REGENERATIVE TOOL CHATTER NEAR A CODIMENSION-2 HOPF POINT

Pankaj Wahi (pankaj@mecheng.iisc.ernet.in)*, Anindya Chatterjee**
*Mechanical Engineering, Indian Institute of Science, Bangalore, 560012, India.
**Mechanical Engineering, Indian Institute of Science, Bangalore, 560012, India.

Summary We study a well known regenerative machine tool vibration model (a delay differential equation) near a codimension-2 Hopf bifurcation point. The method of multiple scales is used directly, bypassing a center-manifold reduction. Both sub- and supercritical bifurcations occur near the reference point, depending on choice of parameters. Analytical approximations are supported by numerics.

INTRODUCTION

The regenerative effect in metal cutting is an important source of undesirable vibrations in machine tools [1]. Mathematical models of the same are delay-differential equations (DDEs). Hopf bifurcations arising in such DDEs have been studied using center-manifold reductions [2, 3] and the method of multiple scales (MMS) [4, 5]. Fofana [3] has also treated double Hopf bifurcation points. We study a double Hopf bifurcation in a tool vibration DDE using MMS, with a definition of the associated small parameter that allows us to *not* treat the vibration amplitude as small. Both sub- and supercritical Hopf bifurcations are observed, depending on how two key parameters vary near the double Hopf point.

MODEL FOR REGENERATIVE TOOL VIBRATION

We consider the non-dimensionalized 1 DOF model for regenerative tool vibrations derived in [2], but with an added stiffening cubic nonlinearity in the restoring force (as in [6]). The governing equation then is:

$$\ddot{x}(t) + 2\,\zeta\,\dot{x}(t) + (1+p)\,x(t) + \alpha_1 x(t)^3 - p\,x(t-\tau) = 3\,p/10\left\{(x(t)-x(t-\tau))^2 - (x(t)-x(t-\tau))^3\right\}\,, \tag{1}$$

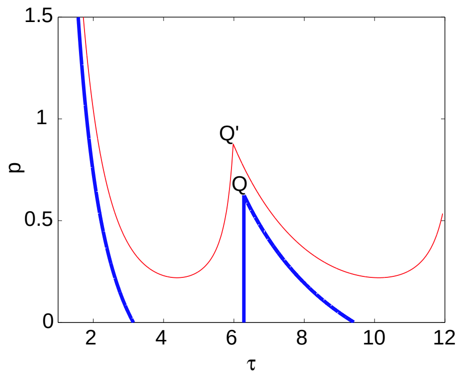

Figure 1. Stability boundary of Eq. (1), thick line: $\zeta = 0$ and thin line: $\zeta = 0.1$.

where $\alpha_1 = \mathcal{O}(\epsilon)$ (where ϵ is defined below), p is a parameter proportional to chip width, and τ, the time-delay, is inversely proportional to the cutting speed. Linear stability analysis of Eq. (1) is given in [2]. The stability boundaries showing the first two lobes in the $p-\tau$ plane for $\zeta = 0$ and $\zeta = 0.1$ are plotted in Fig. 1. Q $= (2\pi, 5/8) = (6.283, 0.625)$ and Q$' = (5.980, 0.877)$ represent double Hopf points for $\zeta = 0$ and $\zeta = 0.1$, respectively. Observe that, though the presence of small nonzero damping affects the stability boundary, the local structures (near Q and Q$'$) are similar.
The damping ratio ζ is often small (e.g., $\zeta = 0.02-0.03$ in [4]), so we consider the *unperturbed* linear equation

$$\ddot{x}(t) + (1+p)\,x(t) - p\,x(t-\tau) \;=\; 0\,. \tag{2}$$

Substituting $x(t) = Ce^{\lambda\,t}$ in Eq. (2), we get the characteristic equation, $\lambda^2 + (1+p) - p\,e^{-\lambda\,\tau} = 0$. When $\tau = 2\pi$ and $p = 5/8$ (as noted above), there are two pairs of pure imaginary roots corresponding to $\lambda = \pm i$ and $\lambda = \pm 3i/2$ with all other roots having negative real parts (verified numerically; details not presented here). We will now study small perturbations of Eq. (2) near this double Hopf point.
Scaling: To perform multiple scales analysis, we could scale $x(t)$ as $\epsilon y(t)$, where ϵ is some "small" parameter (as in [4]). However, this restricts $x(t)$ to be small. Instead we take $3\,p/10 = 3/16 = 0.1875$ as ϵ in Eq. (1), allowing $x(t) = \mathcal{O}(1)$.
Detuning: To indirectly account for variations in τ near the double Hopf point, we stretch time as $\bar{t} = (1+\epsilon\,\Delta)t$ (and then drop the overbar for notational simplicity). An increase in Δ now corresponds to a decrease in the delay.
Finally, letting $p = 5/8 + \epsilon p_1$, we obtain

$$\ddot{x}(t) + \frac{13}{8}x(t) - \frac{5}{8}x(t-2\pi) = \epsilon\left(2\,\Delta\left(\frac{13}{8}x(t) - \frac{5}{8}x(t-2\pi)\right) - 2\,\zeta_1\dot{x}(t) - \alpha\,x(t)^3 - p_1\,D + D^2 - D^3\right) + \mathcal{O}(\epsilon^2)\,, \tag{3}$$

where $D = x(t) - x(t-2\pi)$, $\zeta_1 = \zeta/\epsilon$ and $\alpha = \alpha_1/\epsilon$. Introducing the two time scales t and $T_0 = \epsilon\,t$; writing the solution to Eq. (3) as $x(t) = X(t, T_0) = X_0(t, T_0) + \epsilon X_1(t, T_0) + \cdots$; substituting in Eq. (3); and simplifying, we obtain

$$\frac{\partial^2 X_0}{\partial t^2} + \frac{13}{8}X_0 - \frac{5}{8}X_0(t-2\pi) + \epsilon\left(\frac{\partial^2 X_1}{\partial t^2} + \frac{13}{8}X_1 - \frac{5}{8}X_1(t-2\pi) + 2\frac{\partial^2 X_0}{\partial t\,\partial T_0} + 2\,\zeta_1\frac{\partial X_0}{\partial t} + \frac{5\,\pi}{4}\frac{\partial X_0}{\partial T_0}(t-2\pi) + \alpha X_0^3\right.$$

$$+\frac{5}{4}\,\Delta\,X_0(t-2\pi)-\frac{13}{4}\Delta\,X_0+p_1\,(X_0-X_0(t-2\pi))-(X_0-X_0(t-2\pi))^2+(X_0-X_0(t-2\pi))^3\Big)+\mathcal{O}(\epsilon^2)=0\,. \tag{4}$$

The solution to the lowest order equation is $X_0 = A_1 \sin t + A_2 \cos t + A_3 \sin(3t/2) + A_4 \cos(3t/2)$, where $A_i = A_i(T_0), i = 1, 2, 3, 4$. Here, we have explicitly dropped infinitely many exponentially decaying components in the solution (see [5]). Substituting this at the next order leads to potentially resonant forcing terms. Eliminating these resonant forcing terms as usual, we obtain $\partial A_i/\partial T_0, i = 1, 2, 3, 4$. Now setting $A_1 = R_1 \cos(\phi_1), A_2 = R_1 \sin(\phi_1), A_3 = R_2 \cos(3\phi_2/2)$ and $A_4 = R_2 \sin(3\phi_2/2)$, we obtain

$$\dot{R}_1 = \epsilon\frac{\left(40\,\pi\,\Delta - 64\,\zeta_1 - 15\,\pi\,\alpha\,R_1^2 - 30\,\pi\,\alpha\,R_2^2\right)\,R_1}{25\,\pi^2+64}\,, \tag{5}$$

$$\dot{R}_2 = \epsilon\frac{(40\,\pi\,p_1 - 144\,\zeta_1 - 90\,\pi\,\Delta + 120\,\pi\,R_2^2 + 15\,\pi\,\alpha\,R_2^2 + 30\,\pi\,\alpha\,R_1^2)\,R_2}{25\,\pi^2+144}\,, \tag{6}$$

(along with two equations for $\dot{\phi}_1$ and $\dot{\phi}_2$ that are not presented here for space). The bifurcation structure of the above system in the $p_1/\zeta_1 - \Delta/\zeta_1$ plane is shown in Fig. 2. The plane is divided into 5 regions with distinct behaviors.

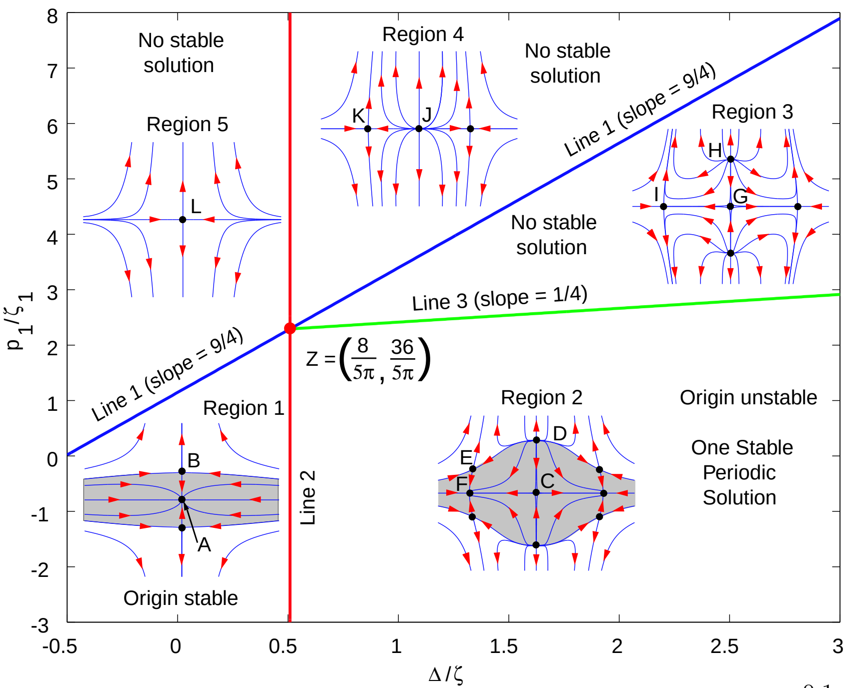

Figure 2. Bifurcation structure of Eqs. (5) and (6). Numerically, for $\zeta = 0.1$ or $\zeta_1 = \dfrac{0.1}{\epsilon} = \dfrac{8}{15}$, Z represents $(p, \tau) = (5.979, 0.854)$, agreeing well with numerics. For each of the phase portraits above, the horizontal axis represents R_1 and the vertical axis represents R_2.

In region 1, there are two qualitatively different fixed points (along with reflections of the same) with the origin being stable. In region 2, there are four qualitatively different fixed points marked as C, D, E and F in Fig. 2. F is a stable node, and represents a stable periodic solution of Eq. (3). Numerical simultions of Eq. (3) for $\zeta_1 = 1$, $\alpha = 1$, $\Delta = 8/(5\pi) + 1/4$ and $p_1 = 36/(5\pi)$ (in region 2) verifies this prediction (the steady state amplitude found numerically is 0.75, comparing reasonably well with 0.81 predicted from first order MMS). Hence, in region 2 we have a possibility of supercritical bifurcation. Regions 3, 4 and 5 are qualitatively similar in that there are no stable fixed points in the amplitude equations; in the original equations, no stable solutions remain.

DISCUSSION

The direct use of MMS without center manifold reductions simplified the calculations near the double Hopf point. Our choice of ϵ allowed $x(t) = \mathcal{O}(1)$, giving a glimpse of the dynamics for not-too-small amplitudes. The supercritical bifurcations observed here are important in the context of *control*: a minor excursion into the unstable region causes small amplitudes of vibration, leaving scope for returning to stable operation. In contrast, in the subcritical regime, any excursion into instability leads immediately to large amplitudes from which return to stable operation is problematic.

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