REGIME SWITCHING AND BOND PRICING

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Abstract
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In this paper we propose an overview of the usefulness of the regime switching approach for building various kinds of bond pricing models and of the roles played by the regimes in these models. The objective of the pricing models in which regimes appear can be to price default-free or defaultable bonds, or to analyse simultaneously credit ratings and defaultable bonds prices. The roles of the regimes can be to capture stochastic drifts and/or volatilities, to represent discrete values of a target rate, to incorporate business cycle or crises effects, to isolate flight-to-quality or contagion effects, to reproduce zero lower bound spells, or to evaluate the impact of standard or non-standard monetary policies. From a technical point of view, we stress the key role of Markov chains, Compound Autoregressive (Car) processes, Regime Switching Car processes and multihorizon Laplace transforms.

Keywords : Term Structure, Regime Switching, Affine Models, Car Process, Multihorizon Laplace Transform, Contagion, Default Risk, Monetary Policy.

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1 INTRODUCTION

Regime switching models have been widely used in Financial Econometrics. The domains of applications include the analysis of stock returns [see e.g. Hamilton, Susmel (1994), Billio, Pelizzon (2000), Ang, Bekaert (2002)a, Ang, Chen (2002)], exchange rates [see e.g. Engel, Hamilton (1990), Bekaert, Hodrick (1993)], asset allocations [see Ang, Bekaert (2002), (2004), Guidolin, Timmerman (2008), Tu (2010)], electricity prices [Huisman, Mahieu (2003), Mount, Ning, Cai (2005), Monfort, Feron (2012)], or systemic risk [Billio, Getmansky, Lo, Pelizzon (2011)]. See also the survey paper by Ang, Timmerman (2011).

However, it is in the modeling of default-free interest rates that the regime switching approach is the more frequent. A first stream of literature does not consider the pricing problem, but shows how the introduction of switching regimes can improve the properties of dynamic models of interest rates in terms of persistence, of fitting or forecasting of the yields, or of their unconditional and conditional moments [see e.g. Hamilton (1988), Garcia, Perron (1996), Ang, Beckaert (2002)b and (2002)c]. A second stream of literature focuses on the pricing problem and incorporates switching regimes in a simultaneous modelings of historical dynamics, risk-neutral dynamics and stochastic discount factor, in order to evaluate market prices of risks, risk premia or term premia [see e.g. Bansal, Zhou (2002), Dai, Singleton, Yang (2007), Monfort, Pegoraro (2007), Ang, Bekaert, Wei (2008), Chib, Kang (2012)].

In both kinds of literature the switching regimes are latent, that is to say not observed by the econometrician. More recently such latent switching has been introduced in the modeling of defaultable bond prices [see Monfort, Renne (2011)a, (2011)b] and credit ratings; in the latter case the latent regimes are introduced in addition to the observed regimes representing the ratings as already considered for instance by Jarrow, Lando, Turnbull (1997).

The present paper focuses also on the applications to interest rate models. More precisely, we propose an overview of the usefulness of the regime switching approach for building various kinds of bond pricing models and of the roles which can be played by the regimes in these models. The objective of the pricing models can be to price default-free or defaultable bonds, or to analyse simultaneously credit ratings and defaultable bonds prices. The roles of the regimes can be to capture stochastic drifts and/or volatilities, to represent discrete values of a target rate, to incorporate business cycle or crises effects, to introduce contagion effects, to reproduce zero lower bound spells, or to evaluate the impact of standard or non-standard monetary policies. From a technical point of view, we stress the key role of Markov chains, Compound Autoregressive (Car) processes, Regime Switching Car processes and multihorizon Laplace transforms.

In a preliminary Section 2, we show that a key tool for pricing both default-free and defaultable bonds in discrete time is the multihorizon Laplace transforms of the underlying risk factors. These Laplace transforms can be computed in closed form for Markov chains and recursively for Compound Autoregressive (Car) processes.

Then we develop regime switching term structure models in various directions. We first consider in Section 3 the pricing of default-free bonds. We carefully distinguish the Regime Switching Term Structure Models (RSTSM), which provide affine formulas for the yields as functions of underlying risk factors, and the RSTSM for which the affine formulas are satisfied by the bond prices. In the latter case, we discuss the respective properties of models with exogenous and endogenous switching regimes and the ability of these models to generate short rate paths staying at a lower bound. We also discuss the practical implementation of these models, where the bond prices can be easily computed recursively, and sometimes under closed form. We also propose numerical illustrations showing the potentialities of these models for reproducing zero lower bound spells, or for evaluating monetary policies.

In Section 4 we consider pricing models for defaultable bonds. In this framework there exist individual (specific) risk factors as well as common (systematic) risk factors including a global regime indicator. When the stochastic discount factor (s.d.f.) depends on the common factors only, the causality features between individual and common factors are the same under the historical and risk-neutral distributions. This is illustrated by an application to sovereign bonds of the Euro-zone countries. A common regime variable is introduced to capture the crisis periods. We try to disentangle the credit and liquidity risks and we evaluate the historical and risk-neutral default probabilities.

Finally, we discuss in Section 5 directions for future research. We explain why the RSTSM are appropriate to capture default contagion, or flight-to-quality. To capture default contagion, the idea is to introduce an intermediate sector level between the individual and global levels in order to avoid a combinatorial explosion. Section 6 concludes. Proof are gathered in Appendices.

2 A TOOLBOX FOR REGIME SWITCHING TERM STRUCTURE MODELS

This section gathers the tools, which are useful for the analysis of RSTSM. We first recall the pricing formulas for default-free and defaultable bond pricing and highlight the key role of the multihorizon Laplace transform of the risk factors. Then, we consider the computation of these

Laplace transforms for Markov chains and for regime switching compound autoregressive processes.

2.1 Bond pricing

We adopt a discrete time setting in which the new information of the investors[5] at date $t, t = 1, 2, \ldots,$ is a n-dimensional factor w_t. The whole *information* of the investors at date t is therefore $\underline{w_t} = (w_t', w_{t-1}', \ldots, w_1')'$. The *historical dynamics* of the factor process $\{w_t\}$ is characterized either by the sequence of conditional probability density functions (p.d.f.) $f^P(w_t|\underline{w_{t-1}})$ (with respect to a dominating measure μ), or by the sequence of conditional Laplace transforms $\varphi_{t-1}^{(w)}(u) = E[\exp(u'w_t)|\underline{w_{t-1}}]$, defined in a convex set containing 0. Let us denote by $p_t[g(\underline{w_{t+h}})]$ the (spot) price at t of an asset providing at $t+h$ the payoff $g(\underline{w_{t+h}})$. Under standard assumptions, including the absence of arbitrage opportunity [see Harrison-Kreps (1979), Hansen-Richard (1987), Bertholon-Monfort-Pegoraro (2008)], there exists a sequence of positive random variables $M_{t-1,t} = M_{t-1,t}(\underline{w_t})$, called *stochastic discount factors* (s.d.f.), such that :

$$p_t[g(\underline{w_{t+h}})] = E[M_{t,t+1}(\underline{w_{t+1}}) \ldots M_{t+h-1,t+h}(\underline{w_{t+h}})g(\underline{w_{t+h}})|\underline{w_t}]. \tag{2.1}$$

In particular the price at date t of a default-free zero-coupon bond with residual maturity h, delivering the unitary payoff 1 at $t+h$, is :

$$B(t,h) = E_t(M_{t,t+1} \ldots M_{t+h-1,h}).$$

The default-free *yield to maturity* h is :

$$R(t,h) = -\frac{1}{h}\log[B(t,h)].$$

In particular for $h = 1$, we get the *short rate* $r_t = R(t,1)$, defined by :

$$r_t = -\log[E_t(M_{t,t+1})] \Longleftrightarrow E_t(M_{t,t+1}) = \exp(-r_t). \tag{2.2}$$

The *risk-neutral* (R.N.) dynamics of $\{w_t\}$ is defined by the sequence of conditional distributions of w_t given $\underline{w_{t-1}}$, whose p.d.f. with respect to the corresponding historical distribution is :

$$M_{t-1,t}\exp(r_{t-1}).$$

[5] We focus more on bond pricing than on the estimation of the dynamic term structure models, which depends on the information available to the econometrician. This information can be different from the information of the investor.

In other words the conditional R.N. p.d.f. of w_t given $\underline{w_{t-1}}$ with respect to dominating measure μ is :

$$f^Q(w_t|\underline{w_{t-1}}) = f^P(w_t|\underline{w_{t-1}})M_{t-1,t}\exp(r_{t-1}), \tag{2.3}$$

This equality is equivalent to :

$$M_{t-1,t} = \frac{f^Q(w_t|\underline{w_{t-1}})}{f^P(w_t|\underline{w_{t-1}})}\exp(-r_{t-1}), \tag{2.4}$$

and implies :

$$E^Q_{t-1}(M^{-1}_{t-1,t}) = \exp(r_{t-1}), \tag{2.5}$$

which is the R.N. analogue of equation (2.2).

Therefore the pricing formula (2.1) can be also written as :

$$p_t[g(\underline{w_{t+h}})] = E^Q_t[\exp(-r_t - \ldots - r_{t+h-1})g(\underline{w_{t+h}})]. \tag{2.6}$$

Thus the spot price is the conditional expectation under the R.N. distribution of the continuously discounted cash-flow $g(\underline{w_{t+h}})$. When the R.N. expectation of the payoff is computed without discounting, we get the associated price of the future contract written on this payoff :

$$p^f_t[g(\underline{w_{t+h}})] = E^Q_t[g(\underline{w_{t+h}})]. \tag{2.7}$$

Formula (2.6) can be used to derive an alternative expression of the (spot) price of the zero-coupon bond :

$$B(t,h) = E^Q_t[\exp(-r_t - \ldots - r_{t+h-1})]. \tag{2.8}$$

When the short rate is an affine function of risk factors w_t :

$$r_t = \beta_0 + \beta_1' w_t,$$

the zero-coupon price $B(t,h)$ becomes :

$$B(t,h) = \exp(-\beta_0 h - \beta_1' w_t)E^Q_t[\exp(-\beta_1' w_{t+1} - \ldots - \beta_1' w_{t+h-1})]. \tag{2.9}$$

For a defaultable zero-coupon bond, with residual maturity h, the payoff at $t+h$ is 1, if the issuing entity n has not defaulted, and 0, otherwise, when the recovery rate is zero. The price of the defaultable bond is (see Section 4) :

$$\begin{aligned} B_n(t,h) &= E_t^Q[\exp(-r_t \ldots - r_{t+h-1} - \lambda^Q_{n,t+1} \ldots - \lambda^Q_{n,t+h})] \\ &= \exp[-h(\beta_0 + \alpha_{0n}) - \beta_1' w_t] \\ &\quad E_t^Q\{\exp[-(\beta_1 + \alpha_{1,n})' w_{t+1} - \ldots - (\beta_1 + \alpha'_{1,n}) w_{t+h-1} - \alpha'_{1,n} w_{t+h}]\} \end{aligned} \tag{2.10}$$

where $\lambda^Q_{n,t} = \alpha_{0,n} + \alpha'_{1,n} w_t$ denotes the R.N. default intensity.

These bond pricing formulas highlight the role of the conditional Laplace transforms of the risk-factors. More precisely, throughout the paper, we will have to compute for any date t, and given sequences $(\gamma_1^{(h)}, \ldots, \gamma_h^{(h)})$, $h = 1, \ldots, H$, the multihorizon conditional Laplace transforms :

$$\varphi^{(w)}_{t,h}(\gamma_1^{(h)}, \ldots, \gamma_h^{(h)}) = E_t[\exp(\gamma_1^{(h)\prime} w_{t+1} + \ldots + \gamma_h^{(h)\prime} w_{t+h})]$$

in an efficient way. In formulas (2.9) and (2.10) the sequences $(\gamma_1^{(h)}, \ldots, \gamma_h^{(h)})$, $h = 1, \ldots, H$ have a *reverse order* structure in which, for any $h \in \{1, \ldots, H\}$, we have :

$$\gamma_h^{(h)} = \delta_1, \gamma_{h-1}^{(h)} = \delta_2, \ldots, \gamma_1^{(h)} = \delta_h \, .$$

for a given sequence $\delta_1, \ldots, \delta_H$. In formula (2.9) we have $\delta_h = -\beta_1, \forall h$, whereas in formula (2.10) we have $\delta_1 = -\alpha_{1,n}$ and $\delta_h = -\beta_1 - \alpha_{1,n}, \forall h \geq 2$.

2.2 Markov chains

Switching regimes are usually represented by Markov chains. When there are J regimes, we can define a Markov chain as a process $\{z_t\}$, whose components $z_{j,t}, j = 1, \ldots, J$, are indicator functions of regime j; in other words, z_t is valued in $\{e_1, \ldots, e_J\}$, where e_j is the J-dimensional vector, whose components are all equal to zero except the j^{th} one which is equal to one. The dynamics of $\{z_t\}$ is characterized by its transition matrices Π_t, whose entries $\pi_{i,j,t}$ are defined by :

$$\pi_{i,j,t} = P(z_t = e_j | z_{t-1} = e_i).$$

These probabilities may depend on time in a deterministic way, in order to incorporate exogenous variables or seasonal dummies.

The conditional distribution of z_t given z_{t-1} can also be characterized by its *conditional Laplace* *transform* :

$$\begin{aligned}\varphi^{(z)}_{t-1}(u) &= E[\exp(u'z_t)|z_{t-1}]. \\ &= \left[\sum_{j=1}^{J}\pi_{1,j,t}\exp(u_j),\ldots,\sum_{j=1}^{J}\pi_{J,j,t}\exp(u_j)\right]z_{t-1} \\ &= e'P'_t(u)z_{t-1}\end{aligned}$$

where $P_t(u) = \Pi_t$ diag $[\exp(u)]$, diag $[\exp(u)]$ is the diagonal matrix with components the exponential of the components of u and $e' = (1,\ldots,1)$. The conditional Laplace transform can alternatively be written as an exponential function of z_{t-1} :

$$\varphi^{(z)}_{t-1}(u) = \exp\{[\log\sum_{j=1}^{J}\pi_{1,j,t}\exp(u_j),\ldots,\log\sum_{j=1}^{J}\pi_{J,j,t}\exp(u_j)]z_{t-1}\}. \tag{2.11}$$

Moreover, it turns out that the multihorizon conditional Laplace transform :

$$\varphi^{(z)}_{t,h}(\gamma^{(h)}_1,\ldots,\gamma^{(h)}_h) = E_t[\exp(\gamma^{(h)'}_1 z_{t+1}+\ldots+\gamma^{(h)'}_h z_{t+h})],$$

has a closed from.

Proposition 1. *: The multihorizon conditional Laplace transform of a Markov chain is :*

$$\varphi^{(z)}_{t,h} = e'P'_{t+h}(\gamma^{(h)}_h)\ldots P'_{t+1}(\gamma^{(h)}_1)z_t,$$

where $P_t(\gamma) = \Pi_t diag[\exp(\gamma)], e' = (1,\ldots,1)$ *and* $diag[\exp(\gamma)]$ *denotes the diagonal matrix with diagonal terms the exponential of the components of* γ.

Proof : see Appendix 1.

According to Proposition 1 the multihorizon conditional Laplace transform is of the form $\alpha' z_t$ (the component α_j of α being positive), i.e. linear in z_t; it can also be considered as an exponential linear function of z_t, since $\alpha' z_t = \exp(\beta' z_t)$, where the components of β are $\beta_j = \log(\alpha_j)$. This remark will be useful for combining Markov chains, with the Car processes considered in Section 2.3.

2.3 Regime Switching Car process

The usefulness of Car processes, or discrete-time affine processes, introduced by Darolles, Jasiak, Gourieroux (2006) is now well documented [see for instance, Gourieroux, Monfort (2007), Gourieroux, Monfort, Polimenis (2006), Monfort, Pegoraro (2007), Le, Singleton, Dai (2011), Monfort, Renne (2011)a and (2011)b]. A Car process of order one, Car (1), is defined as follows :

Definition 1. *: A n-dimensional process $\{w_t\}$ is Car (1) if its conditional log-Laplace transform given the past $\underline{w}_{t-1} = (w'_{t-1}, \ldots, w'_1)$, is affine in w_{t-1}, that is, of the form :*

$$\begin{aligned} \psi^{(w)}_{t-1}(u) &= \log E[\exp(u'w_t)|\underline{w}_{t-1}] \\ &= a_{t-1}(u)'w_{t-1} + b_{t-1}(u), \end{aligned}$$

where a_{t-1} and b_{t-1} may depend on time in a deterministic way.

It is known that we can also define Car processes of order p [Car (p)] and that, by extending the dimension of the process, a Car(p) process is also a Car(1) process. Therefore we only consider Car (1) processes in the next sections. It is also known that the family of Car(1) processes contains many important processes like autoregressive Gaussian processes, autoregressive Gamma processes, compound Poisson processes, autoregressive Wishart processes. Equation (2.11) also shows that a Markov chain is Car(1). Other important stochastic processes are the Regime Switching Car (1) processes [RSCar (1)] defined in the following way:

Definition 2. *: Let us consider :*

i) a family of Car(1) conditional log-Laplace transforms of the form :

$$a_{t-1}(u)'y_{t-1} + b^{(0)}_{t-1}(u)'\delta,$$

where δ is a K-dimensional vector and $b^{(0)}_{t-1}$ a K-dimensional vector of functions;

ii) a J-regime exogenous Markov chain $\{z_t\}$ with transition matrices Π_t;

iii) a set of independent random K-dimensional vectors $\Delta^j_{i,t}, i = 1, \ldots, J, j = 1, \ldots, J$, identically distributed over time.

The stochastic process $\{y_t\}$ such that the conditional log-Laplace transform of y_t given $y_{t-1}, z_t = e_j, z_{t-1} = e_i, \Delta^j_{i,t} = \delta^j_{i,t}$ is given by :

$$a_{t-1}(u)'y_{t-1} + b^{(0)}_{t-1}(u)'\delta^j_{i,t},$$

is called a RSCar(1).

Regime Switching Car(1) processes are similar to diffusion models with jumps encountered in continuous time models. The baseline dynamics corresponds to the baseline diffusion equation and this diffusion equation involves several parameters which can switch. The underlying Markov chain defines the times of the jumps on the different parameters and the components of $\Delta^j_{i,t}$ define the stochastic sizes of the jumps.

Example : *Gaussian autoregressive process with switching drift and volatility.*

Let us consider a baseline Gaussian AR(1) dynamics :

$$w_{1,t} = \mu + \varphi w_{1,t-1} + \sigma\varepsilon_t, \text{where } \varepsilon_t \sim IIN(0,1).$$

We have :

$$a_{t-1}(u) = \varphi u, b_{t-1}(u) = u\mu + \frac{1}{2}u^2\sigma^2 = b^{(0)}_{t-1}(u)'\delta,$$

with :

$$b^{(0)}_{t-1}(u) = (u, \frac{1}{2}u^2)', \delta = (\mu, \sigma^2)'.$$

Therefore we can introduce regime switching drift and volatility parameters.

A RSCar(1) process $\{y_t\}$ is not Car(1), but the extended process $\{y'_t, z'_t\}'$ is Car(1). Indeed, we have the following property :

Proposition 2. *: The process $(y'_t, z'_t)'$, where $\{z_t\}$ is a Markov chain and $\{y_t\}$ an associated RSCar(1), is a Car(1) process; its conditional log-Laplace transform, if the size of the jumps $\Delta^j_{i,t}$ are not observed, is given by :*

$$a'_{t-1}(u)y_{t-1} + [A_1(u,v), \ldots, A_J(u,v)]z_{t-1},$$

with :

$$A_i(u,v) = \log \sum_{j=1}^{J} \{\pi_{i,j,t} \exp[\psi_{i,j}(b^{(0)}_{t-1}(u)) + v_j]\},$$

$\psi_{i,j}(.)$ *being the log-Laplace transform of* $\Delta^j_{i,t}$.

If the size of the jumps is observed, $\Delta^j_{i,t}$ is non random and we have $\psi_{i,j}(b^{(0)}_{t-1}(u)) = b^{(0)}_{t-1}(u)'\Delta^j_{i,t}$.

Proof : see Appendix 2.

As stressed in subsection 2.1 an important issue is the computation of multihorizon conditional Laplace transforms of factor process $\{w_t\}$. The following result shows that if $\{w_t\}$ is Car(1) or, according to Proposition 2, a RSCar(1) process, these computations are easily done recursively.

Proposition 3. *: If the conditional log-Laplace transform of $\{w_t\}$ is $\psi_{t-1}^{(w)}(u) = a_{t-1}(u)'w_{t-1} + b_{t-1}(u)$, the multihorizon conditional Laplace transform :*

$$\varphi_{t,h}^{(w)} = E_t[\exp(\gamma_1^{(h)'} w_{t+1} + \ldots + \gamma_h^{(h)'} w_{t+h})],$$

is equal to :

$$\varphi_{t,h}^{(w)} = \exp(A'_{t,h} w_t + B_{t,h}),$$

where $A_{t,h} = A_{t,h}^{(h)}, B_{t,h} = A_{t,h}^{(h)}$, the $A_{t,i}^{(h)}, B_{t,i}^{(h)}, i = 1, \ldots, h$ are defined recursively by :

$$\begin{cases} A_{t,i}^{(h)} & = & a_{t+h-i}(\gamma_{h+1-i}^{(h)} + A_{t,i-1}^{(h)}), \\ B_{t,i}^{(h)} & = & b_{t+h-i}(\gamma_{h+1-i}^{(h)} + A_{t,i-1}^{(h)}) + B_{t,i-1}^{(h)}, \\ A_{t,0}^{(h)} & = & 0, B_{t,0}^{(h)} = 0. \end{cases}$$

Proof : see Appendix 3.

Therefore, to compute $\varphi_{t,h}^{(w)}$, for $t = 1, \ldots, T$, and given sequences of parameters $(\gamma_1^{(h)}, \ldots, \gamma_h^{(h)})$, $h = 1, \ldots, H$, we have, in general, to apply the above algorithm TH times. However, if functions a_t and b_t do not depend on t, we have to use it H times only. More importantly if the parameters $(\gamma_1^{(h)}, \ldots, \gamma_h^{(h)})$, $h = 1, \ldots, H$, have a *reverse order* structure $\gamma_{h+1-i}^{(h)} = \delta_i$ for $i = 1, \ldots, h$ and $h = 1, \ldots, H$, that is, if we want to compute :

$$E_t\left[\exp(\delta'_h w_{t+1} + \ldots + \delta'_1 w_{t+h})\right], h = 1, \ldots, H, t = 1, \ldots, T,$$

the algorithm has to be used only once for each date t. If, moreover, a_t and b_t do not depend on t, the algorithm has to be used *only once*.

3 REGIME SWITCHING AND DEFAULT-FREE BOND PRICING

In this section we describe two kinds of models for pricing default-free zero-coupon bonds. In the first kind of models (see Section 3.1) the formulas for the *yields* are *affine* with respect to the factor w_t, whereas in the second kind of models (see Section 3.2) the affine structure is obtained for the *prices*. In Section 3.3 we combine both kinds of formulas.

3.1 Regime Switching Affine *Yield* Term Structure Model

3.1.1 Regime switching risk-neutral dynamics and bond pricing

We assume that the new information of the investors at date t is :

$$w_t = (z_t', y_t')',$$

where $\{z_t\}$ is a time homogeneous and exogenous Markov chain and $\{w_t\}$ is Car (1) in the risk-neutral (R.N.) world. In Section 2 we have seen that a Car(1) process can be obtained by allowing some parameters of a non-switching Car(1) to switch.

If the short rate r_t, is an affine function of w_t :

$$r_t = \beta_0 + \beta_1' w_t = \beta + \beta_{11}' z_t + \beta_{12}' y_t, \tag{3.1}$$

the price at t of a default-free zero-coupon bond of residual maturity h is :

$$\begin{aligned} B(t,h) &= E_t^Q \exp(-r_t - \ldots - r_{t+h-1}) \\ &= \exp(-\beta_0 h - \beta_1' w_t) E_t^Q \exp[-\beta_1'(w_{t+1} + \ldots + w_{t+h-1})] \end{aligned}$$

According to Proposition 3, the prices $B(t,h), t = 1, \ldots, T, h = 1, \ldots, H$, are of the form :

$$B(t,h) = \exp(c_h' w_t + d_h),$$

where the c_h, d_h are obtained from a simple recursive scheme. Therefore we obtain the *switching affine yield term structure* :

$$R(t,h) \;=\; -\frac{c_h'}{h} w_t - \frac{d_h}{h} = -\frac{c_{1,h}'}{h} z_t - \frac{c_{2,h}'}{h} y_t - \frac{d_h}{h}. \tag{3.2}$$

Thus the stochastic term structure is obtained as a combination of baseline deterministic term structures, that are the components of $c_{1,h}, c_{2,h}, d_h$, with stochastic coefficients. An interesting property of these affine term structure models is that some components of y_t can be chosen as yields of different residual maturities, while staying compatible with pricing formula (3.2). For instance, if the first component $y_{1,t} = R(t,k)$, we have just to fix $c_{1,k} = 0, c_{2,k} = -ke_1, d_k = 0$, where e_1 is the vector selecting the first component of y_t, in the recursive scheme of Proposition 3. This constraints the R.N. dynamics.

3.1.2 Back to the historical dynamics

Once the R.N. dynamics of $\{w_t\}$ is specified as well as the short rate function $r_t(\underline{w_t})$, the historical conditional p.d.f. of w_t given $\underline{w_{t-1}}$ with respect to the same dominating measure can be specified freely. Equivalently, we can specify any stochastic discount factor satisfying :

$$E^Q_{t-1}[M^{-1}_{t-1,t}(\underline{w_t})] = \exp(r_{t-1}). \tag{3.3}$$

A convenient, flexible specification of the s.d.f. is the *exponential affine* s.d.f. :

$$M_{t-1,t} = \exp\{-r_{t-1} + \gamma'(\underline{w_{t-1}})w_t + \psi^Q_{t-1}[-\gamma(\underline{w_{t-1}})]\}, \tag{3.4}$$

where the vector of *risk sensitivity coefficients* $\gamma(\underline{w_{t-1}})$ are functions of the past value of $w_t = (z'_t, y'_t)'$. This large choice of risk sensitivity coefficients $\gamma(\underline{w_{t-1}})$ implies a large choice of historical dynamics, which in general are not Car. Nevertheless the conditional log-Laplace transform of $\{w_t\}$ in the historical world is easily obtained, since :

$$\begin{aligned}
\psi^P_{t-1}(u) &= \log E^P_{t-1}\left[\exp(u'w_t)\right] = \log\ E^Q_{t-1}\left[M^{-1}_{t-1,t}\exp(-r_{t-1} + u'w_t)\right] \\
&= -\psi^Q_{t-1}\left[-\gamma(\underline{w_{t-1}})\right] + \log E^Q_{t-1}\left\{\exp[u - \gamma(\underline{w_{t-1}})]'w_t\right\},
\end{aligned}$$

where $\psi^Q_{t-1}(u)$ is the R.N. conditional log-Laplace transform of w_t. Therefore :

$$\psi^P_{t-1}(u) = \psi^Q_{t-1}\left[u - \gamma(\underline{w_{t-1}})\right] - \psi^Q_{t-1}\left[-\gamma(\underline{w_{t-1}})\right], \tag{3.5}$$

$$\text{where : } \psi^Q_{t-1}(u) = a^Q_{t-1}(u)'w_{t-1} + b^Q_{t-1}(u),$$

since factor process $\{w_t\}$ is Car (1),

It is deduced from the R.N. conditional Laplace transform by an appropriate "drift adjustment", which can depend on the past.

3.1.3 A Gaussian Switching Affine Yield Term Structure Model

Let us assume that the R.N. dynamics of $w_t = (z_t', y_t')'$ is given by :

$$y_t = \mu(z_t, z_{t-1}) + \Phi y_{t-1} + \Omega(z_t, z_{t-1})\eta_t, \tag{3.6}$$

where $\{\eta_t\}$ is a standard Gaussian white noise and $\{z_t\}$ is a time homogeneous exogenous Markov chain valued in $\{e_1, \ldots, e_J\}$, independent of $\{\eta_t\}$, and with transition matrix Π.

The factor process $\{w_t\}$ is Car(1) under Q and its conditional log-Laplace transform is given by :

$$\psi_{t-1}^Q(u_1, u_2) = \log E_{t-1}^Q[\exp(u_1' z_t + u_2' y_t)] = [A_1(u_1, u_2), \ldots, A_J(u_1, u_2)]z_{t-1} + u_2'\Phi y_{t-1}, \tag{3.7}$$

with $A_i(u_1, u_2) = \log\{\Sigma_j \pi_{i,j} \exp[u_{1,j} + u_2'\mu(e_j, e_i) + \frac{1}{2}u_2'\Sigma(e_j, e_i)u_2]\}$, and $\Sigma(e_j, e_i) = \Omega(e_j, e_i)\Omega'(e_j, e_i)$.

Let us assume that the s.d.f. has the form :

$$\begin{aligned} M_{t-1,t} &= \exp[-r_{t-1} + \frac{1}{2}\nu'(z_t, z_{t-1}, y_{t-1})\nu(z_t, z_{t-1}, y_{t-1}) \\ &+ \nu'(z_t, z_{t-1}, y_{t-1})\eta_t + \delta'(z_{t-1}, y_{t-1})z_t], \end{aligned} \tag{3.8}$$

$$\begin{aligned} \text{with} \quad \nu(e_j, e_i, y_{t-1}) &= \Omega^{-1}(e_j, e_i)[\tilde{\Phi} y_{t-1} + \tilde{\mu}(e_j, e_i)], \\ \delta_j(e_i, y_{t-1}) &= \log[\frac{\pi_{ij}}{\tilde{\pi}(e_j|e_i, y_{t-1})}], \end{aligned}$$

where the matrix $\tilde{\Phi}$, and the functions $\tilde{\mu}(z_t, z_{t-1}), \tilde{\pi}(z_t|z_{t-1}, y_{t-1})$ can be chosen arbitrarily. In this specification of the s.d.f., both the risk coming from the Gaussian white noise $\{\eta_t\}$ and from the stochastic regime $\{z_t\}$ are priced. The adjustment term $\frac{1}{2}\nu'(z_t, z_{t-1}, y_{t-1})\nu(z_t, z_{t-1})\nu(z_t, z_{t-1}, y_{t-1})$ and the form of function μ ensure that the required constraint (3.4) on the s.d.f. $E_t^Q(M_{t-1,t}^{-1}) = \exp(r_{t-1})$ is satisfied. Moreover, the historical dynamics is [see Monfort, Renne (2011)a] :

$$y_t = \mu(z_t, z_{t-1}) - \tilde{\mu}(z_t, z_{t-1}) + (\Phi - \tilde{\Phi})y_{t-1} + \Omega(z_t, z_{t-1})\varepsilon_t, \tag{3.9}$$

where $\{\varepsilon_t\}$ is a standard Gaussian white noise under P, z_t is valued in $\{e_1, \ldots, e_J\}$ and such that $P(z_t = e_j|z_{t-1} = e_i, \underline{y_{t-1}}) = \tilde{\pi}(e_j|e_i, y_{t-1})$.

The specific form of the s.d.f. provides R.N. and historical dynamics, which can differ by their switching drift and autoregressive matrix, but share the same switching volatility matrix processes. Since z_t is valued in $\{e_1, \dots, e_J\}$, the s.d.f. given in (3.9) can be written as :

$$M_{t-1,t} = \exp[-r_{t-1} + \frac{1}{2} z_t' \tilde{\nu}'(z_{t-1}, y_{t-1}) \tilde{\nu}(z_{t-1}, y_{t-1}) z_t + z_t' \tilde{\nu}(z_{t-1}, y_{t-1}) \eta_t + \delta'(z_{t-1}, y_{t-1}) z_t], \tag{3.10}$$

where $\tilde{\nu}(z_{t-1}, y_{t-1})$ is the matrix whose j^{th} column is $\nu(e_j, z_{t-1}, y_{t-1})$.

Therefore the s.d.f. $M_{t-1,t}$ is *exponential quadratic* in (z_t, η_t), and also exponential quadratic in (z_t, y_t)[6] [see Monfort, Pegoraro (2012)]. If $\nu(z_t, z_{t-1}, y_{t-1})$ does not depend on z_t, [7] $\tilde{\nu}(z_{t-1}, y_{t-1})$ is equal to $\nu_0(z_{t-1}, y_{t-1})e'$, where $\nu_0(z_{t-1}, y_{t-1})$ is a vector with the same dimension as y_t, and e the vector of size J whose all components are equal to one, and the s.d.f. becomes :

$$M_{t-1,t} = \exp[-r_{t-1} + \frac{1}{2} \nu_0'(z_{t-1}, y_{t-1}) \nu_0(z_{t-1}, y_{t-1}) + \nu_0'(z_{t-1}, y_{t-1}) \eta_t + \delta'(z_{t-1}, y_{t-1}) z_t] \tag{3.11}$$

which is *exponential affine* in (z_t, η_t).

3.2 Regime Switching Affine *Price* Term Structure Model

The models described in Section 3.1 provide term structures, where the *yields* are affine functions of the factor $w_t = (z_t', y_t')'$. In this section we consider other RSTSM in which the bond *prices* are affine functions of factors. Contrary to the Regime Switching Affine Yields Term Structure Models, these new models are able to reproduce a behavior of the short term rate staying equal to a lower bound during some spells. We distinguish two cases depending whether the Markov chain *is exogenous*, or *endogenous*.

3.2.1 *Exogenous* Markov chain

Let us consider a process $\tilde{w}_t = (z_t', r_t', y_t')'$, where $\{z_t\}$ is an *exogenous* Markov chain, with transition matrices Π_t in the R.N. world. Thus we assume that the conditional distribution of z_t given $\underline{\tilde{w}_{t-1}}$ depends on z_{t-1} only and is characterized by Π_t, which implies that $\{r_t, y_t\}$ does not cause $\{z_t\}$. We assume that the R.N. conditional distribution of r_t given $\underline{z_t}, \underline{r_{t-1}}, \underline{y_{t-1}}$ depends on z_t only and has a conditional Laplace transform given by :

$$E[\exp(u r_t) | \underline{z_t}, \underline{r_{t-1}}, \underline{y_{t-1}}] = \exp[\gamma_t(u)' z_t],$$

[6] The term $z_t' \tilde{\nu}'(z_{t-1}, y_{t-1}) \tilde{\nu}(z_{t-1}, y_{t-1}) z_t$ can also be written in the linear way $\frac{1}{2} \tilde{\nu}^{2'}(z_{t-1}, y_{t-1}) z_t$, where $\tilde{\nu}^2$ is understood componentwise.

[7] This condition is in particular satisfied if there is no instantaneous causality between $\{z_t\}$ and $\{y_t\}$ in both worlds.

where $\gamma_t(u)$ is the vector :

$$[\gamma_{1t}(u), \ldots, \gamma_{Jt}(u)]'.$$

Finally we assume that the R.N. conditional distribution of y_t given $\underline{z_t}, \underline{r_t}, \underline{y_{t-1}}$ depends on $z_t, \underline{r_t}, \underline{y_{t-1}}$ only.

The information of the investors is, either $\underline{\tilde{w}_t}$, if z_t is observed, or $\underline{w_t} = (\underline{r_t}, \underline{y_t})$, if z_t is not observed. If we assume that z_t is not observed by the investors (or hidden), the zero-coupon price $B(t,h)$ is a *linear function* of the transformed factor $\hat{z}_t \exp(-r_t)$, where $\hat{z}_t = E^Q(z_t|\underline{r_t}, \underline{y_t})$. More precisely we have the following result :

Proposition 4. *:*

$$B(t,h) = e' P'_{t+h-1}(\tilde{\gamma}_{t+h-1}) \ldots P'_{t+1}(\tilde{\gamma}_{t+1}) \hat{z}_t \exp(-r_t),$$

where : $P_t(\gamma) = \Pi_t diag[\exp(\gamma)]$ and $\tilde{\gamma}_t = \gamma_t(-1)$, where, for $h = 1$, the product of the P matrices reduces to the identity matrix.

Proof : see Appendix 4.

The price of the short term zero-coupon $B(t,1)$ reduces to $e'\hat{z}_t \exp(-r_t) = \exp(-r_t)$ as expected.

If the Markov chain is homogeneous, i.e. $\Pi_t = \Pi$, and the conditional distribution of r_t given z_t does not depend on t, i.e. $\gamma_t(u) = \gamma(u)$, we get the following result :

Corollary 1. *: If $\Pi_t = \Pi, \gamma_t(u) = \gamma(u)$, we have $B(t,h) = e'P'(\tilde{\gamma})^{H-1}\hat{z}_t \exp(-r_t)$, where $\tilde{\gamma} = \gamma(-1)$.*

The zero-coupon prices are explicit linear functions of the transformed factor $\exp(-r_t)\hat{z}_t$, which is nonlinear in $\underline{r_t}, \underline{y_t}$. Therefore it is important to have a simple way to compute the risk-neutral predictions $\hat{z}_t$. The following proposition shows that $\hat{z}_t$ can be computed recursively using an algorithm similar to the Kitagawa-Hamilton' algorithm.

Proposition 5. *:*

$$\hat{z}_{t+1} = \frac{diag(f_t g_t)\Pi'_t \hat{z}_t}{e' diag(f_t g_t)\Pi'_t \hat{z}_t},$$

where $diag(f_t g_t)$ is the diagonal matrix, with the k^{th} diagonal term given by :

$$f_{k,t}(r_{t+1}) g_{k,t}(y_{t+1}|\underline{r_{t+1}}, \underline{y_t}),$$

where $g_{k,t}$ is the conditional p.d.f. of y_{t+1} given $z_{t+1} = e_k, \underline{r_{t+1}}, \underline{y_t}$, and $f_{k,t}(r_{t+1})$ is the p.d.f. of r_{t+1} given $z_{t+1} = e_k$.

Proof : see Appendix 5.

The proof in Appendix 5 includes the case where the conditional distribution of r_{t+1} given $z_{t+1} = e_1$ (say) is the point mass at a given value, for instance zero. This allows the short rate to stay at some *lower bound* during some spells.

3.2.2 *Endogenous* Markov chain

In the previous model, the Markov chain $\{z_t\}$ is exogenous in the R.N. world that is, it is not caused by the other processes $\{r_t, y_t\}$. In this section we consider a situation in which, the process $\{z_t\}$ is *endogenous*, that is, caused by the other process $\{r_t, y_t\}$.

More precisely, we assume that the risk-neutral conditional distribution of z_t given $(\underline{z_{t-1}}, \underline{r_{t-1}}, \underline{y_{t-1}})$ depends on (r_{t-1}, y_{t-1}), i.e. is characterized by a J-dimensional vector of probabilities $\beta_{t-1}(r_{t-1}, y_{t-1})$. Moreover, we assume that the R.N. conditional distribution of (r_t, y_t) given $(\underline{z_t}, \underline{r_{t-1}}, \underline{y_{t-1}})$ depends on z_t only. We denote by $\alpha_t(r_t, y_t)$ the J-dimensional vector whose j^{th} component $\alpha_{j,t}$ is the p.d.f. of the conditional distribution of (r_t, y_t) given $z_t = e_j$, with respect to a given dominating probability measure. We assume that this dominating probability measure has in turn a p.d.f. $\alpha_{0,t}(r_t, y_t)$ with respect to a given measure μ. In other words, for given values of (r_{t-1}, y_{t-1}), z_t is drawn according to the vector of probabilities β_{t-1} and, then, if $z_t = e_j$, (r_t, y_t) is drawn in the distribution whose p.d.f. with respect to μ is $\alpha_{0,t}\alpha_{j,t}$. We assume that the information of the investors is $\underline{w_t} = (w'_t, \dots, w'_1)'$ with $w_t = (r'_t, y'_t)'$ and, therefore, z_t is not observed (or hidden).

The conditional p.d.f. or (r_t, y_t) given (r_{t-1}, y_{t-1}) w.r.t. μ is :

$$\alpha_{0,t}(r_t, y_t)\alpha'_t(r_t, y_t)\beta_{t-1}(r_{t-1}, y_{t-1}) \tag{3.12}$$

This kind of dynamics has been introduced by Gourieroux and Jasiak (2000) and called Finite Dimensional Dependence (FDD) dynamics.

It is easily seen that the conditional distribution of z_t given its own past $\underline{z_{t-1}}$ depends on z_{t-1} only; in other words, z_t is marginally Markov.

Let us denote by $E_{0,t}$ the expectation operator with respect to the probability distribution $\alpha_{0,t}d\mu$ and by Π_t the R.N. transition matrix of z_t, whose entries are $\pi_{i,j,t} = Q_t(z_{t+1} = e_j | z_t = e_i)$, where Q_t

is a risk-neutral probability. $E_{0,t}$ is an unconditional expectation operator w.r.t. the distribution $\alpha_{0,t}d\mu$ depending on time in a deterministic way.

We have the following results :

Proposition 6. *:*

$$\begin{aligned} \Pi_t &= E_{0,t}(\alpha_t \beta_t'), \\ B(t,h) &= e' \tilde{P}'_{t+h-1} \ldots \tilde{P}_{t+1} \beta_t \exp(-r_t), \end{aligned}$$

where $\tilde{P}_t = E_{0,t}[\exp(-r_t)\alpha_t\beta_t']$, and the product of the $\tilde{P}$ matrices reduces to the identity matrix for $h = 1$.

Proof : see Appendix 6.

The formulas obtained for $B(t,h)$ in the exogenous case (Proposition 4) and in the endogenous case (Proposition 6) are similar. The prices are still *linear functions of factors*, the P_t matrices appearing in Proposition 4 are replaced by the $\tilde{P}_t$ matrices in Proposition 6 and the factors $\exp(-r_t)\hat{z}_t$ are replaced by the factors $\exp(-r_t)\beta_t$. In both cases $B(t,h)$ is, for any h, a linear combination of factors, which are nonlinear in the variables $(\underline{r_t}, \underline{y_t})$ (in the endogenous case the factor β_t are functions of (r_t, y_t) only).

In the stationary case where $\alpha_{0,t}, \alpha_t$ and β_t do not depend on t, we get a simplified formula.

Corollary 2. *: In the stationary case, that is, if α_{0t}, α_t and β_t do not depend on t, we have :*

$$\begin{aligned} B(t,h) &= e' \tilde{P}'^{h-1} \beta(r_t, y_t) \exp(-r_t), \\ \textit{with :} \qquad \tilde{P} &= E_0[\exp(-r_t)\alpha(r_t, y_t)\beta'(r_t, y_t)]. \end{aligned}$$

Two additional remarks are of interest. First, the basic probability measure $\alpha_{0,t}d\mu$ may not be absolutely continuous with respect to the Lebesgue measure, for instance it could be such that the probability of the hyperplane $\{r_t = 0\}$ is strictly positive and, moreover, one of the p.d.f. $\alpha_{j,t}$, say $\alpha_{1,t}$, could be non zero only in this hyperplane. Thus the short rate would be equal to zero in the first regime and would *remain equal to zero* for some time (see the illustration in the next subsection). Second, the FDD dynamics is rather general since it can approximate any Markov dynamics; indeed, any conditional p.d.f. $f(w_t|w_{t-1})$ of $w_t = (r_t, y_t)$ given w_{t-1} can be approximated by the FDD dynamics :

$$\sum_{l=1}^{L} f(w_t|\tilde{w}_l) \frac{\mathcal{K}\left(\frac{w_{t-1}-\tilde{w}_l}{d}\right)}{\sum_{l=1}^{L} \mathcal{K}\left(\frac{w_{t-1}-\tilde{w}_l}{d}\right)}, \tag{3.13}$$

where $\mathcal{K}$ is a kernel, $\tilde{w}_l, l = 1, \ldots, L$ a fixed grid and d a bandwidth.

Finally let us consider the historical dynamics. Since the R.N. and historical conditional distributions of w_t given the past are equivalent, the historical conditional distribution is absolutely continuous with respect to the probability $\alpha_{0,t} d\mu$.

We have the following result:

Proposition 7. *: If the R.N. dynamics is FDD and if the s.d.f. is factorized as $M_{1,t-1,t}(w_t) M_{2,t-1,t}(w_{t-1})$, the historical dynamics is FDD, and conversely.*

Proof : See Appendix 7.

3.2.3 The zero lower bound problem

Both kinds of Regime Switching Affine Price Term Structure Models are able to generate paths of the short rate staying at a lower bound, zero for instance during some endogenous spells. As an illustration, let us consider a FDD model in which $\{y_t\}$ is univariate and the number of states is $J = 3$. The conditional risk-neutral probabilities of the regime are given by :

$$\beta_{j,t-1} = \frac{\varphi\left(\frac{r_{t-1}+y_{t-1}-k_j}{d}\right)}{\sum_{l=1}^{3} \varphi\left(\frac{r_{t-1}+y_{t-1}-k_l}{d}\right)}, \tag{3.14}$$

where φ is the p.d.f. of the standard normal, $k_j, j = 1, 2, 3$ are given values and d a bandwidth.

We assume that r_t and y_t are independent conditionally on (z_t, w_{t-1}) and with the following distributions:

i) for r_t: the point mass at zero, if $j = 1$, and the gamma distribution $\gamma(\nu_j, \mu_j)$ with mean m_j and variance σ_j^2, if $j = 2$ or 3, that is, with $\nu_j = m_j^2/\sigma_j^2$ and $\mu_j = m_j/\sigma_j^2$.

ii) for y_t: the gamma distribution $\gamma(\nu_2, \mu_2)$.

Since we are in a stationary case, the price at time t of a zero-coupon bond of residual maturity h is given by the formula of Corollary 2 :

$$B(t,h) = e' \tilde{P}'^{h-1} \beta(r_t, y_t) \exp(-r_t). \tag{3.15}$$

The matrix $\tilde{P}' = E_0[\exp(-r_t)\beta(r_t, y_t)\alpha'(r_t, y_t)]$ is easily computed by Monte-Carlo. More precisely its first column can be approximated by : $\frac{1}{S}\sum_{s=1}^{S}\beta(0, y^s)$, where the simulated y^s are drawn in $\gamma(\gamma_2, \mu_2)$. The columns $j = 2, 3$, can be approximated by : $\frac{1}{S}\sum_{s=1}^{S}\exp(-r^s)\beta(r^s, y^s)$, where the simulated y^s are drawn in $\gamma(\nu_2, \mu_2)$ and the simulated rates r^s in $\gamma(\nu_2, \mu_2)$, if $j = 2$, and $\gamma(\nu_3, \mu_3)$, if $j = 3$.

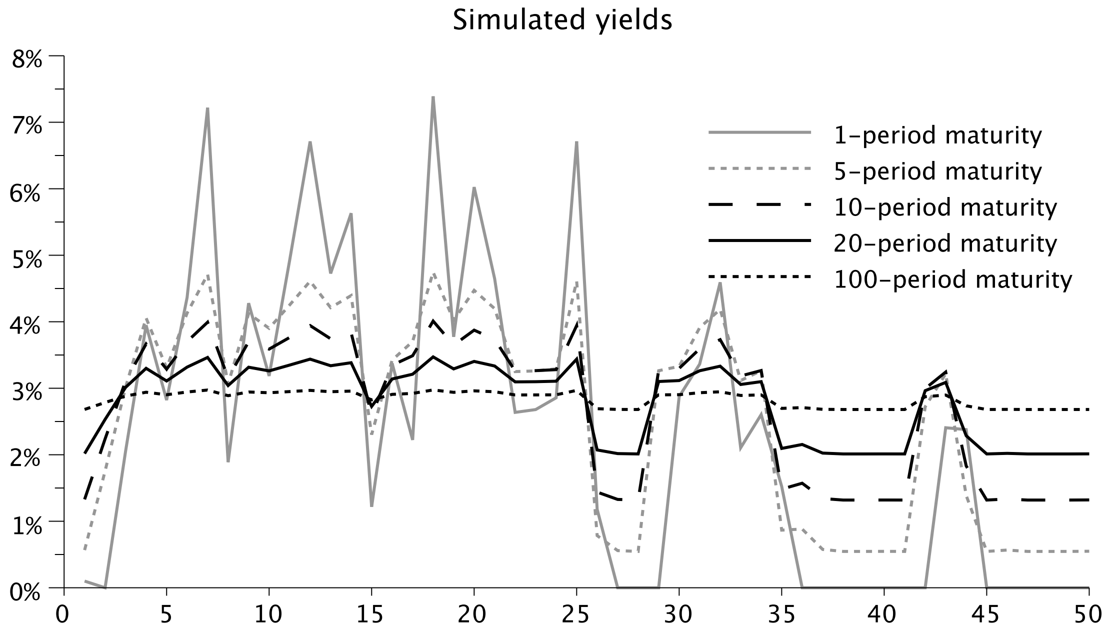
For the Monte-Carlo analysis, we do not distinguish the R.N. and historical dynamics and the numerical values of the parameters are :

$$\begin{aligned} k_1 &= .03, k_2 = .05, k_3 = .07, d = .005, \\ m_2 &= .03, \sigma_2 = .01, m_3 = .04, \sigma_3 = .02. \end{aligned}$$

We simulate paths of length $T = 50$ for the factor (r_t, y_t) and for the yields $R(t,h) = -\frac{1}{h}\log B(t,h)$, for $h = 5, 10, 20, 100$ (and initial values $r_1 = y_1 = .001$). Figure 1 shows such trajectories. The short rate r_t is equal to zero in periods 27 to 29, and 36 to 42. Within these periods, the rest of the yield curve is varying; in particular, within the first period, there is a change of the slope sign.

Figure 1: Interest rates paths and the lower bound

Notes: Simulated paths of yields $R(t,h) = -\frac{1}{h}\log B(t,h)$, for $h = 5, 10, 20, 100$. Initial values: $r_1 = y_1 = .001$.



3.3 A simultaneous use of explicit and recursive pricing formulas

In Sections 3.1, 3.2, we have obtained either explicit, or recursive formulas for the prices of zero-coupon bonds. There are many ways to use simultaneously these results. In this section we consider such an approach and an application.

3.3.1 A flexible framework

In the risk-neutral world we consider two independent Markov chains $\{z_t^{(1)}\}$, $\{z_t^{(2)}\}$ with, respectively, J_1 and J_2 states and transitions matrices $\Pi_t^{(1)}, \Pi_t^{(2)}$. Moreover we consider an independent Car(1) process $\{y_t\}$ and a sequence of $K \times J_2$ matrices Δ_t serially independent and independent of the other processes. Finally we assume that the short rate between t and $t+1$ is given by :

$$r_t = \mu_1' z_t^{(1)} + \mu_2' \Delta_t z_t^{(2)} + \mu_3' y_t, \tag{3.16}$$

If we assume that $z_t^{(1)}, z_t^{(2)}$ and y_t are observed by the investor. The price of the zero coupon bond $B(t,h)$ is :

$$\begin{aligned} B(t,h) &= \exp(-r_t) E_t^Q \exp(-r_{t+1} - \ldots - r_{t+h-1}) \\ &= \exp(-r_t) B_{1,t}(h) B_{2,t}(h) B_{3,t}(h) \end{aligned}$$

$$\begin{aligned} \text{where} \quad B_{1,t}^{(h)} &= E_t^Q \exp(-\mu_1' z_{t+1}^{(1)} - \ldots - \mu_1' z_{t+h-1}^{(1)}) \\ B_{2,t}^{(h)} &= E_t^Q \exp(-\mu_2' \Delta_{t+1} z_{t+1}^{(2)} - \ldots - \mu_2' \Delta_{t+h-1} z_{t+h-1}^{(2)}) \\ B_{3,t}^{(h)} &= E_t^Q \exp(-\mu_3' y_{t+1} - \ldots - \mu_3' y_{t+h-1}) \end{aligned}$$

Using Proposition 1, we see that $B_{1,t}(h)$ is an explicit linear function of $z_t^{(1)}$, or, equivalently, an explicit exponential linear function of $z_t^{(1)}$, since $z_t^{(1)}$ is valued in $\{e_1, \ldots, e_J\}$

$$B_{1,t}(h) = \exp[a_{1,t}'(h) z_t^{(1)}]$$

Similarly, conditioning first by $z_{t+1}^{(2)}, \ldots, z_{t+h-1}^{(2)}$ and taking the expectation in $B_{2,t}(h)$ with respect to $\Delta_{t+1}, \ldots, \Delta_{t+h-1}$, we get a closed form expression for $B_{2,t}(h)$:

$$B_{2,t}(h) = \exp[a_{2,t}'(h) z_t^{(2)}].$$

Using Proposition 3 we get :

$$B_{3,t}(h) = \exp[a_{3,t}'(h) y_t + a_{4,t}(h)],$$

where $a_{3,t}(h)$ and $a_{4,t}(h)$ can be computed recursively. Finally we get :

$$B(t,h) = \exp[a_{1,t}'(h) z_t^{(1)} + a_{2,t}'(h) z_t^{(2)} + a_{3,t}'(h) y_t + a_{4,t}(h)],$$

and :

$$R(t,h) = -\frac{1}{h}[a_{1,t}'(h) z_t^{(1)} + a_{2,t}'(h) z_t^{(2)} + a_{3,t}'(h) y_t + a_{4,t}(h)], \tag{3.17}$$

where $a_{1,t}(h), a_{2,t}(h)$ have closed forms and $a_{3,t}(h), a_{4,t}(h)$ can be computed recursively.

Therefore we get a very flexible framework which is able to take into account simultaneously many features :

- switching regimes with *deterministic values*
- switching regimes with *stochastic values*
- transition matrices *depending on time* in a deterministic way

- *quantitative factors.*

An application using these features is the following multiregime model.

3.3.2 A multiregime model : the euro-area yield curve with discrete policy rates

This application illustrates the flexibility of the short-term rate's specification given in (3.16). The main features and results of the model ar reported here; a complete study can be found in Renne (2012). The time unit is the day. The states of the Markov chain $z_t^{(1)}$ appearing in (3.16) are the Kronecker product of two discrete sets of states $z_{1,t}^{(1)} \otimes z_{2,t}^{(1)}$:

$z_{1,t}^{(1)}$ is valued in the set of the selection vectors of size $K+1$, where $K+1$ is the number of possible values of the target rate of the European Central Bank (ECB): $k \times 0.25\%$, with $k = 0, \dots, K$, 25bp being the basic tick. Here we take $K = 40$ and, hence, the maximal value of the target rate is assumed to be 10%. A key advantage of this model is that the the support of the target rate is discrete and, importantly, positive. Therefore, this model turns out to be appropriate to deal with situations of very low short-term rates.

$z_{2,t}^{(1)}$ is valued in the set of selection vectors of dimension 3, each regime representing a monetary policy phase: tightening (T), status-quo (S) and easing (E). A tightening (resp. easing) monetary policy aims at restricting (resp. weakening) credit conditions.[8]

$z_t^{(2)}$ is a two-regime Markov chain independent of $z_t^{(1)}$, representing two liquidity situations of the banking system: the agregate liquidity situation of the banking system being either normal or in excess. The excess liquidity regime is aimed at accommodating the drop of the overnight interbank rate (with respect to the policy rate) that appeared after October 2008, following changes in the monetary-policy implementation in the euro area in response to the financial crisis.

Δ_t is a bivariate row vector of independent variables whose distributions are mixtures of beta distributions and $\{y_t\}$ is a bidimensional Gaussian VAR(1) in both worlds. Whereas all the variables are observed by the investors, the econometrician observes $z_{1,t}^{(1)}$ only. The short rate, which is the overnight interbank rate, is given by:

$$r_t = \bar{r}_t + \Delta_t' z_t^{(2)} + e' y_t,$$
$$\text{with a target, or policy rate:} \bar{r}_t = D' z_{1,t}^{(1)},$$

where the components of D give the admissible values of the target rates: the i^{th} entry of D is $\log[1 + (i-1) \times 0.25\%/360]$. The process $z_t^{(1)}$ is specified in both worlds as a Markov chain. This Markov chain depends on time in a deterministic way, since the value of the policy rate can change

[8] In other words, a rise (cut) in the policy rate can only take place during a tightening (resp. an easing) regime.

only on days at which a monetary-policy meeting is scheduled.[9] The size of the transition matrices Π_t is 123×123, involving 15006 independent entries. A parcimonious parameterization is obtained by introducing the following assumptions:

(a.1) An easing or tightening regime is necessarily followed by a status-quo regime.

(a.2) Conditional on an easing (E), statu-quo (S), or tightening (T) regime, the target can move in $\{-0.5\%, -0.25\%, 0\}, \{0\}, \{0, +0.25\%, 0.50\%\}$, respectively.

Under (a.1) and (a.2) the transition probabilities matrices of $\{z_t^{(1)}\}$, namely Π_t, can be characterized by eight conditional probabilities. These provide the transitions between monetary-policy regimes and the probabilities of a change of level of the target rate given the regime. They are :

$$p_{E\to S}, p_{T\to S}, p_{S\to E}, p_{S\to T},$$
$$p_{-.25}, p_{-.50}, p_{+.25}, p_{+.50}$$

(a.3) The following eight conditional probabilities are specified parametrically as functions of the previous value of the target rate:

(a.4) The Markov chain $\{z_t^{(2)}\}$ is an homogenous Markov chain.

Under these parametric assumptions, the final number of free parameters is 20 (instead of $123 \times 122 = 15006$). The theoretical prices of the zero-coupon bonds are of the form (3.17), in which $a_{1,t}(h)$ does depend on t, because of the non-homogeneity of $z_t^{(1)}$, whereas the $a_{i,t}(h), i = 2, 3, 4$ do not depend on t. Whereas functions $a_{1,t}(h)$ and $a_{2,t}(h)$ are obtained in closed form, $a_{3,t}^{(h)}$ and $a_{4,t}^{(h)}$ are obtained recursively.

As far as the data are concerned, the short rate r_t is the EONIA (Euro Over-Night Index Average) and the default-free term structure is obtained from the OIS (Overnight Index Swaps). Six maturities are used in the estimation: 1, 3, 6, 12 months and 2, 4 years. The sample period is January 15, 1999 to February 17, 2012 (3416 dates). Figure 2 shows some of these data.

Parameter estimates are obtained by maximizing the likelihood. To compute the latter, we use a methodology based on a joint use of the Kitagawa-Hamilton filter and inversion techniques *à la* Chen and Scott (1993). The resulting fit is satisfying, the standard deviation of the yield pricing errors being of 8 basis points across the different maturities.

Figure 3 displays the smoothed probabilities of being in the different regimes. Interestingly, the approach is able to detect changes in the monetary policy regimes even when the policy rate

[9] For those dates for which no monetary-policy meeting is scheduled, the probabilities of changes in the monetary-policy regime ($z_{2,t}^{(1)}$) may be strictly positive (even if the target, defined by $z_{1,t}^{(1)}$, can not be changed at those dates).

does not move. For instance, while there were no target move between mid-2007 and mid-2008, a period is identified as an easing regime in early 2008. This reflects the fact that the OIS rates were relatively low at that time, indicating that market participants were anticipating future cuts in the policy rate (that did not materialize eventually). In addition, the third panel of Figure 3 shows how the excess-liquidity regime is associated with those periods during which the spread between the overnight interbank rate (EONIA) and the policy rate is persistently low.

As shown in Renne (2012), this model can be used to examine risk premia associated with target moves. The analysis notably suggests that market yields reflect the behavior of a central bank that would tend to rise (respectively cut) the target rate more rapidly than is physically observed when in a tightening (resp. easing) phase. This has implications regarding the common practice that consists in inverting the OIS yield curve to extract market-based short-term forecasts of the policy-rate path. Specifically, it means that such a practice –that assumes that the expectation hypothesis holds at the short-end of the yield curve– is valid in terms of sign of next target changes, but tend to overestimate their size.

In addition, this model is a natural framework to assess the implications of various monetary-policy measures. In particular, the model is exploited to predict the potential effects on the yield curve of a commitment of the central bank to keep its rate unchanged for a given period of time. The simulations support the view that such measures may be particularly effective in a context of low short-term rates (see Renne, 2012).

Figure 2: Estimation data

Notes: The first panel shows the target rate together with the overnight interbank interest rate (EONIA). The dashed lines delineate the monetary-policy "corridor" whose upper bound is the Eurosystem marginal-lending-facility rate and the lower bound is the Eurosystem deposit-facility rate. Since the Eurosystem's banks can lend at the former rate and borrow at the latter rate, the overnight interbank rate evolves between these two rates. The second panel displays the policy rate together with longer-term rates: the 6-month and the 4-year OIS rates.

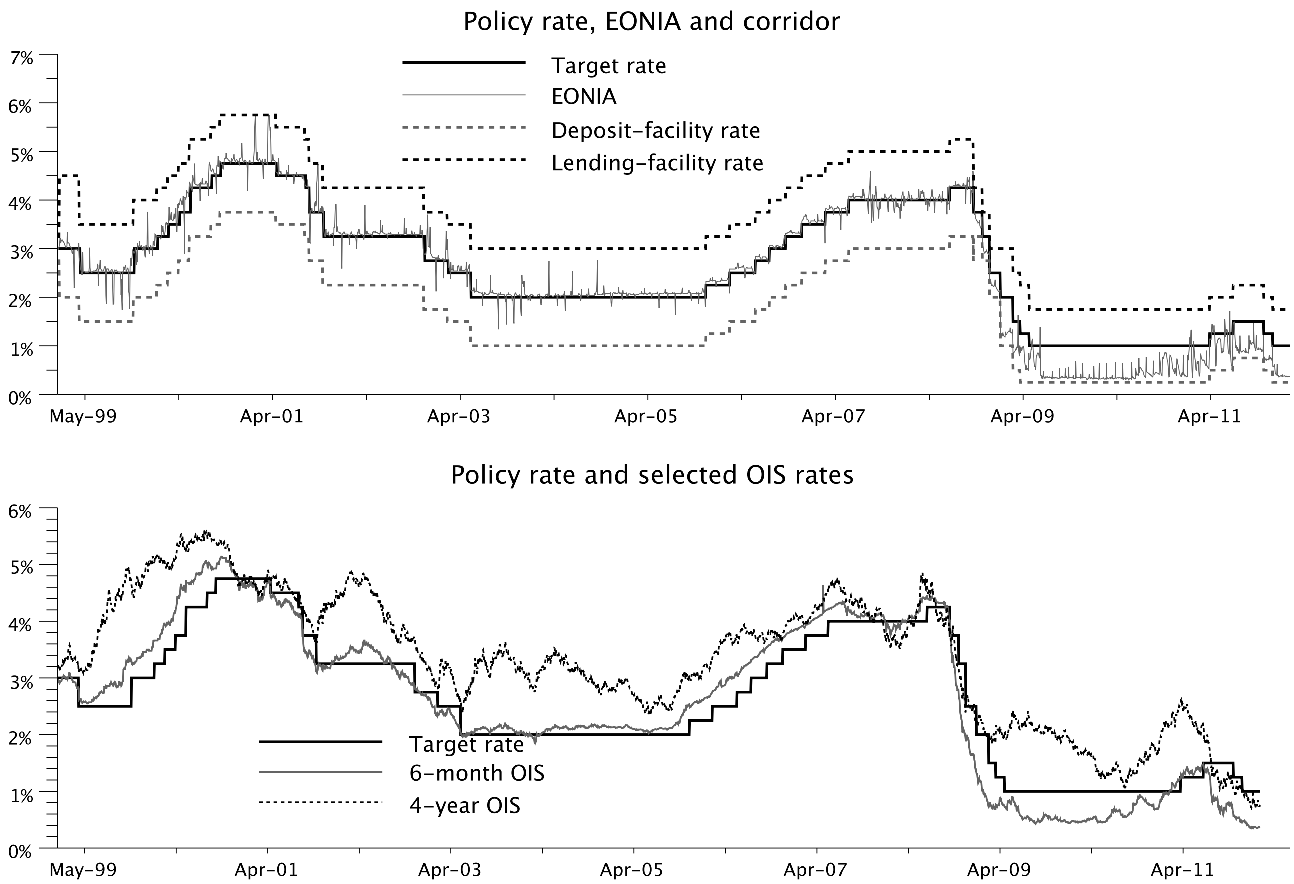
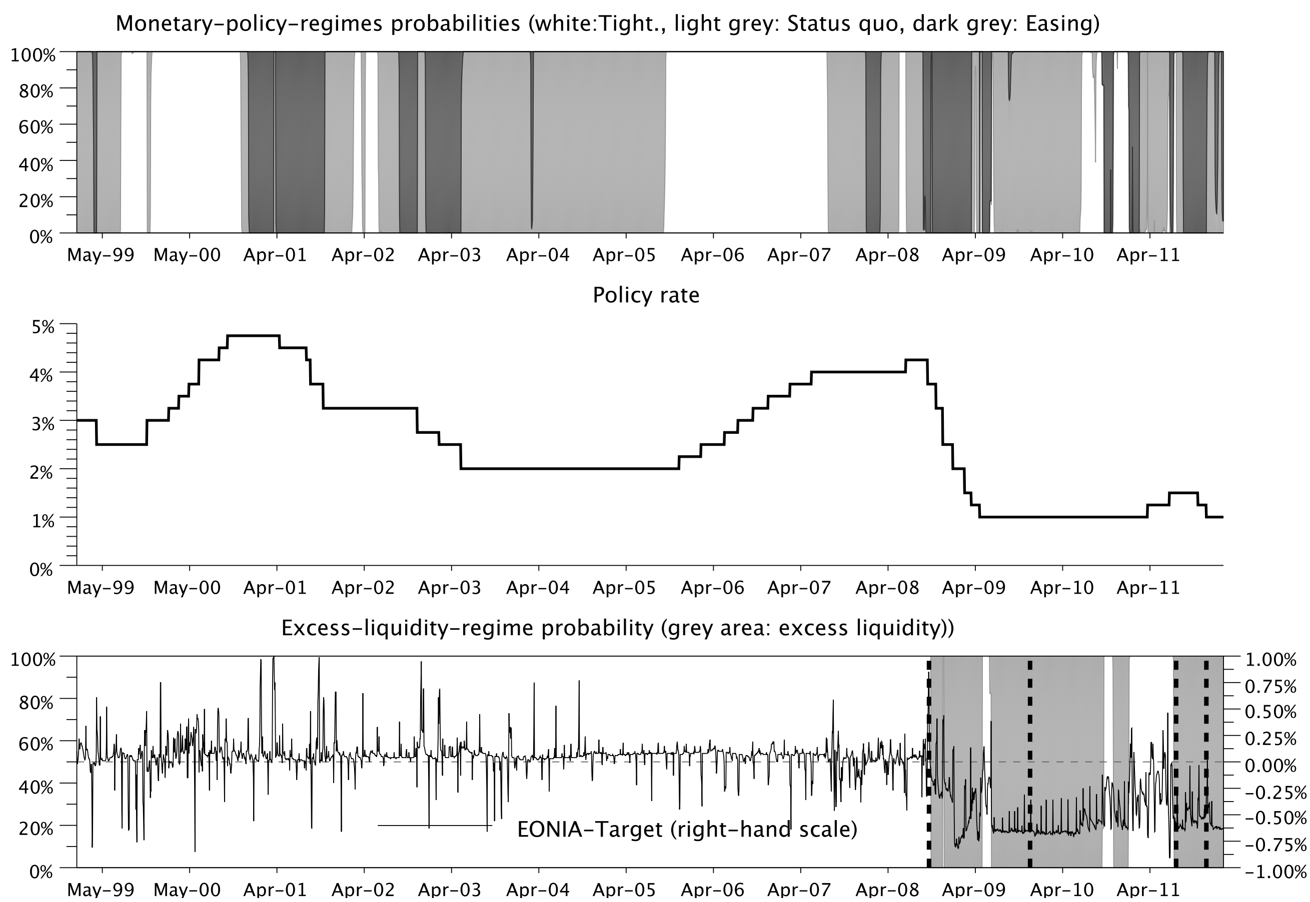


Figure 3: Estimation data

Notes: The first panel shows the smoothed probabilities of being in the different monetary-policy regimes. It can be seen for instance that rises in the policy rate (see second panel) take place during the tightening regime (white areas in the first panel). The third panel displays the probabilities of being in the excess-liquidity regime (grey area). The EONIA spread, that is the spread between the overnight interest rate and the policy rate, is also reported in the third panel. It appears that the excess-liquidity regime correspond to those periods during which the EONIA spread is low. The four vertical bars in Panel B indicate the four following dates, respectively: 8 October 2008 (introduction of Fixed-Rate Full Allotment procedures), 3 December 2009 (announcement of the phasing out of the very long-term refinancing operations), 4 August 2011 (given the renewed financial-market tensions, announcement of supplementary VLTRO), 8 December 2011 (3-year VLTRO). Source: Renne (2012)



4 REGIME SWITCHING AND DEFAULTABLE BOND PRICING

The RSTSM can be extended to the modelling of defaultable bonds. In this framework, we distinguish the individual default indicators and associated individual risk factors from the common risk factors. This modeling is illustrated by an analysis of the Euro-zone sovereign bonds.

4.1 The setting

4.1.1 Risk-neutral dynamics and causality structure

The new information in the economy at date t is $w_t = (z_t', y_t', w_{s,t}', d_t')'$, where z_t is a regime variable valued in $\{e_1, \ldots, e_J\}, y_t$ is a vector of common factors, $w_{s,t}$ is a vector $(w_{s,t}^{1'}, \ldots, w_{s,t}^{n'}, \ldots, w_{s,t}^{N'})'$ of specific variables, $w_{s,t}^n$ corresponding to debtor $n (n = 1, \ldots, N)$ and $d_t = (d_t^1, \ldots, d_t^n, \ldots, d_t^N)'$ is a vector of default indicators, where $d_{n,t} = 1$, if entity n is in default at date t, $d_t^n = 0$, otherwise. Thus there are *two* kinds of regime variables: z_t is a *systematic* regime variable and d_t is a set of *individual binary* regime variables $d_t^n, n = 1, \ldots, N$.

We use below the following notations :

$w_{c,t} = (z_t', y_t')'$ for the common variables, $\tilde{w}_t = (w_{c,t}', w_{s,t}')'$ for all common and specific variables, $\tilde{w}_t^n = (w_{c,t}', w_{s,t}^{\prime n})'$ for common variables and specific variables of entity n only.

We make some assumptions about the R.N. dynamics of process $\{w_t\}$, in particular about its R.N. causality structure. Since these assumptions concern the risk-neutral distribution, their economic interpretation is in terms of prices only, not in terms of sufficient information for historical prediction.

A.1. (R.N. Causality structure) : $(w_{s,t}', d_t')'$ does not Granger cause $\{w_{c,t}\}$, and, $\{d_t\}$ does not cause $\{\tilde{w}_t\}$.

Thus the price of a future contract written on a common qualitative, or quantitative factors, does not depend on the individual observations. Moreover, the prices of these futures on the underlying individual risk factors depend only of the histories of the factors, but not on the observed individual defaults.

A.2.(R.N. Conditional independence of the entity behaviors) :The variables $(w_{s,t}^{n'}, d_t^n)'$, $n = 1, \ldots, N$, are independent conditionally on $(w_{c,t}', \underline{w_{t-1}'})'$, and the conditional distribution of $w_{s,t}^n$ only depends on $(\underline{w_{c,t}'}, \underline{w_{s,t-1}^{n'}})$.

A.3. Process $\{w_{c,t}\}$ is Car(1) and process of individual risk factors $\{w^n_{s,t}\}$ is conditionally Car(1), that is, the conditional Laplace transform of $w^n_{s,t}$ given $\underline{w_{c,t}}$, $\underline{w^n_{s,t-1}}$ is exponential affine in $w_{c,t}, w_{c,t-1}, w_{s,t-1}$ (which implies that $\tilde{w}^n_t$ is Car (1)).

A.4. (R.N. default intensity) : $Q(d^n_t = 0|d^n_{t-1} = 0, \underline{\tilde{w}_t}) = \exp(-\lambda^Q_{n,t})$, with
$\lambda^Q_{n,t} = \alpha_{0,n} + \alpha'_{1,n}\tilde{w}^n_t$, and $Q(d^n_t = 1|d^n_{t-1} = 1, \tilde{w}_t) = 1$, that is, the state $d^n_t = 1$ is an absorbing state.

The exponential expression of the R.N. transition probability ensures its positivity, and the affine expression of the intensity is introduced to facilitate the computation of the term structure. Since the transition probability is also smaller than 1, the intensity has to be nonnegative, which induces restrictions on the R.N. dynamics of $\{\tilde{w}^n_t\}$. The price of a future contract written on the individual default of a given entity does not depend on the individual risk factors of the other entities. Thus all the commonality between prices of future contracts on individual defaults is captured by the common risk factor.

A.5. (Riskfree rate) : The riskfree short rate between t and $t+1$ is :

$$r_t = \beta_0 + \beta'_1 w_{c,t}.$$

Since the individual risk factors do not appear in the expression of the riskfree rate, no entity has an impact on the riskfree prices, that is, there is no concentration effect. Under Assumptions A.1-A.5, the spot price of any derivative written on $w_{c,t}$ depends on the past of the common factor only.

4.1.2 Pricing defaultable bonds

Let us consider the case where the recovery rate is zero. The price at time t of a zero-coupon bond issued by entity n, with residual maturity h, is :

$$B_n(t,h) = E^Q_t[\exp(-r_t \ldots - r_{t+h-1})(1 - d^n_{t+h})]. \tag{4.1}$$

Although $(\tilde{w}'_t, d^n_t)'$ is not Car (1), the causality structure assumed in Assumption A.1 implies that $B_n(t,h)$ can still be expressed as a multihorizon Laplace transform of the process $\{\tilde{w}^n_t\}$, with reverse ordered coefficients (see Subsection 2.1). More precisely we have the following Proposition, which justifies formula (2.6) :

Proposition 8. *: Under Assumptions A.1-A.2, A.4-A.5:*

$$B_n(t,h) = \exp(-r_t)E_t^Q[\exp(-r_{t+1} - \ldots - r_{t+h-1} - \lambda_{n,t+1}^Q - \ldots - \lambda_{n,t+h}^Q)]$$

$$= \exp[-h(\beta_0 + \alpha_{0,n}) - \tilde{\beta}_1' \tilde{w}_t^n]$$
$$+ E_t^Q\{\exp[-(\tilde{\beta}_1 + \alpha_{1,n})'\tilde{w}_{t+1}^n - \ldots - (\tilde{\beta}_1 + \alpha_{1,n})'\tilde{w}_{t+h-1}^n - \alpha_{1,n}'\tilde{w}_{t+h}^n]\},$$

where $\tilde{\beta}_1 = (\beta_1', 0)'$.

Proof : see Appendix 8.

If moreover Assumption A.3 is satisfied, $\{\tilde{w}_t^n\}$ is Car(1), and, since the Laplace transform is with a reverse order structure :

$$\delta_1 = -\alpha_{1,n}, \delta_j = -(\tilde{\beta}_1 + \alpha_{1,n}), \forall j \geq 2,$$

the prices $B_n(t,h), t = 1, \ldots, T, h = 1, \ldots, H$ can be computed recursively by using only once the algorithm of Proposition 3.

So the yield $R_n(t,h)$ of residual maturity h associated with entity n is an *affine function* of $\tilde{w}_t^n$:

$$R_n(t,h) = c_n'(h)\tilde{w}_t^n + b_n(h), \text{ say.} \tag{4.2}$$

The riskfree rate of residual maturity h is obtained by the same algorithm, with $\alpha_{0,n} = 0, \alpha_{1,n} = 0$, and is an affine function of $w_{c,t}$:

$$R^*(t,h) = c'(h)w_{c,t} + b(h), \text{ say,} \tag{4.3}$$

as are the spreads :

$$R_n(t,h) - R^*(t,h) = [c_n(h) - c^*(h)]'\tilde{w}_t^n + b_n(h) - b(h),$$

where $c^*(h) = [c'(h), 0]'$.

Therefore, the riskfree and defaultable term structures are all affine. They differ by the baseline term structures and the set of factors involved in their affine expressions. Also note that a direct impact of the regime variable appears since $w_{c,t} = (z_t', y_t')'$. This result can be extended to the case of a "market value" recovery rate [see Duffie, Singleton (1999), Monfort, Renne (2011), and Appendix 9].

4.1.3 The historical dynamics

Once the R.N. distribution f^Q and the short rate r_{t-1} are specified, the historical p.d.f. f^P can be chosen arbitrarily and the s.d.f. $M_{t-1,t}$ is deduced from (2.4). In this section, we assume that the s.d.f. $M_{t-1,t}$ depends on the common variables $\underline{w_{c,t}}$ only :

Assumption A.6 :$M_{t-1,t}$ is a function of $\underline{w_{c,t}}$.

Assumption A.6 means that the individual variables $w_{s,t}$ and d_t have no impact on the adjustment for risk.

This assumption has important consequences. Let us first show a lemma.

Lemma : *If w_t is partitioned into $w_t = (w'_{1,t}, w'_{2,t})'$ and if the s.d.f. $M_{t-1,t}$ is a function of $(w_{1t}, \underline{w_{t-1}})$:*

i) the conditional R.N. and historical distributions of $w_{2,t}$ given $(w_{1,t}, \underline{w_{t-1}})$ are the same.

ii) the conditional R.N. and historical distributions of $w_{1,t}$ given $\underline{w_{t-1}}$ satisfy the relation :

$$f^P(w_{1,t}|\underline{w_{t-1}}) = f^Q(w_{1,t}|\underline{w_{t-1}})M_{t-1,t}^{-1}(w_{1,t}, \underline{w_{t-1}}) \exp(-r_{t-1}).$$

Proof :

Indeed, equation (2.3) can be written :

$$\begin{aligned} f^Q(w_{1,t}|\underline{w_{t-1}})f^Q(w_{2,t}|w_{1,t}, \underline{w_{t-1}}) &= f^P(w_{1,t}|\underline{w_{t-1}})f^P(w_{2,t}|w_{1,t}, \underline{w_{t-1}}) \\ &\times M_{t-1,t}(w_{1,t}, \underline{w_{t-1}}) \exp(r_{t-1}). \end{aligned}$$

Integrating both sides of this equation with respect to $w_{2,t}$ gives the equality ii) of the Lemma, and i) follows. □

The Lemma above shows the consequences of the absence of some risk factors in the s.d.f. Let us now apply it to see the consequences of the additional Assumption A.6 on the joint R.N. and historical analysis of default.

Proposition 9. *: Under Assumption A.6 on the s.d.f. and Assumption A.1 of non-causality from $(w'_{s,t}, d'_t)'$ to $w_{c,t}$:*

i) the R.N. and historical conditional distributions of $(w'_{s,t}, d'_t)'$ given $(w_{c,t}, \underline{w_{t-1}})$ are the same.

ii) $\{w'_{s,t}, d'_t\}'$ does not cause $\{w_{c,t}\}$ in the historical world.

Proof :

Proposition 9, i) is obtained from Lemma i) by taking $w_{1,t} = w_{c,t}$ and $w_{2,t} = (w'_{s,t}, d'_t)'$, and Proposition 9 ii) is obtained from Lemma ii) noting that $f^Q(w_{1t}|\underline{w_{t-1}})M_{t-1,t}$ and r_t depend on $\underline{w_{t-1}}$ through $\underline{w_{c,t-1}}$ only $\square$.

Proposition 9, i) implies that assumptions A.2 and A.4 are also valid in the historical world. In particular the historical and R.N. default intensities are the same :

$$\lambda^P_{n,t} = \lambda^Q_{n,t} = \alpha_{0,n} + \alpha'_{1,n}\tilde{w}^n_t. \tag{4.4}$$

However, equality (4.4) does not imply that the historical intensity $\lambda^P_{n,t}$ (or $\lambda^Q_{n,t}$) has the same dynamic behavior in both worlds since the R.N. and historical dynamics of common risk factor $w_{c,t}$ are *different,* such that :

$$f^P_c(w_{c,t}|\underline{w_{c,t-1}}) = f^Q_c(w_{c,t}|\underline{w_{c,t-1}})M^{-1}_{t-1,t}(\underline{w_{c,t}})\exp[-r_{t-1}(\underline{w_{c,t-1}})]. \tag{4.5}$$

Once f^Q_c and r_{t-1} have been specified, f^P_c can be chosen arbitrarily and the s.d.f. is deduced from (4.5). However we can want to specify $M_{t-1,t}$ in a way which makes it easily interpretable, while giving a tractable historical dynamics for $w_t = (z'_t, y'_t)'$. The general method based on an *exponential affine* s.d.f. presented in Section 3.1.2. remains valid as well as the case of a switching VAR model described in Section 3.1.3.

4.2 Credit vs liquidity risks in euro-area sovereign yield curves

The following application is detailed in Monfort, Renne (2011)b. Its objective is to model the sovereign yield curves of ten euro-area countries in order, in particular, to disentangle the impacts of the credit and liquidity risks, and to evaluate the historical and risk-neutral evolutions of the probabilities of default. We introduce a hidden Markov chain $\{z_t\}$ with two regimes in order to capture crisis periods.

Estimation data include monthly yields with residual maturities 1,2,5 and 10 years and cover the period between July 1999 and March 2011. The short rate is the one month EONIA swap rate. The considered sovereign issuers are Austria, Belgium, Finland, France, Germany, Ireland, Italy, the Netherlands, Portugal and Spain. The German bonds, known as *Bunds*, are considered as riskfree. The identification of liquidity-pricing effects relies on the spreads between the German sovereign bonds and those issued by KfW (Kreditanstalt für Wiederaufbau), a German agency whose bonds are fully and explicitly guaranteed by the Federal Repubilc of Germany. Therefore, the credit qualities of German sovereign and KfW bonds are the same, implying that the KfW-Bund spread is essentially liquidity-driven.

We use a Regime-Switching VAR (1) model with a five dimensional factor y_t. The observable entries of y_t are: the 10-year German yield, the slope of the German yield curve (10 year-1 month), the convexity the German yield curve (2×3 year - 10 year-1 month), the first and second principal components of the spreads of four countries versus Germany (France, Italy, Spain and the Netherlands, 10-year maturity).

Both the risk-neutral and historical models for the factor y_t are RSCar(1) of the form:

$$y_t = \mu' z_t + \Phi y_{t-1} + \Omega(z_t)\varepsilon_t,$$

where $\{z_t\}$ is a two regime exogenous time homogenous Markov chain. The default-free yields are given by

$$R(t,h) = -\frac{1}{h} \log E_t^Q \exp(-r_t \ldots - r_{t+h-1}), \tag{4.6}$$

where r_t is the one-month riskfree yield. The risky yields are given by:

$$R_n(t,h) = -\frac{1}{h} \log E_t^Q \exp(-r_t \ldots - r_{t+h-1} - \lambda_{n,t+1} \ldots \lambda_{n,t+h}), \tag{4.7}$$

where the intensity $\lambda_{n,t}$ is decomposed into $\lambda_{n,t} = \lambda^c_{n,t} + \lambda^l_{n,t}$, $\lambda^c_{n,t}$ and $\lambda^l_{n,t}$ being the credit (or default) and the liquidity intensities, respectively [see Liu, Longstaff, Mandall (2006), Feldhütter Lando (2008), Fontaine, Garcia (2009)].

The disentangling of the credit and illiquidity effect is based on the above-mentioned interpretation of the KfW-Bund spread and on the assumption according to which the $\lambda^l_{n,t}$ are affine functions of the illiquidity intensity obtained for KfW bonds. The intensities are assumed to be affine functions in z_t and y_t, and, since y_t is RSCar (1), formula (4.6) and (4.7) provide affine functions in z_t and y_t for $R(t,h)$ and $R_n(t,h)$.

The Kitagawa-Hamilton algorithm is used to compute the probabilities of being in the crisis regime. Figure 4 presents the resulting estimated periods of crisis. This figure illustrates that the crisis periods are associated with increasing and highly-volatile sovereign spreads. The approach results in a satisfying fit of the data (see Figure 5), the standard deviations of the yield pricing errors being of 18 bp (the model accounts for 98% of the yields' variances).

Importantly, this framework allows us to compute historical (or real-world) probabilities of default. Figure 6 presents the estimated term structures of probabilities of default (PDs), during two dates of our sample. Here, it is important to note that most of the methods implemented by practitioners to extract market-perceived PDs implicitly assume that historical and risk-neutral probability coincide. However, based on the present methodology, Monfort and Renne (2011)b show that the historical probabilities of default tend to be significantly lower than their risk-neutral counterparts.

Figure 4: The crisis regime

Notes: The grey-shaded areas correspond to crisis periods (estimated as those periods for which the smoothed probabilities of being in the crisis regime are higher than 50%, the smoothed probabilities being basedon the Kitagawa-Hamilton algorithm). The plot also displays the Spanish-German and the Irish-German 10-year sovereign spreads.

Figure 5: Model-implied vs. actual spreads

Notes: The black dotted lines (grey solid lines) correspond to model-implied (actual) spreads.

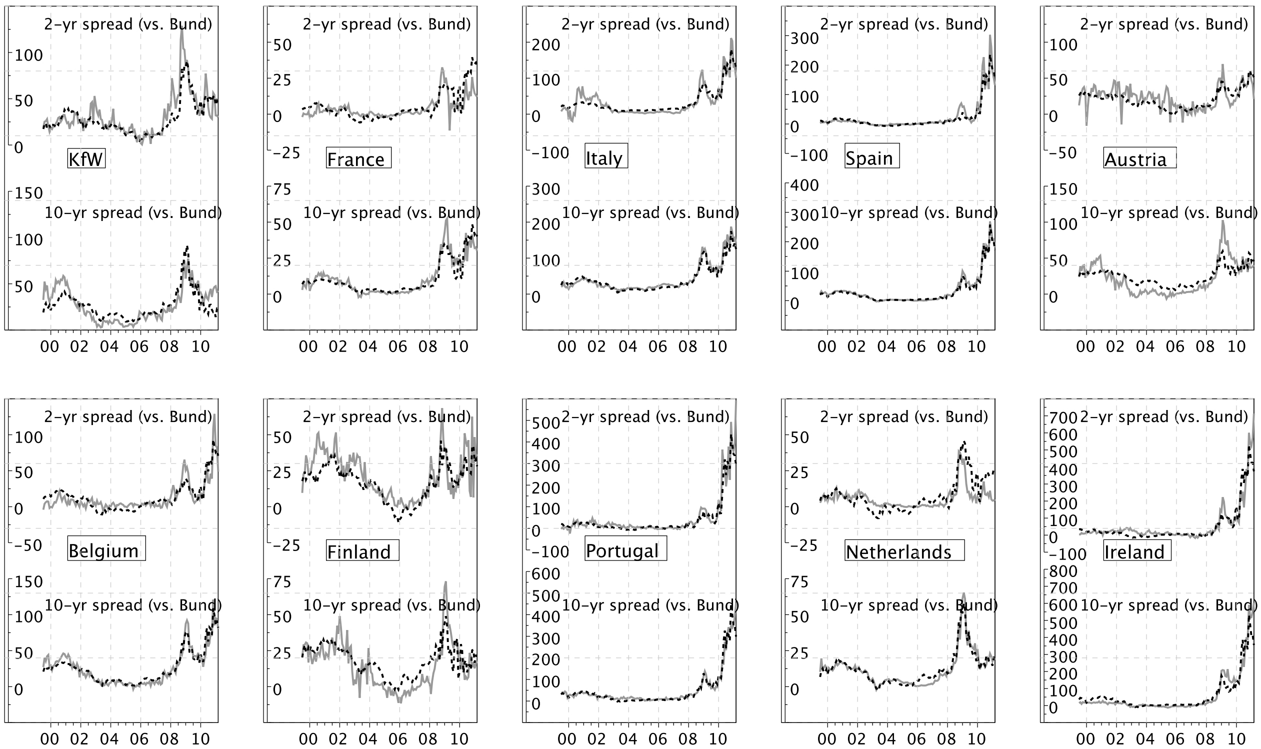
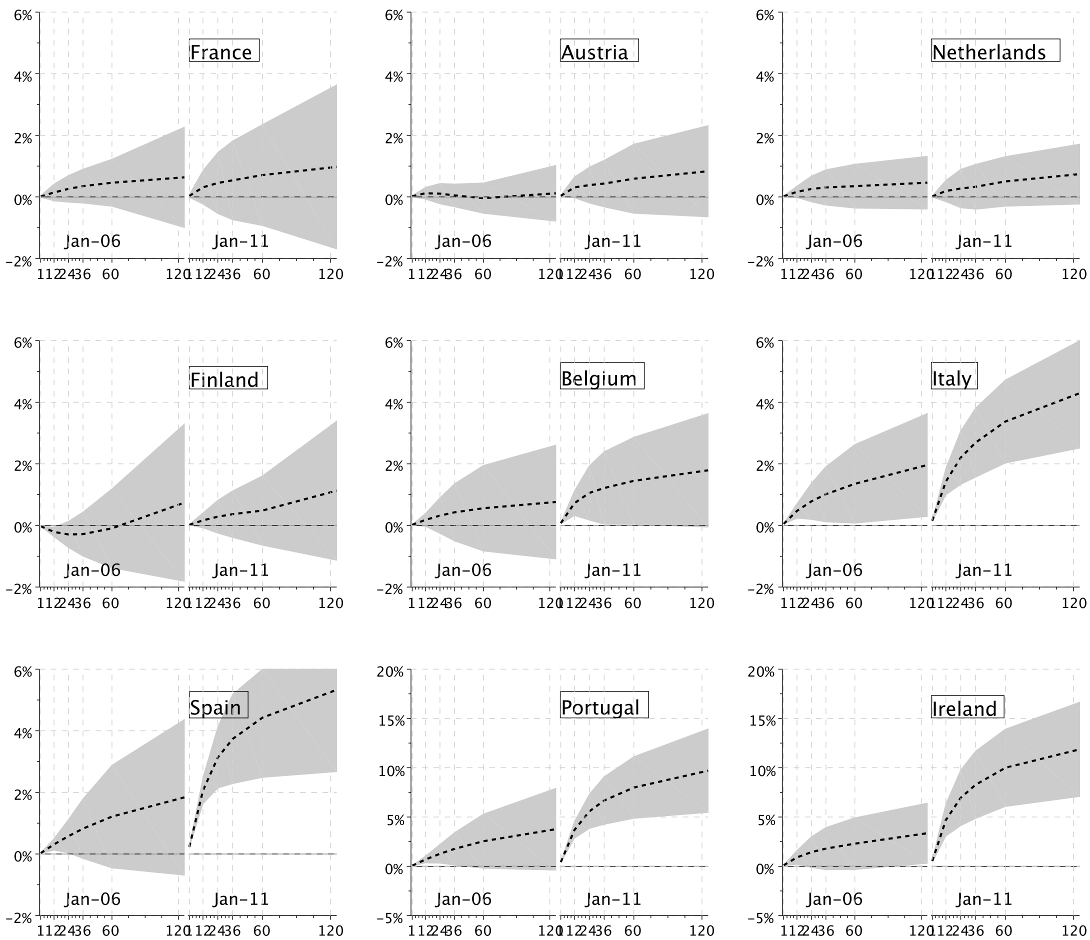


Figure 6: Term structures of (historical-world) probabilities of default

Notes: These plots show the term structures of default probabilities during two different dates for the different countries. For instance, for country n and for the five-year maturity (60 months on the x-axis), the plot reports the model-implied probability that country n defaults in the next five years. Note that these probabilities are historical ones, that is, they are based on the historical dynamics of the factor. 95% confidence intervals are reported.



5 OTHER RESEARCH DIRECTIONS

5.1 Regime switching and contagion between sectors

Under the Assumptions of Section 4, the historical and risk-neutral default intensities of entity n are the same functions of the common factors $w_{c,t}$ and of variables $w^n_{s,t}$, which are specific of entity n. In particular, we assume that the dependence between defaults is only captured by these common factors, which may be observed by the econometrician or latent. In the latter case they are called *"dynamic frailty"* factors [see e.g. Duffie, Eckner, Horel and Saita (2009)].

The s.d.f., defining the bridge between the historical and risk-neutral dynamics, is assumed not to depend on the default indicator functions $d_{n,t}, n = 1, \ldots, N$. In other words it is assumed that only the risk associated with the common factors $w_{c,t} = (z_t', y_t')'$ is priced and that there is no additional risk pricing associated with the default of any given entity.

In order to study the impact of the individual defaults, we could start from a historical dynamics of all the variables in the system and introduce the $d_{n,t}$ in the s.d.f. However, such an approach would lead to complicated R.N. dynamics and untractable pricing formulas. In order to avoid this combinatorial explosion, we introduce an intermediate level, called "sector", between the global and individual levels. Each sector, indexed by $k = 1, \ldots, K$, could be "risk infected", or not, and characterized by regime variables $z_{k,t}$ equal to $(1, 0)'$, if sector k is risk infected, and to $(0, 1)'$, otherwise. The infection state would not be an absorbing state. If the number K of sectors is not too large, the number of regimes 2^K of the global process $z_t = z_{1,t} \otimes \ldots \otimes z_{K,t}$ would remain tractable.

We could then propose parsimonious parameterizations of the historical transitions matrices $\tilde{\Pi}_t$ of z_t, of the historical default intensities $\lambda_{n,t}$ and of the impact of z_t in the s.d.f. $M_{t-1,t}$, given for instance by (3.9).

As far as $\tilde{\Pi}_t$ is concerned, we could assume that, conditionally on z_{t-1}, the variables $z_{k,t}, k = 1, \ldots, K$, are independent, and that the conditional distribution of $z_{k,t}$ given z_{t-1} depends on the $z_{l,t-1}, l = 1, \ldots, K$, to capture contagion between sectors. For instance to have a parsimonious and easily interpretable model, we could assume that this conditional distribution depends on $z_{k,t-1}$ and on the weighted number of infected sectors in $t-1$, which is a known linear function of z_{t-1}, $q'z_{t-1}$ say.

Similarly, we could assume that historical default intensity $\lambda_{n,t}$ is a linear function of $z_{k,t}$, where k is the sector of entity n, and of the weighted number of infected sectors $q'z_t$, and that z_t appears in the s.d.f. $M_{t-1,t}$ through $q'z_t$.

In such a framework all the results of Section 4 remain valid, the new feature being that there is an intermediate sector level appearing in the contagion and pricing modeling. For an illustration of this framework [see Monfort Renne (2011 a)].

5.2 Regime Switching and credit ratings

In their study of credit spreads Jarrow, Lando, and Turnbull (1997) model rating transitions as a time-homogenous Markov chain. However, there exist a strong evidence that transition probabilities are time-varying [see e.g. Lucas, Lonski (1992), Feng, Gourieroux, Jasiak (2008), Banga, Diebold, Kronimus, Schagen and Schuerman (2002).

The framework of Section 4 can be extended in order to model rating migration both in the historical and R.N. worlds. We can still introduce common factors $w_{c,t} = (z_t', y_t')'$ and specific factors $w_{s,t}^n$, for each entity $n = 1, \ldots, N$, with the difference that the default indicator variable $d_{n,t}$ is replaced by a rating variable $\tau_{n,t}$, which can take more than two alternatives, $1, 2, \ldots, K$, alternative K corresponding to default.

With similar assumptions in terms of causality and conditional independence in the R.N. world and in terms of s.d.f., the conditional distribution of $\tau_{n,t}$ given $\underline{z_t}, \underline{y_t}, \underline{w_{s,t}^n}$ is the same in the R.N. and historical worlds. It is characterized by a transition matrix $\Pi(z_t, y_t, w_{s,t}^n)$, which depends only on the current values $(z_t, y_t, w_{s,t}^n)$. Since default state K is absorbing, the last row of transition matrix Π is equal to $(0, \ldots, 0, 1)$. Buiding on Lando's approach [see Lando (1998) and Feldhutter, Lando (2008)], we assume, that this matrix admits a diagonal representation of the form

$$C\psi(z_t, y_t, w_{s,t}^n)C^{-1}$$

where the columns of C are fixed eigenvectors and $\psi(z_t, y_t, w_{s,t}^n)$ is a diagonal matrix with real positive diagonal terms equal to $\exp[-\psi_1(z_t, y_t, w_{s,t}^n], \ldots, \exp[-\psi_{K-1}(z_t, y_t, w_{s,t}^n)], 1$. It turns out that the (historical or R.N.) survival probabilities at horizon h are given by [see Monfort Renne (2011) a] :

$$\begin{aligned} & Q(\tau_{n,t+h} < K | \underline{z_{t+h}}, \underline{y_{t+h}}, \underline{w_{t+h}^n}, \tau_{n,t} = k) \\ = \;& -\sum_{j=1}^{K-1} (c_{k,j}(C^{-1})_{j,K} \exp[-\sum_{p=1}^{h} \psi(z_{t+p}, y_{t+p}, w_{s,t+p}^n)], \end{aligned} \tag{5.1}$$

where $C_{i,j}$ denotes the entries of matrix C and $(C^{-1})_{i,j}$ the entries of C^{-1}.

The price at t of a zero-coupon bond of residual maturity h issued by entity n currently in grade k at date t is given by :

$$B_n(t,h) = E_t^Q[\exp(-r_t - \ldots - r_{t+h-1}) \mathbb{1}_{\{\tau_{n,t+h<K}\}}],$$

in the case of a zero recovery rate. If we assume that r_t and the $\psi_k, k = 1, \ldots, K-1$ are affine functions of the factors, formula (5.1) implies that $B_n(t,h)$ is a linear combination of $K-1$ multihorizon

Laplace transforms. Therefore, if $\{z_t, y_t, w_{s,t}^n\}$ is CaR(1) under the risk-neutral probabilities, the general version of the recursive formulas of Proposition 3 can be used.

6 CONCLUDING REMARKS

In this paper we have stressed the role of the regime switching approach in various kinds of bond pricing models. We have seen that the regimes can capture a wide variety of underlying phenomena, and that the Regime Switching model are able to combine flexibility and tractability.

There are many related topics which have not been treated in this paper, in particular the inference methods adapted to these kinds of models [see Monfort, Renne (2011a)], the simultaneous modeling of nominal and real yield curves [see Ang, Bekaert, Wei (2008)] or the joint modeling of yield curves of several countries and the associated exchange rates [see, among the others, Backus, Foresi and Telmer (2001), Brennan and Xia (2006), Leippold and Wu (2007), Gourieroux, Monfort, Sufana (2010) and Graveline, Joslin (2011) for an approach without switching regimes]. The extensions to contagion and ratings outlined in the last section are also interesting directions of future research.

APPENDIX 1

Proof of Proposition 1

For expository purpose we omit the exponent h in $\gamma_i^{(h)}$. The formula is true for $h = 1$ since :

$$\begin{aligned} E_t \exp(\gamma_1' z_{t+1}) &= e' diag[\exp(\gamma_1)] E_t(z_{t+1}) \\ &= e' diag[\exp(\gamma_1)] \Pi_{t+1}' z_t \\ &= e' P_{t+1}'(\gamma_1) z_t, \end{aligned}$$

since $E_t(z_{t+1}) = (\pi_{i,1,t+1}, \ldots, \pi_{i,J,t+1})' = \Pi_{t+1}' e_i$, if $z_t = e_i$.

Assuming that the formula of Proposition 1 is true for $h - 1$, we get :

$$\begin{aligned} \varphi_{t,h}^{(z)} &= E_t[\exp(\gamma_1' z_{t+1} + \ldots + \gamma_h' z_{t+h})] \\ &= E_t[\exp(\gamma_1' z_{t+1}) E_{t+1} \exp(\gamma_2' z_{t+2} + \ldots + \gamma_h' z_{t+h})] \\ &= E_t[\exp(\gamma_1' z_{t+1}) e' P_{t+h}'(\gamma_h) \ldots P_{t+2}'(\gamma_2) z_{t+1}] \\ &= E_t[e' P_{t+h}'(\gamma_h) \ldots P_{t+2}'(\gamma_2) diag[\exp(\gamma_1)] z_{t+1}], \end{aligned}$$

where $diag[\exp(\gamma)]$ is the diagonal matrix whose diagonal terms are the exponential of the components of γ.

Therefore we have :

$$\begin{aligned} \varphi_{t,h}^{(z)} &= e' P_{t+h}'(\gamma_h) \ldots P_{t+2}'(\gamma_2) diag[\exp(\gamma_1)] \Pi_t' z_t \\ &= e' P_{t+h}'(\gamma_h) \ldots P_{t+1}'(\gamma_1) z_t. \end{aligned}$$

APPENDIX 2

Proof of Proposition 2

$$
\begin{aligned}
& E_{t-1}[\exp(u'w_{2,t}+v'z_t)] \\
= \;& E_{t-1}\{\exp(v'z_t)E[\exp(u'w_{2,t})|w_{2,t-1},\underline{z_t},\Delta_t]\} \\
= \;& E_{t-1}\{\exp[v'z_t+a'_{t-1}(u)w_{2,t-1}+b^{(0)\prime}_{t-1}(u)\Delta_t z_t]\} \\
= \;& \exp\{a'_{t-1}(u)w_{2,t-1}+[A_1(u,v),\ldots,A_J(u,v)]z_{t-1}\},
\end{aligned}
$$

with :

$$
A_i(u,v) = \log \sum_{j=1}^{J} \pi_{i,j,t} \exp[\psi_j(b^{(0)}_{t-1}(u))],
$$

$\psi_j(.)$ being the log-Laplace transform of Δ_t^j. Therefore $(w'_{2,t}, z'_t)'$ is Car (1).

If Δ_t^j is non random we have :

$$
\psi_j(b^{(0)}_{t-1}(u)) = b^{(0)}_{t-1}(u)'\Delta_t^j.
$$

APPENDIX 3

Proof of Proposition 3

Let us still omit the exponent h in the $\gamma_j^{(h)}$. For any $j=1,\ldots,h$ we have :

$$
(i) \qquad \varphi^{(w)}_{t,h} = E_t[\exp(\gamma'_1 w_{t+1} + \ldots \gamma'_j w_{t+j} + A^{(h)\prime}_{t,h-j} w_{t+j} + B^{(h)}_{t,h-j})],
$$

where :

$$(ii)\begin{cases} A^{(h)}_{t,h-j+1} = a_{t+j}(\gamma_j + A^{(h)}_{t,h-j}), \\ \\ B^{(h)}_{t,h-j+1} = b_{t+j}(\gamma_j + A^{(h)}_{t,h-j}) + B^{(h)}_{t,h-j}, \\ \\ A^{(h)}_{t,0} = 0, B^{(h)}_{t,0} = 0. \end{cases}$$

Indeed, we can prove formula (i) by recursion. Formula (i) is true for $j = h$, and, if this is true for j, we get :

$$\begin{aligned} \varphi^{(w)}_{t,h} &= E_t[\exp(\gamma_1' w_{t+1} + \ldots + \gamma_{j-1}' + a_{t+j}'(\gamma_j + A^{(h)}_{t,h-j})w_{t+j-1}) \\ &+ b_{t+j}(\gamma_j + A^{(h)}_{t,h-j}) + B^{(h)}_{t,h-j}]\,. \end{aligned}$$

Therefore formula (i) is true with $j-1$, $A^{(h)}_{t,h-j+1}$, $B^{(h)}_{t,h-j+1}$ being given by formulas (ii) above.

For $j = 1$ we get :

$$\begin{aligned} \varphi^{(w)}_{t,h} &= E_t \exp(\gamma_1' w_{t+1} + A^{(h)'}_{t,h-1} w_{t+1} + B^{(h)}_{t,h-1}) \\ &= \exp(A_{t,h}' w_t + B_{t,h}). \end{aligned}$$

Finally, if we put $h - j + 1 = i$, formula (i) becomes the formula of Proposition 3.

APPENDIX 4

Proof of Proposition 4

$$\begin{aligned} B(t,h) &= \exp(-r_t) E^Q[\exp(-r_{t+1} \ldots - r_{t+h-1}) | \underline{r_t}, \underline{y_t}] \\ &= \exp(-r_t) E^Q\{E^Q[\exp(-r_{t+1} \ldots r_{t+h-1}) | \underline{z_{t+h-1}}, \underline{r_{t+h-2}}, \underline{y_{t+h-2}}] | \underline{r_t}, \underline{y_t}\} \\ &= \exp(-r_t) E^Q\{\exp(\tilde{\gamma}_{t+h-1}' z_{t+h-1} - r_{t+1} \ldots - r_{t+h-2} | \underline{r_t}, \underline{z_t}\} \\ &= \exp(-r_t) E^Q\{\exp(\tilde{\gamma}_{t+h-1}' z_{t+h-1}) E^Q[\exp(-r_{t+1} \ldots - r_{t+h-2}) | \underline{z_{t+h-1}}, \underline{r_{t+h-3}}] | \underline{r_t}, \underline{y_t}\} \end{aligned}$$

Using the non causality from (r_t, y_t) to z_t, we can replace $\underline{z_{t+h-1}}$ by $\underline{z_{t+h-2}}$ in the conditioning and get :

$$B(t,h) = \exp(-r_t) E^Q \{ \exp(\tilde{\gamma}'_{t+h-1} z_{t+h-1} + \tilde{\gamma}'_{t+h-2} z_{t+h-2} - r_{t+1} \ldots - r_{t+h-3}) | \underline{r_t}, \underline{y_t} \},$$

and, by recursion :

$$B(t,h) = \exp(-r_t) E^Q \{ \exp(\tilde{\gamma}'_{t+1} z_{t+1} + \ldots + \tilde{\gamma}'_{t+h-1} z_{t+h-1}) | \underline{r_t}, \underline{y_t} \}.$$

Conditioning first by $z_t, \underline{r_t}, \underline{y_t}$ and using Proposition 1, we get :

$$B(t,h) = \exp(-r_t) E^Q \{ e' P'_{t+h-1}(\tilde{\gamma}_{t+h-1}) \ldots P'_{t+1}(\tilde{\gamma}_{t+1}) z_t | \underline{r_t}, \underline{y_t} \}$$

with $P_t(\gamma) = \Pi_t diag[\exp(\gamma)]$.

Finally :

$$B(t,h) = e' P'_{t+h-1}(\tilde{\gamma}_{t+h-1}) \ldots P'_{t+1}(\tilde{\gamma}_{t+1}) \hat{z}_t \exp(-r_t).$$

APPENDIX 5

Proof of Proposition 5

Let us consider the case where the conditional distribution of $r_{t+1} = e_1$ is the point mass at zero, and define the p.d.f. of $(z_{t+1}, z_t, r_{t+1} y_{t+1})$ given $\underline{r_t}, \underline{z_t}$, with respect to the measure $(\sum_{j=1}^{J} \delta_j)^{\otimes 2} \otimes (\delta_0 + \lambda_1) \otimes \lambda_1^K$, J being the number of states in the Markov chain z_t, K the size of y_t, $\delta_j, j = 1, \ldots, J$ the unit point mass at e_j and δ_0 the unit point mass at 0. This p.d.f can be factorized as :

$$q_t(z_{t+1}|z_t) f_t(r_{t+1}|z_{t+1}) g_t(y_{t+1}|z_{t+1}, \underline{r_{t+1}}, \underline{y_t}) p_t(z_t|\underline{r_t}, \underline{y_t}),$$

where $q_t(z_{t+1}|z_t), f_t(r_{t+1}|z_{t+1}),\ g_t(y_{t+1}|z_{t+1}, \underline{r_{t+1}}, \underline{y_t}), p_t(z_t|\underline{r_t}, \underline{y_t})$ denote the conditional p.d.f. with respect to the appropriate measure. In particular, we have $f_{1,t}(0) = 1$ and $f_{1,t}(r) = 0, \forall r \neq 0$. Therefore, we get :

$$p_{t+1}(z_{t+1}|\underline{r_{t+1}}, \underline{y_{t+1}}) = \frac{\Sigma_{z_t} q_t(z_{t+1}|z_t) f_t(r_{t+1}|z_{t+1}) g_t(y_{t+1}, |z_{t+1}, \underline{r_{t+1}}, \underline{y_t})}{\Sigma_{z_{t+1}} \Sigma_{z_t} q_t(z_{t+1}|z_t) f_t(r_{t+1}|z_{t+1}) g(y_{t+1}|z_{t+1}, \underline{r_{t+1}}, \underline{y_t}) p_t(z_t|\underline{r_t}, \underline{y_t})}.$$

Stacking the different value of $p_{t+1}(e_j|\underline{r_{t+1}}, \underline{y_{t+1}}) = \hat{z}_{j,t+1}$, :

$$\hat{z}_{t+1} = \frac{diag(f_t g_t)\Pi'_t \hat{z}_t}{e' diag(f_t g_t)\Pi'_t \hat{z}_t},$$

where $diag(f_t g_t)$ is the diagonal matrix whose k^{th} diagonal element is the product of $f_{k,t}(r_{t+1}) = f_t(r_{t+1}|e_k)$ by $g_{k,t}(y_{t+1}|\underline{r_{t+1}}, \underline{y_t}) = g_t(y_{t+1}|e_k, \underline{r_{t+1}}, \underline{y_t})$.

APPENDIX 6

Proof of Proposition 6

$$\begin{aligned}
\pi_{i,j,t} &= Q(z_t = e_j | z_{t-1} = e_i) \\
&= \int Q(z_t = e_j | r_t, y_t, z_{t-1} = e_i)\alpha_{i,t}\alpha_{0,t} d\mu \\
&= E_{0,t}(\alpha_{i,t}\beta_{j,t}), \\
B(t,h) &= \exp(-r_t) E_t^Q[\exp(-r_{t+1} - \ldots - r_{t+h-1})].
\end{aligned}$$

We have to show that for $h \geq 2$, :

$$E_t^Q[\exp(-r_{t+1} - \ldots - r_{t+h-1})] = e' \tilde{P}'_{t+h-1} \ldots \tilde{P}'_{t+1}\beta_t, \forall t. \quad (a)$$

The formula is true for $h = 2$, since :

$$\begin{aligned}
E_t^Q[\exp(-r_{t+1})] &= E_{0,t+1}[\exp(-r_{t+1})\alpha'_{t+1}]\beta_t \\
&= e' E_{0,t+1}[\exp(-r_{t+1})\beta_{t+1}, \alpha'_{t+1}]\beta_t \\
&= e' \tilde{P}'_{t+1}\beta_t,
\end{aligned}$$

(since $e'\beta_{t+1} = 1$).

Let us assume that formula (a) is valid for $h - 1$, we get :

$$
\begin{aligned}
& E_t^Q[\exp(-r_{t+1} - \ldots - r_{t+h-1})] \\
= \;& E_t^Q[\exp(-r_{t+1}) e' \tilde{P}'_{t+h-1} \ldots \tilde{P}'_{t+2} \beta_{t+1}] \\
= \;& e' \tilde{P}'_{t+h-1} \ldots \tilde{P}'_{t+2} E_t^Q[\exp(-r_{t+1}) \beta_{t+1}] \\
= \;& e' \tilde{P}'_{t+h-1} \ldots \tilde{P}'_{t+2} E_{0,t+1}[\exp(-r_{t+1}) \beta_{t+1} \alpha'_{t+1} \beta_t] \\
= \;& e' \tilde{P}'_{t+h-1} \ldots \tilde{P}'_{t+1} \beta_t.
\end{aligned}
$$

APPENDIX 7

Proof of Proposition 7

Let us consider the FDD historical dynamics defined by the conditional p.d.f. :

$$\alpha_{0,t}(w_t) \tilde{\alpha}'_t(w_t) \tilde{\beta}_{t-1}(w_{t-1}).$$

In this case the s.d.f. is :

$$M_{t-1,t} = \frac{\alpha'_t(w_t)\beta_{t-1}(w_{t-1})}{\tilde{\alpha}'_t(w_t)\tilde{\beta}_{t-1}(w_{t-1})} \exp(-r_{t-1}).$$

Conversely, let us consider a s.d.f. of the form :

$$M_{t-1,t}(w_t, w_{t-1}) = M_{1,t-1,t}(w_t) M_{2,t,t-1}(w_{t-1})$$

satisfying $E_{t-1}^Q(M_{t-1,t}^{-1}) = \exp(r_{t-1})$.

The historical conditional p.d.f. of w_t given w_{t-1} is given by :

$$M_{t-1,t}^{-1} \alpha_{0,t} \alpha'_t \beta_{t-1} \exp(-r_{t-1})$$

Let us define the p.d.f., w.r. to the distribution $\alpha_{0,t}$:

$$\tilde{\alpha}_{j,t} = \frac{\alpha_{j,t} M_{1,t-1,t}^{-1}}{E_{0,t}(\alpha_{j,t} M_{1,t-1,t}^{-1})} \tag{5.2}$$

and the probabilities :

$$\tilde{\beta}_{j,t-1} = \beta_{j,t-1} M^{-1}_{2,t-1,t} E_{0,t}(\alpha_{j,t} M^{-1}_{1,t-1,t}) \exp(-r_{t-1}) \tag{5.3}$$

which are summing to one since :

$$\begin{aligned}
\sum_{j=1}^{J} \tilde{\beta}_{j,t-1} &= \exp(-r_{t-1}) E_{0,t}(\alpha'_t \beta_{t-1} M^{-1}_{t-1,t}) \\
&= \exp(-r_{t-1}) E^Q_{t-1}(M^{-1}_{t-1,t}) \\
&= 1.
\end{aligned}$$

We have $M^{-1}_{t-1,t} \alpha_{0,t} \alpha'_t \beta_{t-1} \exp(-r_{t-1}) = \alpha_{0,t} \tilde{\alpha}'_t \tilde{\beta}_{t-1}$. $\square$

APPENDIX 8

Proof of Proposition 8

i) By definition the price of the defaultable zero-coupon bond with zero recovery rate is :$B_n(t,h) = E^Q_t[\exp(-r_t - \ldots - r_{t+h-1})(1 - d^n_{t+h})]$.

Conditioning with respect to $\underline{\tilde{w}_{t+h}}$ and using Bayes formula, we get :

$$B_n(t,h) = \exp(-r_t) E^Q_t\{\exp(-r_{t+1} - \ldots - r_{t+h-1}) \times \prod_{j=1}^{h} Q(d^n_{t+j} = 0 | d^n_{t+j-1} = 0, \underline{\tilde{w}_{t+h}})\}.$$

Since $\{d_t\}$ does not cause $\{\tilde{w}_t\}$ we can replace $\underline{\tilde{w}_{t+h}}$ by $\underline{\tilde{w}_{t+j}}$ in the generic term of the product.

Finally, we get:

$$B_n(t,h) = \exp(-r_t) E^Q_t[\exp(-r_{t+1} - \ldots - r_{t+h-1} - \lambda^Q_{n,t+1} - \ldots - \lambda^Q_{n,t+h})].$$

ii) Formula ii) is obtained by replacing r_t and $\lambda^Q_{n,t}$ by their expressions given in Assumptions A4 and A5.

APPENDIX 9

Term structure of recovery adjusted defaultable bonds

If the recovery payoff, when issuer n defaults between $t-1$ and t, is equal to a fraction $F_{n,t}$ (function of $\tilde{w}_t^n$) of the price that would have prevailed without default, $B_n(t,h)$ can still be computed in the same way as in Proposition 8, provided that the R.N. default intensity λ_{nt}^Q is replaced by a R.N. "*recovery adjusted*" default intensity $\tilde{\lambda}_{n,t}^Q$ defined by :

$$\exp(-\tilde{\lambda}_{n,t}^Q) = \exp(-\lambda_{n,t}^Q) + [1 - \exp(-\lambda_{n,t}^Q)]F_{n,t}.$$

The quantity $\exp(\tilde{\lambda}_{n,t}^Q)$ represents the short term R.N. expected gain. If there is no expected default, the recovery rate is equal to 1, which corresponds to the first component. If there is an expected default, with probability $1-\exp(-\lambda_{n,t}^Q)$, the recovery rate is the contractual market value $F_{n,t}$.

If $F_{n,t} = 0$, we get the previous model $\tilde{\lambda}_{n,t}^Q = \lambda_{n,t}^Q$, and, if $F_{n,t} = 1$, we get $\tilde{\lambda}_{n,t}^Q = 0$, that is, the default-free case. If $\lambda_{n,t}^Q$ is small, we get $\tilde{\lambda}_{n,t}^Q \simeq \lambda_{n,t}^Q(1 - F_{n,t})$.

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