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‘Region Calculus’ : Generalization of Newton Calculus

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Abstract

In this paper the author introduces a new type of calculus called by “Region Calculus” which could be considered as a generalization of the classical Newton Calculus. The generalization is done by improving the ‘distance’ concept used in Newton Calculus for measuring the amount of increment/decrement of x-values (and y-values). Besides that, Region Calculus is developed based upon the region concerned, not just based upon a particular region R (set of Real numbers). Another new branch of mathematics called by “Object Geometry” is also introduced as a prerequisite for developing the branch of Region Calculus.

Mathematics Subject Classifications: 26A06, 26A12

Keywords: Chain Region. Partitioned Region. Extended region. Calculus space. Complete region. Positive object. Negative object. Onteger. Object linear continuum line.

1. Introduction

It has been justified in [3] with several examples that many of the simple results, formula, equalities, identities, rules etc. of elementary algebra are not valid in general in a group, ring, field, module, linear space, algebra over a field, associative algebra over a field, division algebra, or in any existing algebraic structure alone, by their respective definitions and by virtue of their respective properties, but are well valid in the algebraic structure 'region'[3]. These very simple observations may apparently seem to be surprising today in this century, but it is fact as established in details in [3]. The minimum platform required for practicing elementary algebra is region, not any division algebra or any other algebraic structure. In [4] a new type of numbers is discovered called by 'compound numbers' which is a very fruitful generalized notion of the concept of 'complex numbers'. Two new numbers **e** and **w** are discovered in [4] as 'imaginary' for the set C of complex numbers, analogous to the concept of the number **i** which is 'imaginary' for the set R of real numbers. Consequently a new topic 'compound algebra' is opened as a generalization of 'complex algebra'. Besides that, a new type of number theory called by "Theory of A-numbers" is introduced in [4] corresponding to any complete region A. These new numbers will surely enrich various branches of mathematics viz. Calculus, Algebra, Geometry, etc., to name a few only out of many.

The new algebraic structure 'Region' encourages to initiate another new direction in mathematics which is the development of a new calculus. In this paper a new type of calculus is developed called by 'Region Calculus', which is a generalization of the classical Newton Calculus. The key concept of this new calculus is that it uses the measure of the distance between two points (say, on real axis) using a metric. We know that there exist an infinite number of metric spaces over the set R of real numbers. It is justified here that while developing a new calculus, it must be based over a region as a minimum platform (not a division algebra). Without a region, no calculus can be developed. No calculus can be developed just over a Division Algebra by virtue of its definition and independently owned properties alone. In 17th century the Newton Calculus was developed mainly over the set R. But it is justified in this work that the Newton Calculus actually happened to have developed over a particular region RR, not over the set R or not over the division algebra R. The work in this paper is sequel to the works [3,4]. It is established in this paper that without region no calculus can be developed. The rich properties of division algebra is not sufficient to fully cooperate the development of a calculus. After the development of different algebraic structures viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, Division Algebras, the development of Newton Calculus has been getting viewed by the world to be based upon the division algebra R (instead of the set R of real numbers, as a set alone cannot offer operations of addition, subtraction, multiplication, division, etc). But in this paper we say that it is even not so, just by virtue of the definition and so rich properties of division algebra. It will be mistake to say that the Newton Calculus took development over a division algebra R by virtue of the definition and properties of the algebraic structure division algebra. It is now unearthed that the Newton Calculus actually in a hidden way took development over a region which is RR (neither over the set R nor even over the division algebra R) by virtue of the definition and independently owned properties of region. An important observation is made regarding the concept of the amount of increment or decrement which is measured by distance. It is obvious that calculus is the mathematics about changes. If the variable x approaches the value a then it means the distance of x from the particular value a decreases gradually. But our observation is that in Newton Calculus, a very particular type of distance formula is being used in general. It is probably because of the reason that the concept of 'metric space' took birth after more than 200 years span of the inception of calculus. Today we have various options of metrics depending upon the nature of domain under consideration, nature of the problems under consideration. Consequently the distance between the two points x and x+Δx on the real axis may be suitably chosen to be ρ(x+Δx, x) using a suitable metric ρ if it serves better and if it is more appropriate compared to the very particular result $|\Delta x|$, in some complex cases of real life problems. Even in Region Calculus we have open options to use two different metrics, one for the x values and the other for the y values, however subject to such requirements in the quest of better results by better computation in some complex cases.

Calculus is one of the most important discoveries in Mathematics. It is all about changes, about increment or decrement by an amount of distance that happen due to changes. Modern calculus was developed in 17^{th} century by Newton and Leibniz (independently by each other). It is observed that the existing rich calculus of Newton and Leibnitz is a particular case of Region Calculus. It is claimed and justified that there could be genuine requirements of this generalized type of calculus to extend the scope of computations required by the mathematicians, scientists, and engineers in the various complex domains of this century. In Newton Calculus the concept of 'distance' (be it infinitesimal small or large) between two real numbers x and x+Δx on the real axis is $|\Delta x|$ and the distance between two real numbers f(x) and f(x+Δx) is $|\Delta y|$ where Δy = f(x+Δx) – f(x). The notion of 'Metric Space' came into Mathematics in 1906, after more than two centuries of years time since the inception of Newton Calculus. Although the notion of 'Metric Space' was not discovered during the tenure of Newton's era, but today it can be observed that the development of Newton Calculus can be appropriately viewed to be based upon one 'particular metric space' ρ over the set R of real numbers, where ρ(p,q) = |p-q|. All the x values are in the metric space (R,ρ) and all the y values are also in the same metric space (R,ρ). However the issue of 'infinity' is also analyzed in this paper. The metric space for both the collections (of x values and of y values) is common. Consequently there is a genuine scope for the branch 'Calculus' of mathematics which can be studied in a generalized way by replacing the particular metric ρ by some other suitable metric depending upon the universe of discourse or its environment under consideration, depending upon the space in which we are interested to analyze the changes. It is expected that a more appropriate metric may provide better scope to analyze the changes of variables in a complex environment. This philosophy has lead here to open the extended concept of the classical calculus called by 'Region Calculus'.

Besides that, Newton calculus uses the beautiful and very useful concept of infinity. The same is true for Region Calculus too, where the concept of infinity is also a generalized concept in the context of the new algebraic structure region. However, let us make a quick visit to the history of infinity, which could be interesting at this stage just on a quick perusal.

The symbol for infinity '∞' was invented by the English mathematician John Wallis in 1657. The concept of infinity is that it describes something without any limit, without any bound, or something larger than any of the natural numbers. Philosophers speculated about the nature of infinity, proposed many interesting paradoxes involving infinity. Eudoxus of Cnidus used the idea of infinitely small quantities in his method of exhaustion, where the meaning of infinity seems to be viewed to define a concept of small positive number which is smaller than any positive real number. Modern mathematics uses the general concept of infinity to find solutions of many mathematical problems, such as in Newton calculus and in set theory. The general concept of infinity is also extensively used in Physics, Chemistry, Statistics and in other branches of natural sciences as well as social sciences. In Region Calculus here a new notion of 'Infinity' is introduced which is an extended concept of the classical 'infinity'. The 'infinity' in a region A describes something larger than any of the objects of that region A. The term 'larger' has been discussed mathematically in a region. Consequently an important term called by 'extended region' is introduced. During the late 19th and early 20th centuries, Cantor formalized many new ideas related to infinity and infinite sets. He conceptualized that there are infinite sets of different sizes (called cardinalities). This concept shook the mathematicians with a natural question that "Can the infinity be of various sizes?". But modern mathematics has the beautiful concepts like : the set of integers is countably infinite, the infinite set of real numbers is uncountable, etc. In Region Calculus too, for every extended region a concept of 'infinity' is introduced. It is not yet explored in this paper whether the various infinities of the various extended regions are same or not, or whether there exist common infinities of two regions. In modern mathematics, the notion of infinity is often treated like a number (in the sense that it does count or it does measure viz. we say that "an infinite number of numbers in this set", "an infinite number of terms in this sequence", etc) but without calling it a number. It is also an acceptable concept to us that infinity is not the same type of number as either a natural or a real number. Similarly in case of extended region too, the notion of infinity is to be treated like an object (in the sense that it is greater than all other objects) but without calling it an object. The notion of 'greater than' and 'infinity' in a region are explained in the next section here.

We will use the region RR [3] extensively in this work. The region RR is the most useful region in Science, Engineering and other areas. Let R be the set of real numbers, '+' be the ordinary addition operator in R and '.' be the ordinary multiplication operator in R. Consider the field (R,+, .) of real numbers, and the linear space (R,+, .) over the field (R,+, .) . Then the algebraic system

(R,+, . , .) forms a region over the outer field (R,+ , .). This region (R,+ , . , .) plays a very important role in our daily life computations, in particular in school level elementary algebra. The content of the syllabus and corresponding instructions at school level algebra is based on the platform of this region (R,+, . , .), not on the platform of any standard algebraic structure like groups, rings, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra R or any existing algebraic structure. In [3], this region (R,+, . , .) is named in short by the word "RR". The region RR is the most useful region in all the branches of Mathematics, Statistics, Science, Engineering and other areas.
The interesting properties of the region RR are that:
(i) its inner field is (R,+, .),
(ii) its outer field is (R,+, .),
(iii) all the three multiplication operators are same, and
(iv) all the three addition operators are same.
For complete details about the definition and rich properties of the algebraic structure 'Region', one could view [3]. In the next section we first of all introduce the concept of Calculus Space, then in Section-3 we introduce a new kind of geometry designated as "Object Geometry", and then in Section-4 we go for developing the new calculus called by "Region Calculus", as a generalization of the classical Newton Calculus.

2. Calculus Space

The Universe is commonly defined as the totality of existence, a part of which we people on this earth have been possibly able to think about so far. The present universe appears to be expanding at an accelerating rate. There are many competing theories about the ultimate fate of the universe. Scientific observation of the universe has led to inferences of its earlier stages too. Physicists remain unsure about what, if anything, preceded the Big Bang. Many refuse to speculate, doubting that any information from any such prior state could ever be accessible. There are various multiverse hypotheses too, in which physicists have suggested that this universe might be one among many universes that likewise exist. One good question arises : Whether every space of this universe is being governed by the same physical laws and constants throughout most of its extent and history?

We consider in this paper an important branch of mathematics which is 'Calculus' (developed independently by Newton and Leibniz), one of the most important discoveries in Mathematics. John von Neumann said : "The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking". But, can we accept the hypothesis that this classical Calculus is valid in every planet of our solar system or at every space of our universe or at every

universe of the multiverse (if exists)? Is our classical Calculus an absolute calculus to be regarded as most appropriate calculus applicable at everywhere in our universe or multiverse? Does it not get influenced at different solar systems or at different spaces of the universe or at different universes of the multiverse where the concept of 'time', 'distance' etc are different? In this work we do not (can not) propose any answer to these questions, but we propose the hypothesis that there could be an useful generalized type of calculus of which our existing classical calculus is a particular case just. Or, there could be a number of new calculus which are siblings to our existing classical calculus. In this quest, we first of all introduce a new mathematical notion called by "Calculus Space", then we introduce a new branch of mathematics called by "Object Geometry", and then we go for introducing the "Region Calculus". A calculus space is a base-platform on which one can develop a new calculus. In other words, a calculus can not be developed without a calculus space, called a base-platform of the calculus.
It is observed that the existing rich calculus of Newton is a beautiful example of Region Calculus (i.e. one instance of Region Calculus).

2.1 Key Role of 'Metric Space' in developing any Calculus

In many computational environment in this giant universe (or multiverse, if exists), the existing rich calculus of Newton may not always work most appropriately. The speculation about the reason is that it may be because of the fact that the Newton Calculus was developed on such a concept of 'distance' which is a particular example of 'distance' of today's volume of mathematics. If it is so then a computational environment in this giant universe (or multiverse, if exists) may need its own calculus.

The **'very particular nature'** of Newton calculus, if analyzed with the help of today's volume of mathematics, can be observed mainly due to the following five Facts:-

No.(1) : The basic platform of Newton Calculus is a very particular set which is the set R of real numbers (along with graphs with the notion of real line axis like X'OX, and the axis Y'OY, etc depending upon the dimension of calculus) and

No.(2) : If we consider the concept of 'change in x' while talking about $\frac{\Delta y}{\Delta x}$, the distance between the two points x_1 and x_2 on the real line axis in the classical Calculus happens to be by a very particular metric ρ defined by $\rho(x_1, x_2) = |x_2 - x_1|$ over the metric space (R, ρ).

No.(3) : the distance between the two points y_1 and y_2 too happens to be by the same particular metric ρ defined by $\rho(y_1, y_2) = |y_2 - y_1|$ over the same metric space (R, ρ).

No.(4) : The metric ρ is absolutely fixed, and usually common to both x-values and y-values which are in R. It is not that : for x-values there is one metric applicable and in y-values there is another metric applicable.

No.(5) : There exist many distinct metric spaces like (R, d) over the set R besides the particular metric space (R, ρ) where ρ is defined by $\rho(x_1, x_2) = |x_2 - x_1|$. But Newton Calculus is observed to be compatible with the use of the metric space (R, ρ).

MNC and MSNC :

In this article we call this particular metric ρ by the name **'Metric of Newton Calculus'** (MNC) and the particular metric space (R, ρ) by the name **'Metric Space of Newton Calculus'** (MSNC).

It is quite obvious that the above mentioned five observations was not of much significance and was not in existence during Newton's era. The author feels that there is a need to make an attempt for developing analogous styled calculus replacing the above particular metric MNC ρ by other suitable metrics if seem to be more appropriate for advance application potential in some complex domains of today's century. In Newton Calculus, there is no concept of using one metric ρ_1 for the x-values over the set R and another metric ρ_2 for the y-values over the set R. This gap too needs attention for thinking for some alternatives, which may sometimes be required to better fit the computational methods in some new complex domains. The requirement is genuine because **Newton calculus can not be applicable** in the universe of discourse R with any arbitrary metric spaces (R,d) except the MSNC (R, ρ). It is known to us that the concept of 'Metric Space' came much later than the discovery of Calculus. The matrix space was developed in 1906, after more than 200 years of the discovery of calculus. The addition operation used in R while developing the Newton Calculus was the usual addition operation('+'), and similarly the other fundamental operations of multiplication, division, scalar multiplication, etc.. And no other addition operation, division operation, multiplication operation, etc over the set R of real numbers are useful to the fundamental theories of Newton Calculus.

And also it is fact, as can be observed by today's volume of mathematics, that if Δx is a positive real number, then

(i) in Newton Calculus the point (x+Δx) is situated at a distance ρ(x, x+Δx) from the point x on the real axis towards the positive direction of it, where ρ(x, x+Δx) <u>happens</u> to be equal to Δx, and ρ is the fixed metric space MNC corresponding to the concept of **'change in x'** or **'change in y'** while talking about $\frac{\Delta y}{\Delta x}$, and

(ii) and similarly, the point (x-Δx) is situated at a distance ρ(x, x-Δx) from the point x on the real axis towards the negative direction of it, where $\rho(x, x-\Delta x)$ happens to be equal to Δx, ρ being the MNC.

In the seventeenth century of Newton, the concept of the term 'change in x' was an absolute concept, a unique concept. But in the present century this concept is a generalized concept, not an absolute concept. It is because of the fact that the concept of 'distance' took an enormous amount of generalizations after the discovery of metric space. In the Region Calculus, the concept of 'change in x' has been generalized with the help of various kind of metrics available over R or over the region for which a calculus being developed. The Region Calculus is a generalized calculus which can widen the application freedom of the mathematician, scientists, statisticians, and engineers, in particular while dealing with very complex problems. The Newton calculus is just a special example of 'Region Calculus'.

Notations :
We use the following notations:
R = set of all real numbers, R^+ = set of all positive real numbers, R^- = set of all negative real numbers, $R^{\geq 0}$ = set of all non-negative real numbers.

First of all we define few new terms called by 'Chain Region', 'Partitioned Region', 'Extended Region' and '2-to-1 bijective mapping'.

2.2 Chain Region, Partitioned Region and Extended Region

In this section we first of all define : Chain Region, Partitioned Region and Extended Region.

2.2.1 Chain Region

Consider a real region $A = (A, \oplus, *, \bullet)$. Suppose that the set A of the region $(A, \oplus, *, \bullet)$ forms a chain with respect to a total order relation (say, denoted by the notation '$\leq$'). Then the real region A is called a chain region with respect to the total order relation '$\leq$'.

2.2.2 Partitioned Region

A real region $A = (A, \oplus, *, \bullet)$ is called a Partitioned Region if the following conditions are satisfied :

(i) A is an infinite region,
(ii) A is a chain region with respect to a total order relation '$\leq$', and
(iii) the characteristic of A is zero.

Here A is called a 'partitioned region' because of the fact that it induces a partition P_A of A into three mutually disjoint non-null sets denoted by A^+, A^- and $\{0_A\}$ such that

(i) $A^{+} = \{ a : a \in A \text{ and } 0_A < a\}$
(ii) $A^{-} = \{ a : a \in A \text{ and } a < 0_A\}$.
Clearly, $\forall a \in A^{+}$, $\sim a \in A^{-}$ and $\forall b \in A^{-}$, $\sim b \in A^{+}$.

(Note : It may be recalled from the properties of the chain that : $a < b$ iff $a \leq b$ and $a \neq b$, where "$\leq$" is the total order relation of the chain A, and similarly $a > b$ iff $b \leq a$ and $b \neq a$).

This partition P_A, once made, is regarded as an absolute partition of the region A corresponding to its total order relation '$\leq$' in the sense that this partition generates the sign of every object of the complete region A, positive or negative, which will remain absolute throughout the complete literature henceforth (complete region is defined in subsection 2.5 below). However for a different type of total order relation defined over the region A we will get a different partition of A. But the set $\{0_A\}$ is common to all such possible partitions.

2.2.3 Extended Region

Consider an infinite region A $= (A, \oplus, *, \bullet)$. The extended region of A is the region A itself with all its infinity objects, if any. The infinity objects are not basically the core member of the region A, but to be included into it. At this point of time we do not consider any method about 'how to find out all the infinity objects of an infinite region'. However for a partitioned region the method is rather easier.

Consider a partitioned region A $= (A, \oplus, *, \bullet)$. If we now include two more objects $+\propto_A$ and $-\propto_A$ in A as two permanent guests, then the set $A^E = A \cup \{+\propto_A, -\propto_A\}$ is the 'extended region' of the region A.
The two guest objects $+\propto_A$ and $-\propto_A$ are called infinities, and are defined as below:

(i) $+\propto_A = \dfrac{x_A}{0_A}$ where x_A ($\neq 0_A$) is any positive object of the region A, and

(ii) $-\propto_A = \dfrac{z_A}{0_A}$ where z_A ($\neq 0_A$) is any negative object of the region A.

The extended region of the partitioned region A is denoted by the notation A^E. However, if there is no confusion then we may use the notation A itself to denote the extended region of A. Note that an extended region is not a region. For a partitioned region, it is just a superset of the set A containing two more objects.

But whenever we say that 'A is an extended region', it will simply mean that A is a region with all its infinities as permanent guests. At this stage we do not explore to study whether there are more infinities other than the two guest objects $+\propto_A$ and $-\propto_A$ for a partitioned region.

An extended region A^E may also be called as 'extended real region' if the corresponding region A is a real region [3] by virtue of the definition of a partitioned region.

For the region C, there are many infinities to be included into it to call it an extended region. For example $\forall$ a, b $\in$ R, the object a+ib is an infinity object for C if either a or b or both are the infinity object of the region R. The extended region of C is denoted by the notation C^E.

2.3 2-to-1 Bijective Mapping

Consider two non-null sets X and Y. A function $f : X \to Y$ is said to be a '2-to-1 Bijective Mapping' if

(i) f is onto, and

(ii) $\forall\, y \in Y$, $\exists$ two and only two distinct (not same) elements x_1 and x_2 in X such that $f(x_1) = y = f(x_2)$.

For example, the function $f : R\text{-}\{0\} \to R^+$ given by $f(x) = x^2$ is a 2-to-1 Bijective Mapping. But the function $g : R \to R^+$ given by $g(x) = x^2$ is not a 2-to-1 Bijective Mapping.

2.4 Calculus Space : the Platform to develop a Calculus

Consider a non-null set S. Surely we can not form a calculus over the set S unless we are aware of any algebraic structure which ensures us about the operations, axioms, properties defined over the set S. Consequently, no calculus can be developed over an algebraic structure like Field, Division Algebra or any of the existing important algebraic structures by virtue of their respective definitions and independently owned properties as analyzed in [3]. It must be necessarily a region [3] at minimum. It is because of the reason that most of the results of elementary algebra are not valid in algebraic structure like Field, Division Algebra or any of the existing important algebraic structures by virtue of their respective definitions and independently owned properties. But a natural question at this point arises: Can we develop a calculus over any given arbitrary region S? We say that if a calculus can be developed over a region S, then this region is to be designated to form a Calculus Space. Thus the same question can be posed in a different way: Can every region form a calculus space? The answer will be clear soon after discussions made in the subsequent subsections.

Definition : Calculus Space

Consider a partitioned region $A = (A, \oplus, *, \bullet)$ with respect to the total order relation '$\leq$'. Then A forms a Calculus Space if the following conditions are satisfied :

(i) A is an extended region (i.e. A is a region and two infinities are also included to it as permanent guests).
(ii) A is a normed complete metric space with respect to a norm $\|.\|$ and the corresponding induced metric $\rho(x, y) = \|x\sim y\|$, (i.e. $\|x\| = \rho(x, 0_A)$).
(iii) The norm $\|.\|$ is a 2-to-1 bijective mapping from $A - \{0_A\}$ to R^+.

2.4.1 'Calculus Space' for Newton calculus

Consider the RR region. This region can be viewed as a partitioned region with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "$\leq$". Choose a norm defined by $\|x\| = |x|$ in RR, and the metric $\rho(x, y) = \|x-y\| = |x-y|$ in RR. Clearly RR forms a region calculus. It can be now observed that this region calculus is nothing but the classical Newton calculus (developed independently by Newton and Leibniz).

The set R of real numbers is so interesting that it very comfortably forms the region RR; and the region RR is so beautiful that it satisfies all the necessary conditions to form a Calculus Space (an eligible platform on which a calculus can be developed). A division algebra by its definition and independently owned properties does not have so much capability. Consequently, it is clear now that the classical calculus developed independently by Newton and Leibniz happens to be on the particular calculus space RR with respect to a particular order relation "Less Than or Equal To" denoted popularly by the notation "$\leq$" and with respect to the norm defined by $\|x\| = |x|$ in RR, where the metric $\rho(x, y) = \|x-y\| = |x-y|$ in RR.

The following interesting facts may be recalled about the metric ρ associated with the norm $\|.\|$ (of the calculus space A here), i.e. the metric $\rho(x, y) = \|x\sim y\|$ has the following special properties :
(i) **'Translation Invariance' :**
$\forall z \in A$ we have $\rho(x \oplus z, y \oplus z) = \rho(x, y) = \|x\sim y\|$ where $x,y \in A$, and
(ii) **'Homogeniety' :**
$\forall r \in R$ we have $\rho(r \bullet x, r \bullet y) = |r|.\|x\sim y\| = |r|.\,\rho(x, y)$ where $x,y \in A$.

Although these two beautiful properties were established much later than the discovery of the Newton calculus, but today these can be observed to be true in the 'Calculus Space' of Newton calculus.

2.5 Complete Region

A real region which forms a calculus space is called a "complete region".

By a complete region, we will always mean one-dimensional complete region (1-D complete region), as a special case of n-d complete region introduced in subsection 4.7 later).

For instance, the region RR is a complete region with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "≤" and the metric ρ(x, y) = $\|x \sim y\|$ = |x-y|, where the norm is the classical norm defined over R.

The collection of all complete regions is called the complete region universe Σ.

2.5.1 How many distinct 1-D complete regions?

An interesting question arises :
How many distinct 1-D complete regions do exist mathematically?
To answer this question, first of all we see that given a region A = $(A, \oplus, *, \bullet)$ over the field (F, +, .) there may (may not) exist more number of regions corresponding to the same set A over the same set F with different operators $\oplus$, *, $\bullet$ and +, ., respectively.
Even if P = $(A, \oplus, *, \bullet)$ be a given fixed complete region with respect to the total order relation '$\leq$' and the norm $\|.\|$, there could be another distinct complete region Q = $(A, \oplus, *, \bullet)$ with respect to a different total order relation or with respect to a different norm or with respect to different pair of total order relation and norm both. There could be many more such complete region $(A, \oplus, *, \bullet)$ in similar ways. However, we will explore this in depth in our future research work.
Thus a given region A = $(A, \oplus, *, \bullet)$ over the field (F, +, .) may produce more than one distinct complete regions (even retaining the set A, retaining the set F and retaining the operators $\oplus$, *, $\bullet$ and +, ., unchanged), but with different total order relations and different norms, subject to fulfillment of the definition of one dimensional region calculus.

For example, consider the Newton Calculus which is based upon the complete region RR but with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "≤" and the classical norm $\|.\|$ defined by $\|x\|$ = |x| in RR, where the corresponding metric is given by ρ(x, y) = $\|x\text{-}y\|$ = |x-y|. Now, for any real number k>0 we can define a new norm $\|.\|_{new}$ over the region RR as: $\|x\|_{new}$ = k |x|.
It can be observed that the region RR in this case forms a new one dimensional calculus space with respect to this new norm $\|.\|_{new}$ and the corresponding metric ρ_{new} given by $\rho_{new}(x, y) = \|x \sim y\|_{new}$ = k |x - y| , even retaining the same crisp order relation "Less Than or Equal To" (≤).

Thus we can define infinite number of distinct norms mathematically and infinite number of distinct corresponding metrics, even retaining the same total order relation. Newton Calculus is an example of 1-D region calculus which is based upon the platform of the real region RR.

3. Object Geometry

In the previous section it is explained that we can mathematically define infinite number of distinct 1-D complete regions in mathematics (by a complete region, we shall always mean 1-D complete region). We are now in a position to initiate a new kind of geometry on a complete region A. We begin the subject by introducing first of all a 2-D Object Geometry developed over an 1-D complete region.

For developing the new geometry called by "Object Geometry", be it in a two dimensional region coordinate system or in an n-dimensional region coordinate system, at least one 1-D complete region A = $(A, \oplus, *, \bullet)$ is required.

3.1 Positive Object and Negative Object

In the work [4], a new theory entitled 'Theory of Objects' is developed. However some of the fundamental concepts of the Theory of Objects are presented here from the work[4] as these will be extensively used in the development of Object Geometry and Region Calculus.

Consider a complete region A = $(A, \oplus, *, \bullet)$. The elements of A^+ are said to be positive objects and the elements of A^- are said to be negative objects. The object 0_A is neither in A^+ nor in A^-, and so we say that 0_A is neither a positive object nor a negative object. The attribute of being positive or negative is called the sign of the object, and 0_A is not considered to have a sign of its own.

For a given calculus space A, a line XX1 can be drawn with all positive objects lying upon it to the right of 0_A, and all negative objects lying upon it to the left of 0_A as shown in Figure 3.1. The concept of 'line' may be assumed to be same as in our classical geometry. Thus the 'positive direction' of X-axis and the 'negative direction' of X-axis can be well understood and the line which the objects of the complete region A is considered to lie upon is called the Object Linear Continuum Line (see Figure 3.1 below) or Object Line in short.

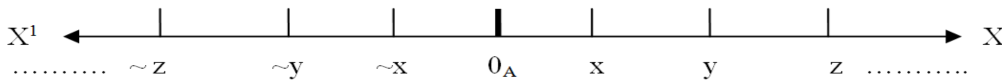

Fig. 3.1. Objects linear continuum line of the complete region A, a general view

Thus, any point on the Object Linear Continuum Line of the complete region A is called an object point of A.

We use the following notations in our work here :

A^+ = set of all positive objects of the complete region A,
A^- = set of all negative objects of the complete region A,
$A^{\geq 0}$ = set of all non-negative objects of the complete region A.

For developing a new calculus, be it in a two dimensional coordinate system or in an n-dimensional coordinate system, at least one calculus space is required as its base. Consider the object linear continuum line and the corresponding X-axis. Since the region A is complete, there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object of A has a unique address on this object linear continuum line and conversely, i.e. corresponding to every address (point) on this object linear continuum line there is a unique object x of the complete region A.

Consider a point x on the X-axis of the object linear continuum line corresponding to the calculus space A. Then for an infinitesimal small positive object Δx of the region A, the point $(x \oplus \Delta x)$ will be at a distance $\|\Delta x\|$ from the point x along the positive direction of X-axis and the point $(x \sim \Delta x)$ will be at a distance $\|\Delta x\|$ from the point x along the negative direction of X-axis. By distance between two objects x and y lying upon the XX^1 Object Linear Continuum Line of the complete region A, we mean the corresponding metric distance $\rho(x,y)$ of the normed complete metric space A.

For example, see a collection of consecutive equi-spaced points on the object line as shown in the Figure 3.2 below.

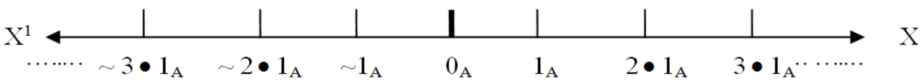

Fig. 3.2. Object Linear Continuum Line of the complete region A showing few consecutive equi-spaced object points.

The term 'equi-spaced' in the caption of Figure 3.2 is well understood in the sense of the corresponding metric (or norm) of the complete region A; i.e. for any real integer r, $\rho(r \bullet 1_A, (r+1) \bullet 1_A)$ = constant (independent of r), in the complete region A by virtue of the beautiful property of 'Homogeneity' possessed by the metric ρ (as mentioned earlier in subsection-2.4.1).

Example 2.1

If we choose the region A to be the RR region which is a partitioned region with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "$\leq$", and if we choose $\|x\| = |x|$ in RR, where $\rho(x, y) = \|x-y\| = |x-y|$, then it can be observed that the X-axis of the region calculus is the classical X-axis popularly used by us in the Cartesian coordinate system in Newton Calculus, the corresponding object linear continuum is the classical real continuum.

It will be mistake if we say that the Newton Calculus is based on the field R of real numbers (of course, considering the extended real-axis). Actually it is neither the field R nor the division algebra R, but it is the region R (which is here the RR region [3]). Interestingly and very fortunately, the particular division algebra R satisfies few additional properties trivially (not by virtue of the definition and independently owned properties of 'division algebra'). And by fulfilling these additional properties, the division algebra R well qualifies [3] to become a real region too, but not by virtue of the definition and independently owned properties of 'division algebra'. Consequently the classical calculus never faced any computational constraints or invalidity even assuming inadvertently R to be a field or division algebra just. Fortunately it is a coincidence that R does also form a complete region! Otherwise the classical calculus would not have reached the extremely rich level of today, rather it would have become blocked somewhere at some time much earlier in its so long beautiful network of journey happened over all branches of academic subjects.

Consider the object linear continuum line and the X_A-axis corresponding to the complete region A. Consider a point x_A (a positive object) on the X_A-axis. Then for an infinitesimal small positive object Δx_A, the point $(x_A \oplus \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the positive direction of X_A-axis and the point $(x_A \sim \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the negative direction X_A^1-axis; and in fact all the objects of the complete region A are well ordered in this sense, as explained in details earlier. Now we incorporate "Y_A-axis" (imagine that a copy of X_A-axis is placed at right angle to the X_A-axis passing through the point 0_A, i.e. rotating through 90^0 anticlockwise about the point 0_A) and thus construct a region coordinate plane in the style of Cartesian coordinate system.

3.2 Region Coordinate Plane

We will observe now that every complete region has its own Object Geometry. We introduce first of all 2-D Object Geometry in a 1-D complete region A = $(A, \oplus, *, \bullet)$. It is a system of geometry where the position of points on the plane is described using an ordered pair of objects, analogous to the case of Cartesian coordinate plane. We call this plane by 'Region Coordinate Plane'. A plane is a flat surface that goes on forever in both directions. If we were to place a point on the plane, region coordinate geometry gives us a way to describe exactly where it is by using two objects. Points are placed on the "region coordinate plane" as shown below in Figure 3.3. It has two scales: one running across the plane called the "X_A-axis" and another at right angles to it called the "Y_A-axis". Both these axes (X_A-axis and Y_A-axis) are thus object linear continuum lines corresponding to the complete region A.

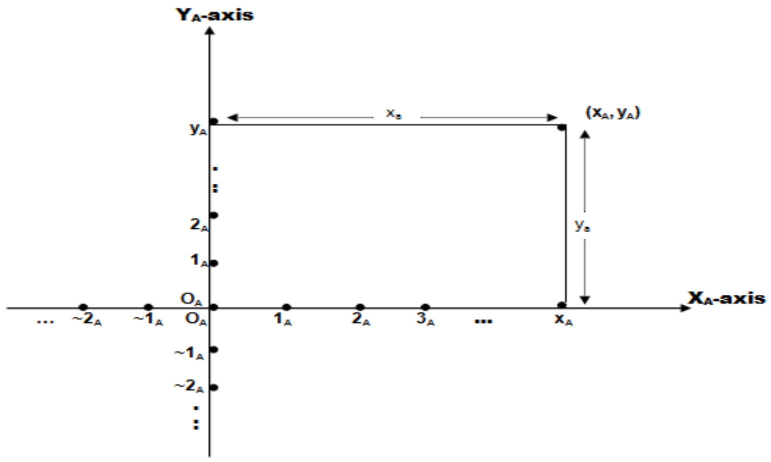

Fig. 3.3 Object coordinates on region coordinate plane of the complete region A

The point where the two axes cross is called the origin denoted by the notation O_A at which both x_A and y_A are the object 0_A. On the X_A-axis, as explained earlier that objects to the right of origin are positive objects and those to the left are negative objects of A. Similarly, on the Y_A-axis, objects above the origin are positive objects and those below the origin are negative objects of A.

A point's location on the region coordinate plane is given by two objects in the form of object coordinates (x_A, y_A), the first coordinate reveals where it is away from the Y_A-axis at parallel to the X_A-axis and the second coordinate reveals where it is away from the X_A-axis at parallel to the Y_A-axis (see Figure 3.3 above). The meaning of the word 'away' is to be drawn with the help of the concerned metric ρ. There are four quadrants, and sign convention rule is same as that of classical Cartesian coordinate geometry. If there is no confusion, we use the word X-axis instead of X_A-axis, Y-axis instead of Y_A-axis in our literature here henceforth.

Consider the Object Geometry corresponding to the 1-D complete region A, and consider also the Object Geometry corresponding to the 1-D complete region RR. Thus there are two sets of Object Geometry we will consider now, and there are two corresponding region coordinate planes. Suppose that the region coordinate plane of A does also represent the region coordinate plane of RR taking same lines as two axes and taking the same location for origin (i.e. O_A and O_{RR} are co-incident points). Thus the X-axis, Y-axis, and the origin are common to both the region coordinate planes.

The results of the following proposition are straightforward.

Proposition 3.1

(1) If P be a point on the X-axis with the coordinates (x_A, 0_A) on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be

(i) ($\|x_A\|$,0) on X-axis, if x_A is a positive object,

(ii) (-$\|x_A\|$,0) on X-axis, if x_A is a negative object.

(i.e. the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(2) If P be a point on the X-axis with coordinates (x, 0) on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be (x_A, 0_A) on the X-axis.
(the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(3) If P be a point on the Y-axis with coordinates (0_A, y_A) on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be

(i) (0, $\|y_A\|$) on Y-axis, if y_A is a positive object,

(ii) (0, -$\|y_A\|$) on Y-axis, if y_A is a negative object.

(the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(4) If P be a point on the Y-axis with coordinates (0, y) on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be (0_A, y_A) on the Y-axis.
(the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(5) If P (x_A, y_A) be a point on the on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be one of the ($\pm\|x_A\|, \pm\|y_A\|$) which is in compliance with the sign retaining rule, as the point P will remain in the same quadrant in both the region coordinate planes.

(6) If P (x, y) be a point on the on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be (x_A, y_A).
(the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

3.3 Slope of an Object Line

An object line passing through two points on the Region Coordinate Plane of the region A is unique. We now compute the slope of an object line for this region coordinate plane.

Slope of an object line passing through the two object points $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ is the real number m_a given by (as shown in Figure 3.4) :

$m_a = \tan\theta = \lambda.\dfrac{\rho(y_{2A}, y_{1A})}{\rho(x_{2A}, x_{1A})}$, where λ is either +1 or -1 as per the usual sign rule followed in classical geometry.

Therefore,
$$\begin{aligned}
m_a &= \lambda.\frac{\|y_{2A} \sim y_{1A}\|}{\|x_{2A} \sim x_{1A}\|} \\
&= \lambda.\frac{\|y_{2a} \bullet 1_A \sim y_{1a} \bullet 1_A\|}{\|x_{2a} \bullet 1_A \sim x_{1a} \bullet 1_A\|} \\
&= \lambda.\frac{\|(y_{2a} - y_{1a}) \bullet 1_A\|}{\|(x_{2a} - x_{1a}) \bullet 1_A\|} \\
&= \lambda.\frac{|y_{2a} - y_{1a}|.\|1_A\|}{|x_{2a} - x_{1a}|.\|1_A\|} \\
&= \lambda.\frac{|y_{2a} - y_{1a}|}{|x_{2a} - x_{1a}|} \\
&= \lambda.\frac{|y_2.1_a - y_1.1_a|}{|x_2.1_a - x_1.1_a|}, \quad y_{2a} \text{ means } (y_2)_a \text{ which is equal to } y_2.1_a \\
&= \lambda.\frac{|y_2 - y_1|}{|x_2 - x_1|}
\end{aligned}$$

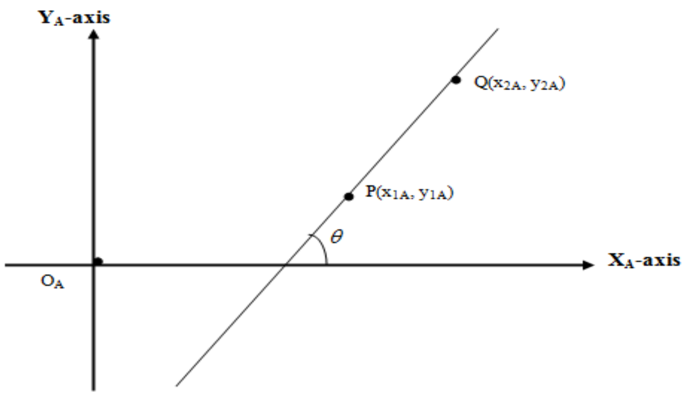

Fig. 3.4. Slope of an objects line

Thus the slope is neither dependent upon the concerned region nor upon the metric of the region.

3.4 Distance between two object points

In Object Geometry, the distance between two object points on a region coordinate plane can be defined in various ways like in classical geometry. However, we follow the style of Eucledian distance here.

Consider the X_AY_A region coordinate plane corresponding to the complete region A. Let $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ be two points on this region plane (see Figure 3.5). Distance PQ between these two points is the positive real number d, where

$$d = \left\{(\rho(y_{2A}, y_{1A}))^2 + (\rho(x_{2A}, x_{1A}))^2\right\}^{\frac{1}{2}}.$$

It can be observed that this distance is an absolute distance in the sense that neither it is dependent upon the concerned region nor upon the metric of the region.

We see that,

$$\begin{aligned}
d^2 &= \|y_{2A} \sim y_{1A}\|^2 + \|x_{2A} \sim x_{1A}\|^2 \\
&= \|y_{2a} \bullet 1_A \sim y_{1a} \bullet 1_A\|^2 + \|x_{2a} \bullet 1_A \sim x_{1a} \bullet 1_A\|^2 \\
&= \|(y_{2a} - y_{1a}) \bullet 1_A\|^2 + \|(x_{2a} - x_{1a}) \bullet 1_A\|^2 \\
&= (y_{2a} - y_{1a})^2 . 1_a^2 + (x_{2a} - x_{1a})^2 . 1_a^2 \\
&= \{(y_{2a} - y_{1a})^2 + (x_{2a} - x_{1a})^2\} \\
&= \{(y_2 - y_1)^2 + (x_2 - x_1)^2\}.
\end{aligned}$$

This implies that d is neither dependent upon the concerned region nor upon the metric of the region.

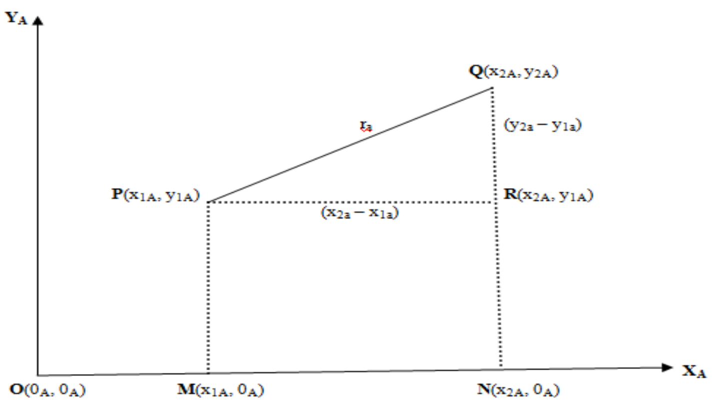

Fig. 3.5 Distance between two object points

The Pythagoras Theorem is thus well valid irrespective of the concerned region or the metric of the region.

Proposition 3.2
Pythagoras Theorem is valid in every Object Geometry, whatever be the corresponding complete region A.

3.5 Equation of an Object Line

Consider the X_AY_A region coordinate plane corresponding to the complete region A (see Figure 3.6). The general equation of an object line whose slope is m_a is

$$y_A = m_a \bullet x_A \oplus c_A .$$

Equation of an object line having slope m_a and passing through the object point $Q(x_{1A},y_{1A})$ is

$$(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A}).$$

Equation of an object line passing through the two object points $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ is

$$(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A}), \quad \text{where } m_a = \frac{y_{2a} - y_{1a}}{x_{2a} - x_{1a}} .$$

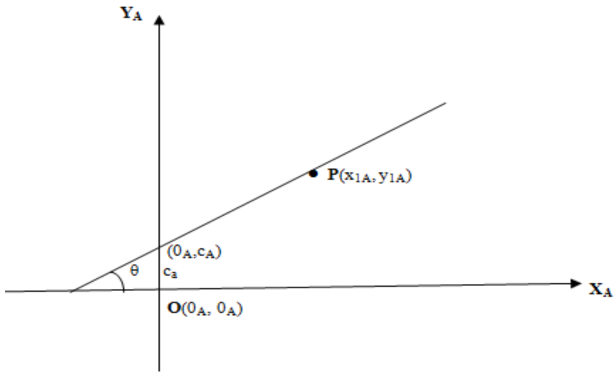

Fig. 3.6 An object line having positive intercept of length c_a on Y_A axis.

If an object line MN has the intercepts of lengths p_a and q_a on the X-axes and Y-axes respectively at the points $(p_A, 0_A)$ and $(0_A, q_A)$, then the equation of this object line MN will be (as shown in Figure 3.7):

$$\frac{x_A}{p_a} + \frac{y_A}{q_a} = 1_A \qquad \text{(using Division Type-2 as introduced in [3]).}$$

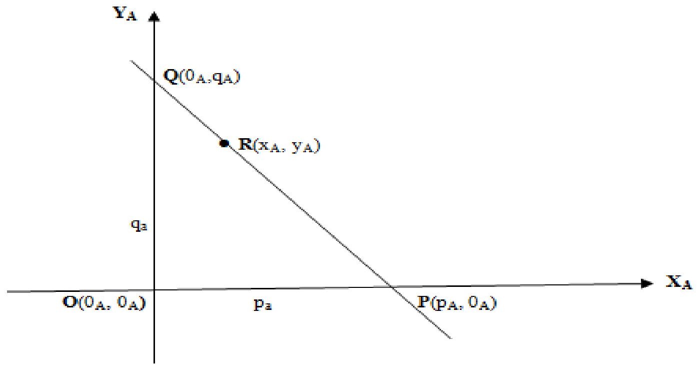

Fig. 3.7 An object line making two intercepts of lengths p_a and q_a.

In the above equations, the variables take values which are objects of the region A. The classical geometry taught at school level is also an instance of Object Geometry based upon a particular region which is RR.

3.6 Object Circle

Consider the X_AY_A region coordinate plane corresponding to the complete region A. Then the equation of an Object Circle (see Figure 3.8) with centre at $(0_A, 0_A)$ and radius r_a (>0) is given by

$$(\rho(x_A, 0_A))^2 + (\rho(y_A, 0_A))^2 = r_a^2,$$

which can be written as

$$\|x_A\|^2 + \|y_A\|^2 = r_a^2.$$

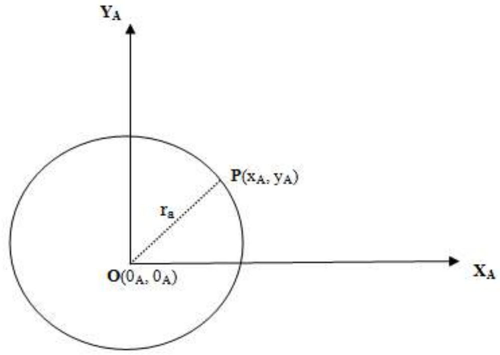

Fig. 3.8 Object circle with centre at origin

And the equation of the Object Circle (see Figure 3.9) with centre at (α_A, β_A) and radius r_a (>0) is given by

$$(\rho(x_A, \alpha_A))^2 + (\rho(y_A, \beta_A))^2 = r_a^2,$$

which can be written as

$\|x_A \sim \alpha_A\|^2 + \|y_A \sim \beta_A\|^2 = r_a^2$.

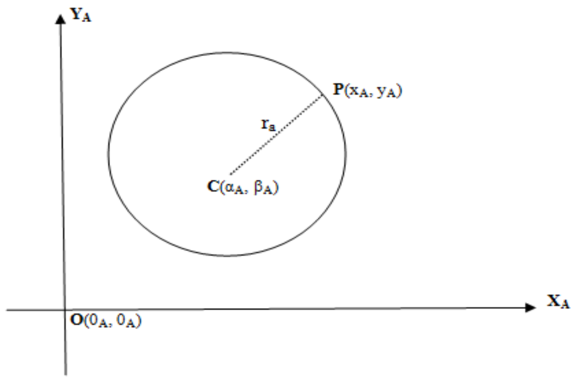

Fig. 3.9 Object circle with centre at the object point **C**(α_A, β_A)

Further study on object geometry can be easily done analogous to the literature of classical geometry.

4. Region Calculus

Let us consider a calculus space A. Suppose that we want to develop now a new calculus in the calculus space A. For this purpose, the basic concepts of any new calculus (of a new differential calculus) are **:** limit, continuity, differentiability of a function of objects, etc. which we need to introduce first of all in the calculus space A analogous to the classical style of Newton calculus.

4.1 Defining "x → a"

We are well aware of the concept of "x → a" in Newton Calculus. In this subsection we define the notion of "x → a" in region calculus.

Consider an object variable x over the calculus space A = (A, ⊕, *, •). Let a ∈ A be a fixed object. Suppose that while approaching the object a from its right side along the Object Linear Continuum Line, the object variable x assumes successive object values, out of which some of them for example are :

(a ⊕ 0.1 • 1_A), (a ⊕ 0.01 • 1_A), (a ⊕ 0.001 • 1_A), (a ⊕ 0.0001 • 1_A), …….

during its course of journey on the object line X^1OX to get close and close to the object a.

Obviously, as x passes through these object points, the value ρ(x, a) becomes less and less and become so small that for any positive real number ε, no matter however small, ρ(x, a) < ε is satisfied. Let us express this situation using the notation "x → a+" which means that the object variable x approaches the fixed object a from the right hand side of a (as shown in Figure 4.1). The meaning of 'right hand side of a' is clear by the explanation presented earlier in Section 2,3.

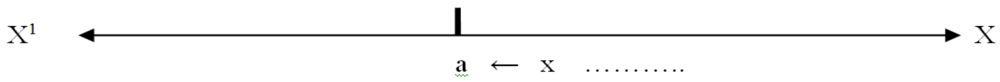

Fig. 4.1. The notion of $x \rightarrow a+$ on the Object Linear Continuum Line

Suppose that while approaching the object a from its left side along the Object Linear Continuum Line, the object variable x assumes successive object values, out of which some of them for example are :

$(a \sim 0.1 \bullet 1_A)$, $(a \sim 0.01 \bullet 1_A)$, $(a \sim 0.001 \bullet 1_A)$, $(a \sim 0.0001 \bullet 1_A)$, ……

during its course of journey on the object line X^1OX to get close and close to the object a.

Obviously, as x passes through these successive values, the value $\rho(x, a)$ becomes less and less and become so small that for any positive real number ε , no matter however small, $\rho(x, a) < \varepsilon$ is satisfied. Let us express this situation using the notation "$x \rightarrow a-$" which means that the object variable x approaches the fixed object a from the left hand side of a (as shown in Figure 4.2).

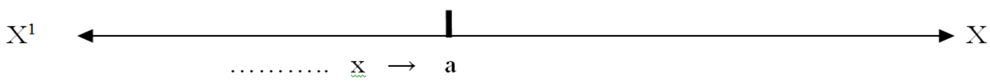

Fig. 4.2. The notion of $x \rightarrow a-$ on the Object Linear Continuum Line

By the expression "x tends to a" symbolically written as "$x \rightarrow a$", we mean that given any real $\varepsilon > 0$ no matter however small, the successive values of x ultimately satisfy the inequality $0 < \rho(x, a) < \varepsilon$. It is to be noted that if "$x \rightarrow a$" then $\rho(x, a) \neq 0$, i.e. $x \neq a$.

4.2 Neighborhood of an Object Point

Consider an object point a on the Object Linear Continuum Line of the calculus space $A = (A, \oplus, *, \bullet)$. Analogous to the concept as in Newton Calculus, we now define the notion of Neighborhood of an Object Point on the object line.
Let $\delta > 0$ be a real number. Then the δ-neighborhood of the object a is defined by the set $N_\delta(a)$ of objects given by $N_\delta(a) = \{ x : x \in A \text{ and } \rho(x, a) < \delta\}$.
Here $N_\delta(a) \subseteq A$, and obviously $N_\delta(a) \neq \varphi$.

4.3 Limit of a Function of Object Variable

Let X and Y be two non-null subsets of the calculus space A = (A, $\oplus$, *, $\bullet$) and let f be a function f : X $\rightarrow$ Y. Thus f is actually an object valued function of object variable. Then f(x) is said to have a limit l in Y if for any pre-assigned real number $\varepsilon > 0$, no matter however small, $\exists$ a real number $\delta > 0$ such that

$$\rho(\,f(x), l) < \varepsilon \quad \text{whenever } 0 < \rho(x, a) < \delta.$$

We write symbolically as:

$$\lim_{x \to a} f(x) = l, \quad \text{i.e. } f(x) \to l \text{ as } x \to a.$$

To understand the concept, let us solve the problems posed below.

Problem 4.1

Show that $\lim_{x \to 2 \bullet 1_A} 5 \bullet x = 10 \bullet 1_A$ in the calculus space A = (A, $\oplus$, *, $\bullet$).

Solution :

Given real $\varepsilon > 0$, no matter however small, we need to find out real $\delta > 0$ such that

$$\rho(\,5 \bullet x, 10 \bullet 1_A) < \varepsilon \quad \text{whenever } 0 < \rho(x, 2 \bullet 1_A) < \delta.$$

i.e. $\| 5 \bullet x \sim 10 \bullet 1_A \| < \varepsilon$ whenever $0 < \| x \sim 2 \bullet 1_A \| < \delta$.

i.e. $5.\| x \sim 2 \bullet 1_A \| < \varepsilon$ whenever $0 < \| x \sim 2 \bullet 1_A \| < \delta$.

Now if we choose $\delta = \varepsilon/5$, our definition is satisfied.

Hence $\lim_{x \to 2 \bullet 1_A} 5 \bullet x = 10 \bullet 1_A$ in the calculus space A.

Problem 4.2

Show that $\lim_{x \to 3 \bullet 1_A} \dfrac{x^2 \sim 9 \bullet 1_A}{x \sim 3 \bullet 1_A} = 6 \bullet 1_A$ in the calculus space A = (A, $\oplus$, *, $\bullet$).

Solution :

Given $\varepsilon > 0$, no matter however small, we need to find out $\delta > 0$ such that

$$\rho\left(\frac{x^2 \sim 9 \bullet 1_A}{x \sim 3 \bullet 1_A},\ 6 \bullet 1_A \right) < \varepsilon \qquad \text{whenever } 0 < \rho(x, 3 \bullet 1_A) < \delta.$$

$$\text{i.e.} \quad \left\| \frac{x^2 \sim 9 \bullet 1_A}{x \sim 3 \bullet 1_A} \sim 6 \bullet 1_A \right\| < \varepsilon \qquad \text{whenever } 0 < \| x \sim 3 \bullet 1_A \| < \delta.$$

Since $x \rightarrow 3 \bullet 1_A$ therefore $x \neq 3 \bullet 1_A$ and hence $(x \sim 3 \bullet 1_A) \neq 0_A$.

Therefore, Cancellation Laws of region algebra can be applied to get the following result :

$$\| (x \oplus 3 \bullet 1_A) \sim 6 \bullet 1_A \| < \varepsilon \quad \text{whenever } 0 < \| x \sim 3 \bullet 1_A \| < \delta.$$

i.e. $\| x \sim 3 \bullet 1_A \| < \varepsilon$ whenever $0 < \| x \sim 3 \bullet 1_A \| < \delta$.

Now if we choose $\delta = \varepsilon$, our definition is satisfied. Hence the result.

4.4 Multi-dimensional Region Calculus

The calculus space discussed so far is basically one dimensional calculus space (1-D calculus space) and the corresponding region calculus is also one dimensional. It is because of the reason that in a calculus space any variable x can vary/move along a straight line only. By the simple terms : calculus space, region calculus, complete region, we shall always mean here the same in an one-dimensional calculus space.

In this section we introduce the concept of 'Multi-dimensional Calculus Space' as a generalization of the concept of 'calculus space'. In a two-dimensional calculus space (2-D calculus space), a variable z can move along a curve on a plane. The corresponding region calculus is called a 2-D region calculus. In a three-dimensional calculus space (3-D calculus space), a variable w can move along a curve on a 3-D space. The corresponding region calculus is called a 3-D region calculus. Similarly, in an n-D calculus space, a variable μ can move along a curve on a n-D hyperspace. The corresponding region calculus is called an n-D region calculus.

We now define n-to-1 Bijective Mapping.

4.5 n-to-1 Bijective Mapping

Consider two non-null sets X and Y. A function $f : X \to Y$ is said to be a 'n-to-1 Bijective Mapping' if

(i) f is onto, and

(ii) $\forall\, y \in Y$, $\exists$ a unique subset S_y of X of cardinality n (> 2) such that $\forall\, x \in S_y$ we have $f(x) = y$.

Here n could be finite positive integer (> 2) or infinity (of the set R).

For example, the function $f :\ C\text{-}\{0\} \to R^+$ given by

$$f(z) = |z|^2$$

is a n-to-1 Bijective Mapping, where C is the set of complex numbers.

4.6 Multi-dimensional Calculus Space

Consider a partitioned region $A = (A, \oplus, *, \bullet)$. Then A forms a Multi-dimensional Calculus Space if the following conditions are satisfied :

(i) A is an extended region.

(ii) A is a normed complete metric space with respect to a norm $\|.\|$ and the corresponding induced metric $\rho(x, y) = \|x \sim y\|$, (i.e. $\|x\| = \rho(x, 0_A)$).

(iii) The norm $\|.\|$ is a n-to-1 bijective mapping from $A-\{0_A\}$ to R^+ for some fixed integer n > 2.

4.7 n-D Complete Region

A real region which can form a n-D calculus space is called a "n-D complete region".
A calculus developed out of n-dimensional calculus space is called by n-dimensional region calculus. It may happen that a region can not form an n_1-dimensional calculus space, but can well form an n_2-dimensional calculus space. In other words, a region may not form an n_1-dimensional region calculus, but may well form an n_2-dimensional region calculus.

It is to be carefully noted that, a Division Algebra is not a region in general. Consequently a Division Algebra can not become a Calculus Space in general even if it satisfies all the conditions of Calculus Space. Given any region G = (G, $\oplus$,*, $\bullet$) over the field (R, +, **.**), one can immediately attempt to explore whether G forms a calculus space with respect to a suitably defined norm $\|.\|$ and one suitably defined total order relation '$\leq$'. If G forms a calculus space, then a new calculus can be developed in G. The set C of complex numbers does not satisfy the required conditions to become a calculus space with respect to its popular norm $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. Consequently, no 1-D region calculus can be developed in the region C with respect to this norm. It is to be carefully noted that the existing rich 'Calculus of Complex Variables' is not an 1-D region calculus.
However, in our future research work we need to explore whether C forms a multi-dimensional calculus space (say, 2-D calculus space) with respect to its popular norm $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. And if so, then will this 2-D region calculus be the same calculus as the existing rich 'Calculus of Complex Variables' of mathematics?

The set of triangular fuzzy numbers (trapezoidal fuzzy numbers) does not form a region with respect to its existing known operators and consequently it can not offer any region calculus of any dimension to the fuzzy mathematicians. This is one of the major demerits of the notion of triangular fuzzy numbers and trapezoidal fuzzy numbers, some of the drawbacks have been justified in [7,8,9].

5. Conclusion

In this paper a generalization of the existing subject 'Calculus' is developed. The generalized calculus is called by 'Region Calculus' which is introduced by defining the notion of 'Calculus Space'. The classical calculus developed independently by Newton and Leibniz is based on the set R of real numbers, extended with two infinities, and then took its shape further with functions of complex variables, vector calculus, tensor calculus, etc. The growth of classical calculus at every stage required fluent applications of various properties of the set

R of real numbers. But it is justified in [3] that using the properties of a 'field' or a 'division algebra' or any existing standard algebra, the classical calculus cannot have the validity of its all fluent results. Fortunately, the set R is a trivial example of the algebraic structure region [3], and the mathematicians enriched the classical calculus using the properties of region R, although 'unknowingly'. It is fact because of the reason that the development of the classical calculus cannot be validated by virtue of the definition and properties of any existing brands of the standard algebraic structures of Abstract Algebra by virtue of their respective definition and independently owned properties. It is only the algebraic system region at minimum which has the capability to validate such elementary and higher level computations.

One of the major breakthrough in this work is that we have precisely identified: 'What are the minimum properties which need to be satisfied by a set A so that a calculus can be developed over A?'. Consequently we have introduced the notion of 'calculus space' as a general minimal platform on which a calculus can be developed. It has been explained how the platform R of classical calculus forms a calculus space. For a non-example, the set of all triangular fuzzy numbers do not form a real region with respect to its commonly used operators, and hence cannot open any platform to develop any fuzzy differential calculus and fuzzy integral calculus over it in the style of the classical calculus. The requirements are precisely identified as a checklist before making any attempt to develop any new calculus over a given set. This work of Region Calculus is initiated with a prior intuitionistic assumption that there is a possibility that the classical calculus may not be most appropriate or even may not be applicable successfully at everywhere of our universe system in some complex situations (if any) or in the multiverse system if exists. We presume that our future computations (be it in this solar system or in other, be it in this universe or in other of the multiverse) may not be sufficiently covered by or compatible with our classical calculus because of a significant limitation in the 'distance' formula used. Consequently, the very first job is to define the general structure of a mathematical space which is a minimum requirement for making an attempt to develop any new calculus over it.

Before going to develop Region Calculus, we have introduced another new branch of Mathematics called by "Object Geometry". Corresponding to every complete region there is a unique Object Geometry. The existing 'classical geometry' is one example of the "Object Geometry" corresponding to the particular region RR. For a non-example, the set of all triangular fuzzy numbers (or the set of all trapezoidal fuzzy numbers) is closed with respect to the addition operator defined over them, but is not closed with respect to the multiplication operator defined over them [7,8]. Thus the set of all triangular fuzzy numbers (or the set of all trapezoidal fuzzy numbers) do not form a real region with respect to its commonly used operators (and cannot open any platform to develop any region calculus), and hence cannot open any type of new Theory of A-Numbers [4] or any new Object Geometry at the present form in the context of Region Mathematics. Similarly the set of all interval numbers, Z-fuzzy numbers, intuitionistic fuzzy numbers, i-v fuzzy numbers, rough numbers, etc do not form a region and hence cannot open any platform to develop any region calculus. Hence these sets cannot open any type of new Theory of A-Numbers or any new Object Geometry in the context of "Region Mathematics" by the present form of their respective literatures developed so far .

It is justified that mathematically there are infinite number of distinct complete regions exist in mathematics, there are infinite number of distinct 1-D region calculus exist. Then we generalize the concept of calculus space by defining 'multi-dimensional calculus space'. The simple term calculus space is basically one dimensional calculus space (1-D calculus space) and the corresponding region calculus is also one dimensional region calculus. In a calculus space any variable x can vary/move along a straight line only, i.e. if $x \rightarrow a$ in a complete region, it means that x is being driven along a straight line. The concept of 'Multi-dimensional Calculus Space' is a generalization of the concept of 'calculus space'. In a two-dimensional calculus space (2-D calculus space), a variable z can move along a curve on a plane. The corresponding region calculus is called a 2-D region calculus. In a three-dimensional calculus space (3-D calculus space), a variable w can move along a curve on a 3-D space. The corresponding region calculus is called a 3-D region calculus. Similarly, in an n-D calculus space, a variable μ can move along a curve on a n-D hyperspace. The corresponding region calculus is called an n-D region calculus (i.e. n-dimensional region calculus). It may happen that a region cannot form an n_1-dimensional calculus space, but can well form an n_2-dimensional calculus space. In other words, a region may not form an n_1-dimensional region calculus, but may form an n_2-dimensional region calculus. It is to be carefully noted that mathematically an arbitrary Division Algebra is not a region in general by virtue of its definition and properties. Consequently an arbitrary Division Algebra cannot qualify to become a Calculus Space in general. The proposed theory of Region Calculus helps us to study for any arbitrary region G = $(G, \oplus, *, \bullet)$ over the field (R, +, **.**) to explore whether G forms a calculus space with respect to a suitable norm $\|.\|$ and a suitable total order relation '$\leq$'. If G forms a calculus space, then a new calculus can be well developed in G. However, the set C of complex numbers does not satisfy the required conditions to become a calculus space with respect to its popular norm $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. Consequently, no 1-D region calculus can be developed in the region C. However, in our future research work we need to explore whether C forms a multi-dimensional calculus space (2-D calculus space) with respect to its popular norm $|z| = \sqrt{z\bar{z}}$ so that a 2-D region calculus can be developed in C. The set of triangular fuzzy numbers (trapezoidal fuzzy numbers) does not form a region with respect to its existing known operations and consequently it cannot offer any region calculus of any dimension to us. This is one of the major demerits of the notion of triangular fuzzy numbers and trapezoidal fuzzy numbers, the drawbacks being justified in details in [7,8].

The set C of complex numbers does not satisfy the required conditions to become a calculus space with respect to its popular norm $|z| = \sqrt{z\bar{z}}$ = $\sqrt{x^2 + y^2}$.
Thus no 1-D region calculus can be developed in C, and hence C is not a 1-D complete region with respect to 2-to-1 bijection. It is to be carefully noted that the existing rich 'Calculus of Complex Variables' is not an 1-D region calculus. Consequently, a number theory of type 'Theory of C-numbers' cannot be developed in C, and due to same reason a Object Geometry too cannot be developed in C; although the concept of prime and composite objects, imaginary and complex objects, compound numbers, etc. can be well studied in C. However if C forms a multi-dimensional complete region (say 2-D complete region or n-D) then C may open a new Theory of C-Numbers with multi-dimensional approach and its own Object Geometry, which is our future course of research work. For this, we first of all need to explore whether C forms a multi-dimensional calculus space (say, 2-D calculus space) with respect to its popular norm $|z| = \sqrt{z\bar{z}}$ = $\sqrt{x^2 + y^2}$ so that a 2-D region calculus can be developed in C. And if so, then will it be the same calculus as the rich 'Calculus of Complex Variables' of existing mathematics?

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