REGGE TRAJECTORIES IN THE QUARK MODEL

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A B S T R A C T

We prove that the large angular momentum behaviour of the leading Regge trajectory of a meson ($q\bar{q}$) or a baryon (qqq) can be obtained by minimizing the classical energy of the system for given angular momentum. A two-body quark-antiquark linear potential plus relativistic kinematics produces asymptotically linear Regge trajectories for mesons. For baryons we take either a sum of two-body potentials with half strength or a string of minimum length connecting the quarks, and find in both cases that the favoured configuration is a quark-diquark system and that the baryon and meson trajectories have the same slope. Short-distance singularities of the potential are shown to be unimportant.

1. INTRODUCTION

In the 1960's, everybody knew about Regge poles. Actually they were thought to explain everything. Now they are out of fashion but they still exist! In the crossed channel they manifest themselves to describe two-body reactions. For instance, the difference between the $\bar{p}p$ and pp total cross-sections is well described from E_{lab} = 10 GeV to E_{lab} = 2000 GeV by

$$\sigma_{\bar{p}p} - \sigma_{pp} = Const \times E^{-0.55}$$

corresponding to the exchange of the ϱ trajectory [1]. In the direct channel it is striking to see trajectories of mesons going up to $J = 6$ and of baryons up to $J = 17/2$. These trajectories are remarkably linear and approximately parallel, i.e. the angular momentum is a linear function of the square of the particle mass.

In the 60's this fact was taken for granted by those who applied Regge's ideas to particle physics [2], even though there was not the faintest justification for this! Regge's original work [3] was done with non-confining potentials, and, at very large energies, trajectories turned around in the complex J-plane to end up at some negative integer [4].

Now we know that hadrons have a composite structure, and that mesons are quark-antiquark pairs and baryons three-quark systems. The use of a potential interaction has met with considerable success in the description of mesons, especially those made of heavy quarks [5], but also to a certain extent those made of light quarks [6-8]. There is more and more support for the belief that the quark-antiquark potential is linear at a large distance, and, to the extent that the quark-antiquark system can be regarded as a relativistic string for large angular momentum, the slope of the Regge trajectory has been connected with the string tension. Here we shall do something slightly heretical and regard the potential as producing an instantaneous interaction between relativistic point-like quarks. In this we follow the point of view of Kang and Schnitzer [8] and more recently Basdevant and Boukraa [9].

Potential models have also been very successful in the description of baryons [10-12]. The most natural prescription is to take two-body forces with $V_{QQ} = 1/2\ V_{Q\bar{Q}}$. However, for large separations there are good reasons to believe that the potential energy between three quarks is proportional to the length of the string of minimum length connecting the three quarks [13, 14]. We shall in fact consider both cases.

Several authors have proposed models in which the baryons are made of a quark-diquark system [15]. Then in particular the parallelism of the meson and baryon trajectories becomes very natural. However, we still have to understand why it might be so.

Although the ground states and low-lying excited states of baryons have been well studied, either by perturbation around the harmonic oscillator potential [10], or by variational methods [12], or with 'exact' solutions using the hyperspherical formalism [11], the excited states with large angular momentum have only been touched upon and do not lend themselves so easily to numerical study. The main remark of the present paper is that, as for mesons-- where, as we shall see, it is completely clear--the leading Regge trajectory of a baryon (i.e. the sequence of ground-state wave functions and energies with increasing angular momentum J) can be obtained in the large J limit, by minimizing the classical energy of the system for given J. Quantum effects only play a role in preventing the collapse of a subsystem caused by short-range singularities of the potential. For a linear two-body potential and for a string, one finds that the configuration minimizing the energy is a quark-diquark system, and this holds for both relativistic and non-relativistic kinematics. With relativistic kinematics one proves that the trajectories tend to become linear, and, unavoidably, since the colour of a diquark system in a baryon has to be a $\bar{3}$, the potential energy is the same as in a meson and the trajectories are parallel.

All these results are asymptotic in J. However, unlike for other asymptotic theorems, one can check by numerical calculations whether precocious linearity appears. In fact these theoretical arguments give indications for the good choice of trial functions in a variational approach.

2. THE TWO-BODY CASE: NON-RELATIVISTIC KINEMATICS

If we take two non-relativistic particles with equal mass m, the reduced Hamiltonian for angular momentum J is

$$H = -\frac{1}{m}\frac{d^2}{dr^2} + \frac{1}{m}\frac{J(J+1)}{r^2} + V(r) \tag{1}$$

If we take into account the *operator* inequality

$$-\frac{d^2}{dr^2} > \frac{1}{4r^2} \quad , \tag{2}$$

that is

$$\int_0^\infty u(r)\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2}\right)u(r)\,dr > 0 \quad , $$

for any continuous u(r) vanishing at r = 0, we have the *operator* inequality

$$H \geqslant \frac{1}{m}\frac{\left(J+\frac{1}{2}\right)^2}{r^2} + V(r) \tag{3}$$

and therefore the ground-state energy $E_Q(J)$ satisfies

$$E_Q(J) \geqslant \inf_{0 < r < \infty}\left[\frac{1}{m}\frac{\left(J+\frac{1}{2}\right)^2}{r^2} + V(r)\right] \tag{4}$$

If we now consider the *classical* energy of the system

$$E_c = 2 \times \frac{p^2}{2m} + V(r_{12}) \tag{5}$$

if particles 1 and 2 had momenta $\vec{p}$ and $-\vec{p}$, the angular momentum is

$$\vec{J} = \vec{r}_{12} \times \vec{p} \tag{6}$$

and

$$|\vec{J}| = r_{12} \cdot p \cdot \sin\theta \tag{7}$$

If we hold $\vec{r}_{12}$ and $|\vec{J}|$ fixed, we minimize p by taking $\sin\theta = 1$, i.e. $\vec{p}$ and $\vec{r}_{12}$ orthogonal. Hence

$$E_c(J) \geqslant \frac{|J|^2}{m\,r^2} + V(r) \geqslant \inf_{0 < r < \infty}\left(\frac{|J|^2}{m\,r^2} + V(r)\right) \tag{8}$$

Except for the replacement of J by (J + 1/2) there is very little difference between formulae (4) and (8). We have

$$E_Q(J) \geqslant \operatorname*{Inf}_{\vec{p},\vec{r}} E_c(J) \tag{9}$$

The extra 1/2 in the inequality for the quantum energy is only really important for a J = 0 state in order to prevent the collapse of the system should the potential be singular at the origin. For three-body and four-body systems, analogous inequalities can be obtained and will be presented in Section 4.

We now claim that, for large J, $\inf_{p,r} [E_c(J)]$ gives the leading behaviour of the ground-state energy. For power potentials $V = \epsilon(v)r^v$, we can get an asymptotic expansion in inverse powers of J in which the first term is the minimum of classical energy. For m = 1/2, it begins as [16]:

$$E_{n,J} = \left(\frac{|v|}{2}\right)^{\frac{2}{v+2}} J^{\frac{2v}{v+2}}\left\{\frac{v+2}{v} + \frac{1}{J}\left[(2n+1)\sqrt{v+2}+1\right] + O\left(\frac{1}{J^2}\right)\right\} \tag{10}$$

where n is the number of nodes, so that the ground state corresponds to n = 0. We see that the correction to the first term has the relative magnitude 1/J.

For 0 < v < 2 it is also possible to obtain strict upper bounds for the quantum energies by using the inequality

$$r^\nu < \frac{(2-\nu)\left(\frac{\nu A}{2-\nu}\right)^{\nu/2}}{2A}\left(A + r^2\right), \quad A \geqslant 0, 0 < \nu < 2.$$

(11)

Then, using the monotonicity of the energies with respect to the potential, we can bound above the energy levels by the energy levels of an harmonic oscillator and then minimize with respect to A. We get, for $0 < \nu < 2$:

$$E_{n,J} \leqslant \left(\frac{\nu}{2}\right)^{\frac{2}{\nu+2}} \times \frac{\nu+2}{\nu}\left(2n + J + 3/2\right)^{\frac{2\nu}{\nu+2}}$$

(12)

For fixed n, the lower bound, given by the classical energy, and the upper bound differ by terms of relative magnitude $1/J$ for $J \to \infty$.

Unfortunately, we see (without surprise!) that if we take a linearly rising potential, we do not get a linearly rising trajectory:

$$E_{0,J} \sim const. \, J^{2/3},$$

that is,

$$J(t) \sim t^{3/4},$$

(13)

where t is the square of the energy. This is because we have used non-relativistic kinematics, whilst the energies, in the limit $J \to \infty$, go to infinity.

3. THE TWO-BODY CASE: RELATIVISTIC KINEMATICS

For simplicity we take only the extreme relativistic case, i.e. the Hamiltonian is, taking $c = 1$,

$$H = |\vec{P_1}| + |\vec{P_2}| + V(r_{12}) , \tag{14}$$

or, in the c.m. system,

$$H = 2p + V(r) \tag{15}$$

Provisionally we restrict ourselves to a purely linear potential,

$$V(r) = \lambda r \tag{16}$$

It will be noticed that unlike Kang and Schnitzer [8], who were the first to show that relativistic kinematics and linear confinement produce linear trajectories, we do not square the Hamiltonian; in this way we avoid the problems due to the absence of a strict energy minimum. Our Hamiltonian is the one used by Basdevant and Boukraa, for instance [9]. It has the defect of not being perfectly relativistic but, from a mathematical point of view, it is perfectly well defined. It has been shown by Herbst [17] that if V(r) is less singular than -2/πr at short distances it is completely satisfactory: lower bounded etc. By a funny accident of nature, the quark-antiquark potential including asymptotic-freedom effects behaves like -1/[r log (1/r)] at short distances and will never violate the condition of Herbst.

Here again the minimum of the classical energy for a given angular momentum will give a lower bound for the energy of the ground state. In fact we can give an explicit proof by generalizing an inequality of Herbst [17], who proves the operator inequality

$$|p| > \frac{2}{\pi} \frac{1}{|r|} \tag{17}$$

to be valid for any state. If we restrict ourselves to states of angular momentum J, we prove in Appendix A the following inequality:

$$\langle J|p|J \rangle > 2 \left[\frac{\Gamma(\frac{J}{2}+1)}{\Gamma(\frac{J}{2}+\frac{1}{2})} \right]^2 \langle J|\frac{1}{r}|J \rangle > (J+\frac{1}{2})\langle J|\frac{1}{r}|J \rangle \tag{18}$$

Therefore, the quantum energy of a system of two particles of momenta $\vec{p}$ and $-\vec{p}$ will satisfy the inequality

$$E_Q(J) \geqslant \mathrm{Inf}\left\{ 4\left[\frac{\Gamma(\frac{J}{2}+1)}{\Gamma(\frac{J}{2}+\frac{1}{2})}\right]^2 \frac{1}{r} + V(r)\right\} > \mathrm{Inf}\left\{\frac{2J+1}{r} + V(r)\right\} \tag{19}$$

while the minimum of the classical energy is

$$E_C(J) = \mathrm{Inf}\left\{\frac{2J}{r} + V(r)\right\} . \tag{20}$$

So if we take the linear potential (16) we get

$$E_C(J) = 2\sqrt{2}\,\sqrt{J}\,\sqrt{a} , \tag{21}$$

and hence if we believe that this gives the leading behaviour of the quantum ground-state energy for large J, then

$$J(t) \simeq \frac{1}{8a}t + \cdots . \tag{22}$$

We can prove that Eq. (21) indeed gives the leading behaviour by bounding above the Hamiltonian (15), by using the operator inequalities

$$p < \frac{1}{2}\left(\frac{p^2}{X} + X\right) \tag{23a}$$

$$r < \frac{1}{2}\left(\frac{r^2}{Y} + Y\right) . \tag{23b}$$

Hence

$$H < \frac{p^2}{X} + X + \frac{a}{2}\left(\frac{r^2}{Y} + Y\right) , \tag{24}$$

and what remains to be done is to minimize the energy levels of the harmonic oscillator Hamiltonian, appearing in the right-hand side, with respect to X and Y. In this way we get

$$2\sqrt{2}\sqrt{\lambda}\left(J+\tfrac{1}{2}\right)^{1/2} < E(n,J) < 2\sqrt{2}\sqrt{\lambda}\left(2n+J+\tfrac{3}{2}\right)^{1/2}, \quad (25)$$

which confirms that except for corrections of the order of unity the leading
Regge trajectory is linear and is given by the minimum of the classical energy.

We can in fact get a more precise but less rigorous estimate of $J(t)$, by
estimating the first-order correction to the minimum of formula (24) in the same
way as was done in the non-relativistic case. We have, if we use 'reduced' wave
functions,

$$H = 2\sqrt{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}} + \lambda r .$$

Neglecting the lack of commutativity of the two operators under the square root,
we write

$$\sqrt{\frac{J(J+1)}{r^2}-\frac{d^2}{dr^2}} \sim \frac{\sqrt{J(J+1)}}{r} - \frac{1}{2}\frac{r_{min}}{\sqrt{J(J+1)}}\frac{d^2}{dr^2}$$

where r_{min} minimizes $2\sqrt{[J(J+1)]}/r + \lambda r$, and making the harmonic oscillator
approximation, we get for $\lambda = 1$,

$$H \sim 2\left[J(J+1)\right]^{1/4}+\frac{(r-r_{min})^2}{\left[J(J+1)\right]^{1/4}} - \frac{1}{2}\frac{1}{\left[J(J+1)\right]^{1/4}}\frac{d^2}{dr^2}$$

and

$$E(n,J) \simeq 2\left[J(J+1)\right]^{1/4} + \frac{(n+\tfrac{1}{2})\sqrt{2}}{\left[J(J+1)\right]^{1/4}} \quad (26)$$

for $J = 1, 2, 3$, and $n = 0$. Equation (26) agrees remarkably well with the exact
calculation of Basdevant and Boukraa [9]:

J	1	2	3
Exact	2.9872	3.5912	4.1084
Approx.	2.973	3.582	4.102

and so

$$J(t) \simeq 4\left(J + \tfrac{1}{2}\right) + 4\sqrt{2}\, n + 2\sqrt{2}\,. \qquad (27)$$

This means that daughter trajectories are asymptotically parallel to the leading trajectory. Their spacing, however, disagrees with what one would expect from naïve WKB, which should hold for n → ∞, J fixed, and for which the only relevant quantity is 2n + J [18].

4. THE THREE-BODY CASE: NON-RELATIVISTIC KINEMATICS

Any sane person is frightened (but not always discouraged!) by the three-body problem. The Faddeev equations or the infinite set of coupled equations arising in the hyperspherical expansion are difficult to handle. However, if we limit our ambition to studying the asymptotic behaviour of the leading Regge trajectory, things remain simple.

Our experience of the two-body case suggests that we should again minimize the classical energy of the three-body system for given J to get the leading Regge trajectory.

For the interaction between the three quarks constituting a baryon, we have taken two extreme cases:

i) A sum of two-body interactions adjusted in such a way that if two quarks are close to one another, the potential between the quark-diquark system is identical to the quark-antiquark potential,

$$V = \frac{1}{2}\left[V(r_{12}) + V(r_{23}) + V(r_{13}) \right], \qquad (28)$$

where V is the $q\bar{q}$ two-body potential, and in the special case (16)

$$V = \frac{9}{2}\left(r_{12} + r_{23} + r_{31} \right). \qquad (29)$$

There is no rigorous justification for this choice. The rule (28) holds for one-gluon exchange contributions. The other remark is that a diquark system has colour $\bar{3}$ and therefore looks like an antiquark. Finally, let us indicate that experience has shown that the application of this rule to the calculation of ground-state energies of baryons has met with remarkable success [19].

ii) The potential energy could be proportional to the minimum length of a Y-shaped string connecting the three quarks. Again the strength is adjusted in such a way that when two quarks coincide it agrees with the quark-antiquark potential,

$$V = \lambda \; \inf_{P} \left(r_{1P} + r_{2P} + r_{3P} \right) . \tag{30}$$

There are good reasons for believing that Eq. (30) holds for large separations between the quarks [13, 14]. In fact the ratio between Eqs. (29) and (30) never differs from unity by more than 15%.

We shall start with the *non-relativistic* case for which an explicit lower bound of the expectation value of the kinetic energy of a system of three or n particles can be obtained. As expected, this lower bound is higher than the minimum of the classical kinetic energy for a given angular momentum. The strategy consists in generalizing (3) by considering, first, two particles, then adding a third one, and so on, and optimizing an intermediate angular momentum to get the least possible kinetic energy for a given angular momentum and given distances between the particles. In Appendix B we derive the *operator* inequality, for n particles of mass m:

$$T > \frac{n}{2m} \frac{\left(J + \frac{n-1}{2} \right)^2}{\sum_{i>j} r_{ij}^2} \tag{31}$$

However, this inequality will not prevent the collapse of the system if the interactions are singular, at short distances, whilst inequality (3) prevents a two-body system from collapsing if the potential is less singular than $-1/4r^2$ at

short distances. In fact, formula (31) is not saturated when one of the distances between the particles is very small, and we derive for instance in Appendix B the inequality:

$$T > \frac{1}{4nm} \left(\sum_{i>j} \frac{1}{(r_{ij})^2} \right) + \frac{n}{2m} \frac{\left(J + \frac{n-2}{2}\right)^2}{\sum_{i>j} (r_{ij})^2} \,. \tag{32}$$

Inequality (32) shows that systems with two-body potentials, such that $\lim r^2 V(r) > 0$, cannot collapse.

The classical inequality corresponding to (31) is, naturally,

$$T > \frac{n}{2m} \frac{J^2}{\sum r_{ij}^2} \,. \tag{33}$$

This inequality is optimal in the sense that for any given space configuration of n points one can find momenta such that formula (33) becomes an equality. This is achieved by taking the successive relative angular momenta parallel and optimizing their lengths.

Hence again,

$$E_Q(J) \geqslant E_C(J) = Inf\left(T + V\right) \,. \tag{34}$$

For the case of the pairwise interactions (29), we can use the inequality (31) directly to get a lower bound and in fact an estimate of the ground-state energy. Indeed, we have

$$r_{12}^2 + r_{23}^2 + r_{13}^2 \leqslant \frac{1}{2} \left(r_{12} + r_{23} + r_{13} \right)^2 \tag{35}$$

where the equality sign holds *if and only if* two of the points coincide. This means that *the minimum configuration is a quark-diquark system.* (Notice that by a generalisation of (35) we would reach the same conclusion for $V(r) = r^v$, with $0 < v < 2$.) The classical energy is then

$$E_C(J) = \left(\frac{3}{4}\right)^{4/3} \frac{(\lambda J)^{2/3}}{m^{1/3}} \tag{36}$$

For the case of the string of minimum length we shall use a *geometrical* method which will also be used in the case of relativistic kinematics. Naturally we wish to minimize the classical energy, but we do not hold the distances between the particles fixed in the first step.

If $\vec{J}$ is the total angular momentum

$$\vec{J} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \vec{r}_3 \times \vec{p}_3 \,, \, (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0),$$

we notice that if we project the points 1, 2, and 3, and the momenta $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ on a plane perpendicular to $\vec{J}$, then $\vec{J}$ is unchanged, the kinetic energy is reduced, and, for *monotonous* two-body potentials as well as for the string interaction (30) the potential energy is reduced. Hence we can restrict ourselves to a motion of the three particles in a *plane*. Now let us call *support of 1* the straight line going through 1 with direction $\vec{p}_1$, etc. If we hold fixed $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, and the supports of 1, 2, and 3, then $\vec{J}$ remains fixed and the kinetic energy remains fixed. We can then minimize the potential energy. In the case of the string (Fig. 1), we have to minimize with respect to 1, 2, and 3, and with respect to P. For a given P we must take $\overrightarrow{P1}$, $\overrightarrow{P2}$, and $\overrightarrow{P3}$ perpendicular to $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$, respectively. Then the interaction is proportional to the sum of the distances of P to the three supports. This quantity is a linear function of the coordinates of P in any system, as long as P is *inside* the triangle delimited by the supports and reaches its minimum when P coincides with one of the summits. Then two of the points 1, 2, 3 coincide, and we therefore have a quark-diquark configuration. Then, necessarily, the energy minima coincide with Eq. (36).

The same *geometrical* argument could be applied to the pair interactions. Holding the supports fixed, we have to minimize the sum of the distances. Holding 1 and 2 fixed, we find that 3 has to be such that the inner bissector of 31 and 32 is orthogonal to the support of 1, and so on. If the triangle made by

the supports has all its angles less than 90° (Fig. 2a), this is possible in one and only one way: 1 has to lie on the perpendicular going through the opposite summit of the triangle, and so on. In fact 1, 2, and 3 are lying on the famous nine-point Euler circle associated with the triangle [20]. If, on the other hand, the triangle has one angle larger than 90°, the configuration minimizing V is the one where two of the points are at one summit of the triangle, and the third one is on the perpendicular to the opposite side (Fig. 2b). We are then back to a quark-diquark configuration.

In the case where all angles are less than 90°, we notice that since 12 and 13 have a bisector perpendicular to $\vec{p}_1$, the force exerted on 1 is perpendicular to $\vec{p}_1$ and hence $d|p_1|/dt = 0$, $d|p_2|/dt = 0$, $d/dt = 0$. This should be true at all times, otherwise one could minimize further. Therefore $|p_1|$, $|p_2|$, and $|p_3|$ are constant, and are proportional to the lengths of the side of the triangle; therefore the triangle has a *fixed shape*. But the energy is conserved, and since the kinetic energy is constant, $r_{12} + r_{23} + r_{31} = $ const., the triangle has sides with *fixed lengths*. The whole motion is just a rotation around the common intersection H of the heights of the triangle. Notice that in all this part of the argument we do not distinguish between relativistic and non-relativistic kinematics. In the case of non-relativistic kinematics *with equal masses* we have, writing Newton's law,

$$m \, \omega^2 \, H3 = \lambda \cos \gamma$$

where

$$\gamma = (31, 3H) \, ,$$

and ω is the angular velocity

$$m \, \omega^2 \, H1 = \lambda \cos \alpha$$

where

$$\alpha = (13, 1H) \, ,$$

but

$$H1 \, \sin \alpha = H3 \, \sin \gamma \, ,$$

and hence

$$\sin 2\alpha = \sin 2\gamma \, ,$$

and the triangle is equilateral.

Then it is easy to do the final minimization:

$$E = \inf \left(\frac{3 p^2}{2m} + \frac{\lambda}{2} 3 \sqrt{3} R \right)$$

with

$$J = 3 p R$$

and hence

$$E = \left(\frac{3}{2} \right)^{5/3} \frac{(\lambda J)^{2/3}}{m^{1/3}} \tag{37}$$

This is larger than what one gets for the quark-diquark configuration given by Eq. (36). So it is not the real minimum! It is, in fact, a saddle point.

What remains to be done is to prove that the quantum energy, for large J, is close to the minimum of the classical energy. This is done quite easily by bounding above r_{ij} by $1/2\{[(r_{ij})^2/A_{ij}] + A_{ij}\}$, reducing the problem to one of harmonic oscillators. We shall not give the explicit calculations since the really interesting case is the relativistic one.

5. THE THREE-BODY CASE: RELATIVISTIC KINEMATICS

Here we shall admit that the minimum of the classical energy indeed gives a lower bound of the ground-state energy, for a given J. To some it may look obviously true because of the Golden-Symanzik inequality [21]. However, the

restriction to given J is non-trivial, but anyway the results previously presented in the two-body relativistic and non-relativistic cases and the three-body non-relativistic case give us complete confidence in this fact.

The geometrical method for minimizing the classical energy applies, with very little change from the non-relativistic case: as long as $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ are held fixed, there is no difference in the following cases:

i) In the case of the string (30) we can prove that the potential energy is minimized when two of the quarks coincide. Then, if $\vec{p}$ is the momentum of particle 3, and $-\vec{p}/2$ and $-\vec{p}/2$ the momenta of particles 2 and 1, we have

$$E = 2p + \lambda\, r_{13}$$

$$J = p\, r_{13} \tag{38}$$

Hence

$$E_< = 2\sqrt{2}\,\sqrt{\lambda}\,\sqrt{J}\ . \tag{39}$$

This coincides with Eq. (21), and hence the baryon trajectory is parallel to the meson trajectory (or at least has the same asymptotic slope!).

ii) In the case of the pair interaction (29) the proof that the triangle (if the angles are less than 90°) made by the supports has sides of fixed length still holds. Then, again, the motion is a rotation, but 1, 2, and 3 being extremely relativistic, they must have the velocity of light. Hence H1 = H2 = H3 = R and the triangle is equilateral. Then we have

$$E = 3p + \frac{\lambda}{2}\, 3\sqrt{3}\, R$$

$$J = 3p\, R\ ,$$

and minimizing we get

$$E_< = 3^{3/4}\,\sqrt{2}\,\sqrt{\lambda}\,\sqrt{J}\ . \tag{40}$$

This is larger than the quark-diquark energy, which is again given by Eq. (39), because

$$3^{3/4} > 2 \ .$$

Therefore the true minimum is again given by the quark-diquark, and the trajectory is the same as in the previous case.

Buchmüller has noticed that there is another possible candidate for the minimum energy (which coincides, in fact, with the old proposal of Szegö and Preparata [22]). Quark 1 has momentum $\vec{p}$, quark 2 momentum 0, quark 3 momentum $-\vec{p}$. However, the mass of the second quark can no longer be neglected and the configuration has a higher energy.

To prove that (39) is a good approximation of the energy for large J, we use the technique of bounding above by harmonic oscillators which has been described previously. In the pair interaction case we have

$$H < \frac{1}{2}\left(\frac{p_1^2}{P/2} + P/2\right) + \frac{1}{2}\left(\frac{p_2^2}{P/2} + P/2\right) + \frac{1}{2}\left(\frac{p_3^2}{P} + P\right)$$
$$+ \frac{\lambda}{2}\left(\frac{r_{12}^2}{R_0} + R_0\right) + \frac{\lambda}{2}\left(\frac{r_{13}^2}{R_1} + R_1 + \frac{r_{23}^2}{R_1} + R_1\right) \ , \tag{41}$$

Now we take the 12 pair to have zero angular momentum, and then

$$H < \left[-\frac{1}{2\,P/4}\frac{d^2}{dr^2} + \frac{\lambda}{2}r^2\left(\frac{1}{2R_0} + \frac{1}{2R_1}\right)\right]$$
$$+ \left[-\frac{1}{2\,P/2}\frac{d^2}{dR^2} + \frac{1}{2\,P/2}\frac{J(J+1)}{R^2} + \lambda\frac{R^2}{R_1}\right] + P \tag{42}$$

where

$$\begin{cases} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{R} = \vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2} \ . \end{cases}$$

Hence we get

$$E(J) < \frac{3}{2} \sqrt{\frac{\lambda\left(\frac{1}{2R_0} + \frac{1}{2R_1}\right)}{P/4}} + \left(J + \frac{3}{2}\right)\sqrt{\frac{\frac{\lambda}{R_1}}{P/2}}$$

$$+ \frac{\lambda R_0}{4} + \lambda R_1 + P \quad . \tag{43}$$

Taking

$$R_0 \text{ fixed}, \quad P = \sqrt{\frac{\lambda(J + \frac{3}{2})}{2}} \quad R_1 = \sqrt{\frac{2(J + \frac{3}{2})}{\lambda}}$$

we get

$$E(J) < \frac{C_1}{\sqrt{J}} + C_2 + 2\sqrt{2}\sqrt{\lambda\left(J + \frac{3}{2}\right)} \tag{44}$$

which proves that Eq. (36) is indeed a good estimate.

A similar upper bound can easily be obtained in the case of the string.

6. CONCLUDING REMARKS

We have shown in this pseudo-relativistic model, and with two types of

plausible three-quark interactions, that the meson and baryon trajectories are

asymptotically linear and parallel because, at least for large J, the favoured

configuration of the baryon has been proved to be a quark-diquark system. It has

long been understood that the quark-diquark configuration was needed in order to

account for the parallelism, but our point is that dynamics favours this confi-

guration. Notice, however, that for a small angular momentum there is not

an enormous difference in energy with the triangle configuration, and this means

that approximation methods might lead to the wrong answer. The weakness of our

result is that it is only asymptotic. Notice, however, that the method of

arriving at the proof gives rise to ideas about the kind of trial wave function

to use in a realistic variational calculation.

One objection might be that we have disregarded the short-range part of the quark-quark interaction, which is singular. Strickly speaking the minimum of the classical energy of a three-body system will be $-\infty$ if $V(r_{12}) \to -\infty$ for $r \to 0$, even for large total angular momentum. However, we know that there are obvious quantum corrections. For instance, in the non-relativistic case, we can use inequality (32). If we take, for example,

$$V\left(r_{ij}\right) = -\frac{\mu}{r_{ij}} + \frac{\lambda}{2} r_{ij} \quad ,$$

we get

$$E > 3 \; \inf\left(\frac{1}{12m\,r^2} - \frac{\mu}{r}\right)$$
$$+ \inf\left[\frac{3}{2m} \frac{(J+1/2)^2}{\sum(r_{ij})^2} + \frac{\lambda}{2} \sum r_{ij}\right] , \qquad (45)$$

so that the asymptotic behaviour for large J is unchanged.

In the three-body relativistic case we could certainly do the same if we had the analogue of the two-body inequality (18), which shows that there is no problem if $\lim_{r \to 0} (\pi/2) \; r \; V(r) > 0$.

Finally, one might object that the model we propose for the mesons does not exactly agree with the slope produced by the two-body string. However, let us remember that our main purpose was a comparison of mesons and baryons. Our hope is that our conclusion will remain valid in a really relativistic model.

Acknowledgements

APPENDIX A

We give proof of the inequality

$$\langle J | \not{p} | J \rangle \geq 2 \left[\frac{\Gamma\left(\frac{J}{2}+1\right)}{\Gamma\left(\frac{J}{2}+\frac{1}{2}\right)} \right]^2 \langle J | \frac{1}{r} | J \rangle . \tag{A1}$$

We define Fourier transforms in the symmetric way:

$$\tilde{f}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{x}) e^{i\vec{p}\cdot\vec{x}} d^3x \tag{A2}$$

$$f(\vec{x}) = \frac{1}{(2\pi)^3} \int \tilde{f}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} d^3p$$

In a state with a wave function $\Psi(\vec{x})$ we have the expectation value of $1/r$, ($|\vec{x}| = r$):

$$\left\langle \frac{1}{r} \right\rangle = \int \Psi^*(\vec{x}) \frac{1}{|\vec{x}|} \Psi(\vec{x}) \, d^3x \tag{A3}$$

with

$$\int |\Psi(x)|^2 d^3x = 1 .$$

In Fourier space we get

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{2\pi^2} \int d^3p \, d^3p' \, \frac{\tilde{\Psi}^*(\vec{p})\tilde{\Psi}(\vec{p}')}{|\vec{p}-\vec{p}'|^2} \tag{A4}$$

We now restrict ourselves to a state Ψ with angular momentum J. Then

$$\tilde{\Psi}(\vec{p}) = \tilde{\psi}(p) Y_J(\hat{p}) ,$$

where Y_J is a spherical harmonic of angular momentum J, with

$$\int |\hat{\psi}(p)|^2 p^2 dp = 1$$

and

$$\int |Y_J(\hat{p})|^2 d\Omega_p = 1 \ .$$

Noticing that

$$\frac{1}{|\vec{p} - \vec{p}'|^2} = \frac{1}{2pp'} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell \left(\cos \hat{p} \cdot \hat{p}' \right) Q_\ell \left(\frac{p^2 + p'^2}{2pp'} \right),$$

(A5)

we get

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{\pi} \int p \, dp \, p' dp' \, Q_J \left(\frac{p^2 + p'^2}{2pp'} \right) \hat{\psi}(p) \hat{\psi}(p') .$$

(A6)

Now using the inequality

$$p^2 \hat{\psi}(p) \, p'^2 \hat{\psi}(p') \leq \frac{1}{2} \left[p^4 |\hat{\psi}(p)|^2 + p'^4 |\hat{\psi}(p')|^2 \right],$$

we get

$$\left\langle \frac{1}{r} \right\rangle \leq \frac{1}{\pi} \int_0^{\infty} |\hat{\psi}(p)|^2 p^3 dp \int_0^{\infty} \frac{dx}{x} Q_J \left(\frac{1+x^2}{2x} \right).$$

The latter integral can be rewritten as

$$2 \int_1^{\infty} \frac{dz}{\sqrt{z^2-1}} Q_J(z) \ ,$$

which can be found in Gradshteyn and Ryzhik [23] in terms of a hypergeometric function whose last argument if fortunately equal to unity. Noticing that the

first integral is precisely the expectation value of p, we get the desired result (A1). To prove that

$$\phi(J) = \frac{2}{J + \frac{1}{2}} \left[\frac{\Gamma\left(\frac{J}{2}+1\right)}{\Gamma\left(\frac{J}{2}+\frac{1}{2}\right)} \right]^2 > 1 ,$$

it suffices to notice that

$$\phi(J) \to 1 \quad \text{for} \quad J \to \infty ,$$

and that

$$\phi(J+2) / \phi(J) < 1 .$$

APPENDIX B

Inequalities on the three- and n-body

non-relativistic quantum kinetic energy

We have seen that for two bodies with equal mass we have the operator inequality

$$T > \frac{1}{m} \frac{\left(J + \frac{1}{2}\right)^2}{r_{12}^2} \tag{B1}$$

where r_{12} is the distance between 1 and 2, and for two bodies with unequal masses m_1 and m_2,

$$T > \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \frac{\left(J + \frac{1}{2}\right)^2}{r_{12}^2} \tag{B2}$$

If we have three bodies with equal mass we group first 1 and 2 with angular momentum j_{12}, and then the system 12 with 3, with angular momentum j_3. We have the constraint that the total angular momentum of the system should be J, and hence

$$\left| j_{12} - j_3 \right| < J < j_{12} + j_3 . \tag{B3}$$

If

$$\vec{r} = \vec{r_2} - \vec{r_1}$$

and

$$\vec{R} = \vec{r_3} - \frac{\vec{r_1} + \vec{r_2}}{3} \tag{B4}$$

the kinetic energy operator reads:

$$T = \left[-\frac{1}{m} \frac{d^2}{dr^2} + \frac{1}{m} \frac{j_{12}(j_{12}+1)}{r^2} \right]$$
$$+ \left[-\left(\frac{1}{m} + \frac{1}{2m}\right)\frac{d^2}{dR^2} + \frac{3}{2m} \frac{j_3(j_3+1)}{R^2} \right]$$

and using inequality (B2) we get the operator inequality

$$T > \frac{1}{m} \frac{\left(j_{12} + \frac{1}{2}\right)^2}{r^2} + \frac{3}{2m} \frac{\left(j_3 + \frac{1}{2}\right)^2}{R^2} \tag{B5}$$

We have to minimize with respect to j_{12} and j_3, holding J fixed. Clearly we must take

$$J = j_{12} + j_3$$

Then, if we forget the extra constraint that j_{12} and j_3 can only be non-negative integers, we can minimize with respect to j_{12} and find

$$j_{12} = -\frac{1}{2} + \frac{\frac{3}{2R^2}}{\frac{1}{r^2} + \frac{3}{2R^2}} \quad , \tag{B6}$$

and

$$T > \frac{(J+1)^2}{m} \frac{1}{r^2 + \frac{2R^2}{3}} \quad)$$

and using the definition of r and R given in (B4),

$$T > \frac{3}{2} \frac{(J+1)^2}{m\left(r_{12}^2 + r_{23}^2 + r_{13}^2\right)} \quad . \tag{B7}$$

Inequality (B7) can easily be generalized by adding an extra particle and splitting the kinetic energy into the three-body internal kinetic energy and the kinetic energy of (123) + 4 systems. Then again the angular momenta of (123) and 4 should be aligned to minimize the kinetic energy. The general result is

$$T > \frac{n}{2m} \frac{\left(J + \frac{n-1}{2}\right)^2}{\sum_{i>j} \left(r_{ij}\right)^2} \quad , \tag{B8}$$

However, these inequalities are not saturated because

i) the successive relative angular momenta are non-negative integers;

ii) the minimization is done for given r_{ij}'s, i.e. for a wave packet with some dimensions infinitesimally small.

Inequality (B6) is insufficient to prevent the collapse of two particles if the potential between these two particles is singular at a short distance. Something very crude can be done to remedy this. From (B5) we deduce

$$T > \frac{1}{4 m R^2} + \left[\frac{1}{m} \frac{(J_{12})^2}{r^2} + \frac{3}{2m} \frac{(J_{3} + \frac{1}{2})^2}{R^2} \right]$$

and applying the usual treatment to the parenthesis we get

$$T > \frac{1}{4 m R^2} + \frac{3}{2m} \frac{(J + \frac{1}{2})^2}{r_{12}^2 + r_{23}^2 + r_{13}^2}$$

and, symmetrizing

$$T > \frac{1}{12 m} \left(\frac{1}{r_{12}^2} + \frac{1}{r_{23}^2} + \frac{1}{r_{13}^2} \right)$$

$$+ \frac{3}{2m} \frac{(J + \frac{1}{2})^2}{r_{12}^2 + r_{23}^2 + r_{13}^2} \; . \tag{B9}$$

The generalization (32) to n bodies is obvious.

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Figure captions:

Fig. 1 : Minimization of the potential energy in the case of a string inter-
action.

Fig. 2 : Minimization of the potential energy in the case of a sum of 2 body
linear potentials.

a) The momenta form a triangle with all angles smaller than
90 degrees.

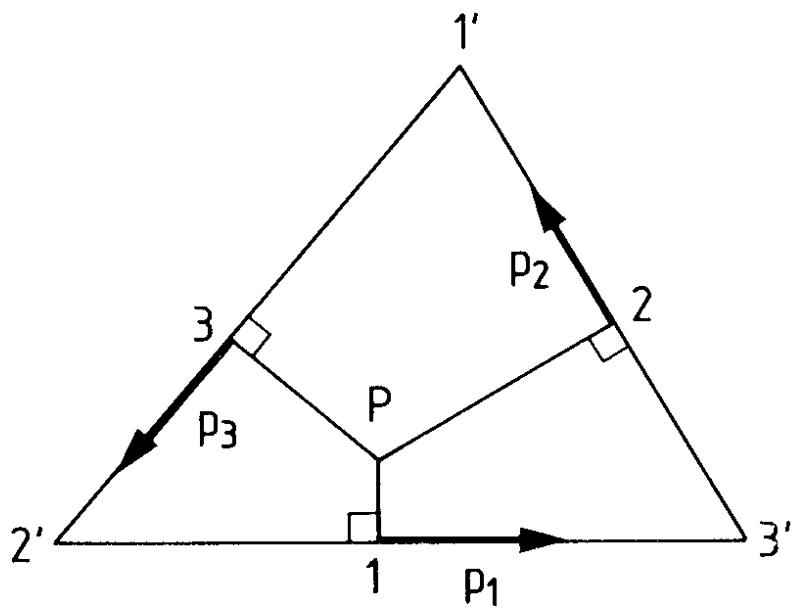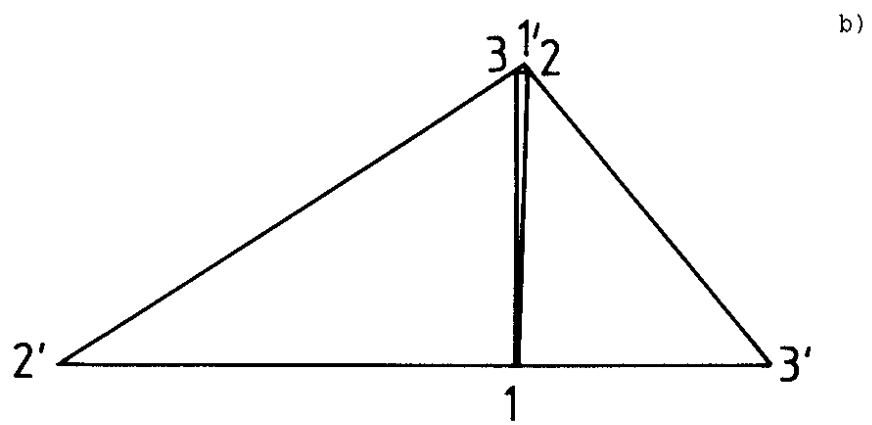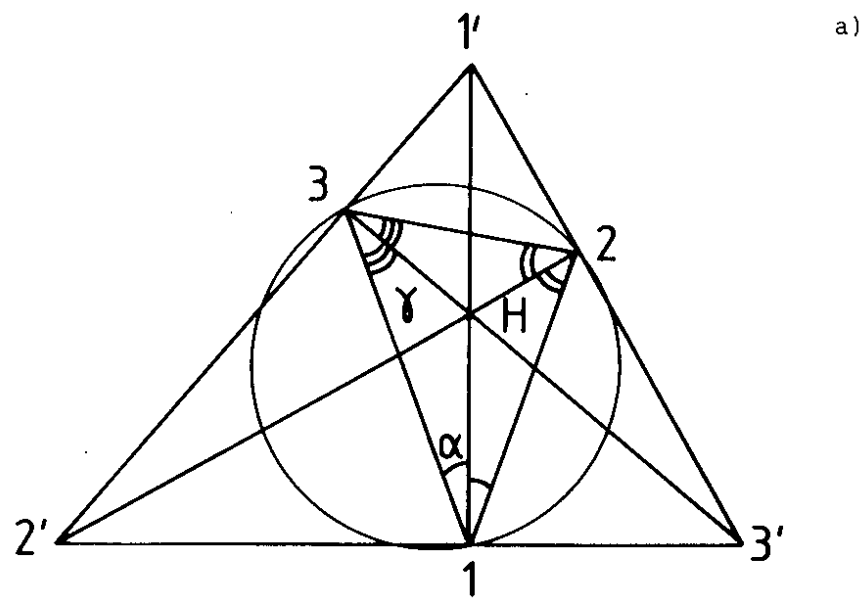b) One of the angles is larger than 90 degrees.

Fig. 1

Fig. 2