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REGISTER SHIFTS VERSUS TRANSITIVE F-CYCLES FOR PIECEWISE MONOTONE MAPS

Abstract

This paper investigates the family of continuous piecewise monotone functions which map a closed interval of the real line into itself. For these maps Preston [1] and Blokh [2] described the asymptotic behavior of the orbit of a "typical" point. Our results show that if the map is expanding on its intervals of monotonicity the dominant role is played by transitive f-cycles. Contrary to this for a "typical" map in a natural closure of the space of these maps there are no transitive f-cycles. Instead the behavior is dominated by the register shifts. This result is illustrated by an example.

1 Introduction

Consider a continuous function which maps a closed interval of the real line into itself. This gives us a simple dynamical system with discrete time. The new state of our system is the image of the old one using the given function. So each starting state determines a whole orbit. We are interested in the asymptotic behavior of the orbit of a "typical" point. For us "typical" has a topological rather than measure theoretic meaning.

A function is piecewise monotone if there is a finite partition of our interval such that the function is monotone on each part. If on each of these

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subintervals the derivative is bigger than some number greater than one, then we show that the "typical" point is attracted by a transitive f-cycle. This means a periodic interval in which some orbit is dense. The orbits inside this transitive f-cycle can be very wild. Moreover such a function does not have any register shift.

On the other hand for a "typical" function from the slightly larger space of piecewise monotone functions with derivatives greater than or equal to one (or strictly greater than one) a "typical" point is attracted by a register shift. Thus the asymptotic behavior of such an orbit is very nice and if we do not notice small differences, then it looks like a periodic orbit. It follows that the "typical" function does not have any transitive f-cycle. So one phenomenon is replaced by the other.

2 Background

Let $I = [0, 1]$ and let $C(I)$ be the space of continuous functions which map I into itself. This space will be endowed with the metric ϱ of uniform convergence.

A function $f \in C(I)$ is called piecewise monotone if there is an $n \geq 0$ and a set of points $0 = d_0 < d_1 < \cdots < d_n < d_{n+1} = 1$ such that f is strictly monotone on $[d_k, d_{k+1}]$ for each $k = 0, \ldots, n$. A point $t \in (0, 1)$ is called a turning point of f if f is not monotone in any neighborhood of t. We denote the set of the turning points of f by $T(f)$. Let M_n be the set of piecewise monotone functions with the number of turning points less than or equal to n. Let $c \geq 0$, $d > 0$ and

$$M_{n,c} = \left\{ f \in M_n;\ \text{if } f|[a,b] \text{ is monotone, then } \left| \frac{f(b) - f(a)}{b - a} \right| > c \right\},$$

$$\tilde{M}_{n,d} = \left\{ f \in M_n;\ \text{if } f|[a,b] \text{ is monotone, then } \left| \frac{f(b) - f(a)}{b - a} \right| \geq d \right\}.$$

For $f \in C(I)$ define f^n (n-th iterate of f) inductively by $f^0(x) = x$ and (for $n \geq 1$) $f^n(x) = f(f^{n-1}(x))$. The orbit of $x \in I$ with respect to f is the sequence $\mathrm{orb}(x) = \{f^n(x)\}_{n=0}^{\infty}$. A closed interval $J \subset I$ is called periodic interval with period $\mathrm{per}(J) = k \in \mathbb{N}$ if $f^k(J) = J$ and $f^i(J) \cap f^j(J) = \emptyset$ for $0 \leq i \neq j < k$. If J is a point, then it is called a periodic point and $\mathrm{Per}(f)$ denotes the set of all periodic points of f. A point $x \in I$ is called eventually periodic if $x \notin \mathrm{Per}(f)$ and $f^k(x) \in \mathrm{Per}(f)$ for some $k \geq 1$.

Recall that $K = \bigcup_{k=0}^{m-1} f^k(J)$ (the orbit of a periodic interval J with period m) is called an f-cycle with period m. This f-cycle is said to be transitive if there is an orbit of a point which is dense in K. Or equivalently K is transitive if for any closed $S \subset K$ such that $f(S) \subset S$ we have either $S = K$ or $\operatorname{int}(S) = \emptyset$. Note that any transitive f-cycle must contain a turning point. Hence a function from M_n can have only n different transitive f-cycles.

Let $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence ($K_{n+1} \subset K_n$) of f-cycles and m_n be a period of K_n. It is easy to see that m_n divides m_{n+1} for each $n \geq 1$. We call the sequence $\{K_n\}_{n=1}^{\infty}$ splitting if $m_{n+1} > m_n$ for each $n \geq 1$. We say that $R \subset I$ is a register shift if there is a splitting sequence of f-cycles $\{K_n\}_{n=1}^{\infty}$ such that $R = \bigcap_{n=1}^{\infty} K_n$. We call $\{K_n\}_{n=1}^{\infty}$ a generator of R. Again note that any register shift must contain a turning point. Hence a function from M_n can have only n different register shifts.

Let K be an f-cycle. We define the set of attraction of K by

$$A(K, f) = \{x \in I : \ f^n(x) \in \operatorname{int}(K) \quad \text{for some } n \geq 0\}.$$

If R is a register shift and $\{K_n\}_{n=1}^{\infty}$ is its generator, then similarly

$$A(R, f) = \bigcap_{n=1}^{\infty} A(K_n, f).$$

Note that $\bigcap_{n=1}^{\infty} A(K_n, f) = \bigcap_{n=1}^{\infty} A(\tilde{K}_n, f)$ for any two generators of R. Hence $A(R, f)$ is well defined. Finally we define the set

$$Z(f) = \{x \in (0,1);\ \exists \varepsilon > 0\ \forall n \geq 0;\ f^n|(x - \varepsilon, x + \varepsilon) \text{ is strictly monotone}\}.$$

Clearly, $A(K, f)$ and $Z(f)$ are open and $A(R, f)$ is a G_δ set. Moreover if K is a transitive f-cycle and R is a register shift, then $A(K, f) \cap A(R, f) = \emptyset$ and $A(K, f) \cap Z(f) = \emptyset$. In general it can happen that $R \cap Z(f) \neq \emptyset$ and so $A(R, f) \cap Z(f) \neq \emptyset$. (For more details about the facts mentioned above see [1] or [2].)

Now we can formulate Theorem A on the asymptotic behavior of a point under a piecewise monotone map.

Theorem A. ([1], [2]). *Let $f \in M_n$ and $K_1, \ldots, K_r$ be transitive f-cycles and $R_1, \ldots, R_s$ be register shifts. Then the set*

$$\Lambda(f) = A(K_1, f) \cup \cdots \cup A(K_r, f) \cup A(R_1, f) \cup \cdots \cup A(R_s, f) \cup Z(f)$$

is of type G_δ dense in I.

The following results give more information about the behavior of a "typical" orbit of $x \in I$ with respect to $f \in \tilde{M}_{n,1}$.

Theorem B.. *Let $f \in \tilde{M}_{n,c}$ for $c > 1$. Then f has no register shift and $Z(f) = \emptyset$.*

If we consider the space $(\tilde{M}_{n,1}, \varrho)$ we have the following contrary results.

Theorem C.. *A typical function from $\tilde{M}_{n,1}$ has no transitive f-cycle and $Z(f) = \emptyset$.*

Theorem D.. *A typical function from $M_{n,1}$ has no transitive f-cycle and $Z(f) = \emptyset$.*

3 Residual set in $(\tilde{M}_{n,1}, \varrho)$

We start this section with some auxiliary results. We will not prove all of these facts. Let $n \geq 1$ be fixed and $\tilde{M}_{n,0}$ be the closure of M_n in the space $(C(I), \varrho)$.

Proposition 3.1.. *$(\tilde{M}_{n,0}, \varrho)$ is a complete metric space.*

Proposition 3.2.. *For $c \geq 0$ the set $\tilde{M}_{n,c}$ is closed in $(\tilde{M}_{n,0}, \varrho)$.*

Proposition 3.3.. *For $c \geq 0$ the set $M_{n,c}$ is of type G_δ dense in $(\tilde{M}_{n,c}, \varrho)$.*

Proof. Let $[a, b]$ be an interval in I. Obviously the sets

$$K_\pm(a, b) = \{f \in \tilde{M}_{n,c};\ f'(x) = \pm c \quad \text{for } x \in (a, b)\}$$

are closed and nowhere dense. Let $\{[a_k, b_k]\}_{k=0}^\infty$ be a sequence of intervals such that for any interval $J \subset I$ there is a $k \geq 0$ such that $[a_k, b_k] \subset J$. Then $M_{n,c} = \tilde{M}_{n,c} \setminus \bigcup_{k=0}^\infty (K_+(a_k, b_k) \cup K_-(a_k, b_k))$ and proof is finished. □

The following assertion is an easy consequence of Propositions 3.1., 3.2., and 3.3..

Lemma 3.4.. *If $f \in M_{n,1}$, then $Z(f) = \emptyset$ and for a typical function from $\tilde{M}_{n,1}$ we have $Z(f) = \emptyset$.*

The following corollary is immediate.

Corollary 3.5.. *If $f \in \tilde{M}_{n,1}$ and R is a register shift, then* $\mathrm{int}(R) = \emptyset$.

Proof. It suffices to observe that $\mathrm{int}(R) \setminus Z(f)$ is countable for any $f \in M_n$. □

There exist functions in $M_{n,c}$ whose turning points are either periodic or eventually periodic points. Hence let

$$P_{n,c} = \{f \in M_{n,c};\ T(f) = A \cup B,\ (A \subset \mathrm{Per}(f))\ \&\ (\forall x \in B\ \exists k \in \mathbb{N};\ f^k(x) \in A)\}.$$

Lemma 3.6.. *If $c \geq 1$, then the set $P_{n,c}$ is dense in $(\tilde{M}_{n,c}, \varrho)$.*

PROOF. Choose an open set U in $\tilde{M}_{n,c}$. By Proposition 3.3. there exists function $f \in U \cap M_{n,c}$ and if we denote the turning points of f by $z_1 < z_2 < \cdots < z_m$ $(m \leq n)$, then without loss of generality we can assume that

$$f(z_i) \notin \{0,1\} \quad \text{for} \quad i \in \{1,\dots,m\}. \tag{$*$}$$

Since $c \geq 1$, the set $\bigcup_{n=0}^{\infty} f^{-n}(T(f))$ is dense in I. Suppose f has at z_1 a local maximum (The opposite case is analogous.) The reader can easily verify that there exists $g_1 \in U \cap M_{n,c}$ such that $T(g_1) = T(f)$, $g_1(z_j) = f(z_j)$ for $j \neq 1$ $g_1(z_1) \geq f(z_1)$ (see $(*)$) and $g_1(z_1) \in \bigcup_{n=0}^{\infty} g_1^{-n}(T(g_1))$. Obviously we have $g_1^k(z_1) = z_i$ for some $k \geq 1$ and $i \in \{1,\dots,m\}$. Moreover g_1 can be chosen such that condition $(*)$ also holds for g_1. So we can repeat this procedure for $z_2,\dots,z_m$ and finally we get a function $g \in U \cap M_{n,c}$ such that $T(g) = T(f)$ and for any $z \in T(g)$ there is a $k \geq 1$ such that $g^k(z) \in T(g)$. Hence obviously $g \in P_{n,c}$. (See figure 1.) □

Remark 3.1. *If $c \in [0,1)$, then the set $P_{n,c}$ is not dense in $(\tilde{M}_{n,c}, \varrho)$.*

For what follows let R_m be a finite set of disjoint closed intervals such that the sum of their lengths is less than $1/m$. Analogous to the definition of $P_{n,c}$ for $f \in \tilde{M}_{n,1}$ let $f \triangle R_m$ denote the statement that there is a partition of $T(f)$ into two disjoint parts A_f, B_f such that

(i) for all $x \in A_f$ there is $J \in R_m$ such that $x \in \mathrm{int}(J)$,

(ii) for all $J \in R_m$ there is $k \in \mathbb{N}$ such that $f^k(J) \subset \mathrm{int}(J)$,

(iii) for all $x \in B_f$ there is $J \in R_m$ and $k \in \mathbb{N}$ such that $f^k(x) \in \mathrm{int}(J)$.

Let $H_m = \{f \in \tilde{M}_{n,1};\ f \triangle R_m \quad \text{for some } R_m\}$.

Lemma 3.7.. *The set $H = \bigcap_{m=1}^{\infty} H_m$ is of type G_δ dense in $(\tilde{M}_{n,1}, \varrho)$.*

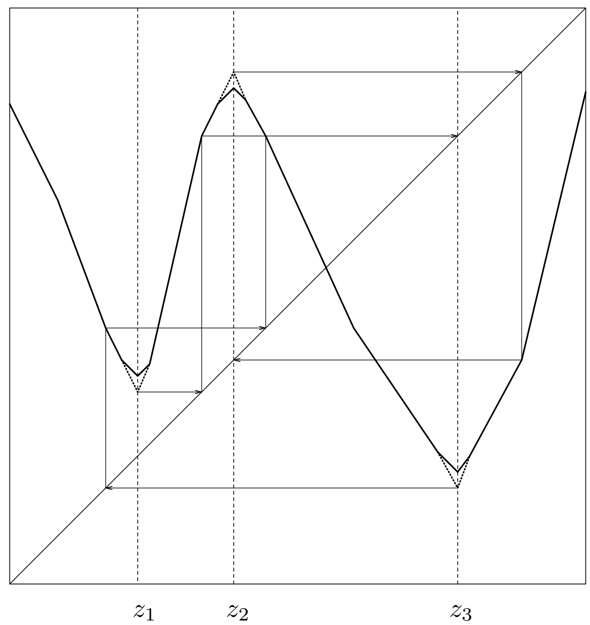

Figure 1: Functions f and g (dotted).

PROOF. By Propositions 3.1. and 3.2. it suffices to show that the set H_m is open and dense in $(\tilde{M}_{n,1}, \varrho)$. The first property is clear from (i)–(iii) and in order to prove the second one we will use Lemma 1.6.

Choose an open set U in $\tilde{M}_{n,1}$. By Lemma 3.6. there is a function $f \in U \cap P_{n,1}$ such that $f(T(f)) \subset (0,1)$. Let $A_f = T(f) \cap \mathrm{Per}(f)$, $B_f = T(f) \setminus A_f$, let $C_f = \{f^k(A_f)\}_{k=0}^{\infty}$ and let $D_f = \{f^k(B_f)\}_{k=0}^{\infty}$. Set $C_f \cup D_f = \{x_1, \dots, x_q\}$ where $q = \mathrm{card}(C_f \cup D_f)$. Then there is a union of disjoint intervals $V = \bigcup_{i=1}^{q}(c_i, d_i)$ such that $x_i \in (c_i, d_i)$ and $\sum_{i=1}^{q}(d_i - c_i) < 1/m$. For $0 < \alpha < \min_{i=1}^{q}\{|x_i - c_i|, |x_i - d_i|\}$ we also have $V_\alpha \subset V$ where $V_\alpha = \bigcup_{i=1}^{q}[x_i - \alpha, x_i + \alpha]$. Let $g \in C(I)$ be such that

(iv) $g(x) = f(x)$ for $\quad x \in C_f \cup D_f$,

(v) $T(g) = T(f)$,

(vi) $|g'(x)| = 1$ for $\quad x \in V_\alpha$,

(vii) $g(x) = f(x)$ for $\quad x \in I \setminus V$,

(viii) $g|J$ is linear for any interval $J \subset V \setminus V_\alpha$.

It is easy to verify that g is unique and $g \in \tilde{M}_{n,1}$. We can choose V and α small enough such that $g \in U$. In addition we have $X_g = X_f$ for $X \in \{A, B, C, D\}$. Let $R_m = \{[x-\alpha, x+\alpha];\ x \in A_g\}$. From (vi) we have $g([x-\alpha, x+\alpha]) \subset [g(x)-\alpha, g(x)+\alpha]$ for any $x \in C_g \cup D_g$ and so from (iv)–(vi) for the partition $T(g) = A_g \cup B_g$ we have that $g \triangle R_m$ is nearly fulfilled. More precisely it is fulfilled except for (ii) where we have only $f^k(J) \subset J$ instead of $f^k(J) \subset \operatorname{int}(J)$.

Observe that C_g is a finite union of orbits of some turning points, so we can write $C_g = \bigcup_{i=1}^{s} \operatorname{orb}(x_i)$ where $x_i \in T(g)$ and $\operatorname{orb}(x_i) \cap \operatorname{orb}(x_j) = \emptyset$ for any $1 \leq i \neq j \leq s$. Now we will modify g in a neighborhood of x_i in order to get new function a h and a set R_m such that $h \triangle R_m$.

Let $k_i = \operatorname{per}(x_i)$ for $1 \leq i \leq s$. Because $x_i \in T(g)$ it is easy to see from (vi) that either

$$g^{k_i}([x_i-\alpha, x_i+\alpha]) = [x_i-\alpha, x_i] \tag{1}$$

or

$$g^{k_i}([x_i-\alpha, x_i+\alpha]) = [x_i, x_i+\alpha]. \tag{2}$$

Suppose (1). Then obviously for any $x \in \operatorname{orb}(x_i) \cap T(g)$ if $g^k([x-\alpha, x+\alpha]) \subset [x_i-\alpha, x_i+\alpha]$ for some $k \geq 1$, then $g^k([x-\alpha, x+\alpha]) \subset [x_i-\alpha, x_i]$. Hence for any $J \in R_m$ there is $k \geq 0$ (we will take the minimal one) and $1 \leq i \leq s$ such that

$$g^k(J) \subset [x_i-\alpha, x_i]. \tag{3}$$

Similarly for (2).

Because $f \in M_{n,1}$ we have that

$$\left|\frac{g(c_i)-g(x_i-\alpha)}{c_i-(x_i-\alpha)}\right| > 1 \quad \text{and} \quad \left|\frac{g(d_i)-g(x_i+\alpha)}{d_i-(x_i+\alpha)}\right| > 1.$$

So there is $0 < \lambda < \min\{\alpha, (x_i-\alpha)-c_i, d_i-(x_i+\alpha)\}$ such that for any $1 \leq i \leq s$

$$\left|\frac{g(c_i)-g(x_i-\alpha)}{(x_i-\alpha+\lambda)-c_i}\right| > 1 \quad \text{and} \quad \left|\frac{g(d_i)-g(x_i+\alpha)}{d_i-(x_i+\alpha-\lambda)}\right| > 1. \tag{4}$$

Now we can define function h and a new set R_m. Let

(ix) $h(x) = g(x)$ for any $x \in I \setminus \bigcup_{i=1}^{s}(c_i, d_i)$,

(x) if (1), then $h(x) = g(x+\lambda)$ for any $x \in [x_i-\alpha-\lambda, x_i+\alpha-\lambda]$,
if (2), then $h(x) = g(x-\lambda)$ for any $x \in [x_i-\alpha+\lambda, x_i+\alpha+\lambda]$,

(xi) if (1), then h is linear on $[c_i, x_i-\alpha-\lambda]$ and $[x_i+\alpha-\lambda, d_i]$,
if (2), then h is linear on $[c_i, x_i-\alpha+\lambda]$ and $[x_i+\alpha+\lambda, d_i]$,

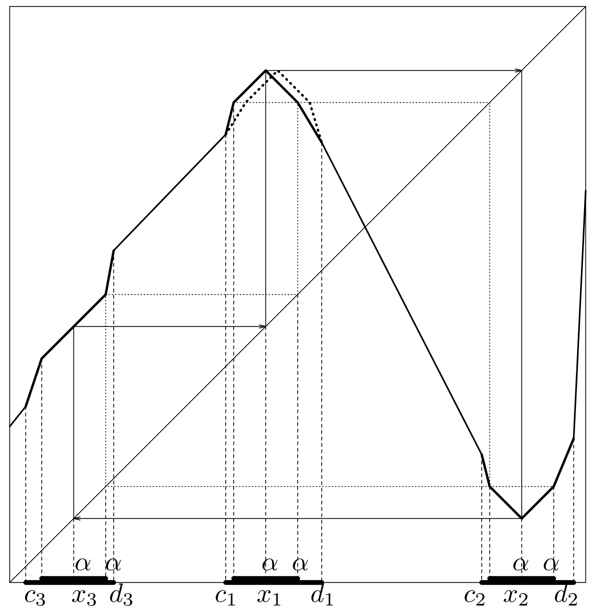

Figure 2: Functions g and h (dotted).

and in R_m we replace interval $[x_i-\alpha, x_i+\alpha]$ by $[x_i-\alpha-\lambda, x_i+\alpha-\lambda]$ in case (1) and by $[x_i-\alpha+\lambda, x_i+\alpha+\lambda]$ in case (2). (See figure 2.)

From (4) it follows that $h \in \tilde{M}_{n,1}$ and we can choose $\lambda > 0$ small enough such that $h \in U$.

For $1 \le i \le s$ let $z_i = x_i - \lambda$ and $J_i = [x_i-\alpha-\lambda, x_i+\alpha-\lambda]$ in case (1) or $z_i = x_i + \lambda$ and $J_i = [x_i-\alpha+\lambda, x_i+\alpha+\lambda]$ in case (2).

Let $B_h = B_g$ and $A_h = (A_g \setminus \bigcup_{i=1}^{s} x_i) \cup \{z_i\}_{i=1}^{s}$. We have $T(h) = A_h \cup B_h$ and $A_h \cap B_h = \emptyset$. Moreover, for any $x \in B_h$ there is $k \ge 1$ (we will take the minimal one) such that $g^k(x) = x_i$ for some $1 \le i \le s$. But then $h^k(x) = x_i$ and $x_i \in \operatorname{int} J$ for some $J \in R_m$. So condition (iii) is fulfilled and condition (i) is obvious. Only (ii) remains.

From (3) we have that for any $J \in R_m$ there is $k \ge 0$ such that $h^k(J) \subset J_i$ for some $1 \le i \le s$. But $h(J) = g(J)$ for any $J \ne J_i$ and $h(J_i) = g([x_i-\alpha, x_i+\alpha])$. Hence if $J = J_i$, then $h^{k_i}(J_i) \subset \operatorname{int}(J_i)$ by (1), (2). If $J \ne J_i$ and $x \in J \cap A_h$, then $h^{k_i}(J) \subset [x-\alpha+\lambda, x+\alpha-\lambda] \subset \operatorname{int}(J)$ by (vi), (x). So we have $h \triangle R_m$. $\square$

4 Proofs of Theorems

Theorem B.. *Let $f \in \tilde{M}_{n,c}$ for $c > 1$. Then f has no register shift and $Z(f) = \emptyset$.*

PROOF. (Compare with [3].) By Lemma 1.4 and Corollary 1.5 it suffices to show that there is $\eta > 0$ such that the length of $f^k(J)$ is greater than η for any interval $J \subset I$ and a suitable $k \in \mathbb{N}$ ($k = k(J)$). Let $c^m > 2$. Then any interval mapped by f^m will expand while it does not contain at least two points of $T(f^m)$. □

Theorem C.. *A typical function from $\tilde{M}_{n,1}$ has no transitive f-cycle and $Z(f) = \emptyset$.*

PROOF. Consider $f \in H$ (see Lemma 3.7.). Assume that f has a transitive f-cycle K. Then $K \cap T(f) \neq \emptyset$. And by (i)–(iii) there exists a closed nondegenerate interval $J \subset K$ such that $f^k(J) \subset J$ for some $k \in \mathbb{N}$ and $\mathrm{orb}(J) \neq K$. But this contradicts our assumption that K is transitive f-cycle. Second part follows from Lemma 3.4.. □

Theorem D.. *A typical function from $M_{n,1}$ has no transitive f-cycle and $Z(f) = \emptyset$.*

PROOF. The assertion easily follows from Propositions 3.1.–3.3. and Theorem C. □

5 Construction of a Function From $M_{n,1}$ That Has No Transitive f-cycle

Let $A = \{a_j\}_{j=1}^{\infty}$ and $p_i = \prod_{j=1}^{i} a_j$. We say that function f has an A-register shift if there is a sequence $\{J_i\}_{i=1}^{\infty}$ of subintervals of I such that J_i is periodic with $\mathrm{per}(J_i) = p_i$ and $J_{i+1} \subset J_i$ for all $i \in \mathbb{N}$.

We denote by $|S|$ the Lebesgue measure of a set S, by $\mathrm{conv}(S)$ the convex hull of S and by $\mathrm{d}(S_1, S_2)$ the distance between the sets S_1, S_2. Moreover, we say $S_1 < S_2$ if $x < y$ for any $x \in S_1$, $y \in S_2$.

Lemma 5.1.. *For any $A = \{a_i\}_{i=1}^{\infty}$ there is $f \in M_{1,1}$ such that f has A-register shift.*

PROOF. Fix an $A = \{a_i\}_{i=1}^{\infty}$. Without loss of generality we can assume that a_i is a prime number for all $i \in \mathbb{N}$.

Let $a \in \mathbb{N}$ be prime. If $a > 2$, then define $\psi\colon \{1, \dots, a\} \to \{1, \dots, a\}$ by

$$\psi(1) = \frac{1}{2}(a+1),$$
$$\psi(i) = a + 2 - i \ \text{ for } \ 1 < i \le \frac{1}{2}(a+1),$$
$$\psi(i) = a + 1 - i \ \text{ for } \ \frac{1}{2}(a+1) < i \le a,$$

and if $a = 2$, then simply $\psi(1) = 2$ and $\psi(2) = 1$.

Let $\{I_i^a\}_{i=1}^a$ be the set of subintervals of $I = [0, 1]$ such that

$$I = \operatorname{conv}\left(\bigcup_{i=1}^{a} I_i^a\right),$$
$$|I_i^a| = |I_j^a| \quad \text{for } \ 1 \le i, j \le a,$$
$$I_i^a < I_{i+1}^a \quad \text{for } \ 1 \le i < a,$$
$$\mathrm{d}(I_i^a, I_{i+1}^a) < \mathrm{d}(I_{\psi(i)}^a, I_{\psi(i+1)}^a) \ \text{for } \ 1 \le i < a.$$

Of course this is possible only if $a > 2$. If $a = 2$ let $I_2^2 = [0, \frac{1}{3}]$, $I_1^2 = [\frac{2}{3}, 1]$. This change in order of indexing saves some troubles when $a = 2$.

We can assume that $\sum_{i=1}^{a} |I_i^a| \le \frac{2}{3}$. Let g_a be a continuous function such that

- $g_a(I_i^a) = I_{\psi(i)}^a \quad$ for $1 \le i \le a$,
- $g_a|I_i^a$ is linear for $i \ne 2$,
- $g_a|J$ is linear for any interval $J \subset I$ with $J \cap I_i^a = \emptyset$ for $i = 1, \dots, a$,
- we do not specify $g_a|I_2^a$,
- g_a can have only one turning point.

(See figure 3.)

Let $f \in C(I)$ and $f^*(x) = f(1-x)$ for $x \in I$. (The graph of f^* is symmetric to the graph of f in the axis $x = \frac{1}{2}$.)

Now for $i \ge 1$ let f_i be a function such that $f_i = g_{a_i}$ and moreover $f_i|I_2^{a_i}$ "looks like" function f_{i+1}^* (this mean that $h_1 \circ f_i|I_2^{a_i} \circ h_2 = f_{i+1}^*$ where h_1 is linear increasing mapping from $I_{a_i}^{a_i}$ onto I and h_2 is linear increasing mapping from I onto $I_2^{a_i}$).

Henceforth, if we say "slope" we mean in fact "absolute value of the slope".

Claim. *Function f_i has the following properties:*

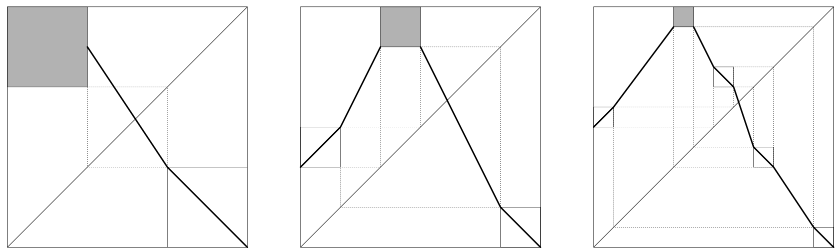

Figure 3: Possible functions g_2, g_3 and g_5.

(1) *$f_i|J$ is linear with slope greater than 1 for any interval $J \subset I$ such that $J \cap I_i^{a_i} = \emptyset$ for all $i = 1, \dots, a_i$,*

(2) *If $A_i = \{a_j\}_{j=i}^{\infty}$, then f_i has A_i-register shift,*

(3) *A_1-register shift of f_1 is generated by $\{J_i\}_{i=1}^{\infty}$ where $|\mathrm{orb}(J_i)| \le (\frac{2}{3})^i$,*

(4) *f_i has a unique turning point.*

PROOF. Part (1) is obvious because $g_a(0) > 0$ and so there is no difficulty even if $a_i = 2$. Interval $I_j^{a_i}$ is periodic with period a_i and $f_i^{a_i}|I_j^{a_i}$ is exactly function f_{i+1} because $I_j^{a_i}$ is mapped once by $f_i|I_2^{a_i}$ which is f_{i+1}^*, once by the order preserving linear homeomorphism $f_i|I_1^{a_i}$ (or zero times if $a_i = 2$) and $a_i - 2$ times by the order reversing linear homeomorphism $f_i|I_j^{a_i}$ for $2 < j \le a_i$ (or once by $f_i|I_1^{a_i}$ if $a_i = 2$). Hence we have part (2). Parts (3) and (4) are obvious. □

Therefore $f_1 \in \tilde{M}_{1,1}$ and it has an A-register shift. Let $J_0 = I$, $J_1 = I_2^{a_1}$ and J_i be the interval corresponding to $I_2^{a_i}$ if we consider only $f_1|J_{i-1}$. More precisely J_i is a periodic interval with period p_i such that $J_i \cap T(f_1) \neq \emptyset$.

Here is our strategy for obtaining a function $f \in M_{1,1}$ such that f has an A-register shift.

1. Let $F_1 = f_1$. Then $\{J_i\}_{i=1}^{\infty}$ is our sequence of periodic intervals which generate A-register shift. Moreover $F_1|J$ is linear with slope greater than 1 for any interval such that $J \cap \mathrm{orb}(J_1) = \emptyset$ (see Claim).

2. Assume that $F_n|J$ is linear with slope greater than 1 for any interval J such that $J \cap \mathrm{orb}(J_n) = \emptyset$. We will modify F_n on the set $\mathrm{orb}(J_{n-1})$ such that we will obtain new intervals J_i for $i \ge n$ and our modified function F_{n+1} will be linear with slope greater than 1 on any interval J such that $J \cap \mathrm{orb}(J_{n+1}) = \emptyset$.

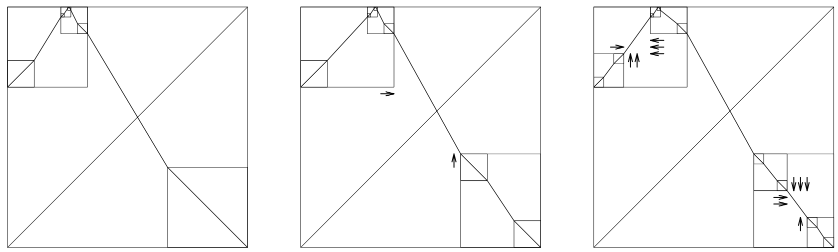

Figure 4: Illustration how to get F_1, F_2, F_3 for $A = \{2, 2, 2, \dots\}$.

3. Finally we will get function $f = \lim_{n\to\infty} F_n$.

Fix $n \in \mathbb{N}$. Let $\mathrm{orb}(J_n) = \{I_i^n\}_{i=1}^{p_n}$ where $I_1^n = J_n$ and $F_n(I_i^n) = I_{i+1}^n$ ($F_n(I_{p_n}^n) = I_1^n$). Let $\mathrm{orb}(J_{n+1})$ equal the set of intervals $\{I_{i,j} = [a_{i,j}, b_{i,j}]\}$ where $1 \le i \le p_n$, $1 \le j \le a_{n+1}$, $I_{i,j} \subset I_i^n$ and $b_{i,j} < a_{i,j+1}$. Of course $I_i^n = [a_{i,1}, b_{i,a_{n+1}}]$.

We have that $|I_i^n| = |I_j^n|$ for any i, j and $F_n|I_i^n$ is either linear with slope ± 1 (if $i > 1$) or "looks like" function f_{n+1} or f_{n+1}^* (if $i = 1$). (All this is true for $n = 1$ and our modification will preserve these properties.)

Let $F_{n+1}|S = F_n|S$ for $S = I \setminus \mathrm{orb}(J_{n-1})$ and we will define F_{n+1} on $\mathrm{orb}(J_{n-1})$. Take $c > 1$ and new intervals $I_i^* \subset \mathrm{orb}(J_{n-1})$ such that $I_2^* = I_2^n$, $I_i^n \subset I_i^*$ for $1 \le i \le p_i$, $|I_{i+1}^*| = c|I_i^*|$ for $2 \le i < p_i$ and $|I_1^*| = c|I_{p_i}^*|$. We can choose c so small that the intervals I_i^* are pairwise disjoint. Let $F_{n+1}(I_i^*) = I_{i+1}^*$ ($F_{n+1}(I_{p_i}^*) = I_1^*$) and let F_{n+1} be linear outside the intervals I_i^*. Choose $a_{i,j}^*, b_{i,j}^* \in I_i^*$ such that

$$\begin{aligned} I_i^* &= [a_{i,1}^*, b_{i,a_{n+1}}^*], \\ a_{i,j}^* < b_{i,j}^* &< a_{i,j+1}^* < b_{i,j+1}^* && \text{for } 1 \le j < a_{n+1}, \\ b_{i,j}^* - a_{i,j}^* &= b_{i,j} - a_{i,j} && \text{for } 1 \le j \le a_{n+1}, \\ a_{i,j+1}^* - b_{i,j}^* &= k(a_{i,j+1} - b_{i,j}) && \text{for } 1 \le j < a_{n+1} \end{aligned}$$

for some constant $k > 1$ and let $I_{i,j}^* = [a_{i,j}^*, b_{i,j}^*]$. Now we can complete the definition of F_{n+1}.

If $F_n(I_{i,j}) = I_{i+1,s}$, then $F_{n+1}(I_{i,j}^*) = I_{i+1,s}^*$ and the graph of F_{n+1} on $I_{i,j}^*$ will be the same as the graph of F_n on $I_{i,j}$. (It will be linear with slope 1 unless $I_{i,j} = J_{n+1}$ when it will "look like" f_{n+2} or f_{n+2}^*.) And let F_{n+1} be linear outside the intervals $I_{i,j}^*$. (See figure 4.)

So F_{n+1} is completely defined. Let the new $J_n = I_1^*$, $J_{n+1} = I_{1,s}^*$ (where old $J_{n+1} = I_{1,s}^*$) and J_i for $i \geq n+2$ be given by $F_{n+1}^{p_{n+1}}|I_{1,s}^*$. Moreover, the slopes of F_{n+1} may be decreased comparing to the slopes of F_n only on the set $(\mathrm{orb}(J_{n-1}) \setminus \mathrm{orb}(J_n)) \cup (J_n \setminus \mathrm{orb}(J_{n+1}))$ where they were bigger than 1 and so it is possible to choose $c > 1$ such that the changes are small enough and the slopes remain greater than 1. And finally it is obvious that the slopes of F_{n+1} on the set $\mathrm{orb}(J_n) \setminus (\mathrm{orb}(J_{n+1}) \cup J_n)$ are now greater than c. Hence we made the required modification.

Moreover we can choose $c > 1$ sufficiently small such that $|\mathrm{orb}(J_n)|$ increases during this modification no more than twice. And obviously for $i > n$ $|\mathrm{orb}(J_i)|$ remains the same. So we made modification on an invariant set S where $|S| \leq 2(\frac{2}{3})^{n-1}$ and this set remains invariant. This proves that $\lim\limits_{n\to\infty} F_n = f$ exists and is continuous. It is obvious that $f \in M_{1,1}$ and f has an A-register shift. □

Corollary 5.2.. *The function f from Lemma 5.1 has no transitive f-cycle.*

PROOF. Each register shift and transitive f-cycle are disjoint and they must contain a turning point. But our f has only one turning point and a register shift. □

Remark 5.1. *For the construction of $f \in \tilde{M}_{n,c}$ for $c \in [0,1]$, see [1], [4].*

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