International Journal of Algebra, Vol. 13, 2019, no. 5, 185 – 238
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2019.9621

‘REGION’ : The Unique Algebraic Structure in Abstract Algebra

Ranjit Biswas

Department of Computer Science & Engineering
Jamia Hamdard University
Hamdard Nagar, New Delhi – 62, India

Abstract

The important algebraic structures viz. group, ring, module, field, linear space, algebra over a field, associative algebra over a field, Division Algebra have made the subject ‘Abstract Algebra’ very rich and well equipped to deal with various algebraic computations at elementary level to higher level of mathematics. In this work it is unearthed that these algebraic structures are not sufficient (i.e. not capable in most of the cases) to support ‘Mathematics’, to support the various branches of Sciences and Engineering. It is unearthed that there are many computational problems and issues in mathematics which do not fall under the jurisdictions of any of these algebraic structures to deal appropriately. It is thus observed that there is a genuine vacuum in the family of all existing standard algebraic structures of Abstract Algebra to provide computational norms to the giant subjects like: mathematics, physics, statistics, etc. and this vacuum was remaining so far in a very hidden way. Consequently, it is strongly justified that this family (of important algebraic structures viz. group, ring, module, field, linear space, algebra over a field, associative algebra over a field, Division Algebra etc.) needs inclusion of an appropriate new member who is well capable and can take the responsibility to deal with the all type of computations being practiced in mathematics, sciences, engineering studies unlike the limited capabilities of the existing standard algebraic structures. It is fact that an algebraist can introduce a number of new algebraic structures if he desires. But the question may arise about the necessity to do so!. A new algebraic structure is not supposed to be a redundant one to the subject ‘Abstract Algebra’ to unnecessarily cater to the existing huge volume of literature of Algebra. It must have some unique as well as advance kind of roles in the mathematical computing of daily practices (at school, college, research levels) which none of the existing

algebraic structures can have by its respective definition and independently owned properties, it must have some unique capabilities to justify the validity of all the mathematical computing of daily practices (at school, college, research levels) which none of the existing algebraic structures can claim. Such kind of statements may apparently appear to many of the readers with very surprise, but it is fact.
With this objective a new and very important algebraic structure called by "Region" is introduced in this paper, and its various properties are studied. The philosophy of the importance of this work is presented in details by making all the justifications with story-based explanations (with hypothetical data/information). The paper finally justifies the fact that Abstract Algebra can not grow without the algebraic structure 'region'. With the introduction of 'region', Abstract Algebra will become more complete and sound as a subject to the mathematicians, scientists and engineers.

Mathematics Subject Classification: 08, 11

Keywords : region, real region, characteristics

1. Introduction

Abstract Algebra is the branch of 'Mathematics' in which we study algebraic structures such as group, ring, module, field, linear space, algebra over a field, associative algebra over a field, Division Algebra etc. This branch has the important responsibility to become sufficiently equipped for validating the rules for manipulating formulae, laws, identities and algebraic expressions involving unknowns, often now called elementary algebra. But at the very outset in Section-3 in this paper we unearth the fact that although there is no error, no self-conflict, no contradiction being faced by the mathematicians with the nature and growth-style of the existing giant subject 'Mathematics', but most of the simple and useful results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra (which are commonly practiced at secondary school level of mathematics and of course at all levels of higher mathematics) are **not valid** in any of the recognized important algebraic structures viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, and even not valid in 'Division Algebras'. They can not be verified in any of the existing standard algebraic structure alone in general, by virtue of their respective definitions and independently owned properties. It is a major gap of Abstract Algebra unearthed today in this work. One could feel the gravity of this fact of vacuum comparing hypothetically with other type of statements as presented below (but with an analogous philosophy):
the drivers have been driving their vehicles very well on the city road network of Calcutta city where many of them are not having any valid official license or any valid official authority to drive on the roads of the city, nevertheless they have been driving for many years without any error, without any accident occurred,

without any conflict, without any collision, but catering to human welfare and human progress as well as to the progress of the city continuously all-time; where it is also fact that the progress of the traffic system and management has been happening significantly with time in good pace. Since there is no collision happening while driving without the driving authority/license, can we accept it to be a valid official act of driving? Since there is no accident occurring in the driving, can we regard this fortunate outcomes to be the ‘sufficient authority’ for the act of driving being practiced on the city roads where it is claimed that the city is under an excellent governance?

Exactly same is the situation with the mathematicians who are computing everyday using so many fundamental and elementary rules/norms/formulas but being not ever authorized by any of the existing algebraic structures of Abstract Algebra. This is a major weakness of the present shape of Abstract Algebra, despite being rich with a huge volume of literature developed so far.

Unearthing gaps and then filling up the gaps by introducing new algebraic structure in Abstract Algebra happened time to time in the last centuries because of genuine requirements only. The past chronological history of Abstract Algebra may be revisited for details. But today the obvious questions now arise : “What is the core algebraic structure out of all the existing standard core algebraic structures based upon which the elementary mathematics and also the higher mathematics, norms/rules/equalities/identities etc. can be practiced fluently with validity?” And if there exists no answer to this question then does the fluent practice of elementary algebra/computation, which have been always producing correct outputs, give the guarantee of official validity to a mathematician to do computations? If not, then how to do justice to the transparency of Mathematics, if the computations in most of the cases be not supported and validated by Abstract Algebra?

Mathematicians discovered the beautiful algebraic structure ‘Group’ in early of 19^{th} century with some genuine objectives and requirements to solve mathematical and scientific issues. Then the mathematicians found some gaps and felt the necessity for obligatory extension of the subject ‘Abstract Algebra’ by defining another new algebraic structure ‘Ring’. After that the mathematicians found further gaps and felt the necessity for obligatory extension of the subject ‘Abstract Algebra’ by defining another new algebraic structure ‘Field’. All these developments happened for smoothness and enrichment of the topic ‘Algebra’, to make it more complete and more sound to validate the practices of computations. Thus, with time the mathematicians defined the algebraic structures groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, ‘Division Algebras’, etc to make the topic ‘Abstract Algebra’ very sound and as best as possible so that the common users (students of school, college, universities, and researchers) can practice elementary mathematics and higher level mathematics with valid authority. But the region mathematics is initiated to grow because it is now unearthed that there is a huge gap and probably highly significant gap that none of the existing important algebraic structures like groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, 'Division Algebras' etc can validate many of the simple and useful results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra (i.e. can not be verified by none by its respective definition and properties). This is a serious incompleteness and major weakness of the topic "Abstract Algebra'. This beautiful important gap is identified in this work and an independently complete and sound algebraic structure called by "Region' is developed to make the 'Abstract Algebra' now much richer and complete. Most important matter in this work is that the very genuine necessity to define the new algebraic structure 'Region' is justified in full length to fill up the gap. It is fact that if an algebraist desires then he can introduce many new algebraic structures, but the 'most important' matter is to justify the necessity of doing so!. The existing Abstract Algebra is so rich and voluminous that it does not need any redundant amount of literature in this century. But it is strongly justified in this work that the new algebraic structure 'Region' is not redundant to 'Algebra' by any amount of it. Rather it is very much genuine and obligatory because it is the **'minimal'** algebraic structure which allows the mathematicians to use the simple and useful results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra. The **'minimal'** property is the extraordinary power of the 'Region' which none of the existing algebraic structures including Division Algebra possesses. Consequently, in this sense it is fact that the algebraic structure 'Region' is the most important algebraic structure of all the existing algebraic structures including Division Algebra, by virtue of their respective definitions and independently owned properties.

Actually speaking, the author was working initially on few suddenly identified issues on the simple and useful results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra; and then eventually unearthed the new algebraic structure 'Region' which is the **'minimal'** algebraic structure having the capability to resolve the issues. It has been thus justified that Region is much more useful algebra compared to Division Algebra to the mathematicians and scientists. One can define several new algebraic structures, but the necessity to define the 'Region' is completely unavoidable, the justification for this claim is thoroughly made and established in this work. Elementary Algebra is one of the main branches of mathematics. It encompasses some of the basic concepts of algebra. It is practiced by all the mathematicians and scientists at their daily works, but as far as teaching is concerned it is typically taught to secondary school students on their understanding of arithmetic.

Since the 'Elementary Algebra' runs over the particular Division Algebra R (in most of the cases) in the existing concept of the academic world, there is no contradiction or error or conflict faced so far by the people dealing with elementary algebra because of the very hidden fact that "this particular Division Algebra R ***fortunately*** *qualifies to be a Region also". It is thus the very fortunate* *but accidental event, because the Division Algebra R satisfies few more additional axioms/conditions which are not covered by virtue of definition and independently owned properties of the algebraic structure 'Division Algebra'. Otherwise, mathematicians would have arrived at red-light stoppages long before in particular while practicing elementary algebra.*

There was no plan or idea to the author initially to develop a new algebraic structure 'Region'. The initialization was ignited with some issues on elementary algebra while working to solve some long standing unsolved problem of mathematics. Let the matter be presented in a simple way, but with the help of some other social domain of our daily life environment. Instead of practicing the subject mathematics, consider another practicing area of daily life, say 'car driving'. If you drive a car, you must have in your pocket a driving license. Almost all the vehicle drivers in the world have valid driving licenses. But if there is no traffic police on your road, if there does not happen any accident by your driving, and if you can commute millions and millions kilometer comfortably without any collision but without having any valid driving license, then quite naturally you may be **confused about the necessity of having your driving license.**

Questions arises : What is wrong with the style and progress of the existing nature of driving although it be without the appropriate official driving license? What is the necessity of having a valid driving license? What is the necessity of acquiring an officially documented authority? Is the 'Driving License' is a redundant document on road today?

An exact situation unearthed to exist in the ground reality on the academic sphere on this earth where instead of roads it is Education and Academic fields in mathematics, and instead of drivers it is the mathematicians, scientists, engineers, etc. The two basic and obvious assumptions are to be that when a driver drives a car then,

(i) "he must be sitting inside the car", and

(ii) "he must be having an official driving authority".

Otherwise he may do mistake anytime today or in future because of the reason that there is no well precise code of conduct at his hand, he may loose discipline on the road.

You can not do the following at a time:

You are sitting inside your non-AC car and during your driving you demand very cool air from outside because of the reason that cool-air is not available(validated in the system) by your car.

In case you need an AC car, then an AC car must be available in the market. But if there is no idea so far developed by any engineer to develop an AC car, then it is a major weakness of the market, and the world automobile engineers need to develop a new advanced car to provide solution in one unique car.

An analogous philosophy is applicable to the people doing mathematical computations, as the basic pre-assumption is that :

"While practicing elementary algebra, you must be sitting upon the platform of one unique algebraic structure and you are free to use all its properties".

Two drivers may have two different category of driving licenses respectively, one may have for driving light vehicles and the other may have for driving heavy vehicles including light vehicles. Consequently the second driver is having the authority to exercise more options while he drives on the roads, whereas the first driver has less amount of options.
Similarly the two basic and obvious assumptions in Region Algebra are that whenever a mathematician practices elementary algebra then,
(i) "he must be sitting on (standing upon the platform of) one very sound and complete appropriate algebraic structure A", and
(ii) "that algebraic structure A must officially allow him to practice elementary algebra", (providing valid rules for manipulating formulae, laws, identities and algebraic expressions, etc by virtue of its own definition and independently owned properties).

Most Important Issue :

The Elementary Algebra of primary/secondary school education (having exercises on simple formulas, identities, equalities, cross-multiplication laws, Componendo & Dividendo Rule, etc.) should not be dependent upon multiple members of the family of core algebraic structures for validity because of non-existence of an appropriate unique algebraic structure in Abstract Algebra. The unique algebraic structure must provide solution by itself by virtue of its own definition and independently owned properties. The non-existence of an appropriate unique algebraic structure is a major weakness of the subject "Abstract Algebra" to justify and validate the practices of computations on Elementary Algebra in primary/secondary school education. This serious observation will be analyzed in this paper in a very rigorous way, as this observation apparently may seem to be a surprising issue to the readers initially.
Every mathematician or every scientist uses elementary algebra fluently during his every kind of mathematical work. The mathematical work could be of school level or college level or of higher level or in some application domains theoretically and/or practically. But whatever be the level of mathematics, it can not be free from elementary algebra. Consequently he must have valid authority to use the formulas, rules, equations, identities, etc of elementary algebra. And it is the subject 'Abstract Algebra' who itself is supposed to issue the appropriate authority to the mathematicians and scientists for fluently practicing elementary algebra. This fact is unearthed (in Section-3 in this paper). On this new light, an introductory explanation is presented in the subsequent section below. It is a revised, updated and more sound version of the work of [7], and therefore it is not required to have a prior knowledge of the work [7].

1.3 About the algebraic structure 'Region'

Let us make an analysis here in a very simple way with some elements of college algebra. But it is fact that behind the simple matters being presented, there exists an element of beautiful mathematics which remained hidden in hibernation so far. And it is now observed that its importance is very high in mathematics. Because of very simple and fundamental nature of the analysis presented below, the readers may have to take some patience to get into the depth of this work from Section-3, because the work is advertently put into a slow pace of progress initially in this paper.

It is obvious that there are more flexible type of computations possible in a ring $(S, \oplus, \otimes)$ compared to the group $(S, \oplus)$ where $\oplus$ and $\otimes$ are two binary operations defined over the set S. For example, if you are standing upon the platform of an algebraic structure which is a group G = $<G, \oplus>$ then you can only add (or subtract) two members of G by virtue of the definition of the addition operation $\oplus$ of the group. You do not have any valid authority (license) to make an attempt for computing a*b where a, b $\in$ G and * is an unknown operation to the group $<G, \oplus>$ (i.e., where * is not the operation $\oplus$ of the group G).

Let us present a specific example.
Consider the group G = $<R, +>$ where R is the set of real numbers and + is the usual addition operation defined over R. Then you surely have the valid authority (license) to compute the expressions: 9+2, 15-3, -8-6, 0+0, etc i.e. to compute an expression like a+b by virtue of the definition of the algebraic structure 'group' where a, b $\in$ R. But on this platform of the group G = $<R, +>$, can you say that $7 \times 3 = 21$? can you say that $0 \times 0 = 0$? It is sure that 'you can not'. Because you do not have the valid authority to do so on this platform as the stranger operation $\times$ is unknown to the platform G = $<R, +>$ which is a group.

Take another similar type of example.
Consider the group G = $<R^+, \times>$ where R^+ is the set of all positive real numbers and $\times$ is the usual multiplication operation defined over R^+. Then you surely have the valid authority (license) to compute the expressions: 9×2, 1.85×3, 5.2×9.64, etc i.e. to compute an expression like $a \times b$ by virtue of the definition of the algebraic structure 'group' where a, b $\in$ R^+. But on this platform of the group G = $<R^+, \times>$ can you say that 7+3 = 10? can you say that 50.7+40.1 = 90.8? It is sure that 'you can not'. Because you do not have the valid authority to do so on this platform as the stranger operation + is unknown to the platform G = $<R^+, \times>$ which is a group.

However, using the atomic operation $\oplus$ of a group G = $<G, \oplus>$ one can define several new composite operations for which valid authority is there inbuilt before us for computing expressions involving these new composite operations. For example, define a new binary operation $[\![+]\!]$ in $<G, \oplus>$ such that $a [\![+]\!] b =$

$a \oplus b \oplus b$ where a, b $\in$ G. Clearly, $[\![+]\!]$ is a valid operation in G because in its definition there is no unknown or unauthorized operations involved. Such type of operation we call a 'Valid Composite Operation' in G.
But the new operation $[+]$ defined as : $a[+]b = a \oplus b \times b$ can not be called to be a 'Valid Composite Operation' in G = $<G, \oplus>$.

The new algebraic structure 'region' is not developed by incorporating new 'Valid Composite Operation(s)' in one of the existing algebraic structure viz. group, ring, modules, field, linear space, algebra over a field, associative algebra over a field, and even not in 'Division Algebra'.

Whenever any mathematician wants to compute an expression involving some basic operations, he must have valid authority over his dealing with those operations before doing so. And this authority he can possess only from a complete and sound algebraic structure upon which he is standing for doing his all mathematical exercises and computations.
For a better understanding consider few very simple statements and analysis made below with some element of imagination for the time being.
It is obvious that there are much more amount of flexible computations possible in the platform of a ring $(S, \oplus, \otimes)$ compared to the group $(S, \oplus)$. **Imagine** now carefully a situation consisting of the following activities:-
(i) the concept of group $(S, \oplus)$ is just discovered in Algebra; and
(ii) the concept of ring $(S, \oplus, \otimes)$ or any other algebraic structure is not yet discovered in the subject Algebra; and
(iii) mathematicians have been talking about groups only, because they do not have any other algebraic structure in existence in the literature of Algebra; and
(iv) mathematicians have been doing a huge volume of computations on the platform of the group $(S, \oplus)$ only, but fluently using the operation like $x \otimes y$ in their daily practices of mathematics inadvertently; and
(v) the people (of all subjects) and mathematicians are so happy because there is not happening anything contradictory, rather producing beautiful results and being in applications in several domains of academic spheres, and all the branches of Science have been growing in excellent ways. No file or document or norm is there to check the validity or authority in the practices of mathematicians, to ban the practices of the operations like $x \otimes y$, but nevertheless the mathematical problems of the world are being solved without any conflict, without any contradiction, without any error, and ultimately all the results catering to excellent benefits in human welfare!

Now mixing with the situation with car driving practices, few questions arise:- You may be driving well having your driving license valid only for Light Motor Vehicle (for the algebraic structure Group), but how can you do justice by driving Heavy Motor Vehicle (for the algebraic structure Ring) with the same driving license, without its corresponding higher driving license? Should you do so with the satisfaction and boldness that there is no accident occurring? Can you say that you have the driving license for Light Motor Vehicle which authorizes you the driving of Heavy Motor Vehicle too?

Consequently it is surely agreeable in the imaginary situation presented above that the exact algebraic structure (we know that it is ring) needs to be unearthed and more important matter is that this exact algebraic structure needs to be identified with an appropriately new title (here it is ring) giving a very precise mathematical definition of it independently, so that the mathematicians feel transparently justified while using the operation like $x \otimes y$ as many times as required. We can not ignore the necessity to study the new algebraic structure (here it is ring) as an independent topic, as an independently entitled one. We can not and should not ignore the necessity of providing a new identity title for it, just by saying that it is a **group $(S, \oplus)$ but along with one more operation $\otimes$ and few more axioms!.**

Let us imagine another hypothetical instance. It is known to us that there are much more amount of flexible computations possible in the platform of a field S compared to a ring $(S, \oplus, \otimes)$ or compared to a group $(S, \oplus)$. **Imagine** now the situation consisting of the following activities:-

(i) the concept of group $(S, \oplus)$ is discovered in the subject Algebra; and then

(ii) the concept of ring $(S, \oplus, \otimes)$ is also discovered in the subject Algebra; and

(iii) the concept of field S or other algebraic structure is not yet discovered; and

(iv) mathematicians have been talking about groups and rings only, because they do not have any other algebraic structure in existence in the literature of the subject Algebra.

(v) mathematicians have been doing huge volume of computations on the platform of ring $(S, \oplus, \otimes)$, but fluently using the operation like x/y in their daily practices of mathematics, in everyday applications in science and engineering areas, etc inadvertently; and

(vi) the people (of all subjects) and mathematicians are so happy because there is not happening anything contradictory, rather producing beautiful results and being in applications in several domains of academic spheres. No file or document or norm is there to check the validity or authority of the mathematicians, to ban the operation like x/y, but nevertheless the mathematical problems of the world are being solved without any conflict, without any contradiction, without any error, and ultimately all the results catering to excellent benefits in human welfare!

Now mixing with the situation with car driving practices, few questions arise:- You may be driving well having your driving license valid for Light Motor Vehicle (for the algebraic structure Ring), but how can you do justice by driving Heavy Motor Vehicle (for the algebraic structure Field) with the same driving license, without any higher driving license? Should you do so with the satisfaction and boldness that there is no accident occurring? Can you say that you have the driving license for Light Motor Vehicle which authorizes you the driving of Heavy Motor Vehicle too?

Consequently it is surely agreeable in the imaginary situation presented above that the exact algebraic structure (we know that it is field) needs to be unearthed and more important matter is that the exact algebraic structure needs to be identified with an appropriately new title (here it is field) giving a very precise mathematical definition of it independently, so that the mathematicians feel transparently justified while using the operation like x/y as many times as required. We can not ignore the necessity to study the new algebraic structure (here it is field) as an independent topic, as an independently entitled one. We can not and should not ignore the necessity of providing a new identity title for it, just by saying that it is a **ring $(S, \oplus, \otimes)$ along with few additional axioms fulfilled!.**

Let me now justify how the above act of imaginations will help the readers to get into the inside of the subject of 'Region'. The idea knocked my mind while I unearthed a peculiar fact that most of the simple and useful results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra, which are commonly practiced at secondary school level of mathematics and of course at all levels of higher mathematics, are not valid (can not be verified) in groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, and **even not in 'Division Algebras';** i.e. are not valid (can not be verified) in any of the existing standard algebraic structures alone in general, by virtue of their respective definitions and independently owned properties.

Consider one simple example. By a careful observation it can be seen that even a simple computation of 'cross-multiplication' like:

$$\text{if } \frac{2 \bullet x}{7 \bullet y} = \frac{5 \bullet z}{3 \bullet t}, \quad \text{then } 6 \bullet x * t = 35 \bullet y * z \text{ (and conversely)}$$

for x, y, z, t $\in$ A,

can 'not be verified' in general in a group A alone, or in a ring A alone, or in a module A, or in a field A, in a linear space A, in an 'algebra A over a field (i.e. F-algebra)', in an 'associative algebra A over a field', and even can 'not be verified' in a 'Division Algebra A' alone, or in any standard existing algebraic structure A alone, by virtue of their respective independent definitions and independently owned properties.

The same issue in simple literature can be stated that, the computation of 'cross-multiplication' practiced at secondary school level elementary algebra like:

$$\text{if } \frac{2x}{7y} = \frac{5z}{3t}, \quad \text{then } 6xt = 35yz \text{ (and conversely) for } x, y, z, t \in R,$$

can 'not be verified' in general in the group R alone, or in the ring R alone, or in the field R, and even can 'not be verified' in the 'Division Algebra' R alone, by virtue of their respective definitions and independently owned properties.

The reason is explained in details in Section-3 in this paper. There are infinite number of such very surprising but interesting cases can be pointed out, but may be surely difficult to believe today in this century. In this regard, an amount of needful justifications and explanations are presented in Example 3.1, in Example 3.2, and Example 3.3 in Section-3 here. **Although apparently it seems to be a simple issue, but the question is not about the 'correctness' of the results, equalities, identities, formulas, cross-multiplication rules, etc. of elementary algebra; the question is how and when they can be accepted to be correct, how they can be stamped as valid, authenticated?.**

Quite naturally a question arises that if many of the simple and useful results, equalities, identities, formulas etc. of elementary algebra are not valid in any of the existing algebraic structure alone (even not in Division Algebra) by virtue of their respective definitions and independently owned properties then 'why the mathematicians do not arrive sometimes at contradictions or at wrong results or at deadlock ends?' or why the physicists, cosmologists, statisticians, engineers, etc do not arrive sometimes at contradictions or at confusion results or at deadlock ends?. I became very much curious to scan the issue thoroughly. Initially I got confused, I got puzzled, I could not find out 'what is the reason, what is wrong or where is the deficit in the existing volume of theories in Abstract Algebra?'. I completely doubted only upon myself as my main area of research is not 'Algebra'. I then revisited my college-life favorite books of Herstein, Jacobson, Lang, Waerden, etc. from my bookshelves. In fact I tried a permutation/combination of the various existing algebraic structures to make out a possible identity of that platform algebraic structure in which all the daily-useful and daily practiced results, identities, formulas, laws, etc. of elementary algebra can be observed to be valid (i.e. can be verified). Finally I became fully confident that there is an excellent something lying hidden so far in the subject 'Abstract Algebra'. I unearthed the hidden beauty, which is a beautiful algebraic structure which I call by 'Region'. In this work in Section-3 the algebraic structure 'Region' is introduced and then various properties of it are studied, to open a new algebra called by 'Region Algebra'.

The enormous and very unique potential of Region Algebra lies upon the fact that this is the **minimal** algebra which, by virtue of the definition and independently owned properties of region, can give full license to the mathematicians to practice all the existing simple and frequently useful results, equalities, identities, formulas etc. of elementary algebra. In other words, region is the **minimal** algebra which can authenticate the elementary algebra. None of the existing important algebraic structures like group, ring, module, field, linear space, algebra over a field, associative algebra over a field, and Division Algebras has this ability.

The beautiful properties and results fulfilled by the set R of real numbers are being so far fluently used by the mathematicians assuming R to be a division algebra, but without knowing that they are actually using the 'region' properties of R, not the properties of division algebra only. The particular Division Algebra R **fortunately** qualifies to be a Region also. It satisfies few more additional axioms/conditions which are not covered by virtue of definition and independent-

ly owned properties of the algebraic structure ‘Division Algebra’.
This is justified in Section-3 in details that the properties of division algebra collectively is not sufficient to use most of the existing useful results of elementary algebra. Fortunately the particular division algebra R fulfils the additional conditions by default (not by virtue of the properties of division algebra) to become qualified to get the status of a region. But at the same sense, it is unfortunate too. Because the above stated fortunate properties of the division algebra R could not happen to unearth the failure of division algebra in practicing the computations of elementary level algebra, and consequently in practicing the mathematics as a totality of college as well as higher level. It may be noted that besides R there are other division algebras in existence; and the fortunate event mentioned above did not (does not) happen with other division algebras as none of the other division algebras is so lucky. Incidentally, the very particular division algebra R is now observed to happen to be a region. And that is the reason why the mathematicians did not face any problem so far while discovering and developing various topics of mathematics (viz. Theory of Numbers, Geometry, Calculus, etc.) on exploiting fluently the infinite number of interesting properties of R. Consequently the mathematicians missed to look at the actual hidden identity of the **minimum** algebraic platform upon which the results stand valid. It is explained in details in Section-3 that division algebra alone can not offer authority to the mathematicians to practice the existing simple and frequently useful results, equalities, identities, formulas etc. of elementary level algebra.
Summarily, it can be stated that in the progress of mathematics achieved so far in all of its branches, the set R of real numbers is being always pre-assumed to be a division algebra. This work justifies with several examples that this pre-assumption is not sufficient (i.e. does not give license) to the practitioners to use many simple results, formula, rules, identities etc. It is a Region Algebra R at minimum, not a Division Algebra R or any of the existing important algebraic structure. The work of “Region Algebra” may apparently seem to be too simple at the first reading, because of the fact that it is truly simple and of very fundamental nature. Because of its very simple initial nature, the readers may have to take patience to read the materials of this work till the last page, even if some of the theories/propositions happen to be unacceptable or debatable or redundant initially.

2. Recollecting the important Algebraic Structures

Since this paper introduces a new algebraic structure which is the most important algebraic structure in the sense of its application potential, a quick visit to the definitions of all the important existing algebraic structures of Abstract Algebra is made in this section which could serve as a ready reference before entering into the literature on Region presented in Section-3 here. However for further details about their properties and for in-depth literature on algebra, any good book like Jacobson [30-32], Hungerford [29], Herstein [28], Hardy et al. [27] etc. may be referred.

Binary Operation

Let S be a non-empty set. An operation * will be called a binary operation over the set S if the following two conditions are satisfied:

(1) if $a, b \in S$ then $a * b \in S$.

(2) $\forall\, a,b \in S$, $a * b$ is unique.

For example, usual addition '+' is a binary operation over the set R of real numbers. Subtraction (-) is a binary operation over the set R of real numbers. But subtraction (-) is not a binary operation over the set N of natural numbers. A non-empty set A equipped with at least one binary operation is an algebraic structure.

It is important information for the readers that throughout the discussion and analysis in this work, the following standard definitions for the important useful algebraic structures are followed.

2.1 Group

A non-empty set G together with a binary operation $*$ is called a Semi-group if the following property is satisfied : if $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.

Example:

The set N of natural numbers forms a Semi-group with respect to the usual binary operation '+'.

A non-empty set G together with a binary operation $*$ is called a Group if the following properties are satisfied:

(1) if $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.

(2) $\exists\, e \in G$ such that $a * e = a = e * a$, $\forall\, a \in G$.

(3) if $a \in G$, $\exists\, b \in G$ such that $a * b = e = b * a$.

We say that $(G, *)$ is a group denoted in short by the notation G simply.

The group $(G, *)$ is said to be an Abelian Group or Commutative Group if the following additional property is satisfied in G:

(4) $\forall\, a,b \in G$, $a * b = b * a$.

The zero element e in property no(2) above is called the identity of the group G. The identity element of a group is unique. The element b in property no(3) above is called the inverse of the element a and is denoted by the notation a^{-1}. The inverse of an element a in the group G is unique. Cancellation laws hold good in a group. The set I of integers, the set R of real numbers, the set Q of rational numbers with respect to the usual addition (+) are examples of group. The set of all $n \times n$ matrices over the set R of real numbers is a group with respect to usual 'matrix addition' operation. The set C of complex numbers is a group with respect to usual 'complex addition' operation.

2.2 Ring

After a strong realization of the importance of Group in mathematics, the possible conceptualization of a new algebraic structure began in the 1870s and was finally shaped in the 1920s in the name of Ring. The main contributors include Dedekind, Hilbert, Fraenkel, and Noether. Actually Rings were initially formalized as a generalization of the Dedekind domains that occur in number theory. Since the inception, rings are proved to be very useful in many other branches of mathematics viz, geometry, mathematical analysis, etc. In mathematics, ring is now one of the most fundamental algebraic structures used in abstract algebra. A group is an algebraic structure equipped with one binary operation only. A ring consists of a set equipped with two binary operations $\oplus$ and $*$ that generalize the arithmetic operations of addition (+) and multiplication ($\times$). Through this beautiful generalization, the theorems and results from arithmetic are extended to non-numerical objects such as polynomials, series, matrices and functions.

A non-empty set R together with two binary operations $\oplus$ and $*$, called 'addition' and 'multiplication' respectively, is called a ring if the following properties are satisfied :

Abelian Group :

(1) if a, b, c $\in$R then $(a\oplus b)\oplus c = a\oplus(b\oplus c)$.

(2) $\exists\ 0_R \in R$ such that $a\oplus 0_R = a = 0_R\oplus a,\ \forall a\in R$.

(3) if $a\in R$, $\exists\ b \in R$ such that $a\oplus b = 0_R = b\oplus a$.

(4) $\forall a,b \in R,\ a\oplus b = b\oplus a$.

Semi-group :

(5) if a, b, c $\in$R then $(a*b)*c = a*(b*c)$.

Distributive Properties :

(6) $\forall$ a, b, c $\in$R then

(i) $a*(b\oplus c) = (a*b)\oplus(a*c)$.

(ii) $(a\oplus b)*c = (a*c)\oplus(b*c)$.

We say that $(R,\oplus,*)$ is a ring denoted in short by the notation R simply. The zero element 0_R in property no(2) above is called the additive identity of the ring R. The additive identity element of a ring is unique. The element b in property no(3) above is called the additive inverse of the element a and is denoted by the notation $\sim a$. The additive inverse of an element a in a ring is unique.

The set I of integers, the set R of real numbers, the set Q of rational numbers with respect to the usual addition (+) and multiplication ($\times$) are examples of ring. The set of all $n\times n$ matrices over R is a ring with respect to usual 'matrix addition' and 'matrix multiplication' operations. The set C of complex numbers is a ring with respect to usual 'complex addition' and 'complex multiplication' operations.

In a ring $(R,\oplus,*)$ it can be verified that

(i) $0_R * a = 0_R = a * 0_R\ \forall a\in R$,

(ii) $\sim(a*b) = (\sim a)*b = a*(\sim * b)$

(iii) $(\sim a)*(\sim b) = a*b$

Commutative Ring

A ring $(R,\oplus,*)$ is said to be commutative if the semigroup $(R,*)$ is commutative, i.e. if $a*b = b*a\ \ \forall a,b \in R$.

Ring with unity
If the semigroup $(R, *)$ has an identity element then it is unique and denoted by the notation 1_R. Then 1_R is called the Unit element of the ring R, and the ring R is called ring with unity.

Zero Divisor
An element $a \in R$ is said to be a left zero divisor if there exists b ($\neq 0_R$) such that $a * b = 0_R$. An element $a \in R$ is said to be a right zero divisor if there exists b ($\neq 0_R$) such that $b * a = 0_R$. An element $a \in R$ is said to be a zero divisor if it is either a left zero divisor or a right zero divisor.

2.3 Field
A field is a non-empty set on which addition, subtraction, multiplication, and division are defined. And hence it is very supportive to the elementary operations practiced over the set R of real numbers, fluently done at school level to college and higher level of mathematics. A field is a fundamental algebraic structure, which is widely used in Algebra, Number Theory and many other areas of Mathematics, Science and Engineering.

A non-empty set F together with two binary operations $\oplus$ and $*$, called 'addition' and 'multiplication' respectively, is called a ring if the following properties are satisfied :

Abelian Group with respect to addition:
(1) if $a, b, c \in F$ then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(2) $\exists\ 0_F \in F$ such that $a \oplus 0_F = a = 0_F \oplus a$, $\forall\, a \in F$.
(3) if $a \in F$, $\exists\ b \in F$ such that $a \oplus b = 0_F = b \oplus a$.
(4) $\forall\, a,b \in F$, $a \oplus b = b \oplus a$.

Abelian Group (excluding the element 0_F) with respect to multiplication:
(5) if $a, b, c \in F$ then $(a * b) * c = a * (b * c)$.
(6) $\exists\ 1_F \in F$ such that $a * 1_F = a = 1_F * a$, $\forall\, a \in F$.
(7) if $a \in F$, $\exists\ b \in F$ such that $a * b = 1_F = b * a$.
(8) $\forall\, a,b \in F$, $a * b = b * a$.

Distributive Properties :
(9) $\forall$ $a, b, c \in F$ then
(i) $a * (b \oplus c) = (a * b) \oplus (a * c)$.
(ii) $(a \oplus b) * c = (a * c) \oplus (b * c)$.

We say that $(F, \oplus, *)$ is a field denoted in short by the notation F simply. The zero element 0_F in property no(2) above is called the additive identity of the field F. The additive identity element of a field is unique. The element b in property no(3) above is called the additive inverse of the element a and is denoted by the notation $\sim a$. The additive inverse of an element a in a field is unique. The unit element 1_F in property no(6) above is called the multiplicative identity of the field F. The multiplicative identity element of a field is unique. The element b in property no(7) above is called the multiplicative inverse of the element a and is denoted by the notation a^{-1}. The multiplicative inverse of an element a in a field is unique. The most useful fields in mathematics are the field (R, +, .) of real numbers, field Q of rational numbers, the field C of complex numbers, etc. Many other fields, such as field of rational functions, algebraic function field, algebraic number field, and p-adic field are studied in mathematics, in particular in studying Theory of Numbers, Algebraic Geometry, etc.

2.4 Linear Space

A linear space consists of a set of elements called points, and a set of elements called lines, with some axioms. Each line is a distinct subset of the points. The points in a line are said to be incident with the line. Any two lines may have no more than one point in common. A linear space is a basic structure in incidence geometry.

A non-empty set X is called a linear space over a field F with respect to two operations called by 'addition (+)' and 'scalar multiplication' if the following axioms are satisfied:

(X, +) is an abelian group :

(1) if $a, b, c \in X$ then $(a+b)+c = a+(b+c)$.

(2) $\exists\ 0_X \in X$ such that $a+0_F = a = 0_F+a$, $\forall\ a \in X$.

(3) if $a \in X$, then $\exists\ b \in X$ such that $a+b = 0_X = b+a$, $\forall\ a \in X$.

(4) $\forall\ a,b \in X$, $a+b = b+a$.

Scalar Multiplication :

(5) Scalar multiplication of $x \in X$ by elements $k \in F$, denoted by kx is to be in X,

(6) $k(ax) = (ka)x$, where $x \in X$, and $k, a \in F$.

(7) $k(x + y) = kx + ky$, $(k + a)x = kx + ax$, where $x, y \in X$, and $k, a \in F$.

Moreover $1\,x = x$ for all $x \in X$, 1 being the unit in F. It follows from the definition that $0x = 0$, $(-1)x = -x$.

A linear space X over the field F is also called a vector space. The elements of a vector space are called vectors, and the elements of the corresponding field F are called scalars. A simple example of vector space is the field F itself. In this space, the vector addition is just the same as field addition, and scalar multiplication is just field multiplication. Any non-zero element of F serves as a basis so that F is a one-dimensional vector space over itself.

Another useful example is F^n. For any positive integer n, the set of all n-tuples of elements of the field F forms an n-dimensional vector space over itself F is called coordinate space and is denoted by F^n. An element of F^n is written as $x = (x_1, x_2, x_3, \ldots\ldots, x_n)$ where each x_i is an element of the field F.

The operations on the vector space F^n are defined as below :

If $x = (x_1, x_2, x_3, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots\ldots, y_n)$ are two vectors in F^n, then

(1) $x + y = (x_1+y_1, x_2+y_2, x_3+y_3, \ldots, x_n+y_n)$,

(2) $kx = (kx_1, kx_2, kx_3, \ldots, kx_n)$ where k is a scalar,

(3) $0 = (0, 0, 0, \ldots., 0)$

(4) $-x = (-x_1, -x_2, -x_3, \ldots, -x_n)$.

As a particular instance, if F is considered to be the field R of real numbers then we obtain the n-dimensional real coordinate space R^n. And if F is the field C of complex numbers then we obtain the n-dimensional complex coordinate space C^n.

However, the general form of an element z = a+ ib of the set C of complex numbers shows that for n = 1 the set C itself is a two-dimensional real vector space with coordinates (a,b). Therefore C^n is a 2n-dimensional real vector space. The quaternions H is four-dimensional real vector space, and the octonions O is eight-dimensional real vector space.

2.5 Module

A module is one of the fundamental algebraic structures. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. In a vector space, the set of scalars is a field and acts on the vectors by scalar multiplication, subject to certain axioms such as the distributive law. In a module, the scalars need only be a ring, so the module concept represents a significant generalization. Clearly a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication.
Suppose that R is a ring and 1_R is its multiplicative identity. A left R-module M consists of an abelian group (M, +) and an operation $\cdot : R \times M \to M$ such that for all r, s in R and x, y in M, we have:
(1) r.(x+y) = r.x + r.y
(2) (r+s).x = r.x + s.x
(3) (rs).x = r.(s.x)
(4) 1_R.x = x
The operation of the ring on M is called scalar multiplication, and is usually written by the action of juxtaposition.
A right R-module M is defined similarly, except that the ring acts on the right; i.e., scalar multiplication takes the form $\cdot : M \times R \to M$, and the above axioms are written with scalars r and s on the right of x and y.
Suppose that R is a ring and 1_R is its multiplicative identity. A right R-module M consists of an abelian group (M, +) and an operation $\cdot : M \times R \to M$ such that for all r, s in R and x, y in M, we have:
(1) (x+y).r = x.r + y.r
(2) x.(r+s) = x.r + x.s
(3) x.(rs) = (x.r).s
(4) x.1_R = x
Any ring R is trivially an R-module over itself. If F is a field, then a vector space over F is a F-module.

2.6 F-Algebra and Associative Algebra

Let F be a field. Let A be a vector space over F equipped with an additional binary operation from A × A to A, denoted here by * (i.e. if x and y are any two elements of A, x*y is the product of x and y). Then A is called an 'algebra over the field F' if the following conditions are satisfied: $\forall$ a, b $\in$ F and $\forall$ x, y $\in$ A,

(1) Right distributivity: $(x + y)*z = x*z + y*z$
(2) Left distributivity: $x*(y + z) = x*y + x*z$
(3) Compatibility with scalars: $(a \bullet x)*(b \bullet y) = (a.b) \bullet (x*y)$.

An algebra over the field F is also called by F-algebra, where F is called the base field of the algebra A. The multiplication (product of two elements) of elements of an algebra A is not necessarily associative.

For example, the set C of complex numbers is an 'algebra over the field R'. Let us make an illustration of this particular example. It is known that a complex number z is represented as $z = a + ib$, where $a, b \in R$ and i is the imaginary unit. In other words, a complex number z is represented by the vector (a, b) over the field R of real numbers.

So the set C of complex numbers forms a two-dimensional real vector space, because

(1) addition : given by $(a, b) + (c, d) = (a + c, b + d)$,
(2) scalar multiplication : given by $k(a, b) = (ka, kb)$,
(3) Vector multiplication : $(a, b) \bullet (c, d) = (ac - bd, ad + bc)$.

If x, y, z are complex numbers and a, b are real numbers, then

(4) $(x + y) \bullet z = (x \bullet z) + (y \bullet z)$.

The complex multiplication is compatible with the scalar multiplication by the real numbers:

(5) $(ax) \bullet (by) = (a.b) (x \bullet y)$.

Example :

Ring of square matrices over a field F with the usual matrix multiplication is an example of F-algebra. Let E/F be an extension of fields of degree n. Then E is an F-algebra of dimension n.

Note that when a binary operation on a vector space is commutative, as in the example of the complex numbers, it is left distributive exactly when it is right distributive. But in general, for non-commutative operations (for example, in the quaternions H) they are not equivalent, and therefore require separate axioms. That means, the multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras and non-associative algebras

In the above notion of 'Algebra over a Field', if we replace the field of scalars by a commutative ring then we get a more general notion called by an 'Algebra over a Ring'. Algebras are not to be confused with vector spaces equipped with a bilinear form like inner product spaces; it is because of the reason that the result of a product $x*y$ is not a member of the space, but rather it is in the field of coefficients.

Associative Algebra over a field F

An associative algebra A over the field F is an algebraic structure with compatible operations of addition, multiplication (assumed to be associative), and a scalar multiplication by elements in the field F. Here the addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a vector space over the

field F. Thus an ‘Associative Algebra over a field F’ is a vector space over the field F which also allows the multiplication of vectors in a distributive and associative manner, having bilinearity of the multiplication. An associative algebra over the field F is also called by ‘Associative F-algebra’.
Given a positive integer n, the ring of real square matrices of order n is an example of an associative algebra over the field R of real numbers under usual operations of matrix addition and matrix multiplication as the matrix multiplication is associative.

Non-associative Algebra over a field F
Three-dimensional Euclidean space with multiplication given by the vector cross product is an example of a non-associative algebra over the field R of real numbers since the vector cross product is non-associative, satisfying the Jacobi identity instead.

2.7 Division Algebra

In Abstract Algebra, a ‘Division Algebra’ is an algebra in which division operation is permissible.
Explicitly, a ‘Division Algebra’ is a non-null set S together with two binary operators $\oplus$ and $*$ denoted by $(S, \oplus, *)$ satisfying the closure properties over both the operations and the following eight conditions :-

1. Additive associativity: $\forall$ a, b, c $\in$ S, $(a\oplus b)\oplus c = a\oplus(b\oplus c)$
2. Additive commutativity: $\forall$ a, b $\in$ S, $a\oplus b = b\oplus a$
3. Additive identity: $\exists$ an element $0_S \in$ S such that
$\forall a \in S, \quad 0_S\oplus a = a\oplus 0 = a$
4. Additive inverse: $\forall a \in S$, $\exists$ an element $\sim a \in S$ such that
$a\oplus(\sim a) = (\sim a)\oplus a = 0_S$
5. Multiplicative associativity: $\forall$ a, b, c $\in$ S, $(a*b)*c = a*(b*c)$
6. Multiplicative identity: $\exists$ an element $1_S \in$ S such that
$\forall a \in S, \quad 1_S * a = a * 1_S = a$
7. Multiplicative inverse: $\forall$ $a\ (\neq 0_S) \in S$, $\exists$ an element $a^{-1} \in S$ such that
$a * a^{-1} = 1_S = a^{-1} * a$
8. Distributivity:
Left distributivity : $\forall$ a, b, c $\in$ S, $a*(b\oplus c) = (a*b)\oplus(a*c)$
Right distributivity : $\forall$ a, b, c $\in$ S, $(b\oplus c)*a = (b*a)\oplus(c*a)$

The set R of real numbers, the set C of complex numbers, the Cayley algebra (Octonion algebra), the set H of quaternions, etc. are examples of division algebra with respect to addition and multiplication operation which are the usual operations defined over them respectively.
Let F be a field. The ring $M_n(F)$ of n×n matrices over the field F is called a matrix algebra) of dimension n^2 over F. Then $M_n(F)$ is neither commutative nor a division algebra for any natural number n > 1.
Thus a division algebra $(S,+,*)$ is a unit ring for which $(S-\{0_S\},*)$ is a group.

A division algebra must contain at least two elements 0 and 1. Every field therefore is also a division algebra, but not conversely.
A Division Ring is a ring A such that for all b and all nonzero a in A there is a unique solution x in A to the equation a*x = b and a unique solution y in A to the equation y*a = b.

Non-commutative Division Algebra
A division algebra is also called a division ring or a skew field, as it is a ring in which every nonzero element has a multiplicative inverse, but multiplication is not necessarily commutative. Even the compatible notion of multiplication may or may not be satisfied in a division algebra.

Non-associative Division Algebra
A Division Algebra may be associative or non-associative. Associative division algebras have no zero divisor. There are algebras which are neither commutative nor division algebra. A division algebra may be commutative or non-commutative. The following are few interesting examples.
The set R of real numbers is an example of associative division algebras. The Cayley algebra (Octonion algebra) is non-associative division algebra.
Consider the set C complex numbers with multiplication defined by taking the complex conjugate of the usual popular way of multiplication, i.e.

$$a * b = \overline{ab}.$$

This is a commutative, non-associative division algebra of dimension 2 over the field R of real numbers.
Let E/F be an extension of fields of degree n. Then E is an F-algebra of dimension n. Let E/F and K/F be field extensions of degrees n and m. Suppose that A = E$\oplus$K, the direct sum as both an F-vector space and as a ring, so addition and multiplication are component wise. Then A is an F-algebra of dimension m+n, which is commutative but not a division algebra.

In the Theory of Numbers, the quaternions H is a number system as an extension of the set C complex numbers. The general form of the quaternions is q = (a, b, c, d) = a + bi + cj + dk where $a, b, c, d \in$ R. In quaternions, the three objects i, j, k are called the fundamental quaternion units. The quaternions were introduced by Hamilton in 1843, and it was very successfully applied in studying 3-D mechanics in mathematics. Quaternions have been applied in both pure and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics, computer vision, and crystallographic texture analysis. In practical applications, they can be used alongside other methods, such as Euler angles and rotation matrices, or as an alternative to them, depending on the application.

The set R of real numbers may be viewed as a one-dimensional vector space with a compatible multiplication, and hence could be regarded as an one-dimensional algebra over itself. The set C of complex numbers form a two-dimensional vector space over the field R of real numbers, and hence could be regarded as a two dimensional algebra over R. Consequently, both R and C are examples of division algebra, because every non-zero vector possesses its own inverse. One is a division algebra in one dimension and the other is a division algebra in two dimension. There is no division algebra in three dimensions. But the quaternions H is a four-dimensional division algebra over R, where one can not only multiply vectors, but also can divide. Consider the two quaternions $q_1 = (0,1,0,0)$ and $q_2 = (0,0,1,0)$. It can be observed that $q_1*q_2 \neq q_2*q_1$ as $q_1*q_2 = (0,0,0,1)$ but $q_2*q_1 = (0,0,0,-1)$. Thus the quaternions H are an example of a non-commutative division algebra over R, unlike the set C of complex numbers.

Elementary algebra is taught in mathematics at secondary school level of education. It introduces the concept of variables representing numbers. Expressions based on these variables are manipulated using the rules of operations, formulas, identities, equalities etc that apply to numbers. Abstract Algebra is much broader than elementary algebra and studies what happens when different rules of operations are used and when operations, formulas, identities, equalities etc are devised for things other than numbers. The two important operations Addition and multiplication are generalized and their precise definitions lead to important algebraic structures like: group, ring, field, vector space, module, F-algebra, associative algebra, division algebra, etc. These algebraic structures say the users (mathematicians, scientists, researchers) which operations are valid and which are not. Consequently, an user apparently gets the license authority from that algebraic structure on which platform the user himself is standing upon to solve his computational problem under consideration. For example, if one user is working in a field then he will have the capability of performing more flexible computations, a lot of varieties of computations, compared to the case of his capability if working in a semi-group or group. For the sake of ready reference, a brief and quick visit to all the existing important algebraic structures is done in this section.

3. Region Algebra

Algebra is one of the most beautiful branches of mathematics and it is about finding the unknowns. Algebra is regarded as one of the broad parts of Mathematics, together with Number Theory, Geometry and Analysis. In its most general form, Algebra is the study of mathematical symbols and the rules for manipulating these symbols. Most important merit point of Algebra is that it is a unifying thread of almost all branches of Mathematics. Consequently, it may sometimes seem to be dry and sometimes very juicy. The basic parts of Algebra are called 'Elementary Algebra' and the abstract parts are called 'Abstract Algebra' or 'Modern Algebra'. Elementary algebra is generally considered to be essential for any study of Mathematics, Statistics, Science, Engineering, as well as applications of it in Computer Science, Social Science, Forensic Science, Information & Communication Technology, Medical Science, and Economics, to list a few only out of many. The Elementary Algebra is being studied from school level to higher levels. Abstract algebra is a major area in advanced mathematics, studied primarily by professional mathematicians. Elementary Algebra differs from Arithmetic in the use of abstractions, such as using letters and/or symbols to stand for numbers that are either unknown or allowed to take on many values (Boyer[17-19]). **The beauty of Algebra is that it provides valid methods, valid formulas, valid rules, identities, equations etc**., and solve equations that are much clearer and easier than the older method of writing everything out in words.

Before introducing the algebraic structure ‘region’, it is shown here by a number of examples that most of the simple and useful results, identities/equalities, formulas or algebraic expressions or equations (commonly practiced at secondary school level of mathematics) of elementary algebra are not valid (i.e. can not be computed/verified) in general in any of the existing standard algebraic structures alone **:** viz. in a group alone, or in a ring alone, or in a field alone , or in module, linear space, algebra over a field, in an associative algebra over a field, or in a division algebra alone, etc.
By the phrase : “the result is valid in the algebraic structure A”, we mean here that the result can be successfully computed and established/verified in the algebraic structure A by virtue of the definition and the independently owned properties of A alone (as mentioned in the previous section).

Consequently, it is unearthed that there is a major gap lying hidden so far in the existing literature on the subject “Abstract Algebra”. To fill-up this gap a new algebraic structure called by “Region” is introduced in this section independently in a unique way. The huge potential and strength of this powerful algebraic structure “Region” is lying in the fact that it is the **minimal** algebraic structure which can validate the simple results, equalities, identities, formulas etc. of elementary algebra which are commonly practiced at secondary school level of mathematics. None of the existing standard algebraic structure (including Division Algebra) possesses this capability except for limited cases. The issue happened to my mind by chance only, by luck, with no prior plan or thinking to proceed for developing a new algebraic structure. Region alone provides the minimal platform on which all elementary algebraic computations practiced by the students, teachers, mathematicians, scientists, engineers, etc. can be done, and in fact unknowingly being done so far. Such a complete and sound platform for practicing ‘elementary algebra’ can not be provided by any of the existing algebraic structures by virtue of its definition and independently owned properties (see Section-2). This important fact was hidden so far, and has been now unearthed here in this section.
With this philosophy, it can be realized that all the existing classical algebraic structures are weaker than the algebraic structure ‘region’ in terms of application potential and caliber. In other words, region is the most important algebraic structure in terms of its infinite volume of applications in mathematics, science

and engineering. An initial development of the theory of regions is done, establishing a number of important properties of the regions.

3.1 Genuine Need for a New Algebraic Structure

In this section let me justify the genuine need to define/identify a new algebraic structure and to introduce it uniquely in an independent way as a subject area in its own right.
A system consisting of a non–null set S and one or more n-ary operators on the set S is called an algebraic structure, denoted by the notation (S, O_1, O_2, …, O_r) where O_i, i = 1,2, …., r, are operators on S.
An algebraist can define an infinite number of new algebraic structures, if he desires. As already mentioned earlier that the objective of the work in this paper is not just for the sake of defining a new algebraic structure, but to fix a major gap unearthed in Abstract Algebra.

In this section we show by few examples that none of the existing algebraic structures viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra or any existing standard algebraic structure, by virtue of their respective definitions and independently owned properties (see Section-2), can not provide a sound and complete environment/platform or algebraic right to the mathematicians for performing many simple algebraic computations, for establishing many useful and simple identities or equalities of two algebraic expressions, and for establishing many useful algebraic results/solutions etc. of elementary algebra. But many of these results/equalities/identities are very much well known even to the secondary school students, and being practiced fluently by the students, teachers, academicians, engineers, scientists, etc.

Let me begin here with a collection of few cases or issues (out of infinite number of available cases) on the various standard algebraic structures **:** groups, rings, modules, fields, etc. These cases (five cases) are mentioned below for the sake of instance only, although they are no doubt very simple and obvious to any algebraist. But special attention of the readers is required on the situations presented in Case-5. And then I will justify the genuine needs for identifying a new kind of atomic, well complete, sound and unique algebraic structure in an independent way with its self-identity.

Few Cases (by examples) :

Case-1
If an expression like $x \oplus y/z$ is known to be a valid expression in an algebraic structure A (while let us suppose that nothing is known to us at this stage about the identity of the algebraic structure A) where x, y, z $\in$ A, then one can say that A is not just a group or a ring in general; however it could be a Division

Algebra or any algebra which is also a division algebra. Thus a group or a ring can not allow us to compute an expression like $x \oplus y/z$ because they are weak algebraic structures in the sense of such type of computations. But a division algebra is a stronger algebraic structure in terms of the power of issuing authorization to the mathematicians to compute an expression like $x \oplus y/z$.

Case-2

If an expression like $x \oplus 2.y \oplus 5.z$ is a valid expression in an algebraic structure A while nothing is known to us at this stage about the identity of the algebraic structure A where x, y, z $\in$ A (assuming that associativity property hold good in A over the operator $\oplus$), then one can say that A is not just a group or ring or a field, in general. However, it could be a linear space over the field R of real numbers, or something else which is also a linear space over the field R of real numbers. Thus a group or a ring or a field can not allow us to compute an expression like $x \oplus 2.y \oplus 5.z$ because they are weak algebraic structures in the sense of such type of computations.

Case-3

If an expression like $x+2.y \odot z$ is a valid expression in an algebraic structure A where x, y, z $\in$ A, then one can immediately say that A can not be just a group or a ring or a field or a linear space in general. However, it could be an 'Associative Algebra over a field', or something else. A group or a ring or a field or a linear space can not allow us to compute an expression like $x+2.y \odot z$ because they are weak algebraic structures in the sense of such type of computations.

Case-4

Suppose that the equality (identity) **I :** $(x+y)^2 = x^2 + 2.(x*y) + y^2$ is valid in an algebraic structure A. While guessing to identify the algebraic structure A, one could observe that **I :** $(x+y)^2 = x^2 + 2.(x*y) + y^2$ is an absurd equality (as it can not be verified) in a group or in a ring/module or in a field or in a linear space or in an associative algebra over a field, by virtue of their respective definitions and independently owned properties. However, this equality (identity) **I** can be well verified in some 'algebra over some field'. Here it may be noted that the LHS of this equality can be evaluated in a ring or in a field, but not the RHS (assuming that the notation t^2 stands for the expression $t*t$).

Now consider few interesting situations presented in Case-5 below :-

Case-5

Let me present few examples here out of infinite number of similar type of available examples.

Example 3.1

Consider a very simple instance from elementary algebra, a type which is very frequently used by the secondary school students, is the equality (identity) **I** of type given by

$$\left(\frac{a}{b}\right)\bullet\left(\frac{x}{y}\right) = \left(\frac{a \bullet x}{b \bullet y}\right),$$

but it is not valid i.e. can 'not be verified' in a group, ring, module, field, linear space, 'algebra over a field' (i.e. F-algebra), 'associative algebra over a field', or in a Division Algebra, or in any existing standard brand of algebraic structure alone by virtue of their respective definitions and independently owned properties (as mentioned in Section-2). See the justification presented below.

Justification

It is because of the reason that **:**

(1) since division operations are involved in both LHS and RHS expressions, it can not be a 'F-algebra' by virtue of its definition and independently owned properties.

(2) on the other hand, if it is not a 'F-algebra' but a division algebra **D**, then the following are fact by virtue of the definition and independently owned properties of division algebra:-

(i) the LHS expression $\left(\frac{a}{b}\right)\bullet\left(\frac{x}{y}\right)$ can be well written to be equal to the expression $\left(a.\frac{1}{b}\right) \bullet (x*y^{-1})$ in the division algebra **D**,

(ii) but the expression $\left(a.\frac{1}{b}\right) \bullet (x*y^{-1})$ **can not be written** in the division algebra **D**, by virtue of its definition and own properties, to be equal to the expression $(a \bullet x) * \left(\frac{1}{b} \bullet y^{-1}\right)$,

(iii) although, it is true that in the division algebra **D,** by virtue of its definition and owned properties,

$(a \bullet x) * \left(\frac{1}{b} \bullet y^{-1}\right)$ is equal to the expression $\left(\frac{a \bullet x}{b \bullet y}\right)$.

Consequently, in a division algebra **D,** the expression $\left(\frac{a}{b}\right)\bullet\left(\frac{x}{y}\right)$ can **not** be equal to $\left(\frac{a \bullet x}{b \bullet y}\right)$, by virtue of its definition and own properties.

Similarly, the equality (identity) **I** of type given by

$$\left(\frac{a}{b}\right)\left(\frac{x}{y}\right) = \frac{ax}{by}$$

is not valid i.e. can 'not be verified' in the Division Algebra R, by virtue of the definition and owned properties of Division Algebra (see Section-2).

Fortunately the set R of real numbers satisfies some additional interesting properties beyond the properties possessed by virtue of the definition and properties of the algebraic structure division algebra, and that is the hidden reason why the mathematics has not been facing any problem and have not been getting any incorrect results or contradictory results in computations.

It can be observed that in the simple elementary expression

$$\left(\frac{a}{b}\right)\bullet\left(\frac{x}{y}\right) = \left(\frac{a \bullet x}{b \bullet y}\right)$$

three multiplication operators '.', '•', and '*' are involved.

Example 3.2

With the same argument as in Example 3.1 above, it can be observed that if an equality (in fact it is an identity) **I** of type given by

$$\left((a \bullet x) \oplus \frac{1}{b \bullet y}\right)^2 = a^2 \bullet x^2 \oplus \frac{1}{b^2 \bullet y^2} \oplus \left(\frac{2a}{b}\right)\bullet\left(\frac{x}{y}\right)$$

is known to be a valid identity (i.e. can be computed and verified) in an algebraic structure A where x, y $\in$ A, a and b are members (scalars) of some field F, then it can be observed that **each** of the following statements are true (unless few additional properties are satisfied beyond their respective definitions and properties owned, or unless few additional properties/axioms are endowed with):

a) A is not a group, or a ring, module, field, linear space, etc by virtue of their respective definitions and independently owned properties.
b) A is not an 'algebra over a field F' (F-algebra) by virtue of its definition and independently owned properties.
c) A is not an 'associative algebra over a field' by virtue of its definition and independently owned properties.
d) A is not a 'Division Algebra' by virtue of its definition and independently owned properties.
e) A is not any standard existing brand of algebraic structure, by virtue of its definition and independently owned properties.

The earlier Section-2 contains a brief introduction of the existing important algebraic structures. However for further details about their properties, any good book like Jacobson (1985,1989), Hungerford(1974), Herstein (2001), Hardy et el (2001) etc. may be re-visited.

Example 3.3

In school algebra, students frequently use the result like :

$$\text{if } \frac{2x}{7y} = \frac{5z}{3t} \text{ then } 6\text{ x t} = 35\text{ y z} \quad \text{(and conversely).}$$

But, by a careful observation it can also be seen that this type of simple computation of 'cross-multiplication' **C** of secondary school level elementary algebra like :

if $\frac{a \bullet x}{b \bullet y} = \frac{c \bullet z}{d \bullet t}$ then $(a.d) \bullet \mathrm{x} * \mathrm{t} = (b.c) \bullet \mathrm{y} * \mathrm{z}$ (and conversely).

can 'not be verified' (i.e. not valid) in a group, ring, module, field, linear space, 'algebra over a field' (i.e. F-algebra), 'associative algebra over a field', or in a Division Algebra alone, or in any standard algebraic structure (assuming that division by zero element is not allowed) by virtue of their respective definitions and independently owned properties. The justification in brief is presented below :

Brief Justification:

The justification is in fact similar to what made in Example 3.1

(1) since division operations are involved in both LHS and RHS expressions, it can not be a 'F-algebra' by virtue of its definition and independently owned properties.

(2) on the other hand, if it is not a 'F-algebra' but a division algebra **D**, then the following are fact by virtue of the definition and independently owned properties of division algebra:-

Suppose that $\frac{a \bullet x}{b \bullet y} = \frac{c \bullet z}{d \bullet t}$ is valid in the division algebra **D.**

Then, $(a \bullet x)(b \bullet y)^{-1} = (c \bullet z)(d \bullet t)^{-1}$ is also valid in the division algebra **D.**

But after this step, the mathematics is blocked and can not proceed further for any next step. Because this can not yield the next step of the computation expected to be as below

$(a.d) \bullet \mathrm{x} * \mathrm{t} = (b.c) \bullet \mathrm{y} * \mathrm{z}$

in the division algebra **D** by virtue of its definition and its independently owned properties, <u>unless it possesses few additional properties/axioms satisfied</u>. But who will provide the division algebra **D** with few additional properties not being possessed by division algebra by its definition?.

Example 3.4

For another example, see that in school algebra the students frequently use the result like :

$$\left(\frac{3 \bullet x}{7 \bullet y}\right)^2 = \frac{9 \bullet x^2}{49 \bullet y^2}.$$

But, by a careful observation it can also be seen that this type of simple square identity **I** like :

$$\left(\frac{a \bullet x}{b \bullet y}\right)^2 = \frac{c \bullet x^2}{d \bullet y^2} \quad (\text{ where } c = a^2 \text{ and } d = b^2)$$

can 'not be verified' (i.e. not valid) in a group alone, or in a ring alone, or in a module, field, linear space, 'algebra over a field' (i.e. F-algebra), 'associative algebra over a field', on in a Division Algebra alone, or in any standard algebraic structure (assuming that division by zero element is not allowed) by virtue of their respective definitions and independently owned properties.

Then, the immediate questions that arose in my mind are :-

- "What algebraic structure is A for the above examples presented in Case-5 above?" Or
- "What could be the minimal algebraic structure in which the above type of identities like **I** or the above type of cross multiplication results like **C** are valid?". Or
- "What algebraic structure the above type of identities like **I** or the above type of cross multiplication results like **C** can be verified in?".

An algebraist can not answer that it is a group, or it is a ring, module, field, linear space, 'algebra over a field (i.e. F-algebra)', 'associative algebra over a field', or it is a Division Algebra, or any standard algebraic structure (assuming that division by zero element is not allowed). For a possible answer, the algebraist has to think of a permutation/combination of the various existing algebraic structures to make out a possible identity of A. But, he might seek to make a unique identity for this algebraic structure A to define it in an independent and atomic way, and then to study the various properties of A, various results valid on A, highlighting its unique importance/role in Mathematics by virtue of its definition and independently owned properties compared to all other standard algebraic structures. It is because of the reason that this algebraic structure A is supposed to be the most appropriate and **minimal** platform for practicing the problems from elementary algebra of secondary school level to higher algebra, compared to any other existing standard algebraic structure, in general. Thus the role of this new algebraic structure to the mathematicians is much more than any other of the existing algebraic structures.

Consequently there a genuine need to identify this minimal algebraic structure, which is hidden so far, unrecognized so far, for practicing the elementary algebra and higher algebra.

3.2 Introducing a new algebraic structure 'Region'

In the previous section it has been justified in length that there is a genuine need to introduce a new algebraic structure having unique independent self-identity in order to provide the **minimal** but sufficient platform based upon which the elementary algebra or higher algebra can be fluently practiced by the mathematicians with valid algebraic right and driving license (i.e. computing license). We call this new algebraic structure by "Region", as defined below.

3.2.1 Region

Consider a non-null set A with three binary operators $\oplus$, * and $\bullet$ defined over it such that for a given field (F, +, **.**), the following three conditions are satisfied:-

(i) $(A, \oplus, *)$ forms a field,

(ii) $(A, \oplus, \bullet)$ forms a linear space over the field (F, +, **.**), and

(iii) A satisfies the property of "Compatibility with the scalars of the field F",

i.e. $(a \bullet x) * (b \bullet y) = (a.b) \bullet (x*y) \quad \forall\, a, b \in F$ and $\forall\, x, y \in A$.

Then the algebraic structure $(A, \oplus, *, \bullet)$ is called a **Region** over the field (F, +, **.**).

If there is no confusion, we may simply use the notation A to represent the region $(A, \oplus, *, \bullet)$, for brevity.

We now study the various interesting properties of a region A.

Two Fields : Inner Field and Outer Field (Base Field)

See that there are two fields used in the definition of region. The field $(A, \oplus, *)$ is called the "inner field" of the region $(A, \oplus, *, \bullet)$; and the field (F, +, .) of the linear space $(A, \oplus, \bullet)$ is called the "outer field" or the "base field" of the region $(A, \oplus, *, \bullet)$.

3.2.2 Three Multiplication Operators in a Region

See that there are three multiplication operators used in the definition of region. Let us call them by First Multiplication Operator, Second Multiplication Operator, and Third Multiplication Operator (or Base Multiplication Operator).

The sequence of the three operators "$\oplus$", " * ", and "$\bullet$" appearing in the notation $(A, \oplus, *, \bullet)$ representing the region A is important in the sense that the operator " * " of the region A which is the multiplication operator of the inner field $(A, \oplus, *)$ is called the "first multiplication" operator of the region A.

The operator "$\bullet$" of the region A which is the multiplication operator of the linear space $(A, \oplus, \bullet)$ is called the "second multiplication" operator of the region A.

The multiplication operator "**.**" of the base field F is called the "third multiplication" operator or the "base multiplication" operator of the region A.

Out of these three multiplication operators, the first two are defined over the set A itself, and the third is defined over the base field F.

3.2.3 Three Addition Operators in a Region

See that there are three addition operators used in the definition of region. Let us call them by First Addition Operator, Second Addition Operator and Third Addition Operator.

The operator "$\oplus$" of the region $(A, \oplus, *, \bullet)$ which is the addition operator of the inner field $(A, \oplus, *)$ is called the "first addition" operator of the region A.

The operator "$\oplus$" of the region $(A, \oplus, *, \bullet)$ which is the addition operator of the linear space $(A, \oplus, \bullet)$ is called the "second addition" operator of the region A.

The operator "+" which is the addition operator of the base field (F, +, **.**) is called the "third addition" operator or the "base addition" operator of the region A.

It may be noted that the First Addition Operator and the Second Addition Operator are same. Consequently there are in general at most two addition operators in the definition of a region which could be distinct.

Thus in a region A, we deal with two distinct addition operators and three distinct multiplication operators, in general. It is obvious from the definition that a region A must have at least two elements. It may also be noted that every region is an 'algebraic structure over a field', but the converse is not true in general.

As a simple instance, it could be now seen that an equality (identity) **I** of type given by

$$\left(\frac{2}{3}\right) \bullet \left(\frac{x}{y}\right) = \left(\frac{2 \bullet x}{3 \bullet y}\right),$$

which can not be verified, <u>in general</u>, in a group or in a ring or in a field or in a linear space or in an associative algebra over a field, or in a division algebra or even not in a simple 'algebra over a field', now can be well verified or established in the algebraic structure 'region' alone.

As another simple instance, it could be now seen that an identity like

$$\left((a \bullet x) \oplus \frac{1}{b \bullet y}\right)^2 = a^2 \bullet x^2 \oplus \frac{1}{b^2 \bullet y^2} \oplus \left(\frac{2a}{b}\right) \bullet \left(\frac{x}{y}\right)$$

which can not be verified in any of the existing standard algebraic structures, now can be well verified or established in the algebraic structure 'region'.

And it can be also be seen now that even a simple computation of 'cross-multiplication' **C** of secondary school level elementary algebra like :

$$\text{if } \frac{2 \bullet x}{7 \bullet y} = \frac{5 \bullet z}{3 \bullet t} \text{ then } 6 \bullet x * t = 35 \bullet y * z \quad \text{(and conversely).}$$

or, a very simple square identity **I** like :

$$\left(\frac{3 \bullet x}{7 \bullet y}\right)^2 = \frac{9 \bullet x^2}{49 \bullet y^2}$$

which can not be verified in any of the existing standard algebraic structures, now can be well verified or established in the algebraic structure ‘region’.
In the region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), one of its component algebraic structures is the algebraic structure $(A, \oplus, *)$ which is a field called by inner field of the region. Thus we see that the region A is a commutative and also a division algebra, but a division algebra is not a region in general. Besides that the other component algebraic structure $(A, \oplus, \bullet)$ of the region A is a linear space over the field F. Considering the distributive properties of the field $(A, \oplus, *)$ along with the condition(iii) mentioned in the definition of region above, it is observed that the region A is also a F-algebra. But a F-algebra is not a region in general. Integrating these three facts, we can see that a region is a composition of commutative property, Division Algebra and F-algebra. Consequently a region can be equivalently regarded as a “commutative division F-algebra”. An algebra satisfying only the property of ‘commutative’ is not sufficient to define the algebra we have been in quest here, satisfying only the properties of ‘division algebra’ is not sufficient to define the algebra we have been in quest here, satisfying only the properties of ‘F-algebra’ is not sufficient to define the algebra we have been in quest here. At minimum it must be a region.

Clearly a region is not a division algebra only, but a lot of things more. However, one could try to view a region by permutation/combination of some of the existing classical algebraic structures in other ways too.

Example 3.4
The region RR : the most useful region in Science, Engineering & Other areas.

Let R be the set of real numbers, ‘+’ be the ordinary addition operator in R and ‘.’ be the ordinary multiplication operator in R. Consider the field (R,+, .) of real numbers, and the linear space (R,+, .) over the field (R,+, .) . Then the algebraic system (R,+, . , .) forms a region over the outer field (R,+ , .).

This region (R,+ , . , .) plays a very important role in our daily life computations, in particular in school level elementary algebra. The content of the syllabus and corresponding instructions at school level algebra is based on the platform of this region (R,+, . , .), not on the platform of any standard algebraic structure like groups, rings, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra or any existing algebraic structure. Let us name this region (R,+, . , .) in short by the word “RR”. The region RR is the most useful region in all the branches of Mathematics, Statistics, Science, Engineering & other areas.
The interesting properties of the region RR are that:

(i) its inner field is (R,+, .),
(ii) its outer field is (R,+, .),
(iii) all the three multiplication operators are same, and
(iv) all the three addition operators are same.

One of the most beautiful qualifications, rich merits and strengths of the Regions is that all the following three important associative properties collectively is not true in a division algebra alone by virtue of its definition and independently owned properties or in any standard algebraic structure alone, but they are well valid in any region. Interestingly they are fluently and freely being used by the mathematicians, scientists and engineers in their daily mathematical works and computations.

3.2.4 Three Associativity Properties

The following three associative properties hold good in a region $(A, \oplus, *, \bullet)$ over the field (F,+, .). They are called "No-Scalar Associative Property", "One-Scalar Associative Property" and "Two-Scalars Associative Property" respectively.

(i) $x * (y * z) = (x * y) * z$: (No-Scalar Associative Property)
(ii) $a \bullet (x * y) = (a \bullet x) * y$: (One-Scalar Associative Property)
(iii) $(a.b) \bullet x = a \bullet (b \bullet x)$: (Two-Scalars Associative Property)
where $a, b \in F$ and $x, y, z \in A$.

Proof :
(i) This follows by inheritance from the properties of the inner field $(A, \oplus, *)$.
(ii) Consider the property of "Compatibility with the scalars of field F" in the region $(A, \oplus, *, \bullet)$ given by : $(a \bullet x)*(b \bullet y) = (a.b) \bullet (x*y)$.
Substituting 1_F for b in the above, we get the result of 'One-Scalar Associative Property', where 1_F is the unit element of the outer field (F, +, .).
(iii) It follows by inheritance from the properties of the linear space $(A, \oplus, \bullet)$.

NOTE 3.1 About Few Conventions
Throughout this section, the following conventions are to be assumed in the context of region $(A, \oplus, *, \bullet)$ over the field (F, +, .), without any confusion :-

(i) By the expression $x \oplus y \oplus z$, we shall mean either side of the equality:
$x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(ii) By the expression $x * y * z$, we shall mean either side of the equality:
$x * (y * z) = (x * y) * z$;
(iii) By the expression $a \bullet x \oplus y$, we shall mean $(a \bullet x) \oplus y$, not $a \bullet (x \oplus y)$;
(iv) By the expression $a \bullet x * y$, we shall mean either side of the equality:
$a \bullet (x * y) = (a \bullet x) * y$
(v) By the expression $a \bullet x * b \bullet y$, we shall mean the expression
$(a \bullet x) * (b \bullet y)$.
(vi) By the expression $a \bullet x \oplus b \bullet y$, we shall mean the expression

$(a \bullet x) \oplus (b \bullet y)$.

(vii) By the expression $a.b \bullet x$, we shall mean the expression $(a.b) \bullet x$.

In the next subsection, we define few terminologies on a region A.

3.2.5 Additive Identity

The additive identity element of the inner field $(A, \oplus, *)$ of a region $(A, \oplus, *, \bullet)$ is called the 'additive identity' element of the region A, and is denoted by the notation 0_A.
Obviously, the 'additive identity' element of a region A is unique, by virtue of inheritance from the properties of the inner field $(A, \oplus, *)$. The additive identity of a region A is also called the 'zero element' of the region A.
It is obvious that the zero-element of the linear space $(A, \oplus, \bullet)$ and the zero element of the region A are the same element.

3.2.6 Multiplicative Identity

The multiplicative identity element of the inner field $(A, \oplus, *)$ of a region $(A, \oplus, *, \bullet)$ is called the 'multiplicative identity' element of the region A, and is denoted by the notation 1_A.
Obviously, 'multiplicative identity' element of a region A is unique, by virtue of inheritance from the properties of the inner field $(A, \oplus, *)$. The multiplicative identity of a region A is also called the 'unit element' of the region A.

3.2.7 Additive Inverse of an element

For an element x of a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, the 'additive inverse' of x is defined to be that element of the region A which is the additive inverse of x in the inner field $(A, \oplus, *)$, and is denoted by the notation $\sim x$.
Obviously, 'additive inverse' of an element of a region is unique, by virtue of inheritance from the properties of the inner field $(A, \oplus, *)$.

3.2.8 Multiplicative Inverse of an element

For a non-zero element x of a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, the 'multiplicative inverse' of x is defined to be that element of the region A which is the multiplicative inverse of x in the inner field $(A, \oplus, *)$, and is denoted by the notation x^{-1}.

Obviously, 'multiplicative inverse' of a non-zero element of a region is unique, by virtue of inheritance from the properties of the inner field $(A, \oplus, *)$. It may be observed that "multiplicative inverse" x^{-1} of an element x of a region A is with respect to the first multiplication operator of the region A. There is no multiplicative inverse of an element x of the region A with respect to the second multiplication operator '$\bullet$' and with respect to the third multiplication operator '.'.

The following proposition on a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**) is obvious, being inherited from its inner field $(A, \oplus, *)$ for the results (i) and (ii), and being inherited from the linear space $(A, \oplus, \bullet)$ for the results (iii) and (iv).

Proposition 3.1
In a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), for $a, b \in F$ and for $x, y \in$ region A,
(i) if $x = y$, then $x \oplus z = y \oplus z$ $\forall z \in$ region A.
(ii) if $x = y$, then $x * z = y * z$ $\forall z \in$ region A.
(iii) if $x = y$, then $a \bullet x = a \bullet y$ $\forall a \in F$.
(iv) if $a = b$, then $a \bullet z = b \bullet z$ $\forall z \in$ region A.

The following result is true in region algebra.

Proposition 3.2
In a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), for $a \in F$ and for $x\ (\neq 0_A)$, $y \in$ region A, if $y*x = a \bullet x$ then $y = a \bullet 1_A$.

Proof : We have $y*x = a \bullet x$

$$\therefore \quad (y * x) * x^{-1} = (a \bullet x) * x^{-1}$$

Applying 'No-scalar Associative Property' on LHS and 'One-scalar Associative property' on RHS, we get

$$y * (x * x^{-1}) = a \bullet (x * x^{-1})$$

$$\therefore \quad y = a \bullet 1_A. \quad \text{Hence proved.}$$

We now define Division operation in a region. It can be seen that there are four types of distinct division operations in a region, unlike in a field or in a division algebra.

3.2.9 Four types of 'Division' in a Region

Let $(A, \oplus, *, \bullet)$ be a region over the field (F, +, **.**). We know that there are three types of multiplication operations in A. Consequently there are supposed to be three division operations in A. But we see that there are four types of division can be performed in this algebraic structure which are mentioned below. For all these four types of division, we use a common notation/style like

$\dfrac{numerator}{deno\min ator}$, (assuming that there is no confusion).

Division Type-(1) :
Division of an 'element of the region A' by another 'element of the region A'

$\forall$ x, y ($\neq 0_A$) $\in$ region A, the division of the element x by the non-zero element y is denoted by the notation $\frac{x}{y}$, and is defined by

$$\frac{x}{y} = x * y^{-1} \quad \text{or} \quad y^{-1} * x \quad \text{(as they are commutative).}$$

This type of division is called by Type-(1) division.

Replacing x by 1_A and y by x in the above, we get the result

$$\frac{1_A}{x} = x^{-1} \quad (\text{where } x \neq 0_A).$$

Division Type-(2) :
Division of 'an element of the region A' by 'an element of the outer field F'

$\forall$ x $\in$A and $\forall$ a ($\neq 0_F$) $\in$F, the division of the region element x by the field element a is denoted by $\frac{x}{a}$, and is defined by $\frac{x}{a} = a^{-1} \bullet x$.

(It may be noted that an expression like $x \bullet a^{-1}$ is not valid here in general, except for some particular regions).
This type of division is called by Type-(2) division.

Replacing a by 1_F, we get the result $\frac{x}{1_F} = x$.

Division Type-(3) :
Division of 'an element of the outer field F' by 'an element of the region A'

$\forall$ a $\in$F and $\forall$ x ($\neq 0_A$) $\in$ A, the division of the field element a by the region element x is denoted by $\frac{a}{x}$, and is defined by $\frac{a}{x} = a \bullet x^{-1}$.
(It may be noted that an expression like $x^{-1} \bullet a$ is not valid here). This type of division is called by Type-(3) division.
Replacing a by 1_F, we get the result $\frac{1_F}{x} = x^{-1}$.

NOTE 3.2
From the two equalities $x^{-1} = \frac{1_A}{x}$ and $\frac{1_F}{x} = x^{-1}$ (where x $\neq 0_A$), we get the result

$\frac{1_A}{x} = \frac{1_F}{x}$.

i.e. $1_A * x^{-1} = 1_F \bullet x^{-1}$

But using Proposition 3.2 we observe that a cancellation-law is not applicable to this result here, and consequently the equality $1_A = 1_F$ does not emerge to be necessarily true. According to Proposition 3.2 we get the result $1_A = 1_F \bullet 1_A$ which is correct. The equality $1_A = 1_F$ is not correct, it is absurd.

Division Type-(4) :

Division of 'an element of the outer field F' by another 'element of the outer field F'.

This is in fact by virtue of the property owned by any field. In the outer field (F,+, **.**) of the region (A, $\oplus$, *, $\bullet$), it is known by field theory that $\forall$ a, b ($\neq 0_F$) $\in$ F, the division of the element a by the non-zero element b is denoted by the notation $\frac{a}{b}$, and is defined by $\frac{a}{b}$ = a**.** b^{-1} or b^{-1}**.**a, (they are commutative).

This type of division is called by Type-(4) division.

NOTE 3.3 Invalid Cancellation : two cases

In Proposition 3.2 we have seen that the equality $a \bullet x = y * x$ does not allow any kind of right-cancellation in the region A in general.

Therefore, the equality $\frac{x}{a} = \frac{x}{y}$ does not allow any kind of cancellation in the region A; and also the equality $\frac{a}{x} = \frac{y}{x}$ does not allow any kind of cancellation in the region A (in general, except for some particular regions).

3.2.10 Characterizations

In this section an initial characterization of region is made. It is earlier shown that the most useful results collectively of elementary algebra are valid in region, but not in any algebraic system by virtue of their respective definitions and independently owned properties. Region is the minimal platform which can provide validity, it is neither division algebra nor any existing algebra alone. May be few results of elementary algebra are true in division algebra, few are true in a F-algebra, few may be true in an Associative Algebra over a field F, **but all the results collectively are valid in the unique algebra 'region'.** That is the reason why region is unearthed and introduced with an independent and unique identity, and thus region is a subject area in its own right.

Results 3.1

In a region (A, $\oplus$, *, $\bullet$) over the field (F,+, **.**), the following results are straightforward $\forall$ x, y $\in$ A and $\forall$ a $\in$ F (keeping in mind that division by 0_A

or by 0_F is not permissible) :-

(i) $x^n * x^r = x^{n+r}$

(ii) $x^n * x^{-r} = x^{n-r}$

(iii) $x^{-n} * x^{-r} = x^{-n-r} = x^{-(n+r)}$

(iv) $(x * y)^n = x^n * y^n$

(v) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

(vi) $\left(\frac{x}{a}\right)^n = \frac{x^n}{a^n}$

(vii) $\left(\frac{a}{x}\right)^n = \frac{a^n}{x^n}$

where n and r are non-negative integers.

Results 3.2

The following results are true in a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), being inherited from the definitions and properties of the inner field $(A, \oplus, *)$ and the outer field (F, +, **.**), and also from the linear Space $(A, \oplus, \bullet)$, and are listed below for the sake of one perusal just : -

$\forall x \in A$ and $\forall a \in F$,

(1) $0_F \bullet x = 0_A$

(2) $a \bullet 0_A = 0_A$

(3) $0_F \bullet 0_A = 0_A$

(4) $0_F \bullet 0_A \neq 0_F$

(5) $1_F \bullet x = x$

(6) $1_A * x = x$

(7) $1_F \bullet 1_A \neq 1_F$

(8) $1_F \bullet 1_A = 1_A$

(9) $a \bullet 1_A \neq a$

(10) $1_F \bullet 0_A \neq 1_F$

(11) $1_F \bullet 0_A = 0_A$

(12) $0_A * x = 0_A$

(13) $0_A * x \neq 0_F$

(14) $1_A \sim 1_A = 0_A$

(15) $1_F - 1_F = 0_F$

(16) $\frac{x}{0_A}$ and $\frac{x}{0_F}$ are meaningless.

(17) $\frac{a}{0_A}$ and $\frac{a}{0_F}$ are meaningless.

Proposition 3.3

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), $\forall x, y \in A$ and $\forall a \in F$

(keeping in mind that division by 0_A or by 0_F is not permissible) the following results are true :-

(i) $\forall x \in A,\ \sim(\sim x) = x$

(ii) $\forall x\ (\neq 0_A) \in A,\ (x^{-1})^{-1} = x$

(iii) $\sim(x \oplus y) = (\sim x) \oplus (\sim y)$

(iv) $\sim(x^{-1}) = (\sim x)^{-1}$

(v) $(x * y)^{-1} = x^{-1} * y^{-1}$

(vi) $\sim(x * y) = (\sim x) * y = x * (\sim y)$

(vii) $\sim \dfrac{x}{y} = \dfrac{\sim x}{y} = \dfrac{x}{\sim y}$

(viii) $\sim \dfrac{x}{a} = \dfrac{\sim x}{a} = \dfrac{x}{-a}$

(ix) $\sim \dfrac{a}{x} = \dfrac{-a}{x} = \dfrac{a}{\sim x}$

(x) $\left(\dfrac{x}{y}\right)^{-1} = \left(\dfrac{x^{-1}}{y^{-1}}\right) = \dfrac{y}{x}$

(xi) $\left(\dfrac{x}{a}\right)^{-1} = \left(\dfrac{x^{-1}}{a^{-1}}\right) = \dfrac{a}{x}$

(xii) $\left(\dfrac{a}{x}\right)^{-1} = \left(\dfrac{a^{-1}}{x^{-1}}\right) = \dfrac{x}{a}$

(xiii) $x * (a \bullet y) = a \bullet (x * y)$

Proof : All the results are straightforward.

Proposition 3.4

In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, $\forall x \in A$,

$$\sim x = (-1_F) \bullet x = (\sim 1_A) * x$$

Proof : We know $0_A = 0_F \bullet x$

or, $0_A = (1_F + (-1_F)) \bullet x$

or, $0_A = 1_F \bullet x \oplus (-1_F) \bullet x$

or, $0_A = x \oplus (-1_F) \bullet x$

or, $(\sim x) \oplus 0_A = (\sim x) \oplus (x \oplus (-1_F) \bullet x)$

or, $\sim x = (-1_F) \bullet x$ which is the result.

Again, we have

$0_A = 0_A * x$

or, $0_A = (1_A \oplus (\sim 1_A)) * x$

or, $0_A = 1_A * x \oplus (\sim 1_A) * x$

or, $0_A = x \oplus (\sim 1_A) * x$

or, $\sim x \oplus 0_A = \sim x \oplus (x \oplus (\sim 1_A) * x)$

or, $\sim x = (\sim 1_A) * x$, which is the result.

Proposition 3.5

In an infinite region A = $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$ where the characteristic of A is zero, if $a \bullet x = 0_A$ then either $a = 0_F$ or $x = 0_A$, where $x \in A$ and $a \in F$.

Proof : We have $a \bullet x = 0_A$.

If $x \neq 0_A$, then

or, $(a \bullet x) * x^{-1} = 0_A * x^{-1}$

or, $a \bullet 1_A = 0_A$

$\Rightarrow a = 0_F$

Otherwise, if $a \neq 0_F$, then

$$a^{-1} \bullet (a \bullet x) = a^{-1} \bullet 0_A$$

or, $(a^{-1} . a) \bullet x = 0_A$

or, $1_F \bullet x = 0_A$

or, $x = 0_A$ Hence the result.

In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, we know that $\forall a \in F$ and $\forall x \in A$, the element $(a \bullet x)$ is in A. Therefore, $(a \bullet x)$ possesses its additive inverse in the region A. Also, if it is not the zero-element of the region A then it possesses its multiplicative inverse too in the region A.

The following proposition defines the additive inverse of the element $(a \bullet x)$ in the region A.

Proposition 3.6

In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, $\forall x \in A$ and $\forall a \in F$,

$$\sim (a \bullet x) = (-a) \bullet x = a \bullet (\sim x)$$

Proof : $(a \bullet x) \oplus ((-a) \bullet x)$

$= (a + (-a)) \bullet x$

$= 0_F \bullet x$

$= 0_A$

$\therefore \sim (a \bullet x) = (-a) \bullet x$

In a similar way, we can also prove that $\sim (a \bullet x) = a \bullet (\sim x)$.

The following proposition defines the multiplicative inverse of the element $(a \bullet x)$ in the region A. This important result is not valid in a division algebra, in general.

Proposition 3.7

In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, $\forall x (\neq 0_A) \in A$ and $\forall a (\neq 0_F) \in F$, $(a \bullet x)^{-1} = a^{-1} \bullet x^{-1}$.

Proof : $(a \bullet x) * (a^{-1} \bullet x^{-1})$

$= (a . a^{-1}) \bullet (x * x^{-1})$, using compatibility property of region A.

$= 1_F \bullet 1_A$

$= 1_A$

Therefore, $(a \bullet x)^{-1} = a^{-1} \bullet x^{-1}$.

The following important result is also not valid in a division algebra, in general.

Proposition 3.8

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, .), for x, y $\in$ A and for a, b $\in$F, if b and y are not zero elements then

$$\frac{a}{b} \bullet \frac{x}{y} = \frac{a \bullet x}{b \bullet y}$$

Proof :
$$\frac{a}{b} \bullet \frac{x}{y} = (a \,.\, b^{-1}) \bullet (x * y^{-1})$$
$$= (a \bullet x) * (b^{-1} \bullet y^{-1})$$
$$= (a \bullet x) * (b \bullet y)^{-1}$$
$$= \frac{a \bullet x}{b \bullet y}$$

The results of the following proposition are straightforward in a region.

Proposition 3.9

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, .), $\forall$ x, y $\in$A and $\forall$ a, b $\in$ F (keeping in mind that a zero element have its inverse),

(i) $(a \bullet (x * y))^{-1} = a^{-1} \bullet x^{-1} * y^{-1}$

(ii) $\sim (a \bullet (x * y)) = (-a) \bullet (x * y)$
$= a \bullet ((\sim x) * y)$
$= a \bullet (x * (\sim y))$

(iii) $(a \bullet x * b \bullet y)^{-1} = (a^{-1} \bullet x^{-1}) * (b^{-1} \bullet y^{-1})$

(iv) $\sim (a \bullet x * b \bullet y) = (\sim (a \bullet x)) * (b \bullet y) = (a \bullet x) * (\sim (b \bullet y))$

(v) $((a.b) \bullet x)^{-1} = (a^{-1} . b^{-1}) \bullet x^{-1}$

(vi) $\sim ((a.b) \bullet x) = ((-a) . b) \bullet x = (a . (-b)) \bullet x = (a.b) \bullet (\sim x)$

Many of the frequently practiced cancellation laws are not valid in a Division Algebra or in any existing standard algebraic structure alone, by virtue of their respective definitions and independently owned properties.
For example, the result

$$\frac{a \bullet x}{a \bullet y} = \frac{x}{y}$$

is not valid in a Division Algebra or in any existing standard algebraic structure alone, by virtue of their respective definitions and independently owned properties. But this is well valid in a region.

The following cancellation laws are valid in a region and thus ensures the superiority of region over any existing algebraic structure as a single brand.

Proposition 3.10 Cancellation Laws

Let $(A, \oplus, *, \bullet)$ be a region over the field $(F, +, .)$. Since $(A, \oplus, *)$ is a field, the following cancellation laws hold good in a region $(A, \oplus, *, \bullet)$ by virtue of inheritance :-

(1) If $x \oplus y = x \oplus z$, then $y = z$ where $x, y, z \in A$.
(2) If $x \oplus y = z \oplus y$, then $x = z$ where $x, y, z \in A$.
(3) If $x * y = x * z$ where $x \neq 0_A$, then $y = z$ where $x, y, z \in A$.
(4) If $x * y = z * y$ where $y \neq 0_A$, then $x = z$ where $x, y, z \in A$.

However, it can be easily shown that the following two cancellation laws too hold good in a region A :-

(5) If $a \bullet x = a \bullet y$ where $a \neq 0_F$, then $x = y$ where $x, y \in A$ and $a \in F$.
(6) If $a \bullet x = b \bullet x$ where $x \neq 0_A$, then $a = b$ where $x \in A$ and $a, b \in F$.

Besides the above six, there are a number of kinds of cancellation operations valid in the region $(A, \oplus, *, \bullet)$ over the field $(F,+, .)$, few of which are quoted below :-

If $x, y \in A$ and $a, b \in F$, then

(7) If $(a.b) \bullet x = (a.c) \bullet y$ where $a \neq 0_F$, then $b \bullet x = c \bullet y$.

(8) $\dfrac{a \bullet x}{a \bullet y} = \dfrac{x}{y}$, where $a \neq 0_F$ and $y \neq 0_A$.

(9) $\dfrac{(a \cdot b) \bullet x}{(a \cdot c) \bullet y} = \dfrac{b \bullet x}{c \bullet y}$, where $a \neq 0_F$.

(10) $\dfrac{(a \cdot c) \bullet x}{(b \cdot c) \bullet y} = \dfrac{a \bullet x}{b \bullet y}$, where $c \neq 0_F$.

(11) $\dfrac{(a \bullet x) * y}{(b \bullet x) * z} = \dfrac{a \bullet y}{b \bullet z}$, where $x \neq 0_A$.

(12) $\dfrac{(a \bullet x) * y}{(a \bullet z) * t} = \dfrac{x * y}{z * t}$, where $a \neq 0_F$.

Proof : Although the proof of all the above results are straightforward, nevertheless we present below the proof of one result (of result-(8)).

$$\begin{aligned}
\frac{a \bullet x}{a \bullet y} &= (a \bullet x) * (a \bullet y)^{-1} \\
&= (a \bullet x) * (a^{-1} \bullet y^{-1}) \\
&= (a.a^{-1}) \bullet (x * y^{-1}), \quad \text{using compatibility property of region A.} \\
&= 1_F \bullet (x * y^{-1}) \\
&= x * y^{-1} \\
&= \frac{x}{y}
\end{aligned}$$

NOTE 3.4

Proposition 3.2 states that in a region $(A, \oplus, *, \bullet)$ over the field (F, +, .), for a $\in$ F and for x $(\neq 0_A)$, y $\in$ region A, if $y*x = a \bullet x$ then $y = a \bullet 1_A$, and thus there is no kind of right cancellation holds good here.

Proposition 3.11

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, .), the following results are true (keeping in mind that division by 0_A or 0_F is not permissible) :-

If x, y, z, t $\in$ A, and a, b, c, d $\in$ F, then

(i) $\frac{x}{y} * \frac{z}{t} = \frac{x*z}{y*t}$

(ii) $\frac{x \oplus y}{z} = \left(\frac{x}{z}\right) \oplus \left(\frac{y}{z}\right)$

(iii) $x^2 \sim y^2 = (x \oplus y) * (x \sim y)$

(iv) $\frac{x}{y} \oplus \frac{z}{t} = \frac{(x*t) \oplus (y*z)}{y*t}$,

(v) $\frac{a \bullet x \oplus b \bullet y}{c \bullet z} = \left(\frac{a \bullet x}{c \bullet z}\right) \oplus \left(\frac{b \bullet y}{c \bullet z}\right) = \left(\frac{a}{c}\right) \bullet \left(\frac{x}{y}\right) \oplus \left(\frac{b}{c}\right) \bullet \left(\frac{y}{z}\right)$

Proof : All the results are straightforward, nevertheless we present the proof of result-(iv) here.

$$\begin{aligned}
&\frac{x}{y} \oplus \frac{z}{t} \\
&= (x * y^{-1}) \oplus (z * t^{-1}) \\
&= ((x * t) * (y^{-1} * t^{-1})) \oplus ((y * z) * (y^{-1} * t^{-1})) \\
&= ((x * t) \oplus (y * z)) * (y^{-1} * t^{-1}) \\
&= ((x * t) \oplus (y * z)) * (y * t)^{-1} \\
&= \frac{x*t \oplus y*z}{y*t}
\end{aligned}$$

The results of the following proposition is also straightforward in any region, but not true in a Division Algebra in general.

Proposition 3.12

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, .), for any non-negative integer n, if x $\in$ A and a, b $\in$ F then the following results are true.

(i) $(a \bullet x)^n = a^n \bullet x^n$

(ii) $\left(\frac{a \bullet x}{b \bullet y}\right)^n = \frac{a^n \bullet x^n}{b^n \bullet y^n} = \frac{a^n}{b^n} \bullet \frac{x^n}{y^n}$ where $b \neq 0_F$ and $y \neq 0_A$.

NOTE 3.5
The equality $a \bullet x = b \bullet y$ implies the following equalities (keeping in mind that division by 0_A or by 0_F is not permissible) :

(i) $\frac{1_F}{b} \bullet x = \frac{1_F}{a} \bullet y$.

(ii) $a \bullet \frac{1_A}{y} = b \bullet \frac{1_A}{x}$.

(iii) $\frac{x}{b} = \frac{y}{a}$

(iv) $\frac{a}{y} = \frac{b}{x}$.

But it is important to observe that the equality $a \bullet x = b \bullet y$ in a region can not imply that $\frac{a}{b} = \frac{y}{x}$. In fact this is an invalid and absurd equality, although both $\frac{a}{b}$ and $\frac{y}{x}$ are individually meaningful. However, it surely implies the following equalities :-

(i) $\frac{y}{x} = \frac{a}{b} \bullet 1_A$ (ii) $\frac{x}{y} = \frac{b}{a} \bullet 1_A$

One of the most useful and most important properties fluently used by the mathematicians in their daily practices is 'Cross Multiplication Property'. The following simple Cross Multiplication Property is not valid in a division algebra alone or in any existing standard algebraic structure alone, by virtue of their respective definitions and independently owned properties. But the 'Cross Multiplication Property' is well valid in a region A.

Proposition 3.13 Cross Multiplication Property
In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, the Cross Multiplication Property is well valid. i.e.

If $\frac{a \bullet x}{b \bullet y} = \frac{c \bullet z}{d \bullet t}$, then $(a.d) \bullet (x * t) = (b.c) \bullet (y * z)$ and conversely,

where $x, y, z, t \in A$, $y \neq 0_A \neq t$ and $a, b, c, d \in F$, $b \neq 0_F \neq d$.

Proof : We have $\frac{a \bullet x}{b \bullet y} = \frac{c \bullet z}{d \bullet t}$

i.e. $(a \bullet x) * (b \bullet y)^{-1} = (c \bullet z) * (d \bullet t)^{-1}$

or, $(a \bullet x) * (d \bullet t) = (b \bullet y) * (c \bullet z)$

or, $(a.d) \bullet (x * t) = (b.c) \bullet (y * z)$.

Proposition 3.14
In a region $(A, \oplus, *, \bullet)$ over the field $(F, +, .)$, if $a \bullet x = b \bullet y$ then

$$\frac{a \bullet x}{b \bullet y} = 1_A \text{ and conversely,}$$

where x, y $\in$ A, a, b $\in$ F and $b \bullet y \neq 0_A$.

The so important result (result-(ii) below) of Componendo & Dividendo Rule is ***not valid*** *in a Division Algebra alone or in any existing standard algebraic structure alone, by virtue of their respective definitions and independently owned properties. But the Componendo & Dividendo Rule is well valid in a region A.*

Proposition 3.15 Componendo & Dividendo Rule

In a region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), the following 'Componendo & Dividendo' rules are well valid**:**

(i) If $\frac{x}{y} = \frac{z}{t}$, then $\frac{x}{y} = \frac{z}{t} = \frac{x \oplus z}{y \oplus t} = \frac{x \sim z}{y \sim t}$,

where x, y, z, t $\in$ A, and denominator $\neq 0_A$.

(ii) If $\frac{x}{y} = \frac{z}{t}$, then

$$\frac{x}{y} = \frac{z}{t} = \frac{(a \bullet x) \oplus (b \bullet z)}{(a \bullet y) \oplus (b \bullet t)} = \frac{a \bullet x \sim b \bullet z}{a \bullet y \sim b \bullet t}$$

where x, y, z, t $\in$ A, and denominator $\neq 0_A$.

Proof :

(i) We have $\frac{x}{y} = \frac{z}{t}$

or, $x * y^{-1} = z * t^{-1}$

or, $(x * y^{-1}) * (y * t) = (z * t^{-1}) * (y * t)$

or, $x * t = z * y$

or, $x * y \oplus x * t = x * y \oplus z * y$

or, $x * (y \oplus t) = (x \oplus z) * y$

or, $\frac{x}{y} = \frac{x \oplus z}{y \oplus t}$

In a similar way we can establish that

$\frac{x}{y} = \frac{x \sim z}{y \sim t}$. Hence the result.

(ii) We have $\frac{x}{y} = \frac{z}{t}$

Now, $\frac{x}{y} = x * y^{-1}$

$= (a.a^{-1}) \bullet (x * y^{-1})$,

$= (a \bullet x) * (a^{-1} \bullet y^{-1})$, using compatibility property of region A.

$= (a \bullet x) * (a \bullet y)^{-1}$

$$= \frac{a \bullet x}{a \bullet y}$$

Similarly, we can also establish that $\frac{z}{t} = \frac{b \bullet z}{b \bullet t}$.

Now, we have

$$\frac{x}{y} = \frac{a \bullet x}{a \bullet y} = \frac{b \bullet z}{b \bullet t} = \frac{z}{t}$$

Applying now the result (i), we have

$$\frac{x}{y} = \frac{a \bullet x \oplus b \bullet z}{a \bullet y \oplus b \bullet t} = \frac{a \bullet x \sim b \bullet z}{a \bullet y \sim b \bullet t} = \frac{z}{t} .$$

3.2.11 Characteristic of a Region

In Region Algebra, the characteristic of a region A denoted char(A) is defined to be the smallest number of times one must use its multiplicative identity 1_A in a sum to get the additive identity element 0_A. A region is said to have characteristic zero if this sum never reaches the additive identity. For example, for the region RR we have Char(RR) = 0.

3.3 Categories of Regions

In this section three special types of regions are discussed which are useful in Region Mathematics.

3.3.1 Real Region

A region $(A, \oplus, *, \bullet)$ over the field (F, +, **.**), is called a Real Region if its outer field F is the classical field R of real numbers.

Example 3.5

The regions RR, C are examples of real region.

The following simple results/formulas (Proposition 3.16, 3.17, 3.18) are very useful and important results valid in regions, but all these collectively are not valid in general in a division algebra alone or in any of the existing classical algebraic structures alone by virtue of their respective definitions and independently owned properties.

These results reduce to the corresponding important classical results of elementary algebra as special cases.

Proposition 3.16

The following results hold good in a real region $(A, \oplus, *, \bullet)$:

(i) $(1_A \oplus x)^2 = 1_A \oplus 2 \bullet x \oplus x^2 , \quad \forall x \in A$

(ii) $(x \oplus y)^2 = x^2 \oplus 2 \bullet x * y \oplus y^2 , \quad \forall x, y \in A$

(iii) $(x \sim y)^2 = x^2 \sim 2 \bullet x * y \oplus y^2 , \quad \forall x, y \in A$

(iv) $(x \oplus y)^3 = x^3 \oplus 3 \bullet x^2 * y \oplus 3 \bullet x * y^2 \oplus y^3, \quad \forall x, y \in A$

(v) $(x \sim y)^3 = x^3 \sim 3 \bullet x^2 * y \oplus 3 \bullet x * y^2 \sim y^3, \quad \forall x, y \in A$

However, the above results in general are not true in a region which is not a real region. A generalized result is given below.

Proposition 3.17
In a real region $(A, \oplus, *, \bullet)$ the following equality is valid $\forall$ x, y $\in$ A :

$$(x \oplus y)^n = \sum_{r=0}^{n} \binom{n}{r} \bullet (x^{n-r} * y^r)$$

where the notation $\sum$ stands for summation over the symbol $\oplus$ of first addition operator of the region A, and n is a positive integer.
(However, this result in general is not true in a region if it is not a real region).

NOTE 3.6
The RR region is the actual algebraic structure in which most of the results, expressions, equalities of school algebra are studied and taught. In RR region, the results of Proposition 3.16 & Proposition 3.17 are being written in traditional style as below :-

(i) $(1 + x)^2 = 1 + 2x + x^2$
(ii) $(x + y)^2 = x^2 + 2xy + y^2$
(iii) $(x - y)^2 = x^2 - 2xy + y^2$
(iv) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
(v) $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
(vi) $(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

which are taught at secondary school level to the students.

The following results are also not valid in a division algebra alone or in any of the existing classical algebraic structures alone by virtue of their respective definitions and properties, but well valid in a real region.

Proposition 3.18
If $(A, \oplus, *, \bullet)$ be a real region, then the following results are true $\forall$ x, y $\in$ A and $\forall$ a, b $\in$ R :

(i) $(a \bullet x \oplus b \bullet y)^2 = a^2 \bullet x^2 \oplus b^2 \bullet y^2 \oplus (2.a.b) \bullet (x * y)$.

(ii) $(a \bullet x \oplus b \bullet y)^n = \sum_{r=0}^{n} \left(\binom{n}{r} a^{n-r}.b^r \right) \bullet (x^{n-r} * y^r)$.

NOTE 3.7
However, in RR region the above results are written in traditional style as below :-

(i) $(ax + by)^2 = a^2x^2 + b^2y^2 + 2abxy$

(ii) $(ax + by)^n = \sum_{r=0}^{n}\left(\binom{n}{r} a^{n-r} b^r x^{n-r} y^r\right)$

3.3.2 Region over a Region (ROR)

Let $(A, \oplus, *, \bullet)$ be a region over a field (F, +, .). If the algebraic system (F, +, . , .) itself be a region over a field (K, ±, .), then we say that A is a 'Region over a Region' (or, ROR). In such case the region F is called the 'base region' of the ROR A.

3.3.3 Region over a Real Region (RORR)

If the base region is a real region, then A is called a 'Region over a Real Region' (or, RORR).
An example of RORR is the region RR.

Proposition 3.19

If $A = (A, \oplus, *, \bullet)$ is a region over a real region F, then $\forall x \in A$

(i) $x \oplus x = (2.1_F) \bullet x$

(ii) $\sum_{r=1}^{n} x = (n.1_F) \bullet x$

Proof :
$$\begin{aligned} x \oplus x &= (1_F \bullet x) \oplus (1_F \bullet x) \\ &= (1_F + 1_F) \bullet x \\ &= (1.1_F + 1.1_F) \bullet x \\ &= (2.1_F) \bullet x \end{aligned}$$
Hence the result.

The result (ii) can be proved similarly.

Proposition 3.20

If $A = (A, \oplus, *, \bullet)$ is a region over a real region F, then $\forall x, y \in A$

(i) $(1_A \oplus x)^2 = 1_A \oplus (2.1_F) \bullet x \oplus x^2$

(ii) $(x \oplus y)^2 = x^2 \oplus (2.1_F) \bullet x * y \oplus y^2$

(iii) $(x \oplus y)^n = \sum_{r=0}^{n}\left(\binom{n}{r}.1_F\right) \bullet x^{n-r} * y^r$ where $x^0 = 1_A$ and $a^0 = 1_F$.

(iv) $(a \bullet x \oplus b \bullet y)^2 = a^2 \bullet x^2 \oplus b^2 \bullet y^2 \oplus (2.(a.b)) \bullet (x * y)$

(v) $(a \bullet x \oplus b \bullet y)^n = \sum_{r=0}^{n}\left(\binom{n}{r} \cdot a^{n-r} \cdot b^r\right) \bullet (x^{n-r} * y^r)$

The following results are straightforward.

Proposition 3.21

If the region $(A, \oplus, *, \bullet)$ is a real region, then $\forall x, y \in A$ the following results are true (not necessarily true in general if the region A is not a real region) :-

(i) $x^2 \oplus y^2 = (x \oplus y)^2 \sim 2.(x * y)$
$= (x \sim y)^2 \oplus 2.(x * y)$

(iii) $(x \sim y)^2 = (x \oplus y)^2 \sim 4.(x * y)$
(iv) $(x \oplus y)^2 = (x \sim y)^2 \oplus 4.(x * y)$
(iv) $x^3 \sim y^3 = (x \sim y) * (x^2 \oplus x * y \oplus y^2)$
$= (x \sim y)^3 \oplus 3. ((x * y) * (x \sim y))$
(v) $x^3 \oplus y^3 = (x \oplus y) * (x^2 \sim x * y \oplus y^2)$
$= (x \oplus y)^3 \sim 3.((x * y) * (x \oplus y))$

There are many algebraic problems of elementary algebra at secondary school level which we solve without knowing the identity of the minimal algebraic structure based upon which we are having our right to solve them. For example, the following problem is a very simple problem of school level 'elementary algebra' which can not be solved in general in groups alone, or in rings alone, or in modules, fields, module, linear spaces, algebra over a field, associative algebra over a field, division algebra alone or in any existing classical algebraic structure alone, by virtue of their respective definitions and independently owned properties. But these problems can well be solved in a region or in an algebraic structure which is at least a region.

Problem 3.1.
Obtain an expression for x in terms of y and t from the following equation in the real region $(A, \oplus, *, \bullet)$:

$$3 \bullet x*y = 2 \bullet y \oplus 3 \bullet t,$$

where $x, y \ (\neq 0_{\mathbf{A}}),\ t \in \mathbf{A}$.

Solution : We have the following equation in the region **A :**

$$3 \bullet x * y = 2 \bullet y \oplus 3 \bullet t$$

Using the properties of region, we then can write

$$\frac{1}{3} \bullet (3 \bullet x*y) = \frac{1}{3} \bullet (2 \bullet y \oplus 3 \bullet t)$$

or, $$\left(\frac{1}{3} \cdot 3\right) \bullet (x*y) = \left(\frac{1}{3} \bullet (2 \bullet y)\right) \oplus \left(\frac{1}{3} \bullet (3 \bullet t)\right)$$

or, $$1_F \bullet (x*y) = \left(\frac{1}{3} \cdot 2\right) \bullet y \oplus \left(\frac{1}{3} \cdot 3\right) \bullet t$$

or, $$x*y = \frac{2}{3} \bullet y \oplus 1_F \bullet t$$

or, $$x*y = \frac{2}{3} \bullet y \oplus t$$

or, $$(x*y)*y^{-1} = \left(\frac{2}{3} \bullet y \oplus t\right) * y^{-1}$$

or, $$x * (y * y^{-1}) = \left(\frac{2}{3} \bullet (y * y^{-1})\right) \oplus (t * y^{-1})$$

or, $\quad x * 1_{\mathbf{A}} = \left(\frac{2}{3} \bullet 1_A\right) \oplus (t * y^{-1})$

or, $\quad x = \left(\frac{2}{3} \bullet 1_A \oplus \frac{t}{y}\right)$, which is the solution.

An interesting analysis of the above solution steps is presented in NOTE 3.8.

NOTE 3.8

Let us analyze now the solution to the above Problem 3.1 slightly in a different way to feel the unique potential of 'Region'. For this, we begin with an element of imagination mentioned in (i) and (ii) below.

(i) Imagine that the identity of the algebraic structure **A** in the above Problem 3.1 is "unknown" to us <u>at this moment</u>, and

(ii) let us also accept that the solution steps presented above are well valid and correct in this "unknown" algebraic structure **A** (without borrowing the cooperation of any other algebraic structure other than A).

Now in the above solution steps, it can be carefully observed that :-

There are few steps which are allowed by virtue of the definition and properties of 'vector space', and there are few steps which are allowed by virtue of the definition and properties of 'division algebra'. It is obvious that a 'division algebra' can not give license to all the steps of the above mentioned solution-method by virtue of its definition and independently owned properties (for example, 'compatibility with scalars' is not a licensed step in division algebra, even not the commutative property). Besides that, see that few division operations are executed in the solution steps. Hence **A** can neither be just an 'algebra over a field' nor an 'associative algebra over a field'.

*Consequently, considering the validity of all the involved operations collectively in the steps of the above mentioned solution-method, it is now obvious that this unknown algebra **A** has to be at minimum a 'region', not less (i.e. not a Division Algebra, not any of the existing standard algebraic structures). Otherwise, the above problem can not be solved for x in the algebraic structure **A**.*

4. Conclusion

The work to introduce the new algebraic structure "Region" was not initiated in my mind with any pre-posed problem or plan. I did not have any pre-proposed synopsis for it. It was an accidental development in my mind while I observed that in general the existing standard algebraic structures viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra, etc. can not validate many of the fundamental and classical equalities, identities, expressions, equations, formulas, results of "elementary algebra" (of secondary school level or higher level) by virtue of their respective definitions and independently owned properties. Few examples are presented and explained in Case-5 in section-3.1, but there exist an infinite number of such examples. The 'Abstract Algebra' as a subject needs to identify an appropriate but minimal algebraic structure, on the platform of which the most practiced classical equalities, identities, expressions, equations, formulas, results of elementary algebra stand valid, can be computed and can be verified to be true. Yes, it is fact that an infinite number of algebraic structures can be defined by an algebraist if he desires, but the objective of this work is not just for introducing a new one. The objective is very much genuine as 'Region' is highly significant to all the branches of mathematics. The existing literature on Abstract Algebra is so rich and voluminous that it does not need any new algebraic structure which is redundant. The sole objective of this work is to introduce "Region" because it provides the **minimal** platform to make the fundamental and classical equalities, identities, expressions, equations, formulas, results, etc. valid (i.e. can be computed and verified); to provide us an <u>algebraic right</u> to use the standard and most practiced equalities, identities, expressions, equations, formulas, results, etc of it fluently in the everyday algebraic computations. Identifying this minimal algebraic structure and then defining it uniquely with an independent self-identity is therefore important for us. Consequently in Section-3 a new but very sound and complete algebraic structure called by "region" has been introduced. Various properties of the algebraic structure 'regions' are studied, and a lot of characterizations is done. Region in a very hidden way has been happening to be the most practiced algebra in the study of Science, Technology, Engineering, etc. Considering the enormous unique potential of "Region" to give license to the mathematicians to practice the existing simple and useful results, equalities, identities, formulas etc. of elementary algebra, we can not ignore the deserving and genuine claim of "Region Algebra" to have a self independent identity. The region algebra is applied in the newly discovered "NR-Statistics" (Biswas[9]) in which the population data are not always real numbers but any kind of other real life objects (viz. a population of 50 paints of beautiful TAJMAHAL by 50 junior artists in a school level competition held at Calcutta High school, a collection of 10 X-ray images of a patient during last ten days in Calcutta Medical Hospital, etc). In "NR-Statistics" various new statistical region measures [9] like: region mean, region standard deviation, region variance, etc. with algebraic approach were studied for real life NR-populations. The region algebra is also applied in the application areas of heterogeneous Data Structure 'r-Atrain' for Big Data (Biswas[10]), in the mathematical Theory of Solid Matrices and Solid Latrices.

The subject Abstract Algebra can not be complete and sound without 'region', the most important algebraic structure of it as justified in length in this work. Philosophically, if we consider the evolution of various algebraic structures, in particular considering their flexible roles and volume of contributing capabilities

towards the subjects from ‘elementary algebra’ to ‘higher algebra’, we could visualize the unique location of “Region” as mentioned below:
Group → Ring → Field → Linear Space → Division Algebra → Region.

References

[1] Althoen, S. C. and Kugler, L. D., When is R^2 a Division Algebra?, *American Math. Monthly*, **90** (1983), 625-635. https://doi.org/10.1080/00029890.1983.11971297

[2] Acheson, David, *The Calculus Story: A Mathematical Adventure,* First Edition, Oxford University Press, UK, 2017.

[3] Adamson, I. T., *Introduction to Field Theory*, Dover Publications, 2007. https://doi.org/10.1017/cbo9780511600593

[4] Apostol, Tom M., *Introduction to Analytic Number Theory*, Springer-Verlag, Heidelberg Berlin, 1976. https://doi.org/10.1007/978-3-662-28579-4

[5] Artin, Michael, *Algebra*, Prentice Hall, New York, 1993. https://doi.org/10.1007/978-3-0348-9124-0

[6] Baron, Margaret E., *The Origins of the Infinitesimal Calculus*, Dover Publications, Inc., Mineola, New York, 2003.

[7] Biswas, Ranjit, Region Algebra, *Information: An International Journal*, **15** (8) (2012), 3195-3228.

[8] Biswas, Ranjit, Birth of Compound Numbers, *Turkish Journal of Analysis and Number Theory*, **2** (6) (2014), 208-219. https://doi.org/10.12691/tjant-2-6-4

[9] Biswas, Ranjit, Introducing Soft Statistical Measures, *The Journal of Fuzzy Mathematics*, **22** (4) (2014), 819-851.

[10] Biswas, Ranjit, “Atrain Distributed System” (ADS): An Infinitely Scalable Architecture for Processing Big Data of Any 4Vs, in: *‘Computational Intelligence for Big Data Analysis Frontier Advances and Applications’*: edited by D.P. Acharjya, Satchidananda Dehuri and Sugata Sanyal, Springer International Publishing Switzerland, 2015, Part-1, 2015, 1-53. https://doi.org/10.1007/978-3-319-16598-1_1

[11] Biswas, Ranjit, *Is ‘Fuzzy Theory’ An Appropriate Tool For Large Size Problems?,* book-series of SpringerBriefs in Computational Intelligence, Springer, Heidelberg, 2016. https://doi.org/10.1007/978-3-319-26718-0

[12] Biswas, Ranjit, Is 'Fuzzy Theory' An Appropriate Tool For Large Size Decision Problems ?, Chapter-8 in Imprecision and Uncertainty in Information Representation and Processing, in the series of STUDFUZZ, Springer. Heidelberg, 2016. https://doi.org/10.1007/978-3-319-26718-0

[13] Biswas, Ranjit, *Intuitionistic Fuzzy Theory for Soft-Computing: More Appropriate Tool Than Fuzzy Theory*, International Journal of Computing and Optimization, **6** (1) (2019), 13-56. https://doi.org/10.12988/ijco.2019.955

[14] Beachy, J.A. and Blair, W.D., *Abstract Algebra*, 2nd Ed., Waveland Press, Prospect Heights, III, 1996.

[15] Bourbaki, Nicolas, *Elements of the History of Mathematics*, Springer-Verlag, Berlin, 1994. https://doi.org/10.1007/978-3-642-61693-8

[16] Bourbaki, Nicolas, *Elements of Mathematics : Algebra I*, New York: Springer-Verlag, 1998.

[17] Boyer, Carl B. and Merzbach, Uta C., *A History of Mathematics*, 3rd Edition, John Wiley & Sons, Hoboken, New Jersey, 2011.

[18] Boyer, Carl B., *History of Analytic Geometry*, Dover Publications, Inc., Mineola, New York, 2004.

[19] Boyer, Carl B., *The History of the Calculus and Its Conceptual Development*, Dover Publications, Inc., Mineola, New York, 1959.

[20] Cajori, Florian, *A History of Elementary Mathematics – With Hints on Methods of Teaching*, Read Books, UK, 2010.

[21] Davenport, Harold; Montgomery, Hugh L., *Multiplicative Number Theory*, Graduate texts in mathematics, Springer-Verlag New York Inc, 1980. https://doi.org/10.1007/978-1-4757-5927-3

[22] Dixon, G. M., Division *Algebras: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics*, Kluwer Academic Publishers, Dordrecht, 2010.

[23] Edwards, Harold M., *Fermat's Last Theorem: a Genetic Introduction to Algebraic Number Theory*, Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[24] Ellis, G., *Rings and Fields*, Oxford University Press, 1993.

[25] Fraenkel, A., Bar-Hillel, Y., and Lev, A.: *Foundations of Set Theory*, North Holland, Amsterdam.

[26] Franklin, Joel L., *Matrix Theory*, Mineola, N.Y., 2000.

[27] Hardy, G.H., Wright, Edward M., Roger Heath-Brown, Silverman, Joseph. and Wile, Andrew, *Introduction to the Theory of Numbers*, 6th Edition, Oxford University Press, UK, 2008.

[28] Herstein, I. N., *Topics in Algebra*, Wiley Eastern Limited, New Delhi, 2001.

[29] Hungerford, T., *Algebra*, Graduate Texts in Mathematics, Vol. 73, Springer-Verlag, New York, 1974.

[30] Jacobson, N., *Basic Algebra I*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco, 1985.

[31] Jacobson, N., *Basic Algebra II*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco, 1985.

[32] Jacobson, N., *The Theory of Rings*, American Mathematical Society Mathematical Surveys. Vol. I. American Mathematical Society. New York, 1943. https://doi.org/10.1090/surv/002

[33] Lam, T. Y., *Exercises in Classical Ring Theory*, Problem Books in Mathematics, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4757-3987-9

[34] Lang, Serge, *Undergraduate Algebra*, 3rd ed., Berlin, New York, Springer-Verlag, 2005.

[35] Loney, S. L., *The Elements of Coordinate Geometry, Part-I*, Macmillan Student Edition, Macmillan India Limited, Madras, 1975.

[36] Lorenz, Falko, *Algebra, Volume II: Fields with Structures*, Algebras and Advanced Topics, Springer, 2008.

[37] Matsumura, Hideyuki, Reid, Miles, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986. https://doi.org/10.1017/cbo9781139171762

[38] Nahin, Paul J., *An Imaginary Tale: The Story of* $\sqrt{-1}$, Walter de Gruyter, 2010. https://doi.org/10.1515/9781400833894

[39] Pierce, Richard S., *Associative Algebras*, Graduate Texts in Mathematics. Studies in the History of Modern Science, Springer-Verlag, 1982. https://doi.org/10.1007/978-1-4757-0163-0

[40] Reyes, Mitchell, The Rhetoric in Mathematics: Newton, Leibniz, the Calculus, and the Rhetorical Force of the Infinitesimal, *Quarterly Journal of Speech*, **90** (2004), 159-184.

[41] Lidl, Rudolf; Niederreiter, Harald, *Finite Fields* (2nd ed.), Cambridge University Press, 2008. https://doi.org/10.1017/cbo9780511525926

[42] Robbins, Herbert and Courant, Richard, *What is Mathematics?: An Elementary Approach to Ideas and Methods,* Second edition, Oxford University Press, New York, 1996.

[43] Rosen, Kenneth H., *Elementary Number Theory* (6th ed.), Pearson Education, USA, 2010.

[44] Rudin, Walter, *Real and Complex Analysis*, McGraw Hills Education, India, 2006.

[45] Saltman, D. D., *Lectures on Division Algebras*, Providence, RI: Amer. Math. Society, 1999. https://doi.org/10.1090/cbms/094

[46] Sardar, Ziauddin., Ravetz, Jerry., Loon, Borin Van Loon, *Introducing Mathematics*, Icon Books Ltd, London, 2015.

[47] Shult, Ernest E., *Points and Lines*, Springer, 2011. https://doi.org/10.1007/978-3-642-15627-4

[48] Simmons, G. F., *Introduction to Topology and Modern Analysis*, McGraw Hill, New York, 1963.

[49] Van der Waerden and Bartel Leendert, *Algebra*, Springer-Verlag, New York, 1991.

[50] Vinogradov, I.M., *Elements of Number Theory* (reprint of the 1954 ed.). Mineola, NY, Dover Publications, Mineola, New York, 2003.

[51] Waerden, Bartel Leenert Van der, *Algebra*, Springer-Verlag, New York, 1991.

Received: July 15, 2019; Published: August 2, 2019