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Region–based theory of discrete spaces: A proximity approach [1]

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Abstract

We introduce Boolean proximity algebras as a generalization of Efremovič proximities which are suitable in reasoning about discrete regions. Following Stone's representation theorem for Boolean algebras, it is shown that each such algebra is isomorphic to a substructure of a complete and atomic Boolean proximity algebra.

Key words: Proximity algebras, discrete spaces, qualitative spatial reasoning

1 Introduction

Qualitative theories of space such as Whitehead [20], its extensions by Clarke [3], or the Region Connection Calculus [12] are based on the assumption that regions

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in space are infinitely divisible and that regions may be arbitrarily close. This is certainly true when the model consists of all regions in Euclidean space or a suitable approximating domain such as the polygonal algebras of Pratt and Schoop [11]. But for applications in the physical world, where observation is limited by the precision of measurement and granularity of knowledge, it is more appropriate to allow regions which may be indivisible and which may have immediate neighbors. Such a scenario will be called *discrete*. Digital topology considers concepts such as *arc*, *connectedness* etc for various topologies on structures which are in some sense discrete, such as $\mathbb{Z}^n$. However, as pointed out by Smyth [13],

> "Standard digital topology ... is not topological in the strict sense, but graph–theoretic. Connectivity is taken in the usual graph sense, and analogies of the notions of arc and closed curve, and of the Jordan curve theorem etc. are developed."

In standard topological spaces, the base elements are points, while in some discrete contexts the base elements are indivisible collections of points, sometimes called *cells*, and binary relations among such cells called *adjacency relations* [6, 7]. For such discrete spaces, Smyth [13] suggests the use of generalized topological structures such as the closure spaces of Čech [2]. Galton's approach is closer to the region based concept of space: He defines regions, contact relations among regions and derivatives of contact relations such as part–of, overlap etc. It is notable that some of the axioms of proximity spaces [10] coincide with some of the axioms of the "contact relation" given in various spatial contexts. This makes it possible to explore proximity theory more deeply in the region–based theories of space. Our previous work [18, 19], as well as [5] can be considered as an application of the proximity approach to the study of some region–based theories of "continuous" space. In the present paper we generalize the notion of proximity space to make it applicable to discrete situations. In particular, we define the notion of *Boolean proximity algebra*, and show that each such structure has a representation as a substructure of a discrete proximity space. Just as the connection algebras of [18] can be considered as a "pointless" formulation of region–based theories of continuous spaces, Boolean proximity algebras can be considered as a "pointless" approach to the region–based theories of discrete spaces.

2 Definitions and notation

Throughout, we suppose that X is a nonempty set. If $A \subseteq X$, we write $-A$ for the set theoretic complement of A in X. If $\leq$ is an ordering of X and $a \in X$, we let $[a) = \{b \in X : a \leq b\}$ be the principal filter and $(a] = \{b \in X : b \leq a\}$ the principal ideal generated by a.

A binary relation R on X is a subset of $X \times X$; we denote the set of all binary relations on X by $Rel(X)$. If $\langle a, b \rangle \in R$ we usually write aRb, and if aRb and bRc we write this as $aRbRc$.

If $f : X \to Y$ is a mapping and $M \subseteq X$, then $f[M] = \{f(m) : m \in M\}$.

Two different elements $a, b \in X$ are called *R–connected* (or just *connected*, if R is understood) if there are $c_0, \ldots, c_k \in X$ such that $a = c_0, b = c_k$, and $c_0 R c_1 R \ldots R c_k$. If a and b are R–connected, we denote thi s by $a \stackrel{R}{\to} b$. A subset W of X is *R–connected* (or just *connected*), if any two different elements of W are connected.

Throughout, $\langle B, +, \cdot, ^*, 0, 1\rangle$ is a Boolean algebra. With some abuse of notation, we will usually identify an algebraic structure with its base set. In particular, 2^X is the Boolean algebra of all subsets of X with the usual set operations. We set $B^+ = B \setminus \{0\}$, and $Ult(B)$ is the set of all ultrafilters of B. A *representation of B* is a monomorphism e from B into a power set algebra 2^X. A representation e is called *reduced* if for all $x, y\ X, x \neq y$ there is some $b \in B$ such that $x \in e(b)$ and $y \notin e(b)$. It is called *perfect*, if for every $\mathcal{F} \in Ult(B)$ there is some $x \in X$ such that $b \in \mathcal{F} \iff x \in e(b)$ for all $b \in B$. The *Stone map* $s : B \to 2^{Ult(B)}$ is defined by $s(b) = \{\mathcal{F} \in Ult(B) : b \in \mathcal{F}\}$.

Theorem 1 *[9, 14] The Stone map s is a reduced and perfect representation of B. Conversely, if $e : B \to 2^X$ is a reduced and perfect representation of B, there is a unique bijection $\varphi : X \to Ult(B)$ such that $e(b) = \varphi^{-1}[s(b)]$.*

The following result is a variant of Stone's Separation Lemma [15]:

Lemma 2 *If I is an ideal of B and F a filter such that $I \cap F = \emptyset$, then there is some $\mathcal{F} \in Ult(B)$ such that $F \subseteq \mathcal{F}$ and $\mathcal{F} \cap I = \emptyset$.*

We will frequently use the *finite co–finite algebra* $FC(\omega)$ of the set ω of natural numbers as a source of examples. $FC(\omega)$ is the subalgebra of 2^ω consisting of all finite and all co–finite sets. The principal ultrafilters of $FC(\omega)$ are of the form $\mathcal{F}_n = \{a \subseteq \omega : n \in a\}$, and there is only one non–principal ultrafilter, namely $\mathcal{F}_\infty = \{a \subseteq \omega : a \text{ is co–finite}\}$[see 9]. For all undefined notions in this paper on Boolean algebras the reader may consult [9].

3 Proximity relations

Let X be a nonempty set and δ a binary relation between subsets of X. We call δ a *minimal proximity* or simply *proximity* if it satisfies the following conditions for all $A, B, C \subseteq X$:

(P1) $A\delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.
(P2) (a) $A\delta(B \cup C) \iff A\delta B$ or $A\delta C$.
(b) $(A \cup B)\delta C \iff A\delta C$ or $B\delta C$.

In this case, the pair $\langle X, \delta\rangle$ will be called a *proximity space*. We also consider various other types of proximity: A proximity δ is called

- a *Čech proximity* or *basic proximity*, if it satisfies

(P3) δ is symmetric.
(P4) $A \cap B \neq \emptyset \Rightarrow A\delta B$,

- a *Efremovič proximity* if it is a basic proximity satisfying

(P5) $A(-\delta)B \rightarrow (\exists C)[A(-\delta)C$ and $(-C)(-\delta)B$,

- a *Pervin proximity* if it satisfies P4 and P5,
- a *connecting proximity* if it satisfies

(P6) If $A \neq \emptyset$ and $-A \neq \emptyset$, then $A\delta - A$.

There are many other notions of proximity, and we invite the reader to consult the fundamental text by Naimpally and Warrack [10] for more examples. Throughout the paper, we will suppose that $\langle X, \delta \rangle$ is a proximity space, and that $A, B, C \ldots$, possibly indexed, are subsets of X.

The proof of the following Lemma follows immediately from P2:

Lemma 3 *If $A\delta B$ and $A \subseteq A_1, B \subseteq B_1$, then $A_1 \delta B_1$.*

4 Discrete proximity spaces

Let X be a non-empty set and R be a binary relation in X. Intuitively we will treat the elements of X as indivisible and small atomic regions called *cells* and the relation R between cells as an adjacency relation. If $x, y \in X$ then xRy means that y is a neighbour of x. Following Galton [6] we call the pair (X, R) adjacency space. This is a slight generalization of Galton's definition, because he assumes that adjacency is a reflexive and symmetric relation. Galton [7] also introduces multiple-adjacency spaces containing several adjacency relations but in this paper we will consider spaces containing only one adjacency relation. A good example is the chess-board desk considering squares as cells. Here we may distinguish four different adjacency relations between squares: y is immediately on the left of x, on the right of x, on the top of x, on the bottom of x and also their compositions.

By Galton, regions in (X, R) are arbitrary subsets of X and the contact relation δ_R between two regions A and B is defined as follows:

$$A\delta_R B \iff (\exists a \in A)(\exists b \in B) aRb. \tag{4.1}$$

Part-of relation is identified with the set-inclusion and the other region relations have their standard definitions by means of contact and part-of.

There is the following general result:

Lemma 4 *[4, 17] $\langle X, \delta_R \rangle$ is a proximity space.*

We will call $\langle X, \delta_R \rangle$ the *discrete proximity space determined by R*. The term "discrete" is chosen for its mereological connotation, and should not be confused with the discrete topology on X.

Just as in correspondence theory for modal style logics, one can ask which properties of δ_R are induced by properties of R. The following Theorem exhibits some of these connections (see for similar situation also [4, 17]):

Theorem 5 *Let $\langle X, \delta_R\rangle$ be a discrete proximity space. Then,*

(1) R is symmetric $\Longleftrightarrow$ δ_R satisfies P3.
(2) R is reflexive $\Longleftrightarrow$ δ_R satisfies P4.
(3) R is transitive $\Longleftrightarrow$ δ_R satisfies P5.
(4) $\langle X, R\rangle$ is R–connected $\Longleftrightarrow$ δ_R satisfies P6.

PROOF. 1. and 2. are obvious from the definition of δ_R.

3: "$\Rightarrow$": Suppose that R is transitive, and let $A(-\delta_R)B$. We need to find some C such that $A(-\delta)C$ and $(-C)(-\delta)B$. Set $C = \{t \in X : (\exists s)[tRs \text{ and } s \in B]\}$. Thus, whenever $s \notin C$, then $s(-R)t$ for all $t \in B$. This shows that $-C(-\delta_R)B$. Next, assume that $A\delta_R C$; then, there are $s \in A,\ t \in C$ such that sRt. By definition of C, there is some $u \in B$ with tRu, and the transitivity of R tells us that sRu. Hence, $A\delta B$, contradicting our hypothesis.

"$\Leftarrow$": Suppose that δ_R satisfies P5, sRt and tRu, and assume that $s(-R)u$. Then we have $\{s\}(-\delta_R)\{u\}$, and, by P5, there is some $C \subseteq X$ such that $\{s\}(-\delta_R)C$ and $-C(-\delta_R)\{u\}$. Since $\{s\}(-\delta_R)C$, we cannot have $t \in C$. On the other hand, if $t \notin C$, then $t(-R)u$ by $-C(-\delta_R)\{u\}$, a contradiction to tRu.

4. "$\Rightarrow$": Suppose that $\langle X, R\rangle$ is R–connected, and $A \subseteq X, A \neq \emptyset, X$. Choose some $a \in A,\ b \notin A$. Since $a \neq b$, and $\langle X, R\rangle$ is R–connected, there is a path $s_0 R s_1 R \dots R s_n$ such that $s_0 = a$ and $s_n = b$. Suppose that t is the smallest index j such that $s_j \in A,\ s_{j+1} \notin A$, and $s_j R s_{j+1}$. Such t exists, since $s_0 \in A$ and $s_n \notin A$. Then, $s_t R s_{t+1}$ exhibits $A\delta_R - A$.

"$\Leftarrow$": Suppose that δ_R is a connecting proximity. If X has only one element, then $\langle X, R\rangle$ is trivially R–connected. Thus, suppose that X has at least two elements. Then, $R(a) \neq \emptyset$ for all $a \in X$, since, by P6, there is some $b \in X \setminus \{a\}$ with aRb. Assume that $a, b \in X, a \neq b$, and there is no R–path from a to b. Let $A = \{c \in X : a \xrightarrow{R} c\} \cup \{a\}$. By our assumption neither A nor $-A$ are empty, and thus, there are $s \in A,\ t \in -A$ such that sRt. It follows that there is a path from some element of A to some element outside of A, which implies that there is a path from a to some element outside A. This contradicts the definition of A.

Let us note that the above theorem motivates why δ in the definition of proximity space is called *minimal proximity*: just because the axioms (P1) and (P2) for the case of discrete proximity space are true for arbitrary relation R.

5 Boolean proximity algebras

Proximities were defined as binary relations between subsets of a given set, satisfying certain properties. The set of all subsets of a set form a Boolean algebra under the set theoretic operations, and one may want to generalize the notion to arbitrary Boolean algebras and to obtain in this way a “pointless" formulation of region-based theory of discrete spaces. This leads to the following definitions: A *Boolean proximity algebra* (BPA) is a structure $\langle B, \delta \rangle$ such that B is a Boolean algebra, and δ is a binary relation on B which satisfies the axioms analogous to P1 and P2, namely,

(P’1) $a \delta b \Rightarrow a \neq 0$ and $b \neq 0$.
(P’2) (a) $a\delta(b+c) \iff a\delta b$ or $a\delta c$.
(b) $(a+b)\delta c \iff a\delta c$ or $b\delta c$.

If $\langle X, \delta \rangle$ is a proximity space, then $\langle 2^X, \delta \rangle$ is a BPA.

We also consider BPAs satisfying some of the following axioms:

(P’3) δ is symmetric.
(P’4) $a \cap b \neq 0 \Rightarrow a\delta b$,
(P’5) $a(-\delta)b \Rightarrow (\exists c)[a(-\delta)c$ and $c^*(-\delta)b]$,
(P’6) If $a \neq 0$ and $a^* \neq 0$, then $a\delta a^*$.

If δ is a proximity on B, we define $R_\delta \in Rel(Ult(B))$ by

(5.1) $$\mathcal{F} R_\delta \mathcal{G} \iff \mathcal{F} \times \mathcal{G} \subseteq \delta.$$

The relation R_δ is called the *canonical relation determined by* δ, and the relational system $\langle Ult(B), R_\delta \rangle$ the *canonical structure of* B.

Conversely, if $R \in Rel(Ult(B))$, we let $\delta_R \in Rel(B)$ be defined by

(5.2) $$a\delta_R b \iff (\exists \mathcal{F}, \mathcal{G} \in Ult(B))[a \in \mathcal{F} \text{ and } b \in \mathcal{G} \text{ and } \mathcal{F} R \mathcal{G}].$$

It is easy to see that δ_R is a proximity on B.

In the sequel, we shall show that all BPAs are of the form $\langle B, \delta_R \rangle$. More precisely, we shall show that each BPA is isomorphic to a substructure of a proximity space $\langle X, \delta \rangle$. Here, a *substructure of* $\langle X, \delta \rangle$ is a BPA $\langle B, \delta' \rangle$ such that B is a Boolean subalgebra of 2^X, and δ' is the restriction of δ to $B \times B$.

As a preparation we require some definitions and one Lemma. For a BPA $\langle B, \delta \rangle$ and a filter F in B, we let

$$I_l(F) = \{b \in B : (\exists a \in F) a(-\delta)b\},$$
$$I_r(F) = \{a \in B : (\exists b \in F) a(-\delta)b\}.$$

Lemma 6 *(1) $I_l(F)$ and $I_r(F)$ are ideals of B.*

(2) If F,G are filters of B, then

$$F\times G\subseteq\delta \iff I_l(F)\cap G=\emptyset \iff F\cap I_r(G)=\emptyset. \tag{5.3}$$

(3) If F,G are filters of B such that $F\times G\subseteq\delta$, then there are $\mathcal{F},\mathcal{G}\in Ult(B)$ such that $F\subseteq\mathcal{F}$, $G\subseteq\mathcal{G}$ and $\mathcal{F}\times\mathcal{G}\subseteq\delta$.

PROOF. 1. This follows immediately from P'1 and P'2.

2. Let $F\times G\subseteq\delta$, and assume that $b\in I_l(F)\cap G$. Then, there is some $a\in F$ such that $a(-\delta)b$, contradicting $F\times G\subseteq\delta$.

Next, let $I_l(F)\cap G=\emptyset$, and assume that there is some $a\in F\cap I_r(G)$. Then, there is some $b\in G$ such that $a(-\delta)b$. Hence, $a\in I_l(F)\cap G$, a contradiction.

Finally, suppose that $F\cap I_r(G)=\emptyset$. Let $\langle a,b\rangle\in F\times G$, and assume that $a(-\delta)b$. Then, $a\in F\cap I_r(G)$, a contradiction.

3. Suppose that F,G are filters of B such that $F\times G\subseteq\delta$. By (5.3), this is the case if and only if $F\cap I_r(G)=\emptyset$. Since $I_r(G)$ is an ideal, by Lemma 1 there is some ultrafilter $\mathcal{F}$ containing F and disjoint from $I_r(G)$. Again by (5.3)., we have $\mathcal{F}\times G\subseteq\delta$, and thus, $I_l(F)\cap G=\emptyset$. By Lemma 1 there is an ultrafilter $\mathcal{G}$ containing G and disjoint from $I_l(\mathcal{F})$. Invoking (5.3) again, we obtain $\mathcal{F}\times\mathcal{G}\subseteq\delta$.

Theorem 7 *Suppose that $\langle B,\delta\rangle$ is a BPA. Then, $\delta=\delta_{R_\delta}$, i.e.*

$$a\delta b \iff (\exists\mathcal{F},\mathcal{G}\in Ult(B))[\mathcal{F}R_\delta\mathcal{G} \text{ and } a\in\mathcal{F}, b\in\mathcal{G}] \tag{5.4}$$

PROOF. "$\subseteq$: Suppose that $a\delta b$, and let F be the principal filter generated by a – i.e. $F=\{c\in B : a\leq c\}$ –, and G be the principal filter generated by b. By P2, $a\delta b$ implies $F\times G\subseteq\delta$, and Lemma 6(3) gives us the desired result.

"$\supseteq$": Suppose that $\mathcal{F},\mathcal{G}\in Ult(B), a\in F, b\in\mathcal{G}$ and $\mathcal{F}R_\delta\mathcal{G}$. By definition of R_δ, we immediately see that $a\delta b$.

The representation Theorem now follows:

Corollary 8 *Each BPA $\langle B,\delta\rangle$ is isomorphic to a substructure of a discrete proximity space.*

PROOF. Suppose that $X=Ult(B)$, and let $S=R_\delta$; then, $\langle X,S\rangle$ is a discrete proximity space. Suppose that $s:B\to 2^{Ult(B)}$ is the Stone map. Since $s[B]$ is a subalgebra of 2^X isomorphic to B, all that remains to show is that $a\delta b \iff s(a)Ss(b)$:

$$\begin{aligned}
a\delta b &\iff a\delta_{R_\delta} b && \text{by Theorem 7}\\
&\iff (\exists \mathcal{F}, \mathcal{G} \in Ult(B))[a \in \mathcal{F}, b \in \mathcal{G}, \mathcal{F} R_\delta \mathcal{G}] && \text{by (5.2)}\\
&\iff (\exists \mathcal{F}, \mathcal{G} \in X)[\mathcal{F} \in s(a), \mathcal{G} \in s(b), \mathcal{F} S \mathcal{G}] && \text{by definition of } s \text{ and } S,\\
&\iff s(a)Ss(b) && \text{by (4.1),}
\end{aligned}$$

which completes the proof.

While we have just shown that each proximity δ on B is of the form δ_R for some $R \in Rel(Ult(B))$, it is not true that each $R \in Rel(Ult))$ is of the form R_δ. This is clear when $|B| \lneq |Ult(B)|$, but it is also true when these cardinalities are equal: Suppose that $B = FC(\omega)$, and let $R \in Rel(Ult(B))$ be defined by $\mathcal{F} R \mathcal{G} \iff \mathcal{F}$ and $\mathcal{G}$ are principal. Then, R is not the universal relation on $Ult(B)$. Assume that $R = R_\delta$ for some proximity δ on B. If $a, b \in B^+$ choose $n \in a, m \in b$, and let $\mathcal{F}$ be the ultrafilter generated by $\{n\}$, and $\mathcal{G}$ be the ultrafilter generated by $\{m\}$. Then, both $\mathcal{F}$ and $\mathcal{G}$ are principal, and $a \in \mathcal{F}, b \in \mathcal{G}$. Hence, $a\delta b$, and it follows that δ is the universal relation on B^+. This implies that R_δ is the universal relation on $Ult(B)$, and shows that $R \subsetneq R_\delta$. This example can also be used to show that it is not possible to carry over the results of Theorem 5. Suppose that R is as above; choose two different principal ultrafilters $\mathcal{F}, \mathcal{G}$ and define

$$S = R \cup \{\langle \mathcal{F}_\infty, \mathcal{F}\rangle\} \cup \{\langle \mathcal{F}, \mathcal{F}_\infty\rangle\} \cup \{\langle \mathcal{F}_\infty, \mathcal{G}\rangle\}.$$

Then, S is not not reflexive, symmetric or transitive; on the other hand, δ_S is the universal relation on B^+, since each nonzero element of B is contained in a principal ultrafilter, and thus it satisfies P'3 – P'5.

In the opposite direction, it is not true for all R that transitivity of R implies that δ_R satisfies P'5: Define R on $Ult(FC(\omega))$ by

$$R = \{\langle \mathcal{F}_n, F_m\rangle : n, m \in \omega \setminus \{0\}\} \cup \{\langle \mathcal{F}, \mathcal{G}\rangle : \mathcal{F}, \mathcal{G} \in \{\mathcal{F}_0, \mathcal{F}_\infty\}\}.$$

Then, R is clearly transitive, but δ_R does not satisfy P'5: Consider $a = \{0\}, b = \{1\}$; then $a(-\delta)b$. Let $c \in FC(\omega)^+$. If c is co–finite, then $c \in \mathcal{F}_\infty$; hence, $\langle a, c\rangle \in \mathcal{F}_0 \times \mathcal{F}_\infty$ shows $a\delta c$. If c is finite, then c^* is co–finite. Choose some $n \in c^*$; then, $\langle c^*, b\rangle \in \mathcal{F}_n \times \mathcal{F}_1$, and it follows that $c^* \delta b$.

Thus, we have to be content with the following, weaker Theorem:

Theorem 9 *Suppose that R is a binary relation in $(Ult(B))$. Then,*

(1) *If R is symmetric, then δ_R satisfies P'3.*
(2) *If R is reflexive, then δ_R satisfies P'4.*
(3) *If $\langle Ult(B), R\rangle$ is R–connected then δ_R satisfies P'6.*

PROOF. 1. Suppose that R is symmetric, and let $a\delta_R b$, i.e. $a \in \mathcal{F}, b \in \mathcal{G}$ and $\mathcal{F}R\mathcal{G}$ for some $\mathcal{F}, \mathcal{G} \in Rel(Ult(B))$. Since R is symmetric, we have $\mathcal{G}R\mathcal{F}$, which implies $b\delta_R a$.

2. Suppose that R is reflexive, and let $a \cdot b \neq 0$. Then, there is some $\mathcal{F} \in Ult(B))$ such that $a, b \in \mathcal{F}$, and $\mathcal{F}R\mathcal{F}$ shows that $a\delta_R b$.

3. Suppose that $\mathcal{F} \xrightarrow{R} \mathcal{G}$ for all $\mathcal{F} \neq \mathcal{G}$, and let $a \in B \setminus \{0,1\}$. Then, there are $\mathcal{F}, \mathcal{G} \in Ult(B)$ such that $a \in \mathcal{F}$ and $a^* \in \mathcal{G}$. By our hypothesis, we can find $\mathcal{F}_0, \dots, \mathcal{F}_n$ such that $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_n = \mathcal{G}$, and $\mathcal{F}_0 R \dots R\mathcal{F}_n$. If j is the smallest index with $a \in \mathcal{F}_j, a \notin \mathcal{F}_{j+1}$ – which exists since $a \in \mathcal{F}_0$ and $a^* \in \mathcal{F}_n$ – then $\mathcal{F}_j R \mathcal{F}_{j+1}$ shows that $a\delta_R a^*$.

If R is of the form R_δ, then things look brighter:

Theorem 10 *Let $\langle B, \delta\rangle$ be a BPA, and set $R = R_\delta$ as defined in* (5.1). *Then,*

(1) R is symmetric $\Longleftrightarrow$ δ satisfies P'3.
(2) R is reflexive $\Longleftrightarrow$ δ satisfies P'4.
(3) R is transitive $\Longleftrightarrow$ δ satisfies P'5.

PROOF. Because of Theorem 9, we only have to show "$\Leftarrow$" for 1. and 2.:

1. Suppose that δ satisfies P'3, and let $\mathcal{F}R\mathcal{G}$. By definition of R, we have $\mathcal{F} \times \mathcal{G} \subseteq \delta$, and P'3 tells us that $\mathcal{G} \times \mathcal{F} \subseteq \delta$. Thus, $\mathcal{G}R\mathcal{F}$.

2. Suppose that δ satisfies P'4, and let $\mathcal{F} \in Ult(B)$. Since $\mathcal{F}$ is a proper filter, we have $a \cdot b \neq 0$ for all $a, b \in \mathcal{F}$, and thus, by P'4, we have $\mathcal{F} \times \mathcal{F} \subseteq \delta$. Hence, $\mathcal{F}R\mathcal{F}$.

3. "$\Rightarrow$: Suppose that R is transitive, and assume that there are $a, b \in B$ such that $a(-\delta)b$, and $a\delta c$ or $c^*\delta b$ for all $c \in B$. Let $I = \{c \in B : a(-\delta)c\}$, and $F = \{c \in B : c^*(-\delta)b\}$; then, I is an ideal, F is a filter, and $b \in I$, $a^* \in F$. If $c \in I \cap F$, then $a(-\delta)c$ and $c^*(-\delta)b$, contradicting our assumption. Thus, with Lemma 2, choose some ultrafilter $\mathcal{F}$ containing F with $\mathcal{F} \cap I = \emptyset$. Our next step is to show that $[a) \times F \subseteq \delta$: Assume otherwise; then, there are $d \geq a$, $c \in \mathcal{F}$ such that $d(-\delta)c$, and, by P'2, we may assume $a = d$. Now, $a(-d)c$ implies $c \in I$, which contradicts $I \cap \mathcal{F} = \emptyset$. By Lemma 6(3), there is some ultrafilter $\mathcal{G}$ such that $a \in \mathcal{G}$ and $\mathcal{G} \times \mathcal{F} \subseteq \delta$. Similarly, it is shown that $\mathcal{F} \times [b) \subseteq \delta$, so that there is some $\mathcal{H} \in Ult(B)$ containing b and $\mathcal{F} \times \mathcal{H} \subseteq \delta$. By the definition of R, we have $\mathcal{G}R\mathcal{F}$, and $\mathcal{F}R\mathcal{G}$. The transitivity of R now implies $\mathcal{G}R\mathcal{H}$, and it follows from $a \in \mathcal{G}, b \in \mathcal{H}$, along with $R = \delta_R$, that $a\delta b$. This contradicts our assumption.

"$\Leftarrow$": Suppose that δ satisfies P'5, and let $\mathcal{F}_0 R\mathcal{F}_1 R\mathcal{F}_2$. Assume that $F_0(-R)F_2$; then, $\mathcal{F}_0 \times \mathcal{F}_2 \not\subseteq \delta$, so there are $a \in \mathcal{F}_0, b \in \mathcal{F}_2$ with $a(-\delta)b$. With P'5, choose some $c \in B$ such that $a(-\delta)c$ and $c^*(-\delta)b$. Together with the fact the $\delta = R_\delta$ by Theorem 7, the latter implies that $c^* \notin \mathcal{F}_1$. Since $\mathcal{F}_1$ is an ultrafilter, we have $c \in \mathcal{F}_1$, and it follows from $\mathcal{F}_0 R\mathcal{F}_1$ that $a\delta c$, a contradiction.

Theorem 10 generalizes a result of [8], see also [16].

To show that P6 does not carry over, we will construct a proximity δ on $B = FC(\omega)$ which satisfies P'6, but for which R_δ is not connected. Define R on $Ult(B)$ by

$$R = \{\langle \mathcal{F}_n, \mathcal{F}_m\rangle : n,m \in \omega, |n-m| = 2\} \cup \{\langle \mathcal{F}, \mathcal{F}\rangle : \mathcal{F} \in Ult(B)\} \tag{5.5}$$

Since, for example, there is no path from $\mathcal{F}_0$ to $\mathcal{F}_1$, R is not connected. Let $\delta = R_\delta$ as defined in (5.2). Clearly, δ satisfies P'1. Thus, let $a\delta(b+c)$; then, there are $\mathcal{F}, \mathcal{G} \in Ult(B)$ such that $\mathcal{F} R \mathcal{G}$, and $a \in \mathcal{F}, b+c \in \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter, we have $b \in \mathcal{G}$ or $c \in \mathcal{G}$, i.e. $a\delta b$ or $a\delta c$. The other case is analogous. Next, suppose that $a \in B \setminus \{0,1\}$, and that a is finite. Let $m = \max a$; then, $a \in \mathcal{F}_m$, $m+2 \in a^*$, and hence, $a^* \in \mathcal{F}_{m+2}$ which implies $a\delta a^*$. Since R is symmetric, the case for co–finite a follows. It remains to show that $R = R_\delta$, i.e. that $\mathcal{F} R \mathcal{G} \iff \mathcal{F} \times \mathcal{G} \subseteq \delta$. By definition, $R \subseteq R_\delta$, so, suppose that $\mathcal{F} \times \mathcal{G} \subseteq \delta$ and $\mathcal{F} \neq \mathcal{G}$. First, let $\mathcal{F} = \mathcal{F}_\infty, \mathcal{G} = \mathcal{F}_n$. Set $b = \{n, n+2\}$ if $n = 0,1$, and $b = \{n, n-2, n+2\}$ otherwise. Then, $b^* \notin \mathcal{F}_{n-2} \cup \mathcal{F}_n \cup \mathcal{F}_{n+2}$ which shows $\mathcal{F}_\infty \times \mathcal{F}_n \not\subseteq \delta$; similarly, $\mathcal{F}_n \times \mathcal{F}_\infty \not\subseteq \delta$. Now, let $|n-m| \neq 2$. Then, $\{n\}(-\delta)\{m\}$ and it follows that $\mathcal{F}_n \times \mathcal{F}_m \not\subseteq \delta$.

6 Outlook

Galton [7] pointed out the need to consider discrete spaces with several adjacency relations. In subsequent work, we will generalize and extend the results of the present paper by investigating proximity spaces and BPAs arising from discrete space with an arbitrary collection $\mathcal{R} = \{R_i : i \in I\}$ of adjacency relations. We will also consider the case when $\mathcal{R}$ is closed under various relational operators such as union, composition, and converse. In this way, we make our vocabulary more expressive and, indeed, similar to dynamic logic.

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