REFLEXIVE MODULES OVER GORENSTEIN RINGS

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Introduction. The aim of this paper is to show the relevance of a class of commutative noetherian rings to the study of reflexive modules. They include integrally closed domains, group algebras over these, and Gorenstein rings. We will be basically concerned with a ring R having the following property: Let M be a finitely generated R-module; then M is reflexive if and only if every R-sequence of at most two elements is also an M-sequence.

In the rest of this note, using the above characterization, we examine the closeness between free modules and Macaulay modules of maximum dimension in a local Gorenstein ring. Finally, it is proved that over a one-dimensional Gorenstein ring only projective modules have projective endomorphism rings.

1. **Reflexive modules.** As a running hypothesis we will assume that all rings considered here are commutative, noetherian and that unspecified modules are finitely generated. Let R be such a ring and M an R-module. Write M^* for the group $\mathrm{Hom}_R(M, R)$ endowed with the usual R-module structure. There is a natural homomorphism $j: M \to M^{**}$ and, after [**5**], M is said to be torsion-less if j is a monomorphism, reflexive if j is an isomorphism. We will also say that M is torsion-free to mean that nonzero elements of M are not annihilated by nonzero divisors of R. In other words, if K denotes the full ring of quotients of R, M is torsion-free if and only if the natural homomorphism $M \to M \otimes K$ is a monomorphism (see Appendix for relation between "torsion-less" and "torsion-free"). Here we want to describe the reflexive modules over a class of rings sensibly vaster than integrally closed domains, but first we recall some notions from [**4**].

Let R be a local ring. R is said to be a Gorenstein ring if it is Cohen-Macaulay [**9**] and whenever $x_1, \cdots, x_n$ is a maximal R-sequence the ideal $(x_1, \cdots, x_n)$ is irreducible, i.e., it is not an intersection of properly larger ideals. Equivalently, R is Gorenstein if as an R-module R has finite injective dimension. For Krull dimension zero, i.e., for artinian rings, this is the same as having all modules (finitely generated!) reflexive and for Krull dimension one, that torsion-less modules are reflexive. Thus, at least at these low dimensions the Gorenstein property and reflexivity are closely related.

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We use some definitions and notation from [2] and [8]. Let R be a noetherian local ring and let M be a finitely generated R-module. An M-sequence is a sequence $x_1, \cdots, x_n$ of elements in the maximal ideal of R such that, if $M \neq 0$, x_i is not a zero divisor of $M/(x_1, \cdots, x_{i-1})M$ for $i \leqq n$. If $M \neq 0$, the largest integer n for which this happens is called the depth of M or, in symbols, Depth M. If $M=0$ we write Depth $M=\infty$. [2] will be used as a reference for the basic properties of Depth, which is there called Codimension.

Definition (1.1). Let R be a commutative noetherian ring, let M be a finitely generated R-module and let k be an integer. M is said to have property S_k if for every prime ideal P, Depth $M_P \geqq \inf(k, \dim R_P)$.

Definition (1.1)′. Let R be a commutative ring and let M be an arbitrary R-module. M is said to satisfy S_k' if every R-sequence $x_1, \cdots, x_n$, with $n \leqq k$, is also an M-sequence.

For Cohen-Macaulay rings it is proved in [8] that the two definitions coincide.

We now come to the basic notion in this paper:

Definition (1.2). Let R be a commutative noetherian ring. R is said to be *quasi-normal* if it has property S_2 and for every prime ideal P of height at most one R_P is a Gorenstein ring.

The following are easy consequences of this definition: If R is quasi-normal, so are $R[x]$ = polynomial ring over R and $R[G]$ = group algebra of the finite abelian group G. Possibly the same is true for the power series ring $R[[x]]$.

Before characterizing reflexive modules over quasi-normal rings we quote [3, Proposition 4.7]:

Proposition (1.3). *Let N be an R-module satisfying S_2'. Then for any R-module M,* Hom(M, N) *also satisfies S_2'.*

Reflexive modules, being duals, enjoy property S_2'. Over quasi-normal rings we claim this to be characteristic.

Theorem (1.4). *Let R be a quasi-normal ring and M a finitely generated R-module. A necessary and sufficient condition for M to be reflexive is that every R-sequence of two or less elements be also an M-sequence.*

Proof. Let M be a finitely generated R-module satisfying S_2'. From the definition of R it is clear that R has no embedded primes. Let K be its total ring of quotients. Since the localization of K at a prime ideal is isomorphic to the localization of R at the corresponding minimal prime, K is a zero-dimensional Gorenstein ring. Consider the exact sequence

$$0 \to M' \to M \xrightarrow{j} M^{**} \to M'' \to 0$$

where j is the natural map. By tensoring with K we get $M' \otimes K = 0$ since K is Gorenstein and so, as remarked earlier, every K-module is reflexive. This implies that some nonzero divisor of R annihilates the submodule M' of M, which is a contradiction, unless $M' = 0$. Thus M is a torsion-less module. Now localize at any height one prime P. Since by hypothesis R_P is Gorenstein, torsion-less are reflexive and so $M''_P = 0$. This means that the only associated primes of M'' have height at least two. Let P be one of them. We get the exact sequence

$$0 \to \mathrm{Hom}(R/P, M) \to \mathrm{Hom}(R/P, M^{**})$$
$$\to \mathrm{Hom}(M/P, M'') \to \mathrm{Ext}(R/P, M).$$

Here $\mathrm{Hom}(R/P, M) = \mathrm{Hom}(R/P, M^{**}) = 0$ for P contains nonzero divisors and M and M^{**} are torsion-free. Since height $P \geqq 2$, by assumption P contains an R-sequence of length two and by standard properties of Depth, $\mathrm{Ext}(R/P, M) = 0$. Hence $\mathrm{Hom}(M/P, M'') = 0$, a contradiction if $M'' \neq 0$, i.e., unless M is reflexive. Q.E.D.

COROLLARY (1.5). *If R is quasi-normal and N is a reflexive R-module, for any R-module M* $\mathrm{Hom}_R(M, N)$ *is also reflexive.*

COROLLARY (1.6). *If R is quasi-normal and*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence with M' and M'' reflexive, then M is also reflexive.

When R is an integral domain the above characterization of reflexive modules can be expressed in still another way (see [8] for the case of an integrally closed domain). For the moment, R does not have to be noetherian. Let M be a not necessarily finitely generated torsion-free R-module. Then

PROPOSITION (1.7). *The following are equivalent:*

(i) *Every R-sequence of at most two elements is an M-sequence.*

(ii) $M = \bigcap_{P \in \mathcal{P}} M_P$, *where $\mathcal{P}$ is the set of all prime ideals associated to principal ideals.*

The proof is implicit in [8]. For an example one can take any flat R-module. Using (1.7) then, if in (1.4) R is a domain, one could state that a torsion-free R-module is reflexive if and only if it is the intersection of its localizations at the height one primes.

2. **Depth and freeness.** In this section R is a Gorenstein local ring and M a finitely generated R-module. If M is free then, of course, Depth $M = \dim R$. Here it is shown some properties of free modules shared by modules of maximum Depth. Only a few aspects related to reflexivity will be examined and these could be viewed as consequences of [**4**, §8]. First we recall that the Depth of an R-module M is given by the following (e.g. [**1**] or [**7**]):

PROPOSITION (2.1). *Let* $M \neq 0$ *and let* $\dim R = d$. *Then* Depth $M = d - r$, *where* r *is the largest integer for which* $\operatorname{Ext}^r(M, R) \neq 0$.

Let P be a prime ideal of the ring R. Since R has finite self-injective dimension, so does R_P. We thus have

COROLLARY (2.2). *If* Depth $M = \dim R$ *and* $M_P \neq 0$, *then* Depth $M_P = \dim R_P$.

COROLLARY (2.3). *If* Depth $M = \dim R$, *then* M *is reflexive and* Depth $M^* = \dim R$.

PROOF. By (2.2) Depth $M_P \geqq \inf(2, \dim R_P)$ for any prime ideal P, i.e. M satisfies S_2. As it was said earlier, since R is Cohen-Macaulay, $S_2 = S_2'$. Now we apply (1.4) for R is quasi-normal. The second statement follows from [**4**, §8].

REMARK. An open question here is whether Depth $M = \dim R$ implies that also Depth $\operatorname{Hom}_R(M, M) = \dim R$.

3. **Gorenstein rings in dimension one.** Let R be an integrally closed domain. Then for every nonzero ideal I, $\operatorname{Hom}_R(I, I) = \operatorname{End}_R(I) = R$. For other rings this is also true provided the ideal I is large enough. For a ring like $Z[G]$ = integral group ring of the abelian group G however, $\operatorname{End}(I) = Z[G]$ if and only if I is an invertible ideal. More generally

THEOREM (3.1). *Let R be a one-dimensional Gorenstein ring and M a finitely generated R-module. Then* $\operatorname{End}_R(M)$ *is a projective module if and only if M itself is projective.*

The proof will follow after a few preliminary steps. One can assume that R is a local ring. Let $M \neq 0$; if $\operatorname{End}(M)$ is projective, to show that M is projective it is enough to prove that M contains a summand isomorphic to R.

(i) If $\operatorname{Hom}(M, M)$ is R-free, it is easy to see that the maximal ideal $\mathfrak{m}$ of R is not associated to M, i.e., $\mathfrak{m}$ is not the annihilator of a nonzero element of M [**3**, Lemma 4.5]. Thus Depth $M = 1$. Let x be an element of $\boldsymbol{m}$ which is not a zero divisor for either R or M. The exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

leads to

$$0 \to \mathrm{Hom}(M, R) \xrightarrow{x} \mathrm{Hom}(M, R) \to \mathrm{Hom}(M, R/(x)) \to \mathrm{Ext}^1(M, R).$$

Since Depth $M=1$, by (2.1) $\mathrm{Ext}^1(M, R)=0$ and we have $\mathrm{Hom}_{R'}(M', R')=\mathrm{Hom}(M, R)\otimes R'$ with $M'=M/xM$ and $R'=R/(x)$.

(ii) Consider the trace map T: $M^*\otimes M \to R$ given by $T(f\otimes m)=f(m)$. It is obvious that M contains a summand isomorphic to R if and only if T is an epimorphism (R is local!). From the preceding step we have the corresponding diagram of trace maps

$$\begin{array}{ccc} M^* \otimes M & \xrightarrow{T} & R \\ \downarrow & & \\ M^* \otimes M \otimes R' & & \downarrow \\ h \downarrow & & \\ (M')^* \otimes M' & \xrightarrow{T'} & R' \end{array}$$

where h is an isomorphism. Thus to show that T is an epimorphism it is enough to show that T' is so.

(iii) The exact sequence

$$0 \to M \xrightarrow{x} M \to M' \to 0$$

leads to the exactness of

$$0 \to \mathrm{Hom}(M, M) \xrightarrow{x} \mathrm{Hom}(M, M) \to \mathrm{Hom}(M', M')$$

and we have that $\mathrm{Hom}(M', M')$ contains a free R'-module. Thus, in particular, M' is R'-faithful. Now R' is a zero-dimensional Gorenstein ring, i.e., R' is self-injective. Here this just means that R' contains a unique nonzero minimal ideal. From this it follows immediately that every faithful R'-module contains a direct summand isomorphic to R'. This completes the proof of the theorem since the converse is trivial.

REMARK. The theorem is not true if dim $R>1$, even if we assume that M is reflexive. This occurs, for instance, in $R[x, y, z]/(xy-z^2)$, where R denotes the real numbers. This ring is normal, Gorenstein but not locally a UFD and so any height one prime which is not invertible gives a counterexample.

Appendix

We shall clarify here the relationship between torsion-less and torsion-free modules. The same finiteness assumptions of before are still in force. If an R-module M is torsion-less, the natural monomorphism $M \xrightarrow{j} M^{**}$ implies that M can be embedded in a free R-module and is thus torsion-free. For the converse

THEOREM (A.1). *For the commutative noetherian ring R, "torsion-free" and "torsion-less" are equivalent notions if and only if R has no embedded primes and R_P is a Gorenstein ring for every minimal prime P.*

PROOF. Assume that R satisfies the ring-theoretic statement above and let M be a torsion-free R-module. Consider the exact sequence

$$0 \to M' \to M \xrightarrow{j} M^{**}$$

for which we want to prove $M' = 0$. If P is a minimal prime, R_P is a zero-dimensional Gorenstein ring and so every R_P-module is reflexive; thus $M'_P = 0$. This says that the annihilator of M' is outside of any minimal prime and so contains a nonzero divisor for R has no embedded prime. Since M' is a submodule of M this implies $M' = 0$.

Conversely, assume "torsion-free" = "torsion-less." Let K be the total ring of quotients of R and let N be a finitely generated K-module. It is clear that we can find a finitely generated R-submodule M of N, such that $K \otimes M = KM = N$. Since M is torsion-free by construction, by assumption it is torsion-less. It follows easily that as a K-module N is also torsion-less. Now it must be shown that K is a zero-dimensional Gorenstein ring, i.e., that K is self-injective. Let I be an ideal of K which is not an annihilator or in other words, $I \neq \mathrm{Ann}(I')$ = annihilator of the ideal I', for any I'. Denote K/I by M. Then $M^* = \mathrm{Hom}(M, K) = \mathrm{Ann}(I) = J$ and $M^{**} = \mathrm{Hom}(J, K)$. Let $L = \mathrm{Ann}(J)$; I is properly contained in L. Pick $a \in L - I$, then in $M \xrightarrow{j} M^{**}$, $j(a) = 0$. Our assumption is then incorrect and thus every ideal is an annihilator. This is however equivalent to K being self-injective [6].

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