1, \mathscr{Y}_1)$ to $(A_2, \mathscr{Y}_2)$. Therefore if $C = \{c_1, c_2, c_3\}$, then either $\mathscr{X} = \{\mathrm{id}_C, \mathrm{id}_C \cup \{(c_1, c_2), (c_1, c_3)\}\}$ or $\mathscr{X}$ is different from this form only by a permutation on the index set $\{1, 2, 3\}$.

Let us suppose that $\mathscr{X} = \{\mathrm{id}_C, \mathrm{id}_C \cup \{(c_1, c_2), (c_1, c_3)\}\}$. Let $a_i\varphi_1 = c_i$, $i = 1, 2$, $b_i\varphi_2 = c_i$, $i = 1, 2, 3$. Evidently $\varphi_1, \varphi_2 \in \mathrm{Mor}\,\mathbf{Y}$. Let $(D, \mathscr{U}) \in \mathrm{Ob}\,\mathbf{Y}$, $D = \{d_1, d_2, d_3\}$, $\mathscr{U} = \{\mathrm{id}_D, \mathrm{id}_D \cup \{(d_1, d_2), (d_1, d_3)\}\}$. Denote $\psi_1 : A_1 \to D$, $\psi_2 : A_2 \to D$ such that $a_i\psi_1 = d_i$, $i = 1, 2$, $b_1\psi_2 = d_2$, $b_2\psi_2 = d_3$, $b_3\psi_2 = d_2$. Then ψ_1 is a morphism from $(A_1, \mathscr{Y}_1)$ to $(D, \mathscr{U})$ and ψ_2 is that from $(A_2, \mathscr{Y}_2)$ to $(D, \mathscr{U})$. Hereby it exists $\psi : (C, \mathscr{X}) \to (D, \mathscr{U}) \in \mathrm{Mor}\,\mathbf{Y}$ such that $\psi_1 = \varphi_1\psi$ and $\psi_2 = \varphi_2\psi$. Indeed, $a_2\psi_1 = d_2$, $(a_2\varphi_1)\,\psi = c_2\psi$ and $b_2\psi_2 = d_3$, $(b_2\varphi_2)\,\psi = c_2\psi$, hence $d_2 = d_3$, a contradiction.

Similarly we obtain a contradiction for all other injections of A_1 and A_2 into C.

In the sequel, Y again means a general system of quasi-identities.

Theorem 3.7. *The category* $\mathbf{Y}$ *has no injective objects.*

Proof. Suppose that $(A, \mathscr{Y}) \in \text{Ob}\,\mathbf{Y}$. Let $b \notin A$, $B = A \cup \{b\}$, $\mathscr{X} = \{\varrho \cup \{(b, b)\};\ \varrho \in \mathscr{Y}\}$. It is evident that $(B, \mathscr{X}) \in \text{Ob}\,\mathbf{Y}$. Let us consider the mappings $1_A : A \to A$ and $1_{A,B} : A \to B$. Then $1_A : (A, \mathscr{Y}) \to (A, \mathscr{Y})$, $1_{A,B} : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \text{Mor}\,\mathbf{Y}$ and $1_{A,B}$ is a monomorphism. However, there exists no injection $\varphi : B \to A$ for which $1_{A,B}\varphi = 1_A$, hence there exists no morphism from $(B, \mathscr{X})$ to $(A, \mathscr{Y})$ with this property. Therefore $(A, \mathscr{Y})$ is not injective in $\mathbf{Y}$.

Theorem 3.8. *An object $(A, \mathscr{Y})$ is projective in $\mathbf{Y}$ if and only if $\mathscr{Y} = Y_0(A)$.*

Proof. Let $(A, \mathscr{Y}) \in \text{Ob}\,\mathbf{Y}$ and let $\mathscr{Y} \neq Y_0(A)$. Clearly $(A, Y_0(A)) \in \text{Ob}\,\mathbf{Y}$. Let $\varphi = \psi = 1_A$. Then $\varphi : (A, Y_0(A)) \to (A, \mathscr{Y})$, $\psi : A, \mathscr{Y}) \to (A, \mathscr{Y}) \in \text{Mor}\,\mathbf{Y}$ and φ is an epimorphism. But $\chi = 1_A$ is not a morphism from $(A, \mathscr{Y})$ to $(A, Y_0(A))$, hence $(A, \mathscr{Y})$ is not projective. Therefore if $(A, \mathscr{Y}) \in \text{Ob}\,\mathbf{Y}$ is projective, then $\mathscr{Y} = Y_0(A)$.

Let A be any set, $(B, \mathscr{X}), (C, \mathscr{Z}) \in \text{Ob}\,\mathbf{Y}$, $\varphi : (B, \mathscr{X}) \to (C, \mathscr{Z})$, $\psi : (A, Y_0(A)) \to$ $\to (C, \mathscr{Z}) \in \text{Mor}\,\mathbf{Y}$ and let φ be an epimorphism. Then ψ is an injection, φ is a bijection and $\mathscr{X} \supseteq \{\varrho\varphi^{-1};\ \varrho \in \mathscr{Z}\}$. If this is the case, then the mapping $\chi : A \to B$ defined by $a\chi = a\psi\varphi^{-1}$ is an injection, and thus the coimage of each relation from $\mathscr{Z}$ is a Y-relation on A. Hence χ is a morphism from $(A, Y_0(A))$ to $(B, \mathscr{X})$ and $\chi\varphi = \psi$ holds. Therefore $(A, Y_0(A))$ is projective in $\mathbf{Y}$.

Theorem 3.9. a) *In the category $\mathbf{Y}$ no separator exists.*

b) $(S, \mathscr{U}) \in \text{Ob}\,\mathbf{Y}$ *is a coseparator in $\mathbf{Y}$ if and only if* card $S = 1$.

Proof. a) Follows from the fact that each morphism in $\mathbf{Y}$ is an injection.

b) If $\varphi, \psi : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \text{Mor}\,\mathbf{Y}$ are such that $\varphi \neq \psi$, then $A \neq \emptyset$. Hence again Theorem 3.1 implies that if $(S, \mathscr{U})$ is a coseparator in $\mathbf{Y}$, then card $S = 1$. Let $\varphi, \psi : (A, \mathscr{Y}) \to (B, \mathscr{X}) \in \text{Mor}\,\mathbf{Y}$, $a \in A$, $a\varphi \neq a\psi$ and let $S = \{s\}$. Then the mapping $\xi : S \to A$ defined by $s\xi = a$ is a morphism from $(S, \mathscr{U})$ to $(A, \mathscr{Y})$ and $\xi\varphi \neq \xi\psi$.

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