Regime Shifts in a Dynamic Term Structure Model of U.S. Treasury Bond Yields

Qiang Dai, Kenneth J. Singleton, and Wei Yang [1]

This draft: April, 2006

Abstract

This paper develops and empirically implements an arbitrage-free, dynamic term structure model with "priced" factor and regime-shift risks. The risk factors are assumed to follow a discrete-time Gaussian process, and regime shifts are governed by a discrete-time Markov process with state-dependent transition probabilities. This model gives closed-form solutions for zero-coupon bond prices, an analytic representation of the likelihood function for bond yields, and a natural decomposition of expected excess returns to components corresponding to regime-shift and factor risks. Using monthly data on U.S. Treasury zero-coupon bond yields, we show a critical role of priced, state-dependent regime-shift risks in capturing the time variations in expected excess returns, and document notable differences in the behaviors of the factor risk component of the expected returns across high and low volatility regimes. Additionally, the state dependence of the regime-switching probabilities is shown to capture an interesting asymmetry in the cyclical behavior of interest rates. The shapes of the term structure of volatility of bond yield changes are also very different across regimes, with the well-known hump being largely a low-volatility regime phenomenon.

1 Introduction

This paper develops and empirically implements an arbitrage-free, dynamic term structure model ($DTSM$) with "priced" factor and regime-shift risks. The risk factors are assumed to follow a discrete-time Gaussian process, and regime shifts are governed by a discrete-time Markov process with state-dependent transition probabilities. Agents are assumed to know both the current state of the economy and the regime they are currently in. This leads to regime-dependent risk-neutral pricing and an equilibrium term structure that reflects the risks of both changes in the state and shifts in regimes.

There is an extensive empirical literature on bond yields (particularly short-term rates) that suggests that "switching-regime" models describe the historical interest rate data better than single-regime models (see, for example, Cecchetti, Lam, and Mark [1993], Gray [1996], Garcia and Perron [1996], and Ang and Bekaert [2002a]).[1] In spite of this evidence, largely for reasons of tractability, most of the empirical literature on $DTSM$s has continued to focus on single-regime models (see Dai and Singleton [2003] for a survey). Recently Naik and Lee [1997], Landen [2000], and Dai and Singleton [2003] have proposed continuous-time regime-switching $DTSM$s that yield closed-form solutions for zero-coupon bond prices, but multi-factor versions of their models have yet to be implemented empirically.

We develop a discrete-time multi-factor $DTSM$ with the following features: (i) within each regime the short-term interest rate follows a three-factor Gaussian model with state-dependent market prices of factor risks;[2] (ii) there are two regimes characterized by low (L) and high (H) volatility, and the transitions between these regimes under the historical measure $\mathbb{P}$ are governed by a Markov process with regime-shift probabilities $\pi_t^{\mathbb{P}ij}$ $(i, j = H, L)$ that depend on the risk factors underlying changes in the shape of the yield curve;[3] and (iii) regime-shift risks are priced. This model yields exact closed-form solutions for bond prices, and an analytic representation of the likelihood function that we use in our empirical analysis of U.S. Treasury zero-coupon bond yields. Expected excess returns are naturally decomposed into two components, which are associated with regime-shift and factor risks, respectively.

Our findings suggest that the omission of regime-shift risk leads single-regime models to understate the fluctuations in excess returns during the periods of transitions between regimes, and to overstate the volatility of factor risk premiums and excess returns during

---

[1]Ang and Bekaert [2002b] suggest that the mixing of regime-dependent state processes inherent in our $DTSM$ can potentially replicate the nonlinear conditional means of short-term yields documented by Ait-Sahalia [1996] and Stanton [1997]. While the non-parametric evidence for non-linearity in the short-rate process is somewhat controversial (see, e.g., Chapman and Pearson [2000]), the findings of Ang and Bekaert for a Gaussian autoregressive model of the short rate suggest that our regime-dependent state process introduces the flexibility to match such nonlinearity if it is present.

[2]More precisely, within each regime, the short rate r_t follows an $A_0(3)$ model (in the notation of Dai and Singleton [2000]): r_t is an affine function of a vector Y_t of three risk factors, Y follows a Gaussian vector-autoregression with constant conditional variances, and the market prices of factor risks depend on Y_t as in Duffee [2002] and Dai and Singleton [2002].

[3]The presence of regime switching implies that, under $\mathbb{P}$, the conditional volatilities of both Y_t and bond yields may be state-dependent; that is, conditional volatilities are stochastic.

less turbulent times.[4] This is illustrated in Figure 1 which displays the one-month ahead expected excess returns on two- and ten-year zero-coupon Treasury bonds implied by a single-regime three-factor Gaussian model (model $A_0(3)$) and our two-regime counterpart (model $A_0^{RS}(3)$). Excess returns in model $A_0^{RS}(3)$ exhibit large spikes (up and down) during the mid-1970's, the period of the "Fed experiment" in the early 1980's, and again during the mid-1980's. These large moves in excess returns are largely missed by the single-regime $A_0(3)$ model. Central to capturing these swings within model $A_0^{RS}(3)$ is priced regime-shift risk with state-dependent market prices of this risk.

From Figure 1 it is also evident that the excess returns implied by model $A_0(3)$ fluctuate much more than their counterparts in model $A_0^{RS}(3)$ during the less turbulent 1990's. We document subsequently that the relative calmness of the excess returns in model $A_0^{RS}(3)$ is due to its accommodation of very different behaviors of factor risk premiums (and the underlying market prices of factor risks) in regimes H and L, a difference that (by construction) is absent from single-regime models. Together these observations suggest that single-regime models fail to capture key dimensions of expected excess returns in U.S. Treasury markets.[5]
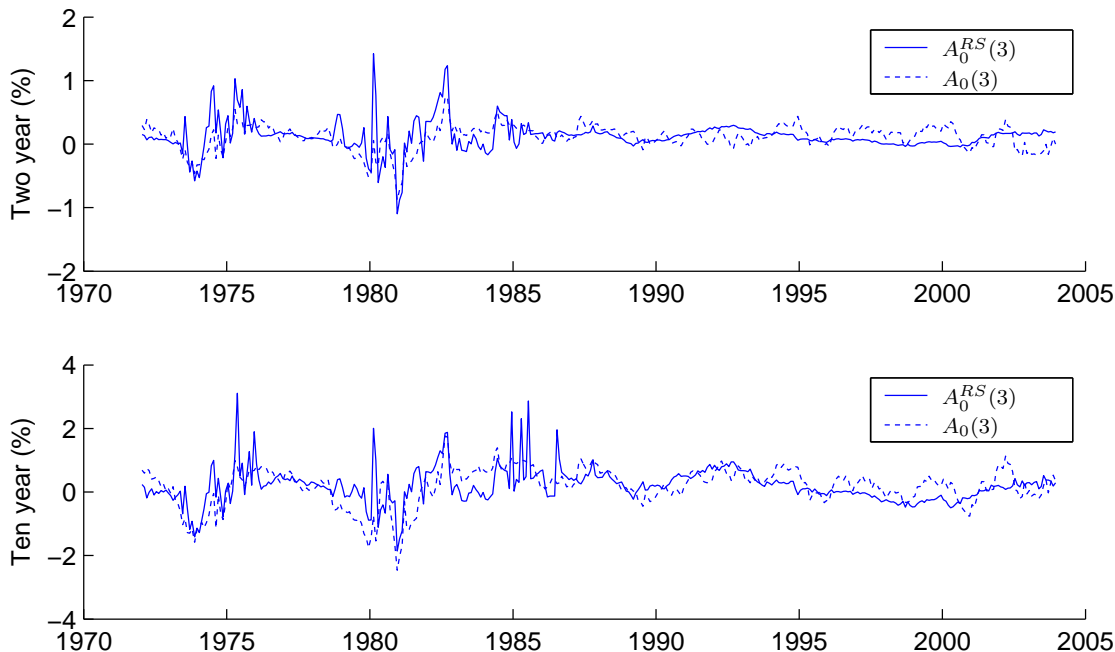

Figure 1: One-month ahead expected excess returns on two- and ten-year zero-coupon Treasury bonds in models $A_0(3)$ and $A_0^{RS}(3)$.

---

[4]The analyses by Pan [2002] and Liu, Longstaff, and Pan [2002] are, in different contexts, premised on a similar point.

[5]Recall that Duffee [2002] and Dai and Singleton [2002] show that sufficiently persistent and variable factor risk premiums in single-regime affine *DTSM*s resolve the expectations puzzles summarized in Campbell and Shiller [1991]. The role for regime shifts in pricing documented here suggests that the tests of the expectations theory may yield different results within the H or L regimes.

Where the state dependence of the market price of regime-shift risk (equivalently, state dependence of $\pi^{\mathbb{P}}$) appears to matter is in modeling the persistence of regimes. A standard result in the empirical literature on regime-switching models of interest rates with constant $\pi^{\mathbb{P}}$ (e.g., Ang and Bekaert [2002b] and Bansal and Zhou [2002]) is that $\pi^{\mathbb{P}HH} \gg \pi^{\mathbb{P}HL}$ and $\pi^{\mathbb{P}LL} \gg \pi^{\mathbb{P}LH}$; i.e., both regimes are highly persistent. With state-dependent $\pi_t^{\mathbb{P}}$, we replicate the finding that $E[\pi_t^{\mathbb{P}LL}] \gg E[\pi_t^{\mathbb{P}LH}]$. On the other hand, though we still find that $E[\pi_t^{\mathbb{P}HH}]$ is larger than $E[\pi_t^{\mathbb{P}HL}]$, the difference is not nearly as large as in models with constant $\pi^{\mathbb{P}}$. In other words, in the presence of priced, state-dependent regime-shift risk, high volatility regimes are less persistent than low volatility regimes. Importantly, this asymmetry is equally present in a descriptive model of Treasury yields, suggesting that models (descriptive or pricing) that impose a constant $\pi^{\mathbb{P}}$ are missing an empirically important asymmetry in the cyclical behavior of interest rates.

In developing our model we build upon a growing literature on *discrete-time DTSM*s by extending the Gaussian, discrete-time *DTSM*s in Bekaert and Grenadier [2001], Ang and Piazzesi [2003], and Gourieroux, Monfort, and Polimenis [2002] to allow for multiple regimes and priced regime-shift risk.[6] This is accomplished by overlaying a switching regime process on the conditional distribution of the risk factors. However, rather than adopting Hamilton [1989]'s convention of specifying the distribution of the state conditional on the *future* regime, we condition on the *current* regime. Under our convention, all of the conditioning variables at date t reside in agents' date t information set, which includes knowledge of the current regime. This leads to an intuitive interpretation of the components of agents' pricing kernel that parallels standard formulations in the continuous-time literature.

Our analysis of a Gaussian *DTSM* is complementary to Bansal and Zhou [2002]'s study of an (approximate) discrete-time "*CIR*-style" *DTSM* with regime shifts. Model $A_0^{RS}(3)$ extends their framework by allowing for state-dependent $\pi_t^{\mathbb{P}}$ (Bansal and Zhou assumed that $\pi_t^{\mathbb{P}} = $ constant), and priced regime-shift risk (they assumed that the market price of regime-shift risk is zero).[7] Furthermore, the added flexibility in the correlation structure of the risk factors in model $A_0^{RS}(3)$ allows us to replicate the well known hump in the term structure of volatility, and to explore the regime dependence of the shape of this hump. The assumption of mutually independent, mean-reverting factors in *CIR*-style models essentially forces downward sloping term structures of volatility in all regimes.

In a concurrent study, with a different objective, Ang and Bekaert [2005] also examine a regime-switching Gaussian *DTSM*.[8] They assume that the regime-shift risk is not priced, $\pi_t^{\mathbb{P}}$ is constant, and the historical rates of mean reversion of the risk factors are the same across

---

[6]To the extent that changes in regimes are related to business-cycle developments, multiple switches within a monthly, or even a quarterly, time frame seem unlikely. Therefore, we see little cost to a discrete-time framework, with the obvious benefit of being able to link our results directly with the descriptive literature on regime shifts in the distributions of interest rates.

[7]As we explain more formally below, neither of our models is nested within the other with regard to the specifications of the market price of factor risks.

[8]In other related studies, Wu and Zeng [2003] derive a general equilibrium, regime-switching model, building upon the one-factor *CIR*-style model of Naik and Lee [1997], with constant $\pi^{\mathbb{P}}$. Veronesi and Yared [2000] develop an equilibrium model of the term structure with regime shifts and constant $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}$.

regimes. Model $A_0^{RS}(3)$ relaxes all of these assumptions thereby facilitating an exploration of the state dependence of $\pi_t^{\mathbb{P}}$ and of the contributions of the market prices of regime-shift and factor risks to expected excess returns.

The remainder of this paper is organized as follows. Section 2 develops our model and derives the arbitrage-free bond pricing relations in the presence of regime shifts. We also compare the nature of the various market prices of risk in our setup to those in previous studies. The likelihood function that is used in estimation is derived in Section 3. Section 4 describes the data, presents the estimates of our models, and interprets the results. The contributions of the regime-shift and factor risks to expected excess returns are explored in more depth in Section 5. Finally, concluding remarks are presented in Section 6.

2   A Regime-Switching, Gaussian *DTSM*

In formulating a *DTSM* for econometric analysis, there is an inherent trade-off between the richness of one's model and the computational complexity that arises in both pricing and estimation. These trade-offs are compounded in our setting by the introduction of multiple regimes for the state vector Y. Just as in the literature on single-regime affine *DTSM*s, we proceed by parameterizing the risk-neutral distribution of Y so as to assure closed-form solutions for bond prices, and then overlay flexible specifications of the market prices of risk to describe the historical distribution of bond yields. This construction also highlights the sources of the added flexibility in our formulation of a regime-switching *DTSM* relative to recent alternative formulations in the literature.

We assume that there are $S + 1$ "regimes" that govern the dynamic properties of the N-dimensional state (factor) vector Y. Formally, the joint process (Y, s) is modelled as a marked point process. Heuristically, the regime variable s_t may be thought of as a $(S + 1)$-state Markov process, with the risk-neutral (hereafter denoted by $\mathbb{Q}$) probability of switching from regime $s_t = j$ to regime $s_{t+1} = k$ given by $\pi_t^{\mathbb{Q}jk}$, $0 \leq j, k \leq S$, with $\sum_{k=0}^{S} \pi_t^{\mathbb{Q}jk} = 1$, for all j. Agents are presumed to know the current and past histories of *both* the state vector and the regime the economy is in. Thus expectations $E_t^{\mathbb{Q}}[\cdot]$ are conditioned on the information set I_t generated by $\{Y_{t-\ell}, s_{t-\ell} : \ell \geq 0\}$. We use the notation $E_t^{\mathbb{Q}}[\cdot | s_t = j]$ in cases where we wish to highlight the current value of $s_t \in I_t$, and use the notation $E^{\mathbb{Q}(j)}[\cdot]$ to denote the unconditional mean of a random variable under the assumption of a single-regime economy governed by the parameters of regime j.[9]

The Markov process governing regime changes is assumed to be conditionally independent of the Y process. In addition, $f^{\mathbb{Q}}(Y_{t+1} | Y_{t-\ell} : \ell \geq 0; s_t = j, s_{t+1} = k) = f^{\mathbb{Q}}(Y_{t+1} | Y_{t-\ell} : \ell \geq 0; s_t = j)$. This differs from (our pricing counterpart to) Hamilton [1989]'s formulation where $f^{\mathbb{Q}}(Y_{t+1} | Y_{t-\ell} : \ell \geq 0; s_t = j, s_{t+1} = k) = f^{\mathbb{Q}}(Y_{t+1} | Y_{t-\ell} : \ell \geq 0; s_{t+1} = k)$. As the length of a unit of time shrinks toward zero (in the continuous time limit), these two formulations are

---

[9]In adopting this notation we are presuming that the unconditional means in each regime are finite under $\mathbb{Q}$. For our empirical implementation, one of the roots of the mean reversion matrix indicates that Y is borderline non-stationary under $\mathbb{Q}$. This finding is of limited practical relevance for our analysis, since we focus on conditional moments under $\mathbb{P}$ and $\mathbb{Q}$ and unconditional moments under $\mathbb{P}$ (which are all finite).

equivalent. We adopt our discrete-time formulation for comparability with the development of continuous-time models, and for the natural interpretation of the market prices of risk that it yields (see below).[10]

Given $s_t = j$, under $\mathbb{Q}$, Y is assumed to follow the process

$$Y_{t+1} = \mu_t^{\mathbb{Q}j} + \Sigma^j \epsilon_{t+1}, \tag{1}$$

where $\mu_t^{\mathbb{Q}j} \equiv E_t^{\mathbb{Q}}[Y_{t+1}|s_t = j]$, $\Sigma^j \equiv \Sigma(s_t = j)$ is a volatility matrix that is regime-dependent but not dependent on time, and $\epsilon_{t+1} \sim N(\mathbf{0}, I)$ is standard normal. It follows that

$$f^{\mathbb{Q}}(Y_{t+1}|Y_{t-\ell} : \ell \geq 0; s_t = j) \sim N(\mu_t^{\mathbb{Q}j}, \Sigma^j \Sigma^{j\prime}). \tag{2}$$

Equivalently, the conditional moment generating function (*MGF*) of Y_{t+1} is, given $s_t = j$,

$$\phi_t^{\mathbb{Q}j}(u) \equiv E_t^{\mathbb{Q}}\left[e^{u' Y_{t+1}} \middle| s_t = j \right] = e^{u' \mu_t^{\mathbb{Q}j} + \frac{u' \Sigma^j \Sigma^{j\prime} u}{2}}, \ u \in \mathbb{R}^N. \tag{3}$$

In order to obtain closed-form solutions for zero-coupon bond prices, we parallel Dai and Singleton [2003]'s construction in continuous time and assume

Assumption $A\mathbb{Q}$: Restrictions on the $\mathbb{Q}$ distribution of (Y, s):

$A\mu^{\mathbb{Q}}$: $\mu^{\mathbb{Q}j}(Y_t) = Y_t + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}j} - Y_t)$, for constant $\theta^{\mathbb{Q}j} \in \mathbb{R}^N$, $j = 0, \ldots, S$ and $N \times N$ constant matrix $\kappa^{\mathbb{Q}}$ with $\kappa_{ij}^{\mathbb{Q}} \in \mathbb{R}$.

$A\pi^{\mathbb{Q}}$: the $\pi^{\mathbb{Q}jk}$ are constants, for all j and k.

An implication of assumption $A\mu^{\mathbb{Q}}$ is that the unconditional mean $E^{\mathbb{Q}(j)}[Y_t]$ differs across regimes. On the other hand, to facilitate pricing, the state-dependent component of $\mu_t^{\mathbb{Q}j}$, $\kappa^{\mathbb{Q}} Y_t$, is assumed to be common across regimes. Assumption $A\pi^{\mathbb{Q}}$ restricts the regime-switching probabilities to be state-independent under $\mathbb{Q}$. Bansal and Zhou [2002] impose assumption $A\pi^{\mathbb{Q}}$, but not $A\mu^{\mathbb{Q}}$ in that they allow $\kappa^{\mathbb{Q}j}$ to be regime-dependent. This takes them outside of a framework with analytic bond prices and, as such, they study approximate prices using a linearization of their model. Like us, Ang and Bekaert [2005] impose assumption $A\mathbb{Q}$ in its entirety.

The continuously compounded yield on a one-period zero-coupon bond in regime j, $r_t^j \equiv r(Y_t, s_t = j)$, is assumed to be related to Y_t according the affine function

Assumption Ar: $r_t^j = \delta_0^j + \delta_Y' Y_t$.

Under assumption Ar, $E^{\mathbb{Q}(j)}[r_t]$ differs across regimes, as a result of the regime dependence of both δ_0^j and $E^{\mathbb{Q}(j)}[Y_t]$. Bansal and Zhou [2002] constrain $\delta_0^j = 0$ in both regimes in their

---

[10]A similar timing convention was adopted by Cecchetti, Lam, and Mark [1993] in their descriptive study of equity returns. In the context of descriptive regime-switching models (i.e., observable variables and no pricing), our and Hamilton's specifications lead to identical likelihood functions, except for the interpretation of the initial values of certain conditional regime probabilities. Once latent factors and pricing are introduced, the interpretations are not the same for reasons discussed subsequently.

CIR-style formulation. Allowing for non-zero δ_0^j seems important in light of the evidence in Pearson and Sun [1994] (for treasury bonds) and Duffie and Singleton [1997] (for interest rate swaps) that, for their single-regime multi-factor *CIR* models, a non-zero δ_0 was essential to fitting the term structure. With regard to the state-dependent component of r_t^j we constrain the "loadings" δ_Y on Y_t to be the same across regimes to facilitate bond pricing. Bansal and Zhou's model is more flexible on this dimension in that δ_Y^j is regime-dependent owing to the volatility coefficient in their approximate *CIR* process changing across regimes.[11]

These assumptions give rise to closed-form solutions for zero-coupon bond prices.[12] Specifically, letting $D_{t,n} \equiv D_n(Y_t, s_t)$ denote the time-t price for a zero-coupon bond with maturity of n periods, and $D_{t,n}^j$ denote the price when the current regime is $s_t = j$, we have:

Proposition 1 (Zero-Coupon Bond Prices) *Assuming that Y_{t+1} follows the process (1) and assumptions $A\mu^{\mathbb{Q}}$, $A\pi^{\mathbb{Q}}$, and Ar hold, zero-coupon bond prices are given by*

$$D_{t,n}^j = e^{-A_n^j - B_n' Y_t}, \tag{4}$$

where,

$$A_{n+1}^j = \delta_0^j + (\kappa^{\mathbb{Q}} \theta^{\mathbb{Q}j})' B_n - \frac{1}{2} B_n' \Sigma^j \Sigma^{j\prime} B_n - \log \left(\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A_n^k} \right), \tag{5}$$

$$B_{n+1} = \delta_Y + B_n - \kappa^{\mathbb{Q}\prime} B_n, \tag{6}$$

with initial conditions: $A_0^j = 0$ and $B_0 = \mathbf{0}$.

Proof: Substituting (4) into the risk-neutral pricing equation

$$D_{t,n+1}^j = E_t^{\mathbb{Q}} \left[e^{-r_t^j} D_{t+1,n} \middle| s_t = j \right]$$

yields

$$e^{-A_{n+1}^j - B_{n+1}' Y_t} = E_t^{\mathbb{Q}} \left[e^{-r_t^j} D_{t+1,n} | s_t = j \right]$$

$$= e^{-r_t^j} \sum_{k=0}^{S} \pi^{\mathbb{Q}jk} E_t^{\mathbb{Q}} \left[D_{t+1,n}^k | s_t = j \right]$$

$$= e^{-r_t^j} \sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A_n^k} E_t^{\mathbb{Q}} \left[e^{-B_n' Y_{t+1}} | s_t = j \right]$$

$$= e^{-r_t^j} \left[\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A_n^k} \right] e^{-B_n' \mu_t^{\mathbb{Q}j} + \frac{1}{2} B_n' \Sigma^j \Sigma^{j\prime} B_n}.$$

---

[11]This is a second contributing factor (besides the regime dependence of $\kappa^{\mathbb{Q}j}$) to their use of approximations in pricing bonds.

[12]More precisely, the dependence of bond yields on Y is known analytically, up to a set of coefficients that solve the recursion equations. These equations can be solved essentially instantaneously.

Equations (5) and (6) are necessary and sufficient for the above equation to hold for any Y_t and $s_t = j$. It is easy to check that $A_0^j = 0$, $B_0 = \mathbf{0}$, $A_1^j = \delta_0^j$, and $B_1 = \delta_Y$ satisfy the recursion. Thus, the recursion can start either at $n = 0$ or at $n = 1$. When n denotes maturities in months, the annualized yields are given by

$$R_{t,n}^j = a_n^j + b_n' Y_t = \frac{A_n^j}{n/12} + \frac{B_n'}{n/12} Y_t. \tag{7}$$

To complete the specification of our model it remains to specify the distribution of (Y_{t+1}, s_{t+1}) conditional on I_t under the historical measure, $\mathbb{P}$. The conditional distributions of (Y_{t+1}, s_{t+1}) under $\mathbb{P}$ and $\mathbb{Q}$ are linked by the Radon-Nikodym derivative $(d\mathbb{P}/d\mathbb{Q})_{t,t+1}$.[13] Equivalently, under the assumption of no arbitrage opportunities, they are linked by the pricing kernel $\mathcal{M}_{t,t+1} = \mathcal{M}(Y_t, s_t; Y_{t+1}, s_{t+1})$ underlying the time-t valuation of payoffs at date $t+1$, as $(d\mathbb{P}/d\mathbb{Q})_{t,t+1} = 1/[e^{r_t} \mathcal{M}_{t,t+1}]$. To accommodate both regime-shift and factor risks we assume that $(d\mathbb{P}/d\mathbb{Q})_{t,t+1}$ is given by

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)_{t,t+1} = e^{\Gamma_{t,t+1} - \frac{1}{2}\Lambda_t'\Lambda_t + \Lambda_t' \Sigma_t^{-1}(Y_{t+1} - \mu_t^{\mathbb{Q}})} = e^{\Gamma_{t,t+1}} \times \frac{e^{\Lambda_t' \Sigma_t^{-1} Y_{t+1}}}{\phi_t^{\mathbb{Q}}((\Sigma_t')^{-1}\Lambda_t)}, \tag{8}$$

where $\Gamma_{t,t+1} = \Gamma(Y_t, s_t; s_{t+1})$ is the market price of regime-shift ($MPRS$) risk from s_t to s_{t+1}, $\Lambda_t = \Lambda(Y_t, s_t)$ is the market price of factor (MPF) risk, and $\phi_t^{\mathbb{Q}}((\Sigma_t')^{-1}\Lambda_t)$ is the MGF of Y_{t+1} under $\mathbb{Q}$ evaluated at $(\Sigma_t')^{-1}\Lambda_t$. The Radon-Nikodym derivative depends implicitly on the regimes (s_t, s_{t+1}), because agents know both the regime s_{t+1} and the regime from which they have transitioned, s_t. As will be discussed in more detail, dependence on s_{t+1} enters solely through the $MPRS$ risk $\Gamma_{t,t+1}$. The MPF risk, on the other hand, takes a form that parallels its counterpart in a continuous-time Gaussian $DTSM$. For later development, we define $\Gamma_t^{jk} \equiv \Gamma(Y_t, s_t = j; s_{t+1} = k)$ and $\Lambda_t^j \equiv \Lambda(Y_t, s_t = j)$.

For $\mathbb{P}$ to be a well-defined probability measure we require that

$$\begin{aligned}
1 &= E_t^{\mathbb{Q}}[(d\mathbb{P}/d\mathbb{Q})_{t,t+1}] \\
&= E_t^{\mathbb{Q}}[e^{\Gamma_{t,t+1}} | s_t = j] = \sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{\Gamma_t^{jk}}, \ 0 \leq \forall j \leq S.
\end{aligned} \tag{9}$$

Since the regime switching probabilities under $\mathbb{P}$ are given by

$$\pi_t^{\mathbb{P}jk} \equiv E_t \left[1_{\{s_{t+1}=k\}} | s_t = j \right] = \pi^{\mathbb{Q}jk} e^{\Gamma_t^{jk}}, \tag{10}$$

the condition in (9) is equivalent to $\sum_{k=0}^{S} \pi_t^{\mathbb{P}jk} = 1$.

The conditional distribution of Y_{t+1} under $\mathbb{P}$ is fully characterized by its MGF:

$$\begin{aligned}
\phi_t^{\mathbb{P}j}(u) &\equiv E_t \left[e^{u'Y_{t+1}} | s_t = j \right] = E_t^{\mathbb{Q}} \left[e^{u'Y_{t+1}} (d\mathbb{P}/d\mathbb{Q})_{t,t+1} | s_t = j \right] \\
&= \frac{\phi_t^{\mathbb{Q}j}((\Sigma^{j'})^{-1}\Lambda_t^j + u)}{\phi_t^{\mathbb{Q}j}((\Sigma^{j'})^{-1}\Lambda_t^j)} = e^{u'\left(\mu_t^{\mathbb{Q}j} + \Sigma^j \Lambda_t^j \right) + \frac{u'\Sigma^j \Sigma^{j'} u}{2}}, \ u \in \mathbb{R}^N.
\end{aligned} \tag{11}$$

---

[13]The $\mathbb{P}$ and $\mathbb{Q}$ distributions are related according to $\mathbb{P}(dY_{t+1}, s_{t+1} | I_t) = (d\mathbb{P}/d\mathbb{Q})_{t,t+1} \mathbb{Q}(dY_{t+1}, s_{t+1} | I_t)$.

Thus, this distribution is also Gaussian with conditional mean

$$\mu_t^{\mathbb{P}} = \mu_t^{\mathbb{Q}} + \Sigma_t \Lambda_t, \tag{12}$$

and variance $\Sigma_t \Sigma_t'$. Moreover, the $\mathbb{P}$ distribution of (Y, s) inherits the property from the $\mathbb{Q}$ distribution that Y and s are conditionally independent processes.

The pricing kernel implied by our choice of Radon-Nikodym derivative in (8) is:

$$\mathcal{M}_{t,t+1} = e^{-r_t - \Gamma_{t,t+1} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\Sigma_t^{-1}(Y_{t+1} - \mu_t^{\mathbb{P}})}. \tag{13}$$

No arbitrage requires that

$$e^{-r_t^j} = E_t[\mathcal{M}_{t,t+1}|s_t = j] = e^{-r_t^j} E_t[(d\mathbb{Q}/d\mathbb{P})_{t,t+1}|s_t = j], \tag{14}$$

which is guaranteed by the requirement that $\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} = 1$.

To motivate our labelling of the component Λ_t of the pricing kernel $\mathcal{M}$ as the market prices of factor risks, consider the security with payoff $e^{-b'Y_{t+1}}$, which has exposure only to factor risks at date $t + 1$. Its price is

$$P_t^j = e^{-r_t^j} E_t^{\mathbb{Q}}[e^{-b'Y_{t+1}}|s_t = j] = e^{-r_t^j} e^{-b'\mu_t^{\mathbb{Q}j} + \frac{1}{2}b'\Sigma^j\Sigma^{j'}b}, \tag{15}$$

and its $\mathbb{P}$-expected payoff is $E_t[e^{-b'Y_{t+1}}] = e^{-b'\mu_t^{\mathbb{P}j} + \frac{1}{2}b'\Sigma^j\Sigma^{j'}b}$ in regime $s_t = j$. Therefore, the log expected return for this security, in excess of the one period zero-coupon bond yield, is[14]

$$\log \frac{E_t[e^{-b'Y_{t+1}}|s_t = j]}{P_t^j} - r_t^j = -b'\Sigma^j\Lambda_t^j. \tag{16}$$

Since $b'\Sigma^j$ is the "risk exposure" or volatility of the security associated with the factor risk, the *MPF* risk in regime $s_t = j$, Λ_t^j, gives the excess log expected return per unit of factor risk exposure.

Turning to the component $\Gamma_{t,t+1}$, consider a security with payoff $1_{\{s_{t+1}=k\}}$, which has exposure only to the risk of shifting to regime k at date $t + 1$. Conditional on the current regime $s_t = j$, its risk-neutral expected payoff is $\pi^{\mathbb{Q}jk}$, and its current price is $P_t^j = e^{-r_t^j}\pi^{\mathbb{Q}jk}$. Thus, its excess log expected return is given by

$$\log \frac{E_t[1_{\{s_{t+1}=k\}}|s_t = j]}{P_t^j} - r_t^j = \Gamma_t^{jk}. \tag{17}$$

That is, $\Gamma_{t,t+1}^{jk}$ is naturally defined as the *MPRS* risk from regime j to regime k.

---

[14]For intuitive interpretation of the market prices of risks, log expected returns in excess of the one period short rate provide a more convenient platform than the conventionally defined, expected excess log returns. The latter, which are related to the market prices of risks in qualitatively very similar manners, will be examined in Section 5.

Importantly, assumption $A\mu^{\mathbb{Q}}$ does *not* constrain the state or regime dependence of $\mu^{\mathbb{P}}$. Given our parametrization of $\mu_t^{\mathbb{Q}j}$ and the regime dependence of Σ^j, we can match any desired state and regime dependence of $\mu_t^{\mathbb{P}j}$ under $\mathbb{P}$,

$$\mu_t^{\mathbb{P}j} = \mu_t^{\mathbb{Q}j} + \Sigma^j \Lambda_t^j, \ 0 \le j \le S, \tag{18}$$

by appropriate choice of the market prices of factor risks, Λ_t^j. In our parametric *DTSM*, we extend Duffee [2002]'s essentially affine, Gaussian model to the case of multiple regimes by assuming that

$$\Lambda_t^j = \left(\Sigma^j\right)^{-1} \left(\lambda_0^j + \lambda_Y^j Y_t\right). \tag{19}$$

Duffee [2002] and Dai and Singleton [2002] found that $A_0(3)$ models with *MPF* risks given by (19) (without the regime index) were able to match many features of historical expected excess returns on bonds. The regime independence of $\kappa^{\mathbb{Q}}$ (assumption $A\mu^{\mathbb{Q}}$) requires that

$$\kappa^{\mathbb{Q}} = \kappa^{\mathbb{P}L} + \lambda_Y^L = \kappa^{\mathbb{P}H} + \lambda_Y^H. \tag{20}$$

To take advantage of the maximal flexibility allowed by this restriction, we set $\kappa^{\mathbb{P}L}$, λ_Y^L, and λ_Y^H as free parameters in our model. Consequently, $\kappa^{\mathbb{P}H}$ as well as $\kappa^{\mathbb{Q}}$ are derived parameters.

Similarly assumption $A\pi^{\mathbb{Q}}$ does *not* restrict the state dependence of $\pi_t^{\mathbb{P}jk}$. Given the $\pi_t^{\mathbb{Q}jk}$, by appropriate choice of the Γ_t^{jk}, we can match any desired state dependence of the $\pi_t^{\mathbb{P}jk}$, subject to the constraint that $\sum_{k=0}^{S} \pi_t^{\mathbb{P}jk} = 1$. Following Gray [1996], Boudoukh, Richardson, Smith, and Whitelaw [1999], and many subsequent studies, we assume that (for the two-regime case)

$$\pi_t^{\mathbb{P}jk} = \frac{1}{1 + e^{\eta_0^{jk} + \eta_Y^{jk} Y_t}}, \ \pi_t^{\mathbb{P}jj} = 1 - \pi_t^{\mathbb{P}jk}, \ j \ne k. \tag{21}$$

Consequently, the *MPRS* risks are

$$\Gamma_t^{jk} = \log\left(\frac{\pi_t^{\mathbb{P}jk}}{\pi^{\mathbb{Q}jk}}\right), \ \forall j, k. \tag{22}$$

The unknown parameters to be estimated are the (constant) risk-neutral regime-shift probabilities $\pi^{\mathbb{Q}jk}$, η_0^{jk}, and η_Y^{jk}. Unlike in descriptive regime-switching models for interest rates, the elements of $\pi^{\mathbb{P}}$ in our *DTSM* depend directly on the latent risk factors Y (rather than on the yields themselves).

Our specification (13) of the pricing kernel $\mathcal{M}$, and the associated market prices of risk, extends the literature on regime-switching models for interest rates along several important dimensions. As in Naik and Lee [1997] and Bansal and Zhou [2002], we assume that the $\pi^{\mathbb{Q}jk}$ are constants (assumption $A\pi^{\mathbb{Q}}$). However, these studies also assume that regime-shift risk is not priced ($\Gamma_t^{jk} = 0$). Regime-shift risk is also not priced (and the $\pi^{\mathbb{Q}jk}$ are constants) in the regime-switching Gaussian model used by Ang and Bekaert [2005] in their study of real returns. The state dependence of the Γ_t^{jk} implied by (21) and (22) is key to achieving our objective of an improved understanding of the nature of market prices of risk, and regime-shift risk in particular. The assumption that $\Gamma_t^{jk} = 0$ means that the state and regime

dependence of Λ_t^j and Σ^j (the volatility matrix of Y) must explain the time-series properties of expected excess returns. By allowing for priced regime-shift risk, we have an additional regime-dependent channel through which risk preferences can affect expected excess returns.

Our formulation of the market prices of factor risks shares with Bansal and Zhou [2002] the features that Λ_t is both state- and regime-dependent. Where we differ is in our focus on different members of the affine family of term structure models. They focus on an approximate *CIR* style model, and assume that the market prices of risk are proportional to factor volatilities (a "completely affine" model). Though our within-regime volatilities are constant (due to the Gaussian framework), our market prices of risk depend directly on the state variables according to (19). Within single-regime models, the latter "essentially affine" models have been found to fit the dynamic properties of yield curves much better than completely affine *CIR*-style models (e.g., Duffee [2002], Dai and Singleton [2002]).

The specification (19) of Λ_t allows both the constant term (λ_0^j) and the coefficients in the state-dependent term (λ_Y^j) to change across regimes. Ang and Bekaert [2005] also allow their counterpart to λ_0^j to change across regimes for one of their two latent factors. However, for this factor, they assume that $\lambda_Y^j = 0$. Further, they assume that the market price of inflation risk is zero at all dates and in all regimes. It follows that, for both of these factors, the *MPF*s have no effect on the time series properties of excess returns within regimes.[15] For their third factor (also latent), the market prices of risk are state-, but not regime-dependent. As such, they constrain the degree of persistence of all three of their risk factors (the state-dependent components of the conditional mean of Y_{t+1}) to be the same across regimes. Given our focus on the relations between the time series properties of excess returns and the market prices of factor and regime-shift risks, we allow the components of λ_Y^j to be non-zero and regime-dependent for all three factors, in order to give maximal flexibility to the market prices of factor risks in our analysis.

A potential weakness of our Gaussian *DTSM*, relative to say multiple-regime versions of $A_M(N)$ *DTSM*s, with $M > 0$, is that the within-regime conditional variances of the Y's are constants. However, our experience with single-regime affine *DTSM*s is that the conditional volatility in bond yields induced by conditional volatility in the Y's is, in fact, very small relative to the volatility of excess returns. Furthermore, by overlaying regime shifts on top of a Gaussian state vector we introduce stochastic volatility into our *DTSM*, perhaps at least to the same degree as in square-root processes. Even in the case of constant $\pi^{\mathbb{P}}$, the conditional variances of bond yields will be time varying due to the possibility of regime shifts. This is the only source of time-varying volatility in Gaussian models that assume that $\pi^{\mathbb{P}}$ is a constant matrix (e.g., Ang and Bekaert [2005]). With the introduction of state-dependent $\pi^{\mathbb{P}}$ (equivalently, state-dependent Γ^{jk}), we allow for an important additional source of time-varying volatility that is absent from extant single- and multiple-regime Gaussian models.

Additionally, assumptions $A\pi^{\mathbb{Q}}$ and (22) imply that our model cannot accommodate state-dependent regime-shift risk that is not priced. This is a consequence of the fact that, since the $\pi^{\mathbb{Q}jk}$ are not state-dependent, any state dependence in the $\pi_t^{\mathbb{P}jk}$ is inherited from

---

[15]The factor risk component of the expected excess returns, as shown in (37), implies that within regimes, time varying expected excess returns result from state-dependent *MPF* risks.

state-dependence of the market prices of regime-shift risks, Γ_t^{jk}, in our model. We nest the special cases of $\pi^{\mathbb{Q}}$ and $\pi^{\mathbb{P}}$ being constants, with Γ^{jk} being either a non-zero constant (priced regime-shift risk) or zero (non-priced regime-shift risk). However, our formulation does not nest the case of state-dependent $\pi_t^{\mathbb{P}}$ with $\Gamma_t^{jk} = 0$. Nevertheless, we view the accommodation of state-dependent $\pi^{\mathbb{P}}$ and rich regime dependence of Λ_t^j as potentially important extensions of the literature on $A_0(3)$ models that are worthwhile exploring empirically.

Finally, a notable difference between our formulation and that in Bansal and Zhou [2002] and Ang and Bekaert [2005] is that we have assumed that $(\Lambda_t^j, \Gamma_t^{jk}) \in I_t$, consistent with the continuous-time regime-switching model developed in Dai and Singleton [2003]. In contrast, using our notation for the one-factor case, these authors adopted the pricing kernel

$$\mathcal{M}_{t,t+1} = \exp\left[-r_{f,t} - (\lambda^{s_{t+1}})^2 \frac{Y_t}{2} - \lambda^{s_{t+1}} \epsilon_{t+1}\right], \tag{23}$$

in which $\lambda^{s_{t+1}}$ depends on s_{t+1}. In our formulation, the components Λ_t^j and Γ_t^{jk} can be directly interpreted as market prices of risk. On the other hand, under the formulation (23), $\lambda^{s_{t+1}}$ is not the *MPF* risk, since it is not in I_t. The *MPF* risk depends on both $\lambda^{s_{t+1}}$ and the regime-switching probabilities $\pi^{\mathbb{P}jk}$.[16]

3 Maximum Likelihood Estimation

Given the Gaussian structure of the risk factors, we proceed with maximum likelihood (*ML*) estimation of the regime-switching *DTSM*s. Following common practice (e.g., Chen and Scott [1995], Duffie and Singleton [1997]), we assume that the yields on a collection of N zero-coupon bonds are priced without error, and the yields on a collection of M zero-coupon bonds are priced with error.

Let $\hat{R}_t$ be the vector of yields for the bonds priced exactly by the model. In regime $s_t = j$, $\hat{R}_t = \hat{a}^j + \hat{b}Y_t^j$, where $\hat{a}^j$ is the $N \times 1$ regime-dependent vector, $\hat{b}$ is the $N \times N$ regime-independent matrix of factor loadings, and Y_t^j is the $N \times 1$ vector of state variables implied by the model. Inverting for fitted yields we obtain

$$Y_t^j = \hat{b}^{-1}(\hat{R}_t - \hat{a}^j). \tag{24}$$

Conditional on $s_t = j$ and $s_{t+1} = k$, we have

$$\begin{aligned}
\hat{R}_{t+1} &= \hat{a}^k + \hat{b}Y_{t+1}^k = \hat{a}^k + \hat{b}\mu_t^{\mathbb{P}j} + \hat{b}\Sigma^j \epsilon_{t+1} \\
&= \hat{R}_t + (\hat{a}^k - \hat{a}^j) + \hat{\kappa}^{\mathbb{P}j}(\hat{\theta}^{\mathbb{P}j} - \hat{R}_t) + \hat{\Sigma}^j \epsilon_{t+1},
\end{aligned} \tag{25}$$

---

[16]More precisely, in their setting, the excess log expected return for a security with regime-independent, time-$(t+1)$ payoff of $e^{-bY_{t+1}}$ is given by

$$\log \frac{\sum_{k=0}^S \pi^{\mathbb{P}jk} e^{-b\mu_t^{\mathbb{P}k} + \frac{Y_t}{2}b\Sigma^k\Sigma^k b}}{\sum_{k=0}^S \pi^{\mathbb{P}jk} e^{-b\mu_t^{\mathbb{P}k} + \frac{Y_t}{2}b\Sigma^k\Sigma^k b + b\lambda^k Y_t}}.$$

where $\mu_t^{\mathbb{P}j} = Y_t + \kappa^{\mathbb{P}j}(\theta^{\mathbb{P}j} - Y_t)$, $\hat{\kappa}^{\mathbb{P}j} = \hat{b}\kappa^{\mathbb{P}j}\hat{b}^{-1}$, $\hat{\theta}^{\mathbb{P}j} = \hat{a}^j + \hat{b}\theta^{\mathbb{P}j}$, and $\hat{\Sigma}^j = \hat{b}\Sigma^j$. It follows that

$$
f(\hat{R}_{t+1}|\hat{R}_t, s_t = j, s_{t+1} = k)
$$
$$
= \frac{e^{-\frac{1}{2}(\hat{R}_{t+1}-\hat{R}_t-(\hat{a}^k-\hat{a}^j)-\hat{\kappa}^j(\hat{\theta}^j-\hat{R}_t))'[\hat{\Sigma}^j\hat{\Sigma}^{j\prime}]^{-1}(\hat{R}_{t+1}-\hat{R}_t-(\hat{a}^k-\hat{a}^j)-\hat{\kappa}^j(\hat{\theta}^j-\hat{R}_t))}}{\sqrt{(2\pi)^N \left|\hat{\Sigma}^j\hat{\Sigma}^{j\prime}\right|}}. \tag{26}
$$

Notice that $f(\hat{R}_{t+1}|\hat{R}_t, s_t = j)$ is obtained by integrating out the dependence of (26) on s_{t+1}, so conditioning only on $s_t = j$ (and $\hat{R}_t$) gives a mixture-of-normals distribution.

The remaining M yields used in estimation are denoted by $\tilde{R}_t$, with corresponding loadings $\tilde{a}^j$ and $\tilde{b}$ when $s_t = j$:

$$
\tilde{R}_t = \tilde{a}^j + \tilde{b}Y_t^j + u_t^j = (\tilde{a}^j - \tilde{b}\hat{b}^{-1}\hat{a}^j) + \tilde{b}\hat{b}^{-1}\hat{R}_t + u_t^j, \tag{27}
$$

where u_t is $i.i.d.$ with zero mean and volatility Ω^j. Thus, the conditional density for $\tilde{R}_{t+1}$, conditional on $\hat{R}_{t+1}$, $s_t = j$ and $s_{t+1} = k$, is given by

$$
f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_t = j, s_{t+1} = k)
$$
$$
= \frac{e^{-\frac{1}{2}(\tilde{R}_{t+1}-(\tilde{a}^k-\tilde{b}\hat{b}^{-1}\hat{a}^k)-\tilde{b}\hat{b}^{-1}\hat{R}_{t+1})'[\Omega^k\Omega^{k\prime}]^{-1}(\tilde{R}_{t+1}-(\tilde{a}^k-\tilde{b}\hat{b}^{-1}\hat{a}^k)-\tilde{b}\hat{b}^{-1}\hat{R}_{t+1})}}{\sqrt{2\pi \left|\Omega^k\Omega^{k\prime}\right|}}. \tag{28}
$$

To construct the likelihood function for the data, we introduce the econometrician's information set $J_t = \{\hat{R}_\tau, \tilde{R}_\tau, \tau \le t\} \subset I_t$, and let $Q_t^j = f(s_t = j|J_t)$ be the probability of regime j given J_t. Define the following matrices:

$$
Q_t = \begin{bmatrix} f(s_t = 0|J_t) & f(s_t = 1|J_t) \end{bmatrix},
$$
$$
f_{t,t+1}^R = \begin{bmatrix} f(\hat{R}_{t+1}|\hat{R}_t, s_t = 0, s_{t+1} = 0) & f(\hat{R}_{t+1}|\hat{R}_t, s_t = 0, s_{t+1} = 1) \\ f(\hat{R}_{t+1}|\hat{R}_t, s_t = 1, s_{t+1} = 0) & f(\hat{R}_{t+1}|\hat{R}_t, s_t = 1, s_{t+1} = 1) \end{bmatrix},
$$
$$
f_{t,t+1}^u = \begin{bmatrix} f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 0) & f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 1) \\ f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 0) & f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 1) \end{bmatrix}.
$$

Using this notation, the conditional density of observed yields is

$$
\begin{aligned}
f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t) &= \sum_j f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t, s_t = j)Q_t^j \\
&= \sum_{j,k} f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t, s_t = j, s_{t+1} = k)Q_t^j \pi_t^{\mathbb{P}jk} \\
&= \sum_{j,k} f(\hat{R}_{t+1}|\hat{R}_t, s_t = j, s_{t+1} = k)Q_t^j \pi_t^{\mathbb{P}jk} f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = k).
\end{aligned}
$$

The regime probability Q_t^j is updated using Bayes rule:

$$
\begin{aligned}
Q_{t+1}^k &= f(s_{t+1} = k | J_{t+1}) \\
&= \frac{\sum_j f(s_{t+1} = k, \hat{R}_{t+1}, \tilde{R}_{t+1} | J_t, s_t = j) Q_t^j}{f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t)} \\
&= \frac{\sum_j Q_t^j f(\hat{R}_{t+1} | \hat{R}_t, s_t = j, s_{t+1} = k) \pi_t^{\mathbb{P}jk} f(\tilde{R}_{t+1} | \hat{R}_{t+1}, s_{t+1} = k)}{f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t)}.
\end{aligned}
$$

Thus, the log-likelihood is given by

$$
\log L = \frac{1}{T-1} \sum_{t=0}^{T-1} \log f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t), \tag{29}
$$

$$
f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t) = Q_t \times \left(f_{t,t+1}^R \odot f_{t,t+1}^u \odot \pi_t^{\mathbb{P}} \right) \times \mathbf{1}, \tag{30}
$$

$$
Q_{t+1} = \frac{Q_t \times \left(f_{t,t+1}^R \odot f_{t,t+1}^u \odot \pi_t^{\mathbb{P}} \right)}{f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t)}, \tag{31}
$$

where $A \odot B$ denotes element by element multiplication of matrix A and B with the same dimensions, and $\mathbf{1}$ is the 2×1 unit vector.

In interpreting our empirical results, we follow the standard practice of using the "smoothed regime probabilities" $q_t^j \equiv f(s_t = j | J_T)$ to classify observations into regimes (recall that we do not observe s_t, or which regime the economy is in at date t). For our case of two regimes, we classify the yield observation at date t into regime j if $q_t^j > 0.5$, where

$$
q_t^j = \frac{g_t^j Q_t^j}{\sum_k g_t^k Q_t^k}, \tag{32}
$$

$g_T^j \equiv 1$ and, for $1 \le t \le T-1$,

$$
\begin{aligned}
g_t^j &\equiv f(\hat{R}_\ell, \tilde{R}_\ell : t+1 \le \ell \le T | s_t = j, J_t) \\
&= \sum_k \pi_t^{\mathbb{P}jk} f(\hat{R}_{t+1}, \tilde{R}_{t+1} | J_t, s_t = j, s_{t+1} = k) g_{t+1}^k.
\end{aligned}
$$

In matrix notation, we have

$$
\begin{aligned}
q_t &\equiv \begin{bmatrix} q_t^0 \\ q_t^1 \end{bmatrix} = \frac{Q_t' \odot g_t}{Q_t g_t}, \ 1 \le t \le T, \\
g_t &\equiv \begin{bmatrix} g_t^0 \\ g_t^1 \end{bmatrix} = \left(\pi_t^{\mathbb{P}} \odot f_{t,t+1}^R \odot f_{t,t+1}^u \right) \times g_{t+1}, \ 1 \le t \le T-1; \ g_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\end{aligned}
$$

4 Empirical Results

In estimating the model, we use the zero-coupon bond yield data constructed from the market prices of coupon bonds with the Fisher-Waggoner (FW) curve fitting method. This

method, together with Unsmoothed Fama-Bliss (UFB), McCulloch-Kwon (MK), Smoothed Fama-Bliss (SFB), and Nelson-Siegel-Bliss (NSB), comprise the most popular zero-coupon yield estimation methods in the empirical literature.[17] One important difference across these methods is the resulting degree of smoothness of the estimated term structure data. At one extreme, the USB method iteratively extracts forward rates from coupon bond prices by building a piece-wise linear discount rate function, and the implied discount rates exhibit kinks at the maturities of the coupon bonds used. At the other extreme, the NSB and the SFB methods approximate the discount rates with exponential functions of time to maturity and the resulting forward rate function is differentiable to infinite order.

The FW method seems to occupy a desirable middle ground. It is based on a cubic spline, similar to the MK method.[18] A large number of knots (as many as 50 to 60 knots) are used when minimizing the fitting errors, and then a penalty is imposed on the excess variability of yields induced by the flexibility of the spline. In contrast to the kinky USB yield curves, the FW method generates a smooth term structure with a continuous first derivative. In contrast to the possibly over-smoothed NSB and SFB data, the flexibility of the 50 to 60 knot points allows the FW data to better track the many small dips and humps in the underlying coupon bond yields.

We estimate a two-regime, three-factor ($N = 3$) model, $A_0^{RS}(3)$, using the FW monthly data on U.S. Treasury zero-coupon bond yields for the period 1972 to 2003.[19] The vector $\hat{R}$ includes the yields on bonds with maturities of 6, 24, and 120 months, and $M = 1$ with $\tilde{R}$ chosen to be the yield on the 60-month bond. The two regimes are denoted L and H, corresponding to "low" and "high" values of the diagonal entries of Σ^j (see below).

In parameterizing model $A_0^{RS}(3)$, we impose several normalizations. Analogous to the normalizations imposed in Dai and Singleton [2000] for single-regime affine $DTSM$s, in regime L, we set the annualized volatility $\sqrt{12}\Sigma^L$ to an identity matrix, $\kappa^{\mathbb{P}L}$ to a lower triangular matrix, and $\theta^{\mathbb{P}L}$ to zero. The normalization of Σ^L is needed, because we have allowed δ_Y to be free and the factors Y are latent. Second, in regime H, Σ^H was set to a lower diagonal matrix, because the Brownian motions in regime H can be rotated independently of any rotations on the Brownian motions in regime L. Beyond these normalizations, the restrictions $\kappa^{\mathbb{Q}H} = \kappa^{\mathbb{Q}L} \equiv \kappa^{\mathbb{Q}}$ and $\delta_Y^H = \delta_Y^L = \delta_Y$ were imposed so that zero-coupon bonds are priced in closed form. Consequently $\kappa^{\mathbb{P}L} + \lambda_Y^L = \kappa^{\mathbb{P}H} + \lambda_Y^H$.

---

[17]Bliss [1997] provides a more detailed description of these methods. The Unsmoothed Fama-Bliss method is documented in Fama and Bliss [1987]. The McCulloch-Kwon method is a modified version of the McCulloch method (McCulloch [1975]). The Fisher-Waggoner method (Waggoner [1997]) is a modified version of the Fisher-Nychka-Zervos method (Fisher, Nychka, and Zervos [1995]). The Nelson-Siegel-Bliss method was originally labeled the extended Nelson-Siegel method in Bliss [1997]. The original Nelson-Siegel method (Nelson and Siegel [1987]) has only four free parameters. The Nelson-Siegel-Bliss method has five parameters free so as to provide a better fit for longer maturities (Bliss [1997]).

[18]In the MK method a cubic spline is used to approximate the discount function, and the spline is estimated with ordinary least squares. The FW method uses a cubic spline to approximate the forward rate function itself.

[19]The FW data is generated using "The Bliss Term Structure Generating Programs" with the filtered "long data set", which filters out bonds with option features and liquidity problems, but otherwise contains all eligible issues. See Bliss [1997].

Even with these normalizations/constraints, the resulting maximally flexible $A_0^{RS}(3)$ model (with restrictions for analytical pricing) involves a high dimensional parameter space: there are 56 parameters in

$$\delta_0^L, \delta_0^H, \delta_Y, \kappa^{\mathbb{P}L}, \theta^{\mathbb{P}H}, \Sigma^H, \lambda_0^L, \lambda_Y^L, \lambda_0^H, \lambda_Y^H, \Omega^L, \Omega^H, \pi^Q, \eta_0^{LH}, \eta_Y^{LH}, \eta_0^{HL}, \eta_Y^{HL}.$$

To facilitate numerical identification of the free parameters, we imposed several additional over-identifying restrictions. The model, together with the normalizations, imply that when economy stays in a regime L or H forever, the long-run mean of the short rate is $E^{(L)}[r_t] = \delta_0^L$ and $E^{(H)}[r_t] = \delta_0^H + \delta_Y' \theta^{\mathbb{P}H}$. Given the challenge of estimating these unconditional means, we discipline our search procedure by fixing them a priori. Specifically, we use the regimes identified from the descriptive regime-switching model and compute the sample means for the one-month Treasury bill yield when the regimes are L (H) for the current month t and months $t - \ell$, where ℓ is at least 1. Using these estimates, we fix $\delta_0^L = 5.30\%/12$ and $\delta_0^H + \delta_Y' \theta^{\mathbb{P}H} = 9.20\%/12$. Also, after a preliminary exploration of model $A_0^{RS}(3)$ we set the parameters $\kappa^{\mathbb{P}L}(2,1)$, $\Sigma^H(2,1)$, $\Sigma^H(3,1)$, $\Sigma^H(3,2)$, $\lambda_Y^L(1,1)$, $\lambda_Y^L(2,1)$, $\lambda_Y^L(2,2)$, $\lambda_Y^L(3,2)$, $\lambda_0^H(1)$, $\lambda_0^H(2)$, $\lambda_0^H(3)$, $\lambda_Y^H(1,3)$, $\lambda_Y^H(2,3)$, $\lambda_Y^H(3,2)$, $\lambda_Y^H(3,3)$, $\eta_Y^{LH}(2)$, and $\eta_Y^{HL}(1)$ to 0, because they were small relative to their estimated standard errors.

A likelihood ratio test of the null hypothesis that $\pi^{\mathbb{P}} = $ constant – regime-shift risk is priced, but the regime-shift probabilities, and hence the $MPRS$ risks, are constants – suggests strong rejection at conventional significance levels (Table 1, row 2).[20] This, in turn, implies a strong rejection of the constraint that $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}$ (= constant) – regime-shift risk is not priced and the historical regime switching probabilities are state-independent (Table 1, row 3). Accordingly, we focus primarily on model $A_0^{RS}(3)$, occasionally comparing the results for this model with those from model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$.

Null	$\log L$	$-2(T-1)\log \frac{L}{L[A_0^{RS}(3)]}$	d.f.	p-value
$A_0^{RS}(3)$	19.54181			
$A_0^{RS}(3)[\pi^{\mathbb{P}} = \text{const}, \pi^{\mathbb{P}} \neq \pi^{\mathbb{Q}}]$	19.52343	14.079	4	0.0070
$A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$	19.52211	15.090	6	0.0196

Table 1: Likelihood ratio tests of constrained versions of model $A_0^{RS}(3)$.

The ML estimates of the parameters of model $A_0^{RS}(3)$ and their associated asymptotic standard errors are reported in Table 2.[21] The diagonal elements of Σ^H are all larger than their counterparts in Σ^L, which motivates our labelling of the two regimes. The estimates of the $\kappa^{\mathbb{P}j}$ show that the rates of mean reversion of the risk factors Y change across regimes. Equivalently, there are statistically significant differences in the state dependence of the MPF risks (in the estimated values of λ_Y^j) across regimes.

---

[20]Since, as noted above, our framework maintains the assumption that $\pi^{\mathbb{Q}}$ is constant, we cannot rule out the possibility that this finding is evidence against this auxiliary assumption. Relaxing the constraint $\pi^{\mathbb{Q}} = $ constant is an interesting topic for future research.

[21]We report $12\delta_0$, $12\delta_Y$, and $\sqrt{12}\Sigma$, i.e., annualized values for ease of interpretation.

$\log L = 19.54181$						
	Regime L			Regime H		
$12\delta_0$	5.30%			6.28%		
$12\delta_Y$	-0.000742 (0.000856)		**-0.00165** (0.00053)	**0.00851** (0.00030)		
$\kappa^{\mathbb{P}}$	**0.0348** (0.0087)			0.1075	0.1886	0.0082
		0.0206 (0.0050)		0.1515	0.3888	-0.0086
	0.1289 (0.0268)	**0.1047** (0.0250)	**0.0407** (0.0112)	0.1211	0.1047	0.0628
$\kappa^{\mathbb{Q}}$		0.0348	0.0361	0.0082		
		0.0206		-0.0086		
		0.1611	0.1047	0.0628		
$\theta^{\mathbb{P}}$				**-3.027** (1.470)	1.118 (0.966)	3.386 (1.836)
$\mathrm{diag}(\sqrt{12}\Sigma)$	1	1	1	**1.505** (0.140)	**2.027** (0.174)	**4.558** (0.314)
λ_0	-0.0635 (0.1389)	-0.0206 (0.0573)	-0.1540 (0.0958)			
λ_Y		**0.0361** (0.0059)	**0.0082** (0.0026)	-0.0727 (0.0592)	-0.1525 (0.0794)	
			-0.0086 (0.0010)	**-0.1515** (0.0728)	**-0.3682** (0.1075)	
	0.0322 (0.0199)		**0.0221** (0.0099)	0.0400 (0.0417)		
Ω	**6.52** bp (0.30 bp)			**18.19** bp (1.53 bp)		
$\pi^{\mathbb{Q}}$			$\begin{bmatrix} 92.20\% & 7.80\% \\ (14.84\%) & \\ 8.28\% & 91.72\% \\ & (12.86\%) \end{bmatrix}$			
	$L \to H$			$H \to L$		
η_0	**5.40** (1.89)			4.98 (9.09)		
η_Y	0.860 (0.616)		-0.442 (0.377)	-6.18 (8.11)	3.64 (4.68)	

Table 2: Maximum likelihood estimates and asymptotic standard errors (in parentheses) for model $A_0^{RS}(3)$. Parameters in bold face are significantly different from 0 at the 5% significance level. Those without standard errors are non-free parameters fixed by normalizations/restrictions.

Within both the H and L regimes, all three factors are stationary stochastic processes under $\mathbb{P}$. The relative magnitudes of the diagonal elements of $\kappa^{\mathbb{P}L}$ and $\kappa^{\mathbb{P}H}$ suggest that there is less mean reversion in regime L. To examine the degree of persistence more formally, we computed the eigenvalues of these matrices for each regime. Two out of the three eigenvalues (sorted in descending order) of $\kappa^{\mathbb{P}L}$ are smaller than those of $\kappa^{\mathbb{P}H}$, with the largest eigenvalue in $\kappa^{\mathbb{P}H}$ being ten times larger than its counterpart for $\kappa^{\mathbb{P}L}$:

$$\text{eig}(\kappa^{\mathbb{P}L}) = \begin{bmatrix} 0.0407 \\ 0.0348 \\ 0.0207 \end{bmatrix}, \quad \text{eig}(\kappa^{\mathbb{P}H}) = \begin{bmatrix} 0.4663 \\ 0.0802 \\ 0.0127 \end{bmatrix}. \tag{33}$$

The relatively faster rate of mean reversion in regime H, which we elaborate on subsequently, is consistent with past studies of descriptive regime-switching models (e.g., Gray [1996] and Ang and Bekaert [2002b]).

Interestingly, the eigenvalues of the mean reversion matrix under the risk-neutral measure,

$$\text{eig}(\kappa^{\mathbb{Q}}) = \begin{bmatrix} -0.00024 \\ 0.0593 + 0.0169i \\ 0.0593 - 0.0169i \end{bmatrix}, \tag{34}$$

suggest that the factors have oscillatory dynamics under $\mathbb{Q}$. More precisely, the first eigenvalue of $I - \kappa^{\mathbb{Q}}$ gives rise to a slightly explosive process (over 30 years or 360 months $1.00024^{360} = 1.09$), while the other two eigenvalues are associated with decaying oscillatory factors with a half-life of 14.6 years ($|1 - (0.0593 \pm 0.0169i)| = 0.941e^{\mp 0.0180i}$). The regime-switching pricing model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ also gives rise to similar complex eigenvalues for $\kappa^{\mathbb{Q}}$.

That one of the factors exhibits near or slightly explosive behavior under $\mathbb{Q}$ is a quite common finding in the estimation of dynamic term structure models. An illustrative example is the nonlinear, single-regime model in Duarte [2004]. The empirical feature of the data underlying this finding is the high degree of volatility of long-term bond yields. In order to sustain this level of volatility at long maturities, at least one of the latent risk factors must exhibit very slow mean reversion or, as we find, slightly explosive mean repulsion under $\mathbb{Q}$. Under $\mathbb{P}$, as noted above, all three factors are mean reverting.

With regard to the oscillatory behavior under $\mathbb{Q}$, we conjecture that it arises in part due to our assumption $A\mathbb{Q}$. As discussed more extensively below, our regime-switching model captures notable cyclical fluctuations under $\mathbb{P}$ that are captured in part through the regime dependence of several of the key parameters of the $\mathbb{P}$ distribution. Our requirement that $\kappa^{\mathbb{Q}}$ in particular be fixed across regimes largely forces the eigenvalues of $\kappa^{\mathbb{Q}}$ to capture persistent, oscillatory factor movements under $\mathbb{Q}$. A half-period of 14.5 years aligns well with the time frame for a complete transition from one regime to another (see Figure 3 below).

Within single-regime $A_0(N)$ models it is common practice, following Dai and Singleton [2000], to normalize $\kappa^{\mathbb{Q}}$ to be lower or upper triangular. Assuming that the elements of $\kappa^{\mathbb{Q}}$ are real, this normalization precludes complex eigenvalues. Viewed in the context of the present discussion, this convention in the single-regime literature may lead, as an unintended

consequence, to the inability of single-regime $A_0(N)$ models to capture the rich cyclical patterns documented here within our $A_0^{RS}(3)$ model.

The estimated values of the "intercepts" a_n^j and factor loadings b_n for yields are displayed in Figure 2. The regime dependence of the a_n^j contributes to different levels and slopes of the mean yield curves across regimes. The inverse of the matrix of the factor loadings for R^6, R^{24}, and R^{120} (or $\hat{b}^{-1}$ in (24)) indicates that

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \text{const} + \begin{bmatrix} 67.03 & -69.25 & -75.61 \\ 86.80 & -227.13 & 187.42 \\ 211.40 & -139.61 & 47.98 \end{bmatrix} \times \begin{bmatrix} R^6 \\ R^{24} \\ R^{120} \end{bmatrix}. \tag{35}$$

From the first row of the matrix pre-multiplying the yields we see that the first factor is approximately the negative of the sum of the short-term slope $R^{24} - R^6$ and the 10-year yield. The second factor displays a "curvature" characteristic with a loading ratio of 2:-5:4, and the third factor is approximately the six-month yield minus the short-term slope. This "rotation" of the factors obtained from the pricing model is different from what is obtained in standard principal component analyses, where the three factors are typically *level*, *slope* (e.g., $R_t^{120} - r_t$), and *curvature*. Of course, unlike in a principal component analysis, the components of Y exhibit substantial correlation induced by the non-zero off-diagonal elements in $\kappa^{\mathbb{P}}$.
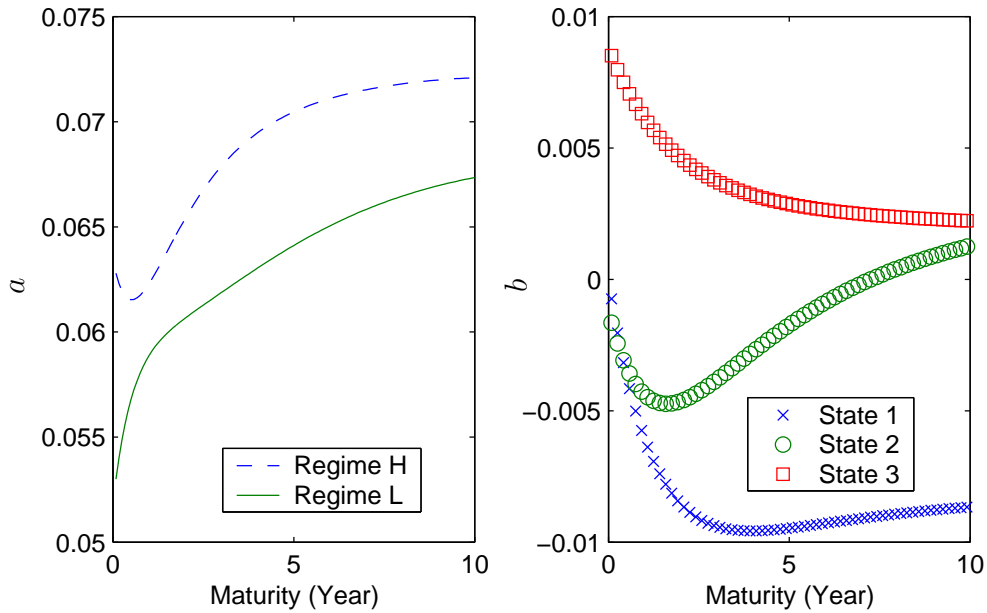

Figure 2: Estimates of the factor loadings, the a^j and b in the affine expression for bond yields.

4.1 Regime Probabilities

The filtered regime probabilities $Q_t^H = f(s_t = H | J_t)$ for models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ are displayed in Figure 3. For comparison we also plot (dotted lines) the corresponding
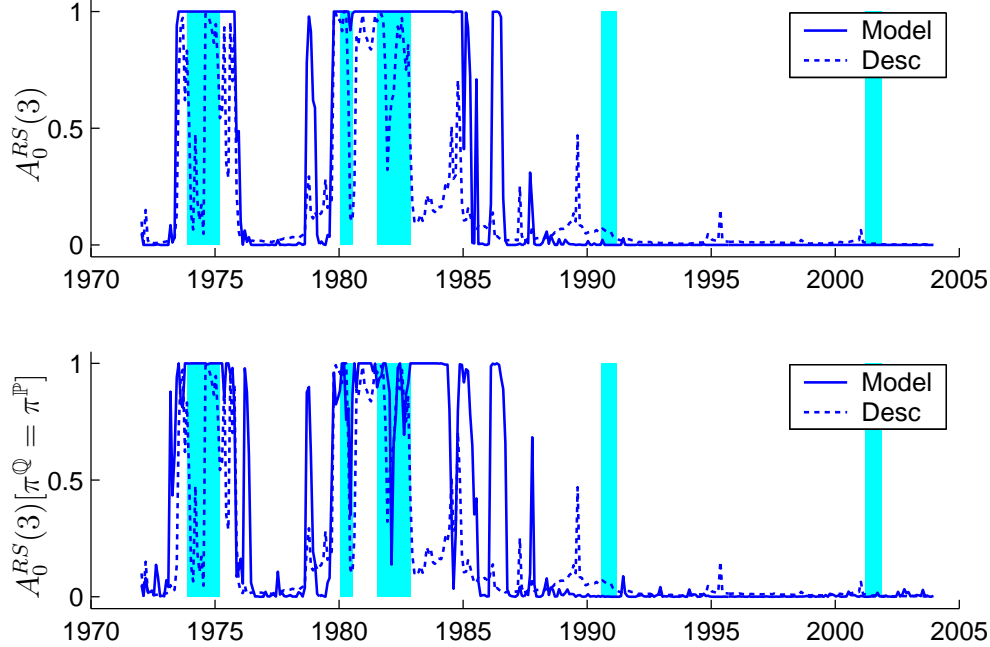
Figure 3: Filtered probabilities Q_t^H from models $A_0^{RS}(3)$ (top panel) and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ (lower panel). In each panel we have overlayed the periods of recessions according to dating by the NBER (shaded portions) and the implied Q_t^H from our descriptive model (dotted lines).

filtered probabilities from a descriptive regime-switching (DRS) model. To estimate the descriptive regime-switching model the vector PC_t of the first three principal components was computed using the covariance matrix of the 6-, 24-, and 120-month zero-coupon bond yields. Then a descriptive model for PC_t in which the state-dependent regime-switching probabilities π_t^{PC} were assumed to depend on PC_t as in (21) was estimated. The shaded periods in Figure 3 represent the periods of recessions according to NBER business-cycle dating.

These plots confirm the widely documented observation that regime H tends to be associated with recessions: both the pricing and descriptive models show that Q_t^H is larger during recessions. The two pricing models display stronger signals in the sense that Q_t^H is closer to one during recessions than that computed from model DRS. In particular, model $A_0^{RS}(3)$ appears to generate the most clear-cut predictions for both the H and L regimes. This is most evident during the period from 1980 to 1985. Both pricing models predict that regime H extended well beyond the end of 1982, when the NBER judged the recession to be over, and the Federal Reserve ended the monetary experiment. However, model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ signals a few brief instances of increased likelihood of being in regime L, while $A_0^{RS}(3)$ suggests an unambiguously persistent H regime throughout. On the other hand, although model DRS suggests that the economy was in regime H in 1985, it indicates an earlier return to the L regime in 1983. We will return to this point later.
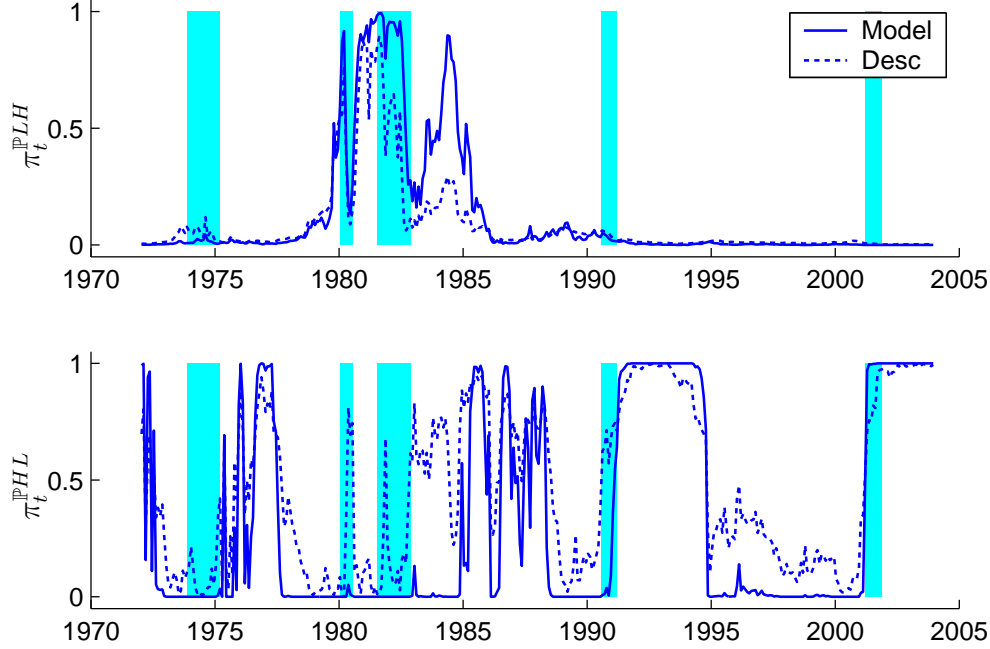
Figure 4: Transition probabilities $\pi_t^{\mathbb{P}LH}$ and $\pi_t^{\mathbb{P}HL}$, evaluated at the ML estimates, from model $A_0^{RS}(3)$. In each panel we have overlayed the periods of recessions according to dating by the NBER (shaded portions) and the model DRS implied transition probabilities (dotted lines).

The parameters governing $\pi_t^{\mathbb{P}ij}$ (Table 2) and the factor loadings in (35) imply that

$$\pi^{\mathbb{P}LH} = \left[1 + e^{13 - 122 \times R^6 - 86 \times (R^{120} - R^6) + 2 \times R^{24}}\right]^{-1},$$

$$\pi^{\mathbb{P}HL} = \left[1 + e^{3 + 234 \times R^6 - 983 \times (R^{120} - 0.91 \times R^{24})}\right]^{-1}.$$

That is, the probability of switching from regime L to regime H increases as the short-term yields or the slope of the yield curve increase. The relative magnitudes of the short rate and the slope imply that $\pi^{\mathbb{P}LH}$ is driven largely by R^6 (the correlation between $\pi^{\mathbb{P}LH}$ and R^6 is 0.80). At the same time, the probability of switching from regime H to regime L increases as the short-term yield declines and the long-term slope increases.

Figure 4 displays the probabilities $\pi_t^{\mathbb{P}LH}$ and $\pi_t^{\mathbb{P}HL}$, evaluated at the ML estimates, from models $A_0^{RS}(3)$ and DRS. Both the $A_0^{RS}(3)$-based and DRS-based estimates of $\pi_t^{\mathbb{P}LH}$ are higher during the recessionary periods in our sample. Pursuing our interpretation of regimes H and L as different stages of the business cycle, towards the end of an expansionary phase of the economy, short-term rates are often rising faster than long-term rates as a central bank's concerns about inflation puts upward pressure on short-term yields. Consistent with these observations, our econometric model shows $\pi^{\mathbb{P}LH}$ increasing as the short rate increases. On the other hand, if we are already in regime H (a recession), then short-term rates typically

have to come down far enough to induce an expansion. This is consistent with $\pi^{\mathbb{P}HL}$ rising as short-term rates fall and the long-term slope increases.

During the recessionary periods in our sample, $\pi^{\mathbb{P}LH}$ and $\pi^{\mathbb{P}HL}$ tend to move in opposite directions. That is, when the U.S. economy was in a recession, the conditional probability of moving from regime H to regime L was lower. As noted above, $\pi^{\mathbb{P}HL}$ was driven by the short-term rate R^6 and the long-term slope. During 1984 the Federal Reserve temporarily tightened monetary policy. Then in late 1984 and throughout 1985 there was a monetary easing and concurrent decline in short-term interest rates. Additionally, the striking decline in U.S. inflation rates, instigated by Volker's anti-inflation policy of the early 1980's, continued. These events show up in our model as an increase in $\pi^{\mathbb{P}HL}$ from near zero in 1984 to near unity by the end of 1985.[22]

During much of the period between 1983 and 1985, $\pi^{\mathbb{P}HL}$ is larger in model DRS than in model $A_0^{RS}(3)$. That is, the pricing model shows much more persistent risk of staying in regime H during this period, suggesting that bond markets did not view the announced shift in monetary policy in 1982 as fully credible. In addition, there were substantial swings in $\pi^{\mathbb{P}HL}$ from 1985 until early 1988. We find this interesting in the light of the fact that the Federal Reserve only weakened its dedication to monetary growth targets in October 1982 (the ending date for the "monetary experiment") and, in fact, maintained a target for $M1$ until 1987 (Friedman [2000]).[23] Consistent with these observations, the filtered probability Q_t^H from the pricing models indicates that a persistent H regime extended beyond 1983 until 1985, and was followed by another increase in Q_t^H in 1986.

For the period after 1990 the time-series of Q_t^H suggests that the economy has stayed in the L regime. On the other hand, there were a few swings in $\pi^{\mathbb{P}HL}$ during the period, with increases occurring in early 1991 and in 2001. Both of these increases were associated with increases in the short rate concurrent with the two most recent recessions dated by the NBER.

The relative sensitivities of the $\pi^{\mathbb{P}}$ to the level and slope of the yield curve may also be relevant for recent findings on the predictability of GDP growth using yield curve variables. Ang, Piazzesi, and Wei [2003] find that both *level* and *slope* have predictive content within a single-regime $DTSM$, and in particular, the short rate contains more information about the GDP growth than the slope. Our two-regime model suggests that the relative predictive contents of these variables may vary with the stage of the business cycle, and reveals a strong role of the short rate in driving the transition probabilities.

Table 3 displays the sample means of the time-varying $\pi_t^{\mathbb{P}ij}$ for the three models. Notably, with $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}} = $ constant, $\pi^{\mathbb{P}HH}$ is much larger than $\pi^{\mathbb{P}HL}$. This finding is similar to those in previous studies of both regime-switching descriptive and pricing models (e.g., Ang and Bekaert [2002b] and Bansal and Zhou [2002]) with constant transition probabilities. However,

---

[22]One concern we had was that our findings might have been affected by imposition of (rarely used) credit controls from mid-March to June in 1980. We found that our estimates changed little after we removed the abrupt dips in interest rates induced by these controls by a linear interpolation between February and July.

[23]Based on their statistical analyses, Friedman and Kuttner [1996] argue that deviations from the Federal Reserve's target for $M1$ remained a significant determinant of their monetary policy rule until mid-1984, and deviations from $M2$ were significant until mid-1985.

Model	$\bar{\pi}^{\mathbb{P}}$		$x^{\mathbb{P}}$	$\bar{Q}$
$A_0^{RS}(3)$	$\begin{bmatrix} 87.58\% & 12.42\% \\ 31.96\% & 68.04\% \end{bmatrix}$		$\begin{bmatrix} 72.00\% \\ 28.00\% \end{bmatrix}$	$\begin{bmatrix} 71.91\% \\ 28.09\% \end{bmatrix}$
DRS	$\begin{bmatrix} 91.38\% & 8.62\% \\ 46.88\% & 53.52\% \end{bmatrix}$		$\begin{bmatrix} 84.36\% \\ 15.64\% \end{bmatrix}$	$\begin{bmatrix} 83.30\% \\ 16.70\% \end{bmatrix}$
$A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$	$\begin{bmatrix} 97.72\% & 2.28\% \\ 5.52\% & 94.48\% \end{bmatrix}$		$\begin{bmatrix} 70.75\% \\ 29.25\% \end{bmatrix}$	$\begin{bmatrix} 71.92\% \\ 28.08\% \end{bmatrix}$

Table 3: Sample means of the transition probabilities $\pi^{\mathbb{P}}$, the stable probability distribution $x^{\mathbb{P}}$ implied by the mean transition matrices, and the sample means of the fitted probabilities (Q_t^L, Q_t^H). For model $A_0^{RS}(3)$ and the descriptive model DRS the transition probabilities are time-varying (state-dependent), while for model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$, are constant.

our model with time-varying regime-switching probabilities ($A_0^{RS}(3)$) for U.S. treasury data gives a smaller difference between $\pi_t^{\mathbb{P}HH}$ and $\pi_t^{\mathbb{P}HL}$ on average, and suggests that regime H was notably less persistent on average than regime L. If we view regime H as capturing periods of downturns and regime L as periods of expansions, consistent with our previous discussion of NBER business cycles and the probability Q_t^H, then this finding can be viewed as a manifestation of the well documented asymmetry in U.S. business cycles: recoveries tend to take longer than contractions (see, e.g., Neftci [1984] and Hamilton [1989]). Model $A_0^{RS}(3)$ with priced, state-dependent regime shift risk captures this asymmetry, but model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ with constant regime-shift probabilities does not.[24]

Table 3 also reveals a close match between $x^{\mathbb{P}}$, the stable probabilities implied by the mean transition matrix[25] and $\bar{Q}$, the sample means of the fitted probabilities (Q_t^L, Q_t^H). Interestingly, both pricing models generate very similar $x^{\mathbb{P}}$ and $\bar{Q}$, although the mean transition matrices are dramatically different. In particular, both $\pi^{\mathbb{P}LL}$ and $\pi^{\mathbb{P}HH}$ are higher in model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ than in model $A_0^{RS}(3)$. Hence, models with constant transition probabilities not only overstate the persistence of the H regime, but also exaggerate the persistence of the L regime in order to match the historical distribution of "residence" in the two regimes.

The estimated risk-neutral transition probabilities from model $A_0^{RS}(3)$ (shown in Table 2) imply an invariant distribution of $x^{\mathbb{Q}} = [51.50\% \quad 48.50\%]'$. Comparing the stable probabilities $x^{\mathbb{Q}}$ and $x^{\mathbb{P}}$, it is seen that the economy spends much more time in regime H and much less time in regime L under $\mathbb{Q}$ than under $\mathbb{P}$. This is intuitive since, with risk-averse bond investors, risk-neutral pricing will recover market prices for bonds only if we treat the

---

[24]We also confirm $\pi^{\mathbb{P}HH} \gg \pi^{\mathbb{P}HL}$ in the model with constant transition probabilities but priced regime shifts (row 2 in Table 1).

[25]For a constant transition matrix Π, the stable (stationary, invariant) distribution x is defined by the equation $\Pi'x = x$. Equivalently, x is the limit of $\Pi'^n x$, as $n \to \infty$. Finite-state Markov chains for which all of the elements of the transition matrix are positive are positive recurrent and irreducible, and have unique invariant distributions.

"bad" H regime as being more likely to occur than in actuality. The diagonal elements of $\pi^{\mathbb{Q}}$ are statistically different from the means of the corresponding elements in $\pi^{\mathbb{P}}$.

4.2   Model-Implied Means and Volatilities of Bond Yields

Figure 5 compares the sample and the model-implied means of the Treasury yields and standard deviations (volatilities) of the monthly yield changes. To obtain the model-implied means and volatilities, we treated the ML estimates as the true population parameters and simulated 1000 time series of yields, each with the same length as that of our historical data (384 months). Then, conditional on either the L or H regime, we computed the mean of the yields and the volatility of the monthly yield changes for each simulated series, and plotted the average and two standard deviation bands for these 1000 means and volatilities.

To construct a sample counterpart, we compute the smoothed probabilities q_t^j given by (32), and then classify a date as being in regime L if $q_t^L \geq 0.5$ or in regime H if $q_t^H > 0.5$. After sorting the dates, we compute the sample means and volatilities of the yields in each regime. These are reported as *Sample* in the plots. Figure 5 suggests that the model does a very good job at matching the first and second unconditional moments in the data, as the sample curves fall well within the two standard deviation bands of the simulated curves. The mean yield curves are upward sloping in both regimes, with the yields being notably higher in regime H.

Of particular note are the shapes of the volatility curves in the two regimes. It is well known that in many U.S. fixed-income markets (e.g., Treasury bond, swaps, etc.), the term structures of unconditional yield volatilities are hump-shaped (see, e.g., Litterman, Scheinkman, and Weiss [1991]), with the peak of the hump being approximately at two years to maturity.[26] Under our classification of dates into regimes, the hump in volatility is an L-regime phenomenon. Fleming and Remolona [1999] present evidence linking the hump to market reactions to macroeconomic announcements. Through the lens of our model, it appears that these, and possibly other, sources of yield volatility show up as a hump in volatility primarily during relatively tranquil, expansionary phases of the business cycle. When the economy is in regime H, volatility is high and the risk factors mean-revert to their long-run means relatively quickly ($\kappa^{\mathbb{P}H}$ in Table 2). The fast mean reversion in regime H swamps a humped reaction (if any) to macroeconomic news, and induces the steeply downward sloping term structure of (unconditional) volatility.

Pursuing the latter point, it is the interaction between the factor correlations and their rates of mean reversion that largely induce humped-shaped term structures of volatility in dynamic term structure models (Dai and Singleton [2000]). Indeed, when we estimated a restricted version of model $A_0^{RS}(3)$ with diagonal $\kappa^{\mathbb{P}j}$, Σ^j, and λ_Y^j matrices, simulations confirmed that mean reversion induces downward sloping term structures of volatility in both the H and L regimes. This is why we highlight the flexibility associated with correlated

---

[26]Figure 5 also shows the "snake" shaped pattern in historical yield volatilities for very short-term bonds. This pattern is partially captured by our three-factor model. The findings in both Longstaff, Santa-Clara, and Schwartz [2001] and Piazzesi [2005] suggest that the addition of a fourth factor would allow our model to replicate this pattern even better.
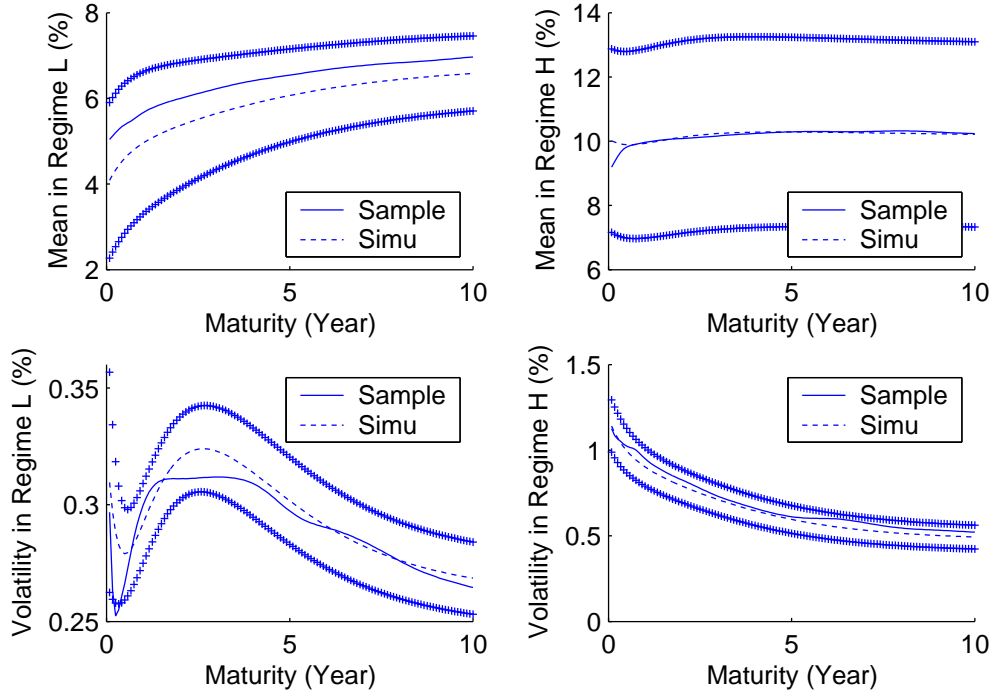
Figure 5: Term structures of unconditional means of Treasury bond yields and volatilities of monthly yield changes implied by model $A_0^{RS}(3)$. The solid lines show the sample results, obtained by computing sample means and volatilities after allocating dates to regimes based on the smoothed probabilities q_t^j. The dotted lines and the crosses plot the averages and two standard deviations for the means and volatilities computed from the time series of yields simulated from the model. Both means and volatilities are annualized.

factors in Gaussian affine models relative to multi-factor *CIR* models with independent factors. Ang and Bekaert [2005] also constrain the "level" and "slope" (latent) factors in their regime-switching Gaussian models to be mutually independent within all regimes.

In unreported results, we also confirm that the simulated curves from model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$, in which regime-shift risk is not priced and regime-switching probabilities are state-independent, perform similarly well in matching the sample mean and volatility curves. By and large, there is not a large difference between the model-implied first and second unconditional moments across these two models.

We examine the model-implied conditional volatilities in Section 6 as part of our assessment of the robustness of the properties of model $A_0^{RS}(3)$ to the presence of within-regime time-varying volatility.

5 Excess Returns and Market Prices of Risk

In this section we return to one of the primary motivations for our analysis, namely, an investigation of the contributions of factor and regime-shift risk premiums to the temporal

variation in expected excess returns.

We start by presenting the decomposition of expected excess returns into components associated with regime-shift and factor risks. Let $p_{t,n}^j \equiv \log D_{t,n}^j$ denote the log price of a n-period bond at time t and in regime $s_t = j$. The one-period expected excess return on the n-period bond is (see appendix A for more details)

$$E_t[p_{t+1,n-1}|s_t = j] - p_{t,n}^j + p_{t,1}^j = \rho_{t,n}^{RSj} + \rho_{t,n}^{Fj}, \tag{36}$$

where

$$\rho_{t,n}^{RSj} = \log \frac{e^{\sum_{k=0}^{S} \pi_t^{\mathbb{P}jk}(-A_{n-1}^k)}}{\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A_{n-1}^k}}, \tag{37}$$

$$\rho_{t,n}^{Fj} = -\frac{1}{2} B_{n-1}' \Sigma^j \Sigma^{j'} B_{n-1} - B_{n-1}' \Sigma^j \Lambda_t^j. \tag{38}$$

Since econometricians do not observe the regimes, we evaluate the expected excess returns conditional on J_t:

$$\begin{aligned} E_t[p_{t+1,n-1} - p_{t,n} + p_{t,1}|J_t] &= \sum_j E_t[p_{t+1,n-1} - p_{t,n} + p_{t,1}|s_t = j, J_t]Q_t^j \\ &= \sum_j \rho_{t,n}^{RSj} Q_t^j + \sum_j \rho_{t,n}^{Fj} Q_t^j \equiv \rho_{t,n}^{RS} + \rho_{t,n}^F, \end{aligned}$$

where the regime-specific components $\rho_{t,n}^{RSj}$ and $\rho_{t,n}^{Fj}$ are weighted by regime probabilities Q_t^j.

The regime-shift component $\rho_{t,n}^{RSj}$ is determined largely by the difference between the historical and risk neutral transition probabilities and the regime dependence of A_{n-1}^k. This can be seen more clearly from its linear expansion,

$$\rho_{t,n}^{RSj} \approx \sum_{k=0}^{S} (\pi^{\mathbb{Q}jk} - \pi_t^{\mathbb{P}jk}) A_{n-1}^k \tag{39}$$

$$= \begin{cases} (\pi^{\mathbb{Q}LH} - \pi_t^{\mathbb{P}LH})(A_{n-1}^H - A_{n-1}^L), & \text{if } s_t = L, \\ (\pi^{\mathbb{Q}HL} - \pi_t^{\mathbb{P}HL})(A_{n-1}^L - A_{n-1}^H), & \text{if } s_t = H. \end{cases} \tag{40}$$

Though $\rho_{t,n}^{RSj}$ is nonzero even if $\pi^{\mathbb{Q}jk} = \pi^{\mathbb{P}jk}$, due to the convexity effect associated with continuously compounded returns, the quantitative importance of this convexity effect is negligible (see below). Thus, the within-regime variation in $\rho_{t,n}^{RSj}$ is determined largely by time variation in $\pi_t^{\mathbb{P}jk}$ and, hence, the historical probabilities of a change in regime potentially play a central role in the temporal variation in expected excess returns.

The convexity effect also produces the first term in $\rho_{t,n}^{Fj}$. The second term, on the other hand, is proportional to the market price of factor risk, Λ_t^j, and the amount of exposure to the factor risk, $B_{n-1}' \Sigma^j$. Substituting $\Sigma^j \Lambda_t^j = \lambda_0^j + \lambda_Y^j Y_t$ into (38), we see that a constant MPF risk ($\lambda_0^j \neq 0, \lambda_Y^j = 0$) would only induce regime specific, constant expected returns.

Consequently, with $\lambda_Y^j = 0$, the only source of temporal variation in the component $\rho_{t,n}^F$ of expected excess returns is the variation in the Q_t^j. By allowing for $\lambda_Y^j \neq 0$ we introduce within-regime variation in ρ_t^{Fj} and, thereby, allow for much richer historical patterns in the variation of the component $\rho_{t,n}^F$.

Figure 6 plots $\rho_{t,n}^{RS}$ and $\rho_{t,n}^F$ for $n = 24$ and 120 months. During extended L (H) regimes we observe persistent positive (negative) levels of the regime shift component of the expected excess returns. Intuitively, during the L regime the physical probability of switching to the H regime is extremely low (almost zero), lower than the $\pi^{\mathbb{Q}LH}$. The bonds are priced in the markets as if the probabilities of going into recessions are higher under the risk-neutral measure. This pushes down the current bond prices, yielding a positive expected return component. Similarly, during the H regime, the relatively higher risk neutral probability of switching to the L regime pushes up the current bond prices and yields a negative expected return. The magnitudes of these persistent levels of the regime-shift components are about 0.2 to 0.3% (monthly) on a ten-year bond, in comparison to a 0.6% standard deviation of the factor risk component.

The large spikes around the mid-1970's and mid-1980's in the regime-shift risk component are attributable to $\rho_{t,n}^{RS\,H}$, the H regime component, and thus are associated with the swings in the $\pi^{\mathbb{P}HL}$ during these two periods. We have noted earlier that the mid-1980's episode suggests investors doubted the credibility of the Federal Reserve's announced change in monetary policy. These spikes are completely missed in the model-implied expected returns for the single-regime $A_0(3)$ model (see Figure 1).

The bottom panel of Figure 6 decomposes the factor risk component into values during the L and H regimes based on model $A_0^{RS}(3)$ (the $\rho_{t,n}^{Fj}$ in (38)). Consistent with the view that expected returns should not fluctuate dramatically under "normal" circumstances, the curves are much smoother in regime L (thick line) than in regime H (thin line).[27] A very different impression comes from inspection of the expected excess returns from the corresponding single-regime Gaussian $A_0(3)$ model displayed in Figure 1 (induced solely by factor risks). They look much more like the choppy patterns during regime H than the relatively smooth behavior during regime L. This finding lends support to a basic premise of this paper; namely, omission of the regime-switching process tends to distort the model-implied (factor risk component of) excess returns both in tranquil and turbulent times.

To demonstrate the critical role of state-dependent $MPRS$ risks in capturing the variations in excess returns, Figure 7 plots the regime-shift component of expected returns on a ten-year bond implied by three models, ordered from the least to the most flexible specifications of the $MPRS$ risk. Model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ restricts $\Gamma_t^{jk} = 0$, and the regime-shift component of excess returns is comprised solely of the convexity term. As anticipated, the magnitudes are negligible, with maximums of about 0.003%. Allowing for a constant, nonzero Γ_t^{jk} in model $A_0^{RS}(3)[\pi^{\mathbb{P}} = \text{const}, \pi^{\mathbb{P}} \neq \pi^{\mathbb{Q}}]$ gives rise to persistent positive (negative) contributions to excess returns within the L (H) regime. However, relative to our most flexi-

---

[27]The regime dependent characteristics of ρ_t^F are attributable to the corresponding market prices of factor risks in L and H regimes. We confirm in unreported results that the market prices of factor risks are much smoother in the L regime than in the H regime.
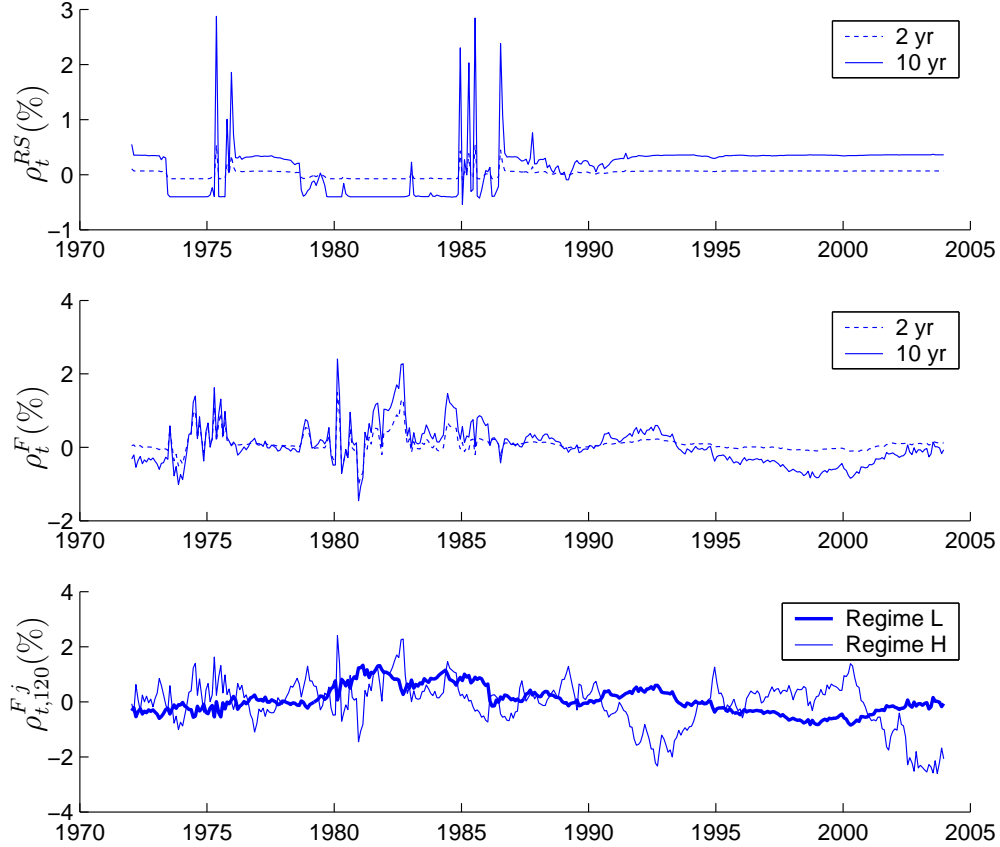
Figure 6: Regime-shift (top) and factor (middle) risk components of one-month ahead expected excess returns for two- and ten-year bonds implied by model $A_0^{RS}(3)$. The bottom panel further displays the factor risk component of the expected excess returns for ten-year bond in the L and H regimes.

ble model $A_0^{RS}(3)$, model $A_0^{RS}(3)[\pi^{\mathbb{P}} = \text{const}, \pi^{\mathbb{P}} \neq \pi^{\mathbb{Q}}]$ understates the average contributions of regime-shift risk to excess returns, and completely misses the large spikes in excess returns during regime transitions. Model $A_0^{RS}(3)$ suggests that there is a substantial contribution of the state-dependent $MPRS$ risk to excess returns, particularly during transitions between turbulent and tranquil periods.

Finally, regarding the predictability of excess returns on bonds, the empirical results in Duffee [2002] and Dai and Singleton [2002] suggest that, within the family of single-regime affine $DTSM$s, the rich state dependence of the market prices of factor risks accommodated by Gaussian models is essential for predictability puzzles associated with violation of the "expectations theory" of the term structure (e.g., Campbell and Shiller [1991]). Since our $A_0^{RS}(3)$ model nests single-regime Gaussian models it is not surprising that it also does a reasonable job of matching the Campbell-Shiller evidence against the expectations theory.
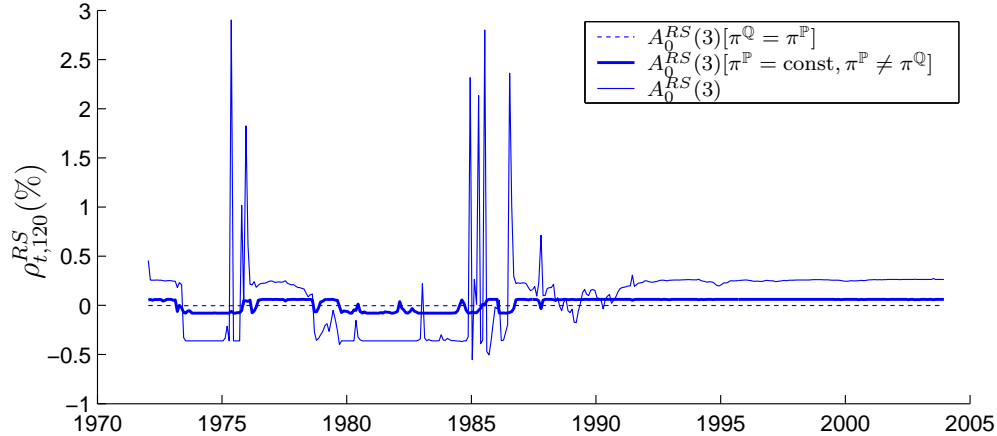
Figure 7: Regime-shift risk component of one-month ahead expected excess returns for ten-year bonds implied by various models.

6   Concluding Remarks

In this paper, we show that regime switching term structure models in which regime transition probabilities are constant and equal under both physical and risk-neutral measures may potentially give a mis-leading impression of the dynamics of expected bond returns and the relationship between the shape of the term structure and business cycle fluctuations. Likelihood ratio tests formally reject the case of constant regime transition probabilities in favor of a model with state-dependent regime transition probabilities and market prices of regime-shift risk. In concluding this paper, we point out some limitations/caveats of our analysis.

First, in order to price bonds analytically, we have imposed some parametric restrictions on the joint dynamics of the state vector and the Markov regime switching process under the risk-neutral measure. These restrictions preclude examination of a model in which regime-shift risk is priced and the regime transition probabilities are state-dependent under both physical and risk-neutral measures (as in Boudoukh, Richardson, Smith, and Whitelaw [1999]), or a model in which factor loadings on bond yields are allowed to be regime-dependent. We could relax these constraints, but at the cost of introducing approximations to both pricing and likelihood functions. Following the tradition of the large single-regime term structure literature, it seemed worthwhile to explore how far one could go in improving the fits over single-regime affine models, while preserving the analytical tractability of this family.

Perhaps of greater concern is the fact that our empirical study is based on the assumption that the state vector conditional on a regime is an autoregressive Gaussian state process. The regime dependence of both the level and the volatility of the short-term interest rates in model $A_0^{RS}(3)$ induce time varying, and in particular level dependence, of the volatilities of bond yields of all maturities. However, we are unable to accommodate level dependence of volatilities within each regime, as incorporated in the models of Naik and Lee [1997] and
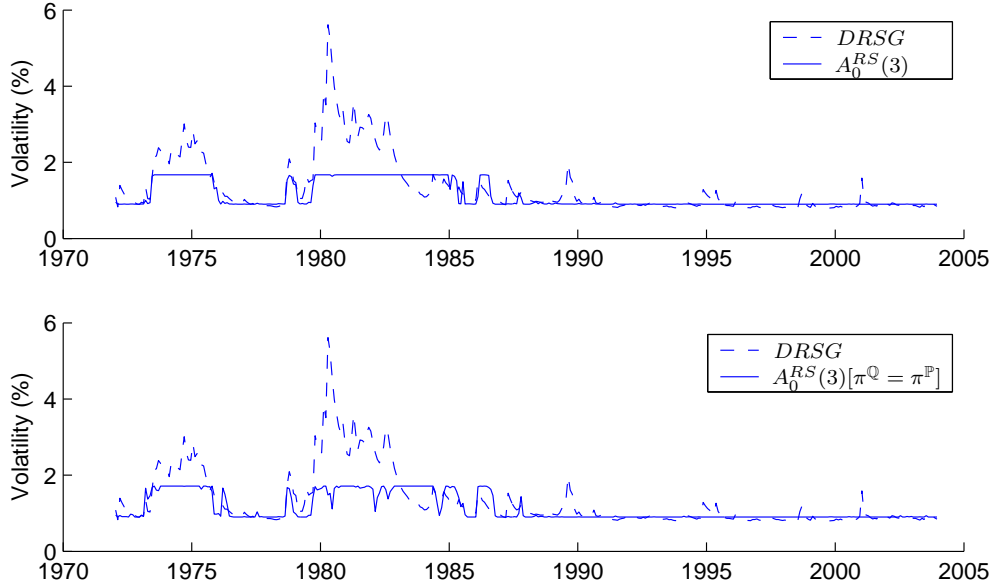
Figure 8: Conditional volatilities of ten-year bond yields from models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ plotted against the implied volatility from a descriptive regime-switching $GARCH$ model ($DRSG$).

Bansal and Zhou [2002].

To gain some insight into how models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ perform relative to a model with time-varying volatility within each regime, we extended our descriptive model for the first three principal components of bond yields to allow the volatility of each principal component in each regime to follow a $GARCH(1,1)$ process (model $DRSG$).[28] Figure 8 displays the one-month ahead conditional volatilities for the ten-year bond yield from our pricing models against those from model $DRSG$.[29] Perhaps the most striking feature of this figure is the fact that our pricing models understate conditional volatility relative to model $DRSG$ during the monetary experiment of the early 1980's. (This is also true, but to a lesser degree, for the spike up in volatility around 1975.)

Of particular concern to us was the robustness (to the presence of time-varying volatility) of our finding that regime-switching $DTSM$s with state-independent regime switching probabilities (constant $\pi^{\mathbb{P}}$) are over-stating the persistence of the high volatility regime H. Equation (41) presents the average value of $\pi^{\mathbb{P}}$ from the descriptive model $DRSG$. The esti-

---

[28]The parameters of the $GARCH$ processes were allowed to differ both across principal components and across regimes. The spirit of this analysis is a multi-variate version of the switching $GARCH$ model examined by Gray [1996]. However, we set up our switching $GARCH$ model using the same timing conventions as in our pricing model.

[29]The analogous pictures for yields on bonds with shorter maturities show higher levels of volatility, but very similar temporal patterns.

mates are very similar to those from the model DRS:

$$\bar{\pi}^{DRSG} = \left[\begin{array}{cc} 91.39\% & 8.61\% \\ 49.11\% & 50.89\% \end{array} \right]. \tag{41}$$

This extended descriptive analysis with model $DRSG$ does not, of course, allow us to assess the implications of within-regime time-varying volatility for the structure of the market prices of factor or regime-shift risks. Such an assessment would require a regime-shifting $DTSM$ that allows for both within regime stochastic volatility and state-dependent regime-shift probabilities. Nevertheless, given the similarity between the results for models DRS and $DRSG$, we are reassured that some of our key findings– in particular, the asymmetry in the persistence of regimes– are robust to extended specifications of volatility beyond what is inherent in model $A_0^{RS}(3)$.

Finally, a natural question is whether our findings are sensitive to our choice of sample period. The nature of a regime-switching model is such that the answer has to be (a qualified) yes. As documented above, in both our pricing and descriptive regime-switching models, the regimes identified by our model are related to stages of the business cycle. For such an identification to be feasible, it is essential that the sample period span a sufficient number of cycles. Otherwise, the flexibility of a regime-switching model will largely be used to capture relatively minor within-cycle variations in the conditional distributions of bond yields.

This was confirmed upon re-estimation of model $A_0^{RS}(3)$ over the post-1987 sample. For this shorter sample, the two-regime model associated the period 1987–1992 with the H regime, and the post-1992 with the L regime. The pre-1992 H regime spans the recession of the early 1990's. Not unexpectedly, the differences in the volatilities of the factors across the H and L regimes are much smaller than those obtained for the full sample period (the diagonal elements of $\sqrt{12}\Sigma^H$ are $(0.96, 1.06, 1.26)$ in the shortened sample compared to $(1.51, 2.03, 4.56)$ in the full sample). This is consistent with the full-sample results, which treat the post-1987 era as a homogeneous L regime. Many other features of the full-sample results, including faster rates of mean reversion in the H regime and asymmetry in the matrix $\pi^{\mathbb{P}}$ when these probabilities are state-dependent, remain qualitatively the same in the post-1987 period. These findings provide further assurance that our key results are robust.

A Decomposition of Expected Excess Returns

The price of a n-period bond is

$$D_{t,n}^j = e^{-A_n^j - B_n' Y_t}.$$

The log price is

$$
\begin{aligned}
p_{t,n}^j &\equiv \log D_{t,n}^j = \log E_t^{\mathbb{Q}} \left[e^{-r_t^j} D_{t+1,n-1} | s_t = j \right] \\
&= -r_t^j + \log \left(\sum_{k=0}^S \pi^{\mathbb{Q}jk} E_t^{\mathbb{Q}} \left[D_{t+1,n-1}^k | s_t = j \right] \right) \\
&= -r_t^j + \log \left(\sum_{k=0}^S \pi^{\mathbb{Q}jk} e^{-A_{n-1}^k} E_t^{\mathbb{Q}} \left[e^{-B_{n-1}' Y_{t+1}} | s_t = j \right] \right) \\
&= -r_t^j + \log \left(\sum_{k=0}^S \pi^{\mathbb{Q}jk} e^{-A_{n-1}^k} \right) - B_{n-1}' \mu_t^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}' \Sigma^j \Sigma^{j'} B_{n-1}.
\end{aligned}
$$

The expected value of the log price is

$$
\begin{aligned}
& E_t[p_{t+1,n} | s_t = j] \\
&= \sum_{k=0}^S \pi_t^{\mathbb{P}jk} E_t[p_{t+1,n}^k | s_t = j] \\
&= \sum_{k=0}^S \pi_t^{\mathbb{P}jk} \left(-A_n^k - B_n' E_t[Y_{t+1} | s_t = j] \right) \\
&= - \left(\sum_{k=0}^S \pi_t^{\mathbb{P}jk} A_n^k \right) - B_n' \mu_t^{\mathbb{P}j}.
\end{aligned}
$$

The expected excess return is then

$$
\begin{aligned}
& E_t[p_{t+1,n-1} | s_t = j] - p_{t,n}^j + p_{t,1}^j \\
&= - \left(\sum_{k=0}^S \pi_t^{\mathbb{P}jk} A_{n-1}^k \right) - B_{n-1}' \mu_t^{\mathbb{P}j} \\
&\quad + r_t^j - \log \left(\sum_{k=0}^S \pi^{\mathbb{Q}jk} e^{-A_{n-1}^k} \right) + B_{n-1}' \mu_t^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}' \Sigma^j \Sigma^{j'} B_{n-1} \\
&\quad - r_t^j \\
&= \log \frac{e^{\sum_{k=0}^S \pi_t^{\mathbb{P}jk} (-A_{n-1}^k)}}{\sum_{k=0}^S \pi^{\mathbb{Q}jk} e^{-A_{n-1}^k}} - \frac{1}{2} B_{n-1}' \Sigma^j \Sigma^{j'} B_{n-1} - B_{n-1}' \Sigma^j \Lambda_t^j.
\end{aligned}
$$

References

Ait-Sahalia, Y. (1996). Testing Continuous-Time Models of the Spot Interest Rate. *Review of Financial Studies 9*(2), 385 – 426.

Ang, A. and G. Bekaert (2002a). Regime Switches in Interest Rates. *Journal of Business and Economic Statistics 20*, 163–182.

Ang, A. and G. Bekaert (2002b). Short Rate Nonlinearities and Regime Switches. *Journal of Economic Dynamics and Control 26*, 1243–1274.

Ang, A. and G. Bekaert (2005). The Term Structure of Real Rates and Expected Inflation. Working paper, Columbia University.

Ang, A. and M. Piazzesi (2003). A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables. *Journal of Monetary Economics 50*, 745–787.

Ang, A., M. Piazzesi, and M. Wei (2003). What Does the Yield Curve Tell us About GDP Growth? Working Paper, Columbia University.

Bansal, R. and H. Zhou (2002). Term Structure of Interest Rates with Regime Shifts. *Journal of Finance 57*, 1997 – 2043.

Bekaert, G. and S. Grenadier (2001). Stock and Bond Pricing in an Affine Equilibrium. Working Paper, Stanford University.

Bliss, R. R. (1997). Testing Term Structure Estimation Methods. *Advances in Futures and Options Research 9*, 197–231.

Boudoukh, J., M. Richardson, T. Smith, and R. F. Whitelaw (1999). Regime Shifts and Bond Returns. Working paper, New York University.

Campbell, J. and R. Shiller (1991). Yield Spreads and Interest Rate Movements: A Bird's Eye View. *Review of Economic Studies 58*, 495–514.

Cecchetti, S., P. Lam, and N. Mark (1993). The Equity Premium and the Risk-free Rate: Matching the Moments. *Journal of Monetary Economics 31*, 21–45.

Chapman, D. and N. Pearson (2000). Is the Short-Rate Drift Actually Nonlinear? *Journal of Finance 55*, 355–388.

Chen, R. and L. Scott (1995). Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure. *Journal of Fixed Income Winter*, 53–72.

Dai, Q. and K. Singleton (2000). Specification Analysis of Affine Term Structure Models. *Journal of Finance 55*, 1943–1978.

Dai, Q. and K. Singleton (2002). Expectations Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure. *Journal of Financial Economics 63*, 415–441.

Dai, Q. and K. Singleton (2003). Term Structure Dynamics in Theory and Reality. *Review of Financial Studies 16*, 631–678.

Duarte, J. (2004). Evaluating an Alternative Risk Preference in Affine Term Structure Models. *Review of Financial Studies 17*, 379–404.

Duffee, G. R. (2002). Term Premia and Interest Rate Forecasts in Affine Models. *Journal of Finance 57*, 405–443.

Duffie, D. and K. Singleton (1997). An Econometric Model of the Term Structure of Interest Rate Swap Yields. *Journal of Finance 52*, 1287–1321.

Fama, E. F. and R. R. Bliss (1987). The Information in Long-Maturity Forward Rates. *American Economic Review 77*(4), 680–692.

Fisher, M., D. Nychka, and D. Zervos (1995). Fitting the Term Structure of Interest Rates with Smoothing Splines. Working Paper 95-1, Finance and Economics Discussion Series Federal Reserve Board.

Fleming, M. J. and E. Remolona (1999). The Term Structure of Announcement Effects. FRB New York Staff Report No. 76.

Friedman, B. (2000). The Role of Interest Rates in Federal Reserve Policy Making. NBER Working Paper 8047.

Friedman, B. and K. Kuttner (1996). A Price Target for U.S. Monetary Policy? Lessons from the Experience with Monetary Growth Targets. *Brookings Papers on Economic Activity 1*, 77–146.

Garcia, R. and P. Perron (1996). An Analysis of the Real Interest Rate Under Regime Shifts. *Review of Economics and Statistics*, 111–125.

Gourieroux, C., A. Monfort, and V. Polimenis (2002). Affine Term Structure Models. Working paper, University of Toronto, Canada.

Gray, S. (1996). Modeling the Conditional Distribution of Interest Rates as a Regime Switching Process. *Journal of Financial Economics 42*, 27–62.

Hamilton, J. (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica 57*, 357–384.

Landen, C. (2000). Bond Pricing in a Hidden Markov Model of the Short Rate. *Finance and Stochastics 4*, 371–389.

Litterman, R., J. Scheinkman, and L. Weiss (1991). Volatility and the Yield Curve. *Journal of Fixed Income 1*, 49–53.

Liu, J., F. Longstaff, and J. Pan (2002). Dynamic Asset Allocation with Event Risk. forthcoming, *Journal of Finance*.

Longstaff, F., P. Santa-Clara, and E. Schwartz (2001). The Relative Valuation of Caps and Swaptions: Theory and Empirical Evidence. *Journal of Finance 56*, 2067–2109.

McCulloch, J. (1975). The Tax-Adjusted Yield Curve. *Journal of Finance 30*, 811–830.

Naik, V. and M. H. Lee (1997). Yield Curve Dynamics with Discrete Shifts in Economic Regimes: Theory and Estimation. Working paper, Faculty of Commerce, University of British Columbia.

Neftci, S. (1984). Are Economic Time Series Asymmetric Over the Business Cycle? *Journal of Political Economy 92*, 307–328.

Nelson, C. and A. Siegel (1987). Parsimonious Modelling of Yield Curves. *Journal of Business 60*, 473–489.

Pan, J. (2002). The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study. *Journal of Financial Economics 63*, 3–50.

Pearson, N. D. and T. Sun (1994). Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model. *Journal of Finance 49*, 1279–1304.

Piazzesi, M. (2005). Bond Yields and the Federal Reserve. *Journal of Political Economy 113*, 311–344.

Stanton, R. (1997). A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk. *Journal of Finance 52*, 1973–2002.

Veronesi, P. and F. Yared (2000). Short and Long Horizon Term and Inflation Risk Premia in the U.S. Term Structure: Evidence from an Integrated Model for Nominal and Real Bond Prices under Regime Shifts. Working Paper, University of Chicago.

Waggoner, D. (1997). Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices. Working Paper, Federal Reserve Bank Atlanta.

Wu, S. and Y. Zeng (2003). A General Equilibrium Model of the Term Structure of Interest Rates under Regime-switching Risk. Working Paper, University of Kansas.