REGULAR CLOSURE OPERATORS

G. Castellini [1], J. Koslowski, G. E. Strecker

Dedicated to Professor Dieter Pumplün, on his 60th birthday

ABSTRACT: In an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category $\mathscr{X}$ for sinks, we identify necessary conditions for Galois connections from the power collection of the class of (composable pairs) of morphisms in $\mathscr{M}$ to factor through the "lattice" of all closure operators on $\mathscr{M}$, and to factor through certain sublattices. This leads to the notion of *regular* closure operator. As one byproduct of these results we not only arrive (in a novel way) at the Pumplün–Röhrl polarity between collections of morphisms and collections of objects in such a category, but obtain many factorizations of that polarity as well. (One of these factorizations constituted the main result of an earlier paper by the same authors). Another byproduct is the clarification of the Salbany construction (by means of relative dominions) of the largest idempotent closure operator that has a specified class of $\mathscr{X}$-objects as separated objects. The same relation that is used in Salbany's relative dominion construction induces classical regular closure operators as described above. Many other types of closure operators can be obtained by this technique; particular instances of this are the idempotent and modal closure operators that in a Grothendieck topos correspond to the Grothendieck topologies.

KEY WORDS: Galois connection, polarity, closure operator, separated object, dense morphism, composable pair of morphisms, factorization structure for sinks.

CLASSIFICATION: Primary: 18A32
Secondary: 06A15, 54B30

0 INTRODUCTION

The goal of this paper is to identify common principles that underlie the construction of various types of closure operators, from regular closure operators induced by some class of objects by means of Salbany's relative dominion construction (cf. [13] and [10]), to idempotent modal closure operators induced by some Grothendieck topology, and to pinpoint the differences between these constructions. Our analysis will also shed additional light on the Pumplün–Röhrl connection and its factorizations, previously dealt with in [5], and allows us to generalize this connection (together with its factorizations) in two different directions.

Section 1 introduces the orthogonality relation $\perp$ that underlies much of the theory. In addition, we recall some background material on Galois connections.

In Section 2 for a category $\mathscr{X}$ with a factorization structure $\langle \boldsymbol{E}, \mathscr{M} \rangle$ for sinks we recall some basic material concerning closure operators on $\mathscr{M}$. In particular the Galois connection $\dot{\omega}$ from the power collection of the class $\mathscr{M} \diamond \mathscr{M}$ of composable pairs of morphisms in $\mathscr{M}$ to

[1] Research supported by the Faculty of Arts and Sciences – University of Puerto Rico – Mayagüez Campus during a sabbatical visit at Kansas State University.

the lattice $\mathscr{M}$-$\boldsymbol{CL}$ of all closure operators on $\mathscr{M}$, and its close relative, the Galois connection $\dot{\Delta}$ from the power collection of $\mathscr{M}$ to the lattice $\mathscr{M}$-$\boldsymbol{wCL}$ of all weakly hereditary closure operators on $\mathscr{M}$ are described. Moreover, $\mathscr{Z}$-modal closure operators are introduced, where $\mathscr{Z}$ is a class of morphisms in the category $\mathscr{M}$ (viewed as a full subcategory of $\mathscr{X}/\mathscr{X}$) that is stable under pullbacks. We obtain the new result that the interaction between idempotent closure operators and weakly hereditary closure operators can be relativized to the $\mathscr{Z}$-modal setting, thereby exhibiting $\dot{\omega}$ and $\dot{\Delta}$ as functors into the category of Galois connections.

In Section 3 we introduce the concept of regular closure operators relative to arbitrary Galois connections of the form $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathscr{Z}} \xrightarrow{\rho\ \bullet} \mathscr{D}$ and $\mathscr{M}\boldsymbol{P}_{\mathscr{Z}} \xrightarrow{\tau\ \bullet} \mathscr{D}$. This notion depends upon ρ and τ factoring through the Galois connections $\dot{\omega}_{\mathscr{Z}}$ and $\dot{\Delta}_{\mathscr{Z}}$ of Section 2, respectively. We analyze the conditions under which this happens.

Section 4 identifies certain restrictions and modifications of the orthogonality relation $\perp$ such that the induced specific variants of the polarities ρ and τ factor as described in Section 3. In particular, we determine which types of conditions (like specific closed morphisms, or specific separated objects) may be imposed such that there exists a largest $\mathscr{Z}$-modal closure operator that satisfies these conditions, and when this operator is idempotent. Such conditions may be viewed as pairs $\langle h, \boldsymbol{h}\rangle$ consisting of a morphism h and a source $\boldsymbol{h}$ with matching codomain and domain. A key observation is that the Pumplün–Röhrl separating relation (cf. [12]) can be expressed in terms of the orthogonality relation if one identifies objects with sources consisting of two identity morphisms (rather than with empty sources or empty sinks, as is usually done). Our shift of perspective allows us to obtain Salbany's classical construction (cf. [13]) of the largest closure operator with a specific class of separated objects without resorting to equalizers of intersections. Consequently, we gain new insights into the Pumplün–Röhrl polarity and its factorizations, which enables us to strengthen results obtained in [5] and to put them into a broader perspective.

The types of conditions that may be imposed in section 4 are limited in as far as the pairs $\langle h, \boldsymbol{h}\rangle$ have to be right-orthogonal to certain sinks in $\boldsymbol{E}$. This initially eliminates the sheaf relation (known from topos theory) and the corresponding variant of τ from consideration. In Section 5 we show that for modal closure operators (i.e., when $\mathscr{Z}$ consists of all cartesian morphisms in the category $\mathscr{M}$) that are idempotent this restriction on $\boldsymbol{h}$ may be dropped, as long as certain sinks in $\boldsymbol{E}$ are colimit sinks. The proof indicates that other values of the parameter $\mathscr{Z}$ are unlikely to work, and that idempotency is essential. This emphasizes the very special role idempotent modal closure operators play.

1 PRELIMINARIES

Our main tool will be a notion of orthogonality that generalizes the one introduced by Tholen (cf. [15]) and encompasses part of the defining properties of factorization structures for sinks and for sources as well as one of the essential features of closure operators (cf. Definition 2.00).

1.00 DEFINITION (cf. [11])

In a category $\mathscr{X}$ a pair $\langle \boldsymbol{l}, l\rangle$ consisting of a sink $\boldsymbol{l} = \langle iA \xrightarrow{il} A'\rangle_I$ and a morphism $A' \xrightarrow{l} A''$ is called **left orthogonal to** a pair $\langle r, \boldsymbol{r}\rangle$ consisting of a morphism $B \xrightarrow{r} B'$ and a source $\boldsymbol{b} = \langle B' \xrightarrow{jr} jB''\rangle_J$, written as $\langle \boldsymbol{l}, l\rangle \perp \langle r, \boldsymbol{r}\rangle$, iff for any sink $\boldsymbol{f} = \langle iA \xrightarrow{if} B\rangle_I$ and any source $\boldsymbol{f}'' = \langle A'' \xrightarrow{jf''} jB''\rangle_J$ with the property that for each $i \in I$ and each $j \in J$ the outer square of following diagram commutes

$$\begin{tikzcd}
iA \arrow[rrr, "if"] \arrow[dr, "il"] \arrow[dd, "il\,;\,l"'] & & & B \arrow[dl, "r"'] \arrow[dd, "r\,;\,jr"] \\
& A' \arrow[r, "f'"] \arrow[dl, "l"] & B' \arrow[dr, "jr"'] & \\
A'' \arrow[rrr, "jf''"'] & & & jB''
\end{tikzcd} \tag{1-00}$$

there exists a unique $\mathscr{X}$-morphism $A' \xrightarrow{f'} B'$ such that all inner trapezoids commute. In this case the pair $\langle r, \boldsymbol{r}\rangle$ is called **right orthogonal to** $\langle \boldsymbol{l}, l\rangle$. We write $\boldsymbol{l} \perp \boldsymbol{r}$ rather than $\langle \boldsymbol{l}, id\rangle \perp \langle id, \boldsymbol{r}\rangle$, i.e., we suppress the morphism-part of a pair in case it is an isomorphism.

$\boldsymbol{L}$ denotes the collection of all pairs $\langle \boldsymbol{l}, l\rangle$ consisting of an $\mathscr{X}$-sink $\boldsymbol{l}$ and an $\mathscr{X}$-morphism l with matching codomain and domain, and $\boldsymbol{R}$ stands for the corresponding collection of all pairs consisting of an $\mathscr{X}$-morphism and an $\mathscr{X}$-source.

We say that $\langle \boldsymbol{l}, l\rangle$ is **separated from** an object X iff $\langle \boldsymbol{l}, l\rangle$ is left-orthogonal to the 2-source $X \xleftarrow{id_X} X \xrightarrow{id_X} X$.

1.01 PROPOSITION

(0) *If $\mathscr{X}$ is an $\langle \boldsymbol{E}, \mathscr{M}\rangle$-category for sinks, then a sink $\boldsymbol{a}$ belongs to $\boldsymbol{E}$ iff $\boldsymbol{a} \perp m$ for every $m \in \mathscr{M}$, and conversely, a morphism b belongs to $\mathscr{M}$ iff $\boldsymbol{e} \perp b$ for every $\boldsymbol{e} \in \boldsymbol{E}$.*

(1) *A morphism a and an object X are separated in the sense of Pumplün and Röhrl (cf. [12]), if the singleton sink consisting of a is separated from X.*

(2) *A sink $\boldsymbol{a}$ is an epi-sink iff $\boldsymbol{a}$ is separated from every $\mathscr{X}$-object.* $\square$

Next we recall some facts about Galois connections between pre-ordered classes, especially between power collections, and introduce some convenient notation.

1.02 DEFINITION

For pre-ordered classes $\mathscr{A} = \langle A, \leq \rangle$ and $\mathscr{B} = \langle B, \sqsubseteq \rangle$ a **Galois connection** $\mathscr{A} \xrightarrow{\pi=\langle \pi_*, \pi^* \rangle} \mathscr{B}$ consists of two order-preserving functions $\mathscr{A} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathscr{B}$ that satisfy $id_{\mathscr{A}} \leq \pi_* \,;\, \pi^*$ and $\pi_* \,;\, \pi^* \sqsubseteq id_{\mathscr{B}}$. Then π^* is the **right adjoint** part and π_* is the **left adjoint** part of π. We write $\mathscr{A}^\pi$ and $\mathscr{B}_\pi$ for the classes of **left fixed points** $\{\, a \in A \mid a \cong a\pi_*\pi^* \,\}$ and **right fixed points** $\{\, b \in B \mid b\pi^*\pi_* \cong b \,\}$ with the induced orders, respectively. π is called a (**co**)**reflection** iff π^* (respectively π_*) is a one-to-one function, and an **equivalence** iff $\langle \pi^*, \pi_* \rangle$ is a Galois connection from $\mathscr{B}$ to $\mathscr{A}$. (Notice that π restricts to an equivalence from $\mathscr{A}^\pi$ to $\mathscr{B}_\pi$.)

The **composite** $\mathscr{A} \xrightarrow{\pi \,;\, \rho} \mathscr{C}$ of two Galois connections $\mathscr{A} \xrightarrow{\pi} \mathscr{B}$ and $\mathscr{B} \xrightarrow{\rho} \mathscr{C}$ is defined as $\langle \pi_* \,;\, \rho_*, \rho^* \,;\, \pi^* \rangle$, and the **dual** $\mathscr{B}^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathscr{A}^{\mathrm{op}}$ of a Galois connection $\mathscr{A} \xrightarrow{\pi} \mathscr{B}$ is given by $\pi^{\mathrm{op}} = \langle \pi^*, \pi_* \rangle$. This defines a category $\mathscr{GAL}$ with a contravariant involution $(-)^{\mathrm{op}}$.

If $\mathscr{A} \xrightarrow{\varphi} \mathscr{C}$ factors as $\mathscr{A} \xrightarrow{\dot\varphi} \mathscr{B} \xrightarrow{\ddot\varphi} \mathscr{C}$ and if $\mathscr{B}$ is equivalent to both fixed point classes $\mathscr{A}^\varphi$ and $\mathscr{C}_\varphi$, then we call $\dot\varphi \,;\, \ddot\varphi$ an **essentially canonical factorization** of φ with **center** $\mathscr{B}$. The dot notation will be employed throughout to indicate essentially canonical factorizations.

1.03 PROPOSITION

If $\mathscr{A} \xrightarrow{\dot\varphi} \mathscr{B} \xrightarrow{\ddot\varphi} \mathscr{C}$ is an essentially canonical factorization of $\mathscr{A} \xrightarrow{\varphi} \mathscr{C}$ with center $\mathscr{B}$, then $\dot\varphi^* \,;\, \dot\varphi_* \cong id_{\mathscr{B}} \cong \ddot\varphi_* \,;\, \ddot\varphi^*$. □

1.04 PROPOSITION (cf. [9])

A relation $R \subseteq A \times B$ between classes A and B defines Galois connections $A\boldsymbol{P} \xrightarrow{\varphi} B\boldsymbol{P}$, called an **axiality**, *and $A\boldsymbol{P} \xrightarrow{\psi} B\boldsymbol{P}^{\mathrm{op}}$, called a* **polarity**, *via*

$$\begin{array}{lll}
U\psi_* := \{\, b \in B \mid \exists_{a \in A}\ \langle a, b \rangle \in R \text{ and } a \in U \,\} & & \textit{for } U \subseteq A \\
V\psi^* := \{\, a \in A \mid \forall_{b \in B}\ \langle a, b \rangle \in R \text{ implies } b \in V \,\} & & \textit{for } V \subseteq B \\
U\varphi_* := \{\, b \in B \mid \forall_{a \in A}\ a \in U \text{ implies } \langle a, b \rangle \in R \,\} & & \text{for } U \subseteq A \\
V\varphi^* := \{\, a \in A \mid \forall_{b \in B}\ b \in V \text{ implies } \langle a, b \rangle \in R \,\} & & \text{for } V \subseteq B
\end{array}$$
□

2 CLOSURE OPERATORS

The topologically-motivated notion of a closure operator for a category $\mathscr{X}$ depends on a class $\mathscr{M}$ of $\mathscr{X}$-morphisms (corresponding to the embeddings in $\boldsymbol{Top}$). We regard $\mathscr{M}$ as a full subcategory of the arrow category of $\mathscr{X}$. By an $\mathscr{M}$**-morphism** $\langle f, g \rangle$ from $m \in \mathscr{M}$ to $n \in \mathscr{M}$ we mean a pair of $\mathscr{X}$-morphisms that satisfy $m \,;\, g = f \,;\, n$. The **domain functor** $\mathscr{M} \xrightarrow{U} \mathscr{X}$ maps $\langle f, g \rangle$ to f, while the **codomain functor** V maps $\langle f, g \rangle$ to g. An $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g \rangle} n$ is called **cartesian** iff it is V**-initial** (cf. [0]). This is equivalent to $m \xrightarrow{\langle f,g \rangle} n$ constituting a pullback square in $\mathscr{X}$.

2.00 DEFINITION

A **closure** (resp. **density**) **operator** $F = \langle (\,)_F, (\,)^F \rangle$ on $\mathscr{M}$ maps each $m \in \mathscr{M}$ to a pair $\langle m_F, m^F \rangle$ with $m = m_F \,;\, m^F$ and $m^F \in \mathscr{M}$ (resp. $m_F \in \mathscr{M}$) such that $mF \perp nF$ for all $n \in \mathscr{M}$.

If F is a closure or density operator, $m \in \mathscr{M}$ is called F**-closed** (resp. F**-dense**) if m_F (resp. m^F) is an isomorphism. $F\nabla_*$ and $F\Delta^*$ denote the classes of F-closed and F-dense members of $\mathscr{M}$, respectively.

2.01 REMARKS

(0) A succinct categorical formulation of the concept of closure operator, first proposed by Dikranjan and Giuli [7], views $(\,)^F$ as an endofunctor $\mathscr{M} \xrightarrow{(\,)^F} \mathscr{M}$ that satisfies $(\,)^F V = V$, and views $(\,)_F$ as the domain-part of a natural transformation $id_{\mathscr{M}} \xrightarrow{\delta} (\,)^F$ that satisfies $\delta id_V = id_V$. The uniqueness part of the orthogonality condition then says that $\langle \delta, (\,)^F \rangle$ is a pre-reflection in the sense of Börger [1], cf. also [16].

(1) If $\mathscr{M}$ satisfies the cancellation condition

$$n \,;\, p \in \mathscr{M} \quad \text{and} \quad p \in \mathscr{M} \quad \text{implies} \quad n \in \mathscr{M} \tag{2-00}$$

then every closure operator on $\mathscr{M}$ is automatically a density operator on $\mathscr{M}$. Furthermore, if F is a closure operator on a class $\mathscr{M}$ of monos, as we will assume later, the uniqueness part of the orthogonality condition is automatically satisfied.

In order to have analogues to the complete subspace lattices of topological spaces, for the remainder of the paper we assume that $\mathscr{X}$ is an $\langle \boldsymbol{E}, \mathscr{M} \rangle$-category for sinks. This insures that $\mathscr{X}$ is sufficiently nice to support certain constructions (cf. Proposition 1.05 of [3]; for proofs see Section 15 of [0]). In particular, $\mathscr{M}$ then consists of monos and satisfies the cancellation condition (2-00). $\leq$ denotes the usual pre-order on $\mathscr{M}$-subobjects. To minimize problems resulting from the fact that this pre-order need not be antisymmetric, we assume that for every sink $\boldsymbol{s}$ in $\mathscr{X}$ a specific $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorization has been chosen. (This choice need not be canonical in any sense.) Now we can speak about *the* infimum and *the* supremum of a sink consisting of $\mathscr{M}$-elements, as well as of *the* pullback of an $\mathscr{M}$-element. We continue to use the term **lattice** for a pre-ordered class that is finitely complete and finitely cocomplete. A lattice is called **complete** if every subclass has an infimum or, equivalently, every subclass has a supremum.

The codomain functor V now is a bi-fibration in the sense that all inverse images exist as do all direct images. More precisely, for an $\mathscr{X}$-morphism $X \xrightarrow{f} Y$ the V**-inverse image functor** $V/Y \xrightarrow{f^{\leftarrow}} V/X$ maps an $\mathscr{M}$-subobject of Y to its pullback along f , while its left adjoint, the V**-direct image functor** $V/X \xrightarrow{f^{\rightarrow}} V/Y$, maps an $\mathscr{M}$-subobject m of X to the $\mathscr{M}$-component of the chosen $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorization of $m \,;\, f$.

2.02 DEFINITION

Pulling V back along U induces a category $\mathscr{M} \diamond \mathscr{M}$ with composable pairs $\langle n, p\rangle \in \mathscr{M} \times \mathscr{M}$ as objects. Morphisms from $\langle n, p\rangle$ to $\langle q, r\rangle$ are triples $\langle a, b, c\rangle$ of $\mathscr{X}$-morphisms with $a ; q = n ; b$ and $b ; r = p ; c$. The functor $\mathscr{M} \diamond \mathscr{M} \xrightarrow{W} \mathscr{M}$ maps $\langle n, p\rangle \xrightarrow{\langle a,b,c\rangle} \langle q, r\rangle$ to the $\mathscr{M}$-morphism $n ; p \xrightarrow{\langle a,c\rangle} q ; r$. For $m \in \mathscr{M}$ a pre-order on the fiber W/m is given by $\langle n, p\rangle \ll \langle q, r\rangle$ iff there exists a (necessarily unique) $\mathscr{X}$-morphism b with $n ; b = q$ and $b ; r = p$.

The W-fibers form (possibly large) complete lattices with respect to $\ll$. Intersections and $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorizations of the (collections of) second components yield infima and suprema, respectively. W is also a bi-fibration. Given $m \xrightarrow{\langle f,g\rangle} n$, the **$W$-inverse image functor** $W/n \xrightarrow{\langle f,g\rangle^{\leftarrow}} W/m$ maps $\langle s, t\rangle \in W/n$ to the unique $\langle q, r\rangle \in W/m$ whose second component is the chosen pullback of t along g. The **W-direct image functor**, $W/m \xrightarrow{\langle f,g\rangle_{\rightarrow}} W/n$, maps $\langle q, r\rangle \in W/m$ to the unique $\langle s, t\rangle \in W/n$ for which there exists an $\mathscr{X}$-morphism d such that $\langle\langle d, s\rangle, t\rangle$ is the chosen $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorization of the 2-sink $\bullet \xrightarrow{r\,;\,g} \bullet \xleftarrow{n} \bullet$. This yields a Galois connection from W/m to W/n with left adjoint $\langle f, g\rangle_{\rightarrow}$ and right adjoint $\langle f, g\rangle^{\leftarrow}$. Details can be found in [3]. The smallest class containing $C \subseteq \mathscr{M} \diamond \mathscr{M}$ and stable under the formation of W-direct images is denoted by C^{di}.

2.03 DEFINITION

A closure operator F is called

(0) **idempotent** iff m^F is F-closed for every $m \in \mathscr{M}$, i.e., iff $(\,)^F(\,)^F \cong (\,)^F$;

(1) **weakly hereditary** iff m_F is F-dense for every $m \in \mathscr{M}$, i.e., iff $(\,)_F(\,)_F \cong (\,)_F$.

(2) **hereditary** iff n^F is a pullback of m^F along p whenever $\langle n, p\rangle \in W/m$;

(3) **modal** iff n^F is a pullback of m^F along g whenever $n \xrightarrow{\langle f,g\rangle} m$ is cartesian, i.e., iff F preserves cartesian $\mathscr{M}$-morphisms.

$\mathscr{M}$-$\boldsymbol{CL}$ denotes the collection of all closure operators on $\mathscr{M}$, pre-ordered by $F \sqsubseteq G$ iff $mF \ll mG$ for all $m \in \mathscr{M}$, while $\mathscr{M}$-$\boldsymbol{iCL}$, $\mathscr{M}$-$\boldsymbol{wCL}$, $\mathscr{M}$-$\boldsymbol{hCL}$, $\mathscr{M}$-$\boldsymbol{mCL}$, and $\mathscr{M}$-$\boldsymbol{iwCL}$ stand for the subcollections of idempotent, weakly hereditary, hereditary, modal, and idempotent weakly hereditary closure operators, respectively.

Notice that under our assumptions arbitrary suprema and infima of closure operators exist. They are formed pointwise in the fibers. In particular, $\mathscr{M}$-$\boldsymbol{wCL}$ is stable under the formation of suprema and $\mathscr{M}$-$\boldsymbol{iCL}$ is stable under the formation of infima in $\mathscr{M}$-$\boldsymbol{CL}$. Thus every closure operator F has an **idempotent hull** (i.e., reflection) $F^{\mathrm{i}} \in \mathscr{M}$-$\boldsymbol{iCL}$ as well as a **weakly hereditary core** (i.e., coreflection) $F_{\mathrm{w}} \in \mathscr{M}$-$\boldsymbol{wCL}$. The first construction of these hulls and cores that did not rely on smallness properties of $\mathscr{X}$ with respect to $\mathscr{M}$ can be found in [11], Theorem 1.12, cf. also [5], Lemma 1.10. For additional background on closure operators see, e.g., [2], [7], and [8].

2.04 REMARK

From [3] we recall the following commuting diagrams of Galois connections the individual parts of which are explained below.

$$
\begin{tikzcd}
\mathscr{M}\boldsymbol{P} \arrow[r, "\dot\Delta"] \arrow[d, "\gamma"'] & \mathscr{M}\text{-}\boldsymbol{wCL} \arrow[d, "\ddot\Delta"] \\
(\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P} \arrow[r, "\dot\omega"'] & \mathscr{M}\text{-}\boldsymbol{CL}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathscr{M}\text{-}\boldsymbol{wCL} \arrow[r, "\dot\epsilon"] \arrow[d, "\ddot\Delta"'] & \mathscr{M}\text{-}\boldsymbol{iwCL} \arrow[d, "\ddot\epsilon"] \\
\mathscr{M}\text{-}\boldsymbol{CL} \arrow[r, "\dot\nabla"'] & \mathscr{M}\text{-}\boldsymbol{iCL}
\end{tikzcd}
\qquad (2\text{-}01)
$$

(0) $\mathscr{M}\boldsymbol{P} \xrightarrow{\gamma} (\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P}$ is the axiality induced by the opposite of the graph of the first projection $\mathscr{M}\diamond\mathscr{M} \longrightarrow \mathscr{M}$, i.e., $A\gamma_* = \{\,\langle n,p\rangle \in \mathscr{M}\diamond\mathscr{M} \mid n \in A\}$ for $A \subseteq \mathscr{M}$.

(1) The polarity $(\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P} \xrightarrow{\omega} (\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$ induced by restricting $\perp$ to $\mathscr{M}\diamond\mathscr{M}$ admits an essentially canonical factorization $\omega = \dot\omega\,;\ddot\omega$. The first factor is given by $mC\dot\omega_* := \big(C^{\mathrm{di}} \cap W/m\big)\mathrm{sup}_{\ll}$ and $F\dot\omega^* := \{\,mF \mid m \in \mathscr{M}\,\}\omega^*$, and the second factor $\mathscr{M}\text{-}\boldsymbol{CL} \xrightarrow{\ddot\omega} (\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$ is defined symmetrically. Elements of $F\dot\omega^*$ are called **relatively** F**-dense** pairs $\langle n,p\rangle \in \mathscr{M}\diamond\mathscr{M}$, and they satisfy $\langle n,p\rangle \perp mF$ for each $m \in \mathscr{M}$. Both Galois fixed point lattices of ω are equivalent to $\mathscr{M}\text{-}\boldsymbol{CL}$.

(2) Δ^* maps a closure operator to its class of dense $\mathscr{M}$-elements (cf. Definition 2.00). The induced Galois connection $\mathscr{M}\boldsymbol{P} \xrightarrow{\Delta} \mathscr{M}\text{-}\boldsymbol{CL}$ factors through $\dot\omega$ by means of γ: if $A \subseteq \mathscr{M}$ then $A\Delta_*$ is weakly hereditary and maps $m \in \mathscr{M}$ to the supremum of all those $\langle n,p\rangle \in W/m$ that are a W-direct image of some $\langle q,r\rangle \in \mathscr{M}\diamond\mathscr{M}$ with $q \in A$. Both fixed point lattices of Δ are equivalent to $\mathscr{M}\text{-}\boldsymbol{wCL}$, hence we get an essentially canonical factorization $\Delta = \dot\Delta\,;\ddot\Delta$. This makes more precise the observation in [11] that (up to isomorphism) a weakly hereditary closure operator F is determined by $F\Delta^*$.

(3) $\dot\nabla_*$ maps a closure operator F to its idempotent hull, and the composite $\ddot\Delta\,;\dot\nabla$ is a Galois connection denoted by $\mathscr{M}\text{-}\boldsymbol{wCL} \xrightarrow{\epsilon} \mathscr{M}\text{-}\boldsymbol{iCL}$. It has the property that its equivalent Galois fixed point lattices in $\mathscr{M}\text{-}\boldsymbol{wCL}$ and in $\mathscr{M}\text{-}\boldsymbol{iCL}$ actually coincide: they are equal to $\mathscr{M}\text{-}\boldsymbol{iwCL}$.

For the remainder of this paper $\mathcal{Z}$ is a collection of $\mathscr{M}$-morphisms that is stable under pullbacks in $\mathscr{M}$. Under ordinary inclusion these collections form a partial order category $\mathscr{Z}$. Of particular interest are the following collections of cartesian $\mathscr{M}$-morphisms:

$$
\begin{aligned}
\mathcal{H} &= \{\,\langle f,g\rangle \in \mathscr{M}\text{-}\boldsymbol{Mor} \mid f \text{ is iso and } g \in \mathscr{M}\,\} \\
\mathcal{C} &= \{\,\langle f,g\rangle \in \mathscr{M}\text{-}\boldsymbol{Mor} \mid \langle f,g\rangle \text{ is cartesian }\} \\
\mathcal{I} &= \mathscr{M}\text{-}\boldsymbol{Iso}
\end{aligned}
$$

The full subcategory of $\mathscr{Z}$ spanned by pullback-stable collections of cartesian $\mathscr{M}$-morphisms is denoted by $\mathscr{C}$. We wish to interpret the diagrams in (2-01) as instances at $\mathcal{Z} = \mathcal{I}$ of similar diagrams of natural transformations between functors with codomain $\mathscr{GAL}$. In the right diagram $\mathscr{Z}$ may be used as domain of the functors, but in the left diagram we need to restrict our attention to $\mathscr{C}$.

2.05 DEFINITION

(0) A closure operator F on $\mathscr{M}$ is called $\mathcal{Z}$**-modal** if F commutes with W-inverse images along members of $\mathcal{Z}$; i.e., if $m \xrightarrow{\langle f,g\rangle} n$ belongs to $\mathcal{Z}$, then $mF \cong nF\langle f,g\rangle^{\leftarrow}$. We write $\boldsymbol{CL}_{\mathcal{Z}}$ for the collection of all $\mathcal{Z}$-modal closure operators on $\mathscr{M}$, and $\boldsymbol{wCL}_{\mathcal{Z}}$, $\boldsymbol{iCL}_{\mathcal{Z}}$, $\boldsymbol{iwCL}_{\mathcal{Z}}$ for the corresponding collections of $\mathcal{Z}$-modal closure operators that are either weakly hereditary, or idempotent, or both.

(1) $C \subseteq \mathscr{M} \diamond \mathscr{M}$ is called $\mathcal{Z}$**-stable** if whenever $m \xrightarrow{\langle f,g\rangle} n$ belongs to $\mathcal{Z}$, then the W-inverse image of any $\langle s,t\rangle \in C \cap W/n$ along $\langle f,g\rangle$ belongs to C. A collection $A \subseteq \mathscr{M}$ is called $\mathcal{Z}$**-stable**, if $A\gamma_*$ has this property. $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}}$ and $\mathscr{M}\boldsymbol{P}_{\mathcal{Z}}$ denote the corresponding subcollections of the power collections, respectively.

(2) $\langle s,t\rangle \in \mathscr{M} \diamond \mathscr{M}$ is called **left-**$\mathcal{Z}$**-orthogonal** to $\langle r,\boldsymbol{r}\rangle \in \boldsymbol{R}$, written as $\langle s,t\rangle \perp_{\mathcal{Z}} \langle r,\boldsymbol{r}\rangle$, if for every $\langle f,g\rangle \in \mathcal{Z}$ with codomain $q\,;r$ any W-inverse image $\langle n,p\rangle$ of $\langle s,t\rangle$ along $\langle f,g\rangle$ is left-orthogonal to $\langle r,\boldsymbol{r}\rangle$. The polarity from $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}$ to $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$ induced by $\perp_{\mathcal{Z}}$ restricts to a Galois connection $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\omega_{\mathcal{Z}}}\!\!\bullet\, (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$.

(3) W is said to satisfy the **Beck-Chevalley condition relative to** $\mathcal{Z}$ if for any pullback $m \xleftarrow{\langle a',b'\rangle} o \xrightarrow{\langle f',g'\rangle} p$ of $m \xrightarrow{\langle f,g\rangle} n \xleftarrow{\langle a,b\rangle} p$ in $\mathscr{M}$ with $\langle f,g\rangle \in \mathcal{Z}$ and for every $\langle q,r\rangle \in W/m$ we have

$$(\langle q,r\rangle)\,\langle f,g\rangle_{\rightarrow}\langle a,b\rangle^{\leftarrow} \cong (\langle q,r\rangle)\,\langle a',b'\rangle^{\leftarrow}\langle f',g'\rangle_{\rightarrow}$$

2.06 REMARKS

(0) If $\mathcal{Y}$ is the compositive hull of $\mathcal{Z}$, then $\boldsymbol{CL}_{\mathcal{Y}}$ and $\boldsymbol{CL}_{\mathcal{Z}}$ clearly coincide. Hence without loss of generality one may assume that $\mathcal{Z}$ is closed under composition.

(1) Since each $\mathcal{Z} \in \mathscr{Z}$ is stable under pullbacks, $\mathcal{Z}$-stability of $A \subseteq \mathscr{M}$ is equivalent to the requirement that whenever $m \xrightarrow{\langle f,g\rangle} n$ belongs to $\mathcal{Z}$ and $n \in A$, then $m \in A$.

(2) For the specific values of $\mathcal{Z}$ described above we have

$$\mathscr{M}\text{-}\boldsymbol{hCL} = \boldsymbol{CL}_{\mathcal{H}} \quad , \quad \mathscr{M}\text{-}\boldsymbol{mCL} = \boldsymbol{CL}_{\mathcal{C}} \quad \text{and} \quad \mathscr{M}\text{-}\boldsymbol{CL} = \boldsymbol{CL}_{\mathcal{I}}$$

Hereditary closure operators are well-known to be weakly hereditary. Hence for $\mathcal{H} \subseteq \mathcal{Z}$ we have $\boldsymbol{CL}_{\mathcal{Z}} = \boldsymbol{wCL}_{\mathcal{Z}}$ and $\boldsymbol{iCL}_{\mathcal{Z}} = \boldsymbol{iwCL}_{\mathcal{Z}}$.

(3) Since for any $\mathscr{M}$-morphism $\langle f,g\rangle$ the W-inverse image functor $\langle f,g\rangle_{\leftarrow}$ is right adjoint and hence preserves infima, $\boldsymbol{CL}_{\mathcal{Z}}$ clearly is closed under the formation of infima in $\boldsymbol{CL}_{\mathcal{I}} = \mathscr{M}\text{-}\boldsymbol{CL}$. In particular, every closure operator F has a $\mathcal{Z}$**-modal hull** $F^{\mathcal{Z}}$.

Since $\boldsymbol{iCL}_{\mathcal{I}} = \mathscr{M}\text{-}\boldsymbol{iCL}$ also is closed under the formation of infima in $\boldsymbol{CL}_{\mathcal{I}}$, this is true for $\boldsymbol{iCL}_{\mathcal{Z}} = \boldsymbol{CL}_{\mathcal{Z}} \cap \boldsymbol{iCL}_{\mathcal{I}}$ as well. Hence every closure operator F has an idempotent $\mathcal{Z}$-modal hull $F^{(\mathrm{i}\mathcal{Z})}$ as well.

2.07 LEMMA

(0) *If F is idempotent, so is its weakly hereditary core F_{w}.*

(1) *If F is weakly hereditary, so is its idempotent hull F^{i}.*

(2) *If F is $\mathcal{Z}$-modal, so is its weakly hereditary core F_{w}.*

(3) *If F is weakly hereditary, so is its $\mathcal{Z}$-modal hull $F^{\mathcal{Z}}$.*

(4) *If F is idempotent and $\mathcal{Z}$-modal, so is its weakly hereditary core F_{w}.*

(5) *If F is weakly hereditary, so is its idempotent and $\mathcal{Z}$-modal hull $F^{(\mathrm{i}\mathcal{Z})}$.*

Proof:

(0) and (1) are well-known, cf. [7] and [11].

(2) Let F be $\mathcal{Z}$-modal, and let $\langle q, r\rangle$ be the W-inverse image of nF_{w} along an $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g\rangle} n$ in $\mathcal{Z}$. Write d for the corresponding pullback of g along $n^{F_{\mathrm{w}}}$. The $\mathscr{M}$-morphism $\langle f, d\rangle$ as a pullback of $\langle f, g\rangle$ in the category $\mathscr{M}$ belongs again to $\mathcal{Z}$. Since $n_{F_{\mathrm{w}}}$ is F-dense and F is $\mathcal{Z}$-modal, q is F-dense as well. By Theorem 1.12 of [11] $mF_{\mathrm{w}} = \{\, \langle s,t\rangle \in W/m \mid s \text{ is } F\text{-dense}\,\}\mathrm{sup}_{\ll}$. In particular, $\langle q, r\rangle \ll mF_{\mathrm{w}}$. On the other hand, by the definition of closure operator and the universal property of the pullback we have an $\mathscr{M} \diamond \mathscr{M}$-morphism from mF_{w} to nF_{w}, which implies that $mF_{\mathrm{w}} \ll \langle q, r\rangle$. Hence $\langle q, r\rangle \cong mF_{\mathrm{w}}$, and therefore F_{w} is $\mathcal{Z}$-modal.

(3) Let F be weakly hereditary. Since $(F^{\mathcal{Z}})_{\mathrm{w}}$ by (0) is $\mathcal{Z}$-modal, and since $F \cong F_{\mathrm{w}} \sqsubseteq (F^{\mathcal{Z}})_{\mathrm{w}} \sqsubseteq F^{\mathcal{Z}}$, it follows that $(F^{\mathcal{Z}})_{\mathrm{w}} \cong F^{\mathcal{Z}}$, i.e., $F^{\mathcal{Z}}$ is weakly hereditary.

(4) and (5) follow by combining (0) with (2) and (1) with (3), respectively. $\square$

2.08 PROPOSITION

(0) *There exists functors $\boldsymbol{CL}$ and $\boldsymbol{wCL}$ from $\mathscr{X}$ to $\mathscr{GAL}$ such that for any inclusion $\mathcal{Y} \subseteq \mathcal{Z}$ in $\mathscr{X}$ the left adjoints*

$$\boldsymbol{CL}_{\mathcal{Y}} \xrightarrow{(\boldsymbol{CL}_{\mathcal{Y},\mathcal{Z}})_*} \boldsymbol{CL}_{\mathcal{Z}} \qquad \text{and} \qquad \boldsymbol{wCL}_{\mathcal{Y}} \xrightarrow{(\boldsymbol{wCL}_{\mathcal{Y},\mathcal{Z}})_*} \boldsymbol{wCL}_{\mathcal{Z}}$$

map a closure operator F to its $\mathcal{Z}$-modal hull $F^{\mathcal{Z}}$, and the corresponding right adjoints are inclusions.

(1) *There exists functors $\boldsymbol{iCL}$ and $\boldsymbol{iwCL}$ from $\mathscr{X}$ to $\mathscr{GAL}$ such that for any inclusion $\mathcal{Y} \subseteq \mathcal{Z}$ in $\mathscr{X}$ the left adjoints*

$$\boldsymbol{iCL}_{\mathcal{Y}} \xrightarrow{(\boldsymbol{iCL}_{\mathcal{Y},\mathcal{Z}})_*} \boldsymbol{iCL}_{\mathcal{Z}} \qquad \text{and} \qquad \boldsymbol{iwCL}_{\mathcal{Y}} \xrightarrow{(\boldsymbol{iwCL}_{\mathcal{Y},\mathcal{Z}})_*} \boldsymbol{iwCL}_{\mathcal{Z}}$$

map a closure operator F to its idempotent $\mathcal{Z}$-modal hull $F^{(\mathrm{i}\mathcal{Z})}$, and the corresponding right adjoints are inclusions.

(2) *There exists natural transformations $\ddot{\Delta}$ and $\ddot{\epsilon}$ that are point-wise coreflections, and there exist natural transformations $\dot{\epsilon}$ and $\dot{\nabla}$ that are point-wise reflections such that the following diagram commutes*

$$\begin{tikzcd}
\boldsymbol{wCL} \arrow[r, "\ddot{\Delta}"] \arrow[d, "\dot{\epsilon}"'] & \boldsymbol{CL} \arrow[d, "\dot{\nabla}"] \\
\boldsymbol{iwCL} \arrow[r, "\ddot{\epsilon}"'] & \boldsymbol{iCL}
\end{tikzcd}$$

Specifically, for $\mathcal{Z} \in \mathscr{X}$ the left adjoints $(\dot{\epsilon}_{\mathcal{Z}})_*$ and $(\dot{\nabla}_{\mathcal{Z}})_*$ map a closure operator to its idempotent $\mathcal{Z}$-modal hull, and the right adjoints $(\ddot{\Delta}_{\mathcal{Z}})^*$ and $(\ddot{\epsilon}_{\mathcal{Z}})^*$ map a closure operator to its weakly hereditary core. The corresponding right adjoints resp. left adjoints are inclusions. Moreover, $\dot{\epsilon}\,;\ddot{\epsilon}$ is point-wise essentially canonical with center $\boldsymbol{iwCL}$.

Proof:

Consider $\mathcal{Y} \subseteq \mathcal{Z}$ in $\mathscr{X}$.

(0) The existence of $\mathcal{Z}$-modal hulls (cf. Remark 2.06(3)) yields the Galois reflection $\boldsymbol{CL}_{\mathcal{Y},\mathcal{Z}}$. Similarly, part (3) of Lemma 2.07 induces the Galois reflection $\boldsymbol{wCL}_{\mathcal{Y},\mathcal{Z}}$. The functoriality of these assignments is trivial.

(1) The existence of idempotent $\mathcal{Z}$-modal hulls (cf. Remark 2.06(3)) yields the Galois reflection $\boldsymbol{iCL}_{\mathcal{Y},\mathcal{Z}}$. Similarly, part (5) of Lemma 2.07 induces the Galois reflection $\boldsymbol{iwCL}_{\mathcal{Y},\mathcal{Z}}$. The functoriality of these assignments again is trivial.

(2) The Galois coreflections $\ddot{\Delta}_{\mathcal{Z}}$ and $\ddot{\epsilon}_{\mathcal{Z}}$ are induced by parts (2) and (4) of Lemma 2.07, respectively. Forming the idempotent $\mathcal{Z}$-modal hull preserves $\mathcal{Z}$-modality (trivially), which yields the Galois reflection $\dot{\nabla}_{\mathcal{Z}}$, and weak hereditariness, cf. 2.07(5), which yields the Galois reflection $\dot{\epsilon}_{\mathcal{Z}}$. The fact that $\dot{\epsilon}_{\mathcal{Z}}\,;\ddot{\epsilon}_{\mathcal{Z}}$ is essentially canonical with center $\boldsymbol{iwCL}_{\mathcal{Z}}$ is immediate.

To complete the proof, it now suffices to show that the following cube commutes:

$$\begin{tikzcd}
\boldsymbol{wCL}_{\mathcal{Z}} \arrow[rrr, "\ddot{\Delta}_{\mathcal{Z}}"] \arrow[ddd, "\dot{\epsilon}_{\mathcal{Z}}"'] & & & \boldsymbol{CL}_{\mathcal{Z}} \arrow[ddd, "\dot{\nabla}_{\mathcal{Z}}"] \\
& \boldsymbol{wCL}_{\mathcal{Y}} \arrow[ul, "\boldsymbol{wCL}_{\mathcal{Y},\mathcal{Z}}"'] \arrow[r, "\ddot{\Delta}_{\mathcal{Y}}"] \arrow[d, "\dot{\epsilon}_{\mathcal{Y}}"'] & \boldsymbol{CL}_{\mathcal{Y}} \arrow[ur, "\boldsymbol{CL}_{\mathcal{Y},\mathcal{Z}}"] \arrow[d, "\dot{\nabla}_{\mathcal{Y}}"] & \\
& \boldsymbol{iwCL}_{\mathcal{Y}} \arrow[dl, "\boldsymbol{iwCL}_{\mathcal{Y},\mathcal{Z}}"] \arrow[r, "\ddot{\epsilon}_{\mathcal{Y}}"'] & \boldsymbol{iCL}_{\mathcal{Y}} \arrow[dr, "\boldsymbol{iCL}_{\mathcal{Y},\mathcal{Z}}"'] & \\
\boldsymbol{iwCL}_{\mathcal{Z}} \arrow[rrr, "\ddot{\epsilon}_{\mathcal{Z}}"'] & & & \boldsymbol{iCL}_{\mathcal{Z}}
\end{tikzcd} \tag{2-02}$$

The commutativity of the upper trapezoid follows, since both $(\boldsymbol{wCL}_{\mathcal{Y},\mathcal{Z}})_*\,;(\ddot{\Delta}_{\mathcal{Z}})_*$ and $(\ddot{\Delta}_{\mathcal{Y}})_*\,;(\boldsymbol{CL}_{\mathcal{Y},\mathcal{Z}})_*$ map a weakly hereditary $\mathcal{Y}$-modal closure operator to its $\mathcal{Z}$-modal hull. A similar argument works for the lower trapezoid, where both left adjoints map an idempotent weakly hereditary $\mathcal{Y}$-modal closure operator to its idempotent $\mathcal{Z}$-modal hull.

Since $\boldsymbol{iCL}_{\mathcal{Z}}$ is the intersection of $\boldsymbol{iCL}_{\mathcal{Y}}$ and $\boldsymbol{CL}_{\mathcal{Z}}$, both of which are closed under the formation of infima in $\boldsymbol{CL}_{\mathcal{Y}}$, we immediately get that $\boldsymbol{iCL}_{\mathcal{Z}}$ is closed under the formation of infima both in $\boldsymbol{iCL}_{\mathcal{Y}}$ and in $\boldsymbol{CL}_{\mathcal{Z}}$. This that the right trapezoid commutes. A very similar argument together with Lemma 2.07(5) works for the left trapezoid.

Since all trapezoids commute, the known commutativity of the square at $\mathcal{Z}=\mathcal{I}$ can be used to derive the commutativity of all other squares. $\square$

Now we turn to the left diagram in (2-01). The question as to which subclasses of $\mathscr{M}\diamond\mathscr{M}$ (resp $\mathscr{M}$) under $(\dot{\omega}_{\mathcal{I}})_*$ (resp. $(\dot{\Delta}_{\mathcal{I}})_*$) give rise to $\mathcal{Z}$-modal closure operators was already addressed in Proposition 3.02 of [4]. Originally that result had been proved under the assumption that $\mathcal{Z}$ consists of cartesian morphisms. Unfortunately the hypothesis of this proposition was not adjusted after that assumption had been dropped. Here is the corrected version of that proposition, the proof of which requires only minor adjustments.

2.09 PROPOSITION

Suppose that W satisfies the Beck-Chevalley condition relative to $\mathcal{Z}$, and that $\langle f,g\rangle^{\leftarrow}$ is left adjoint (i.e., preserves suprema) for each $\langle f,g\rangle\in\mathcal{Z}$.

(0) *If $C\subseteq\mathscr{M}\diamond\mathscr{M}$ is $\mathcal{Z}$-stable, then $C(\dot{\omega}_{\mathcal{I}})_*\in\mathscr{M}$-$\boldsymbol{CL}$ is $\mathcal{Z}$-modal.*

(1) *If $A\subseteq\mathscr{M}$ is $\mathcal{Z}$-stable, then $A(\dot{\Delta}_{\mathcal{I}})_*\in\mathscr{M}$-$\boldsymbol{wCL}$ is $\mathcal{Z}$-modal.* $\square$

How can we be certain that W satisfies the Beck-Chevalley condition for, if not every, then at least a reasonable number of collections $\mathcal{Z}\in\mathscr{Z}$?

2.10 PROPOSITION

If $\boldsymbol{E}$ is stable under pullbacks and $\mathcal{Z}$ consists of cartesian $\mathscr{M}$-morphisms then W satisfies the Beck-Chevalley condition relative to $\mathcal{Z}$ and for each $\langle f,g\rangle\in\mathcal{Z}$ the W-inverse image functor $\langle f,g\rangle^{\leftarrow}$ is left adjoint and hence preserves suprema.

Proof:

The Beck-Chevalley condition relative to $\mathcal{Z}$ follows since the collection $\boldsymbol{E}_2$ of 2-sinks with one member in $\mathscr{M}$ is stable under pullbacks along g whenever $\langle f,g\rangle\in\mathcal{Z}$. The fact that the collection $\boldsymbol{E}_{\mathscr{M}}$ of supremum sinks (which consist entirely of $\mathscr{M}$-elements) are stable under the same types of pullbacks implies that $\langle f,g\rangle^{\leftarrow}$ is left adjoint for each $\langle f,g\rangle\in\mathcal{Z}$. $\square$

The hypothesis on $\boldsymbol{E}$ holds in every famillialy regular category in the sense of Street [14], and hence in every Grothendieck topos. It can even be weakened by requiring only $\bar{\boldsymbol{E}} := \boldsymbol{E}_2 \cup \boldsymbol{E}_{\mathscr{M}}$ to be stable under pullbacks. In Sections 4 and 5 this will be useful.

In view of Proposition 2.10 it is no longer surprising that all collections $\mathcal{Z}$ that occur in practice consist of cartesian $\mathscr{M}$-morphisms. This justifies restricting our attention to the subcategory $\mathscr{C}$ of $\mathscr{Z}$. We use the same names for the correspondingly restricted functors and natural transformations. Notice that the collection $\mathcal{C}$ of all cartesian $\mathscr{M}$-morphisms is is the terminal object of $\mathscr{C}$.

2.11 PROPOSITION

(0) *If $\mathcal{Y} \subseteq \mathcal{Z}$ in $\mathscr{C}$ the inclusions $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Y}} \longleftarrow (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}}$ and $\mathscr{M}\boldsymbol{P}_{\mathcal{Y}} \longleftarrow \mathscr{M}\boldsymbol{P}_{\mathcal{Z}}$ preserve intersections, hence induce Galois reflections*

$$(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Y}} \xrightarrow{(\mathscr{M}\diamond\mathscr{M})\boldsymbol{P}_{\mathcal{Y},\mathcal{Z}}} \bullet\, (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \quad \text{and} \quad \mathscr{M}\boldsymbol{P}_{\mathcal{Y}} \xrightarrow{\mathscr{M}\boldsymbol{P}_{\mathcal{Y},\mathcal{Z}}} \bullet\, \mathscr{M}\boldsymbol{P}_{\mathcal{Z}}$$

with left adjoint parts that map $C \subseteq \mathscr{M} \diamond \mathscr{M}$ and $A \subseteq \mathscr{M}$ to the smallest $\mathcal{Z}$-stable subclasses of $\mathscr{M} \diamond \mathscr{M}$ and $\mathscr{M}$ containing C and A, respectively. Thus $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}$ and $\mathscr{M}\boldsymbol{P}$ may be viewed a functors from $\mathscr{C}$ to $\mathscr{GAL}$.

(1) *The Galois connections $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\omega_{\mathcal{Z}}} \bullet\, (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$, $\mathcal{Z} \in \mathscr{C}$, constitute a natural transformation $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} \xrightarrow{\omega} (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$ with constant codomain.*

(2) *For any $\mathcal{Z} \in \mathscr{C}$, if $C \subseteq \mathscr{M} \diamond \mathscr{M}$ is $\mathcal{Z}$-stable and $D \subseteq \mathscr{M} \diamond \mathscr{M}$ satisfies $C(\omega_{\mathcal{Z}})_* = D$ and $D(\omega_{\mathcal{Z}})^* = C$ the closure operator $C(\dot{\omega}_{\mathcal{Z}})_* = C(\dot{\omega}_{\mathcal{I}})_*$ is isomorphic to the $\mathcal{Z}$-modal hull of $D(\ddot{\omega}_{\mathcal{I}})^*$. Hence the essentially canonical factorization $\omega_{\mathcal{I}} = \dot{\omega}_{\mathcal{I}}\,;\ddot{\omega}_{\mathcal{I}}$ (cf. Remark 2.04(1)) extends to a factorization $\omega = \dot{\omega}\,;\ddot{\omega}$ that is point-wise essentially canonical with center $\boldsymbol{CL}$.*

(3) *There exists natural transformations $\dot{\Delta}$, γ, $\ddot{\Delta}$, and $\dot{\omega}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{M}\boldsymbol{P} & \xrightarrow{\dot{\Delta}} & \boldsymbol{wCL} \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\ddot{\Delta}} \\ (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} & \xrightarrow[\dot{\omega}]{} & \boldsymbol{CL} \end{array}$$

In addition, $\dot{\Delta}\,;\ddot{\Delta}$ is point-wise essentially canonical with center $\boldsymbol{wCL}$. □

3 REGULAR CLOSURE OPERATORS

In the following $\mathcal{Z}$ is an arbitrary collection of cartesian $\mathscr{M}$-morphisms that is pullback stable, $\mathscr{D}$ is an arbitrary pre-ordered class, and $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\ \rho\ }\!\!\bullet\ \mathscr{D}$ as well as $\mathscr{M}\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\ \tau\ }\!\!\bullet\ \mathscr{D}$ are arbitrary but fixed Galois connections. In the applications these will in fact be restrictions of polarities induced by suitable relations.

3.00 DEFINITION

A closure operator $F \in \boldsymbol{CL}_{\mathcal{Z}}$ (resp. $F \in \boldsymbol{wCL}_{\mathcal{Z}}$) is called ρ**-regular** (resp. τ**-regular**), if

(0) there exists a Galois connection $\boldsymbol{CL}_{\mathcal{Z}} \xrightarrow{\ \mu_{\mathcal{Z}}\ }\!\!\bullet\ \mathscr{D}$ (resp. $\boldsymbol{wCL}_{\mathcal{Z}} \xrightarrow{\ \nu_{\mathcal{Z}}\ }\!\!\bullet\ \mathscr{D}$) such that

$$
\begin{tikzcd}
(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \arrow[rr, "\rho", -*] \arrow[dr, "\dot{\omega}_{\mathcal{Z}}"', -*] & & \mathscr{D} \\
& \boldsymbol{CL}_{\mathcal{Z}} \arrow[ur, "\mu_{\mathcal{Z}}"', -*] &
\end{tikzcd}
\quad \text{(resp.} \quad
\begin{tikzcd}
\mathscr{M}\boldsymbol{P}_{\mathcal{Z}} \arrow[rr, "\tau", -*] \arrow[dr, "\dot{\Delta}_{\mathcal{Z}}"', -*] & & \mathscr{D} \\
& \boldsymbol{wCL}_{\mathcal{Z}} \arrow[ur, "\nu_{\mathcal{Z}}"', -*] &
\end{tikzcd}
\quad) \qquad \text{(3-00)}
$$

is a commuting diagram of Galois connections; and

(1) F is a left fixed point of $\mu_{\mathcal{Z}}$ (resp. of $\nu_{\mathcal{Z}}$).

3.01 REMARKS

(0) Proposition 2.11 shows that $\gamma\,;\dot{\omega} = \dot{\Delta}\,;\ddot{\Delta}$. Hence the right triangle of (3-00) can be constructed from the left one, and the left fixed points of $\mu_{\mathcal{Z}}$ that belong to $\boldsymbol{wCL}_{\mathcal{Z}}$ are precisely the left fixed points of $\nu_{\mathcal{Z}}$ whenever τ is taken to be $\gamma_{\mathcal{Z}}\,;\rho$.

(1) To recover classical regular closure operators, one sets $\mathcal{Z} = \mathcal{I}$, considers for $\mathscr{D}$ the power collection of all $\mathscr{X}$-objects ordered by $\supseteq$, and takes for ρ the polarity induced by a variant of the *separatedness relation* of Pumplün and Röhrl. Section 4 addresses this special case in detail.

Let us first consider the question of the existence of Galois connections $\mu_{\mathcal{Z}}$ and $\nu_{\mathcal{Z}}$ that make the diagrams in (3-00) commute. It will turn out that if such $\mu_{\mathcal{Z}}$ and $\nu_{\mathcal{Z}}$ do exist, they are unique up to isomorphism. To find suitable candidates for such Galois connections, we use the fact that $\dot{\omega}_{\mathcal{Z}}$ and $\dot{\Delta}_{\mathcal{Z}}$ as first factors of an essentially canonical factorization are well-behaved in the sense of the following lemma.

3.02 LEMMA

If Galois connections α, β and γ are such that α satisfies $\alpha^*\,;\alpha_* \cong id$, and $\alpha\,;\beta = \gamma$, then $\alpha^*\,;\gamma_* \cong \beta_*$ and $\gamma^*\,;\alpha_* \cong \beta^*$. □

Since by Proposition 1.03 $(\dot\omega_{\mathcal{Z}})^* ; (\dot\omega_{\mathcal{Z}})_* \cong id_{\boldsymbol{CL}_{\mathcal{Z}}}$ and $(\dot\Delta_{\mathcal{Z}})^* ; (\dot\Delta_{\mathcal{Z}})_* \cong id_{\boldsymbol{wCL}_{\mathcal{Z}}}$, if μ (resp. ν) exists, then up to isomorphism we must have

$$\mu_{\mathcal{Z}} = \langle (\dot\omega_{\mathcal{Z}})^* ; \rho_*, \rho^* ; (\dot\omega_{\mathcal{Z}})_* \rangle \quad \text{(resp.} \quad \nu_{\mathcal{Z}} = \langle (\dot\Delta_{\mathcal{Z}})^* ; \tau_*, \tau^* ; (\dot\Delta_{\mathcal{Z}})_* \rangle \quad \text{)}$$

and the notion of regularity in Definition 3.00 indeed is well-defined.

3.03 PROPOSITION

If we set $(\mu_{\mathcal{Z}})_\circ := (\dot\omega_{\mathcal{Z}})^* ; \rho_*$, $(\mu_{\mathcal{Z}})^\circ := \rho^* ; (\dot\omega_{\mathcal{Z}})_*$, $(\nu_{\mathcal{Z}})_\circ := (\dot\Delta_{\mathcal{Z}})^* ; \tau_*$, *and* $(\nu_{\mathcal{Z}})^\circ := \tau^* ; (\dot\Delta_{\mathcal{Z}})_*$ *then both* $(\mu_{\mathcal{Z}})_\circ ; (\mu_{\mathcal{Z}})^\circ$ *and* $(\nu_{\mathcal{Z}})_\circ ; (\nu_{\mathcal{Z}})^\circ$ *are increasing.*

Proof:

ρ and τ are Galois connections, so $\rho_* ; \rho^*$ and $\tau_* ; \tau^*$ are increasing. By Proposition 1.03 $(\dot\omega_{\mathcal{Z}})^* ; (\dot\omega_{\mathcal{Z}})_* \cong id_{\boldsymbol{CL}_{\mathcal{Z}}}$ and $(\dot\Delta_{\mathcal{Z}})^* ; (\dot\Delta_{\mathcal{Z}})_* \cong id_{\boldsymbol{wCL}_{\mathcal{Z}}}$, and all the maps preserve order. □

However, $(\mu_{\mathcal{Z}})^\circ ; (\mu_{\mathcal{Z}})_\circ$ and $(\nu_{\mathcal{Z}})^\circ ; (\nu_{\mathcal{Z}})_\circ$ can fail to be decreasing. If this happens, there is no factorization of ρ through $\dot\omega_{\mathcal{Z}}$ (resp. of τ through $\dot\Delta_{\mathcal{Z}}$). As the next proposition shows, this problem cannot occur if ρ and τ are reasonably well-behaved.

3.04 PROPOSITION

If for every element D *of* $\mathscr{D}$

(0) $D\rho^*$ *is a left fixed point of* $\dot\omega_{\mathcal{Z}}$, *then* $\mu_{\mathcal{Z}} = \langle (\mu_{\mathcal{Z}})_\circ, (\mu_{\mathcal{Z}})^\circ \rangle$ *is a Galois connection;*

(1) $D\tau^*$ *is a left fixed point of* $\dot\Delta_{\mathcal{Z}}$, *then* $\nu_{\mathcal{Z}} = \langle (\nu_{\mathcal{Z}})_\circ, (\nu_{\mathcal{Z}})^\circ \rangle$ *is a Galois connection.*

Proof:

(0) If $D\rho^*$ is a left fixed point of $\dot\omega_{\mathcal{Z}}$, then $D\rho^*(\dot\omega_{\mathcal{Z}})_*(\dot\omega_{\mathcal{Z}})^*\rho_* \cong \rho^*\rho_*$. Since ρ is a Galois connection, we have $D\rho^*\rho_* \sqsupseteq D$, i.e., $(\dot\omega_{\mathcal{Z}})_* ; \rho^* ; \rho_* ; (\dot\omega_{\mathcal{Z}})^*$ is decreasing

(1) Similar. □

3.05 REMARK

By Theorem 2.08(4) of [3] $\dot\omega_{\mathcal{I}}$ and the polarity ω induced by the orthogonality relation on $\mathscr{M} \diamond \mathscr{M}$ have the same left fixed points. Hence we can use their characterization in Theorem 2.05 of [3] when applying part (0) of Proposition 3.04 for $\mathcal{Z} = \mathcal{I}$. By Proposition 2.02 of [4] we then know that for $\mathcal{Z} \in \mathscr{C}$ the left fixed points of $\dot\omega_{\mathcal{Z}}$, i.e., the classes $F(\dot\omega_{\mathcal{I}})^*$ for $\mathcal{Z}$-modal closure operators F, are precisely the left fixed points of $\dot\omega_{\mathcal{I}}$ that are $\mathcal{Z}$-stable (cf. Definition 2.05). Recall that $C \subseteq \mathscr{M} \diamond \mathscr{M}$ is a left fixed point of ω (and hence of $\dot\omega_{\mathcal{I}}$) iff C satisfies the conditions

(C0) C is stable under the formation of W-direct images.

(C1) $\big(C \cap W/m\big)\sup_{\ll} \in C$ for every $m \in \mathscr{M}$.

(C2) $C \cap W/m$ is a lower segment for every $m \in \mathscr{M}$.

We proceed by characterizing the left fixed points of $\dot{\Delta}_{\mathcal{Z}}$. Using Proposition 2.02 of [4] again, these are the left fixed points of $\dot{\Delta}_{\mathcal{I}}$, i.e., the classes $F(\dot{\Delta}_{\mathcal{I}})^*$ with F weakly hereditary, that are $\mathcal{Z}$-stable.

3.06 THEOREM

$A \subseteq \mathscr{M}$ is a left fixed point of $\dot{\Delta}_{\mathcal{I}}$ iff A satisfies the following conditions

($\bar{\mathrm{C}}0$) *$A(\gamma_{\mathcal{I}})_*$ is stable under the formation of W-direct images;*

($\bar{\mathrm{C}}1$) *$\big(A(\gamma_{\mathcal{I}})_* \cap W/m\big)\mathrm{sup}_{\ll} \in A(\gamma_{\mathcal{I}})_*$ for every $m \in \mathscr{M}$.*

Proof:

($\Rightarrow$). Let F be a weakly hereditary closure operator, and set $A = F(\dot{\Delta}_{\mathcal{I}})^*$, i.e., A is the class of F-dense elements of $\mathscr{M}$.

($\bar{\mathrm{C}}0$) For an $\mathscr{M}$-morphism $m \xrightarrow{\langle f,g \rangle} n$ consider $\langle q,r \rangle \in A(\gamma_{\mathcal{I}})_* \cap W/m$ and let $\langle\langle d,s \rangle, t\rangle$ be the $\langle \boldsymbol{E}, \mathscr{M} \rangle$-factorization of the 2-sink $\bullet \xrightarrow{r\,;\,g} \bullet \xleftarrow{n} \bullet$. Thus $\langle s,t \rangle \in W/n$ is the W-direct image of $\langle q,r \rangle$ along $\langle f,g \rangle$. It suffices to show that $s \in A$. Since $q \in A$ we have that $\langle q, id \rangle \perp sF$. So there exists a unique w with $q\,;w = f\,;s_F$ and $w\,;s^F = d$. Now the fact that $\bullet \xrightarrow{d} \bullet \xleftarrow{s} \bullet$ belongs to $\boldsymbol{E}$ and hence is left-orthogonal to $s^F \in \mathscr{M}$ implies the existence of a unique z with $s\,;z = s_F$, $d\,;z = w$, and $z\,;s^F = id$. In particular, $s^F \in \mathscr{X}$-$\boldsymbol{Mono}$ is a retraction and hence is an iso, i.e., s is F-dense. Consequently, $\langle s,t \rangle \in A(\gamma_{\mathcal{I}})_*$.

($\bar{\mathrm{C}}1$) Proposition 1.09(1) of [11] shows $\big(A(\gamma_{\mathcal{I}})_* \cap W/m\big)sup_{\ll} \cong mF \in A(\gamma_{\mathcal{I}})_*$ for each $m \in \mathscr{M}$, since $A = F(\dot{\Delta}_{\mathcal{I}})^* = F(\Delta_{\mathcal{I}})^*$. By ($\bar{\mathrm{C}}0$) the supremum belongs to $A(\gamma_{\mathcal{I}})_*$ as well.

($\Leftarrow$). Suppose that $A \subseteq \mathscr{M}$ satisfies ($\bar{\mathrm{C}}0$) and ($\bar{\mathrm{C}}1$), and set $F = A(\dot{\Delta}_{\mathcal{I}})_* = A\Delta_*$. Since $(\dot{\Delta}_{\mathcal{I}})_*\,;(\dot{\Delta}_{\mathcal{I}})^*$ is increasing, it suffices to show $A \supseteq F(\dot{\Delta}_{\mathcal{I}})^* = F(\Delta_{\mathcal{I}})^*$. The hypothesis and Remark 2.04(2) yield $mF = \big((A(\gamma_{\mathcal{I}})_*)^{\mathrm{di}} \cap W/m\big)sup_{\ll} = \big(A(\gamma_{\mathcal{I}})_* \cap W/m\big)sup_{\ll} \in A(\gamma_{\mathcal{I}})_*$ for each $m \in \mathscr{M}$. If m is F-dense, then $m \cong m_F$ and by ($\bar{\mathrm{C}}0$) belongs to A . $\square$

It is important to notice that ($\bar{\mathrm{C}}0$) is not equivalent to A being stable under the formation of V-direct images. If $m = n\,;p$, then $\langle p, id_{pV} \rangle$ always is an $\langle n, id_{pV} \rangle$-direct image of $\langle m, id_{mV} \rangle$, but p need not be an id_{mV}-direct image of m .

Translating (C2) for $A(\gamma_{\mathcal{I}})_*$ in terms of A results in the following cancellation property

($\bar{\mathrm{C}}2$) $n\,;p \in A$ and $p \in \mathscr{M}$ implies $n \in A$.

But this is just the condition that characterizes hereditary closure operators, cf. Proposition 3.02 of [4]. The fact that this condition is not needed to characterize the left fixed points of $\dot{\Delta}_{\mathcal{I}}$ indicates that A may be a left fixed point of $\dot{\Delta}_{\mathcal{I}}$ without $A(\gamma_{\mathcal{I}})_*$ being a left fixed point of ω . This is no real surprise, since the inclusion $F(\dot{\Delta}_{\mathcal{I}})^*(\gamma_{\mathcal{I}})_* \subseteq F(\dot{\omega}_{\mathcal{I}})^*$ is only reversible if F is hereditary. In particular, for every weakly hereditary closure operator F that is not hereditary there exist relatively dense pairs $\langle n, p\rangle \in F(\dot{\omega}_{\mathcal{I}})^*$ for which n is not dense. An example of such a closure operator is the weakly hereditary core of the classical regular closure operator induced by class of Hausdorff spaces, cf. Example 3.08 of [4] and Section 4.

We conclude this section with a characterization of the classes $F(\dot{\omega}_{\mathcal{I}})^*$ for idempotent closure operators F as well as the classes $G(\dot{\Delta}_{\mathcal{I}})^*$ for idempotent weakly hereditary closure operators G.

3.07 PROPOSITION

(0) *$C \subseteq \mathscr{M} \diamond \mathscr{M}$ is of the form $F(\dot{\omega}_{\mathcal{I}})^*$ for some idempotent closure operator F iff in addition to conditions (C0) – (C2) of Remark 3.05 we have*

(C3) *C is stable under* **left-shifting**, *i.e., $\langle l, q\rangle \in C \cap W/m$ and $\langle n, p\rangle \in C \cap W/q$ implies $\langle l\,;n, p\rangle \in C \cap W/m$.*

(1) *$A \subseteq \mathscr{M}$ is of the form $G(\dot{\Delta}_{\mathcal{I}})^*$ for some idempotent weakly hereditary closure operator F iff in addition to conditions ($\bar{C}$0) and ($\bar{C}$1) of Theorem 3.06 we have*

($\bar{\mathrm{C}}$3) *A is closed under composition.*

Proof:

(0) ($\Rightarrow$). Let F be an idempotent closure operator with $F(\dot{\omega}_{\mathcal{I}})^* = C$. In particular then C as a left fixed point of $\dot{\omega}$ satisfies (C0) – (C2). If $\langle l, n\,;p\rangle, \langle n, p\rangle \in F(\dot{\omega}_{\mathcal{I}})^*$, for any $m \in \mathscr{M}$ we have $\langle l, n\,;p\rangle \perp mF$ and $\langle n, p\rangle \perp m^F F$. Since by the idempotency ${m^F}_F$ is an isomorphism, this implies $\langle l\,;n, p\rangle \perp mF$, and so $\langle l\,;n, p\rangle \in F(\dot{\omega}_{\mathcal{I}})^* = C$.

($\Leftarrow$). Suppose that $C \subseteq \mathscr{M} \diamond \mathscr{M}$ satisfies (C0) – (C3), and set $F = C(\dot{\omega}_{\mathcal{I}})_*$. Since C by Remark 3.05 is a left fixed point of $\dot{\omega}$, for $m \in \mathscr{M}$ both mF and $m^F F$ belong to C. Thus $\langle m_F\,; {m^F}_F, m^{FF}\rangle \in C$ by (C3), and hence $\langle m_F\,; {m^F}_F, m^{FF}\rangle \ll mF$ (cf. Remark 2.04(1)). Therefore ${m^F}_F$ is an isomorphism and so m^F is F-closed.

(1) ($\Rightarrow$). Let G be an idempotent weakly hereditary closure operator with $G(\dot{\Delta}_{\mathcal{I}})^* = A$. In particular then A as a left fixed point of $\dot{\Delta}_{\mathcal{I}}$ satisfies ($\bar{\mathrm{C}}$0) and ($\bar{\mathrm{C}}$1). If $\langle l, n\rangle \in (\mathscr{M} \diamond \mathscr{M}) \cap (A \times A)$, then for $m = l\,;n$ we have $\langle l, n\rangle \perp mG$ and $\langle n, id\rangle \perp m^G G$. Since by the idempotency ${m^G}_G$ is an isomorphism, this implies $\langle l\,;n, id\rangle \perp mG$. But this means that $l\,;n$ is G-dense and hence belongs to A.

($\Leftarrow$). Suppose that $A \subseteq \mathscr{M}$ satisfies ($\bar{\mathrm{C}}$0), ($\bar{\mathrm{C}}$1) and ($\bar{\mathrm{C}}$3), and set $G = A(\dot{\Delta}_{\mathcal{I}})_*$. By Theorem 3.06 we then have $A = G(\dot{\Delta}_{\mathcal{I}})^*$. Thus for $m \in \mathscr{M}$ both m_G and ${m^G}_G$ and hence $m_G\,; {m^G}_G$ are G-dense. But $mG = \{\, \langle s, t\rangle \in W/m \mid s \text{ is } G\text{-dense} \,\}\mathrm{sup}_{\ll}$ by Theorem 1.12 of [11], so $m^{GG} \leq m^G$. The other inequality always holds, which implies $m^{GG} \cong m^G$. Thus ${m^G}_G$ is iso, and hence m^G is G-closed. $\square$

4 APPLICATIONS OF PUMPLÜN-RÖHRL-SALBANY-TYPE

Since the restriction of the orthogonality relation $\perp$ to $\mathscr{M} \diamond \mathscr{M}$ and the induced polarity $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} \overset{\omega}{\longrightarrow\!\bullet} (\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}^{\mathrm{op}}$ lie at the heart of the theory of closure operators (after all $\mathscr{M}$-$\boldsymbol{CL}$ is equivalent to the fixed point lattices of ω) one can clearly construct a largest closure operator F with a prescribed class $C \subseteq \mathscr{M} \diamond \mathscr{M}$ of composable pairs that are relatively F-closed. In particular, it is possible to construct a largest closure operator for which the members of a given class $B \subseteq \mathscr{M}$ of $\mathscr{M}$-elements are F-closed.

Grothendieck topologies on a Grothendieck topos can be identified with idempotent modal closure operators on the class of monomorphisms. Although one is usually interested in the largest Grothendieck topology that has a certain class of presheaves as sheaves, occasionally the question arises as to whether a largest Grothendieck topology can be constructed for which all members of a given class B of monos are closed. A moment's thought will convince the reader that this indeed is possible, and that the obvious construction works (declare to be dense every mono that has the property that each of its pullbacks is left-orthogonal to every member of B). Nevertheless, both problems are fundamentally different, and in this section and the following one we shall explain why.

Another type of condition that may be imposed on a closure operator is evident in two other motivating examples that we wish to explain: namely the factorization of the Pumplün–Röhrl polarity established in [5], and the Salbany construction of a closure operator induced by a class of objects by means of relative dominions, cf. [13] and [10]. In both cases one considers the class $\{\, X \xleftarrow{id_X} X \xrightarrow{id_X} X \mid X \in \mathscr{X}\text{-}\boldsymbol{Ob}\}$ of identity 2-sources, which we temporarily identify with $\mathscr{X}$-$\boldsymbol{Ob}$. Using the notation of the present paper, in [5] we established the commutativity of the diagram

$$\begin{tikzcd}
(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P} \arrow[rr, "\sigma", "\bullet"' near end] \arrow[dr, "\varsigma"'] & & (\mathscr{X}\text{-}\boldsymbol{Ob})\,\boldsymbol{P}^{\mathrm{op}} \ , \\
& \mathscr{M}\text{-}\boldsymbol{iCL} \arrow[ur, "\kappa"'] &
\end{tikzcd} \tag{4-00}$$

provided that $\mathscr{X}$ has equalizers and $\mathscr{M}$ contains all regular monomorphisms, or equivalently that $\boldsymbol{E}$ consists of epi-sinks. σ $(=(\alpha,\beta)$ in [5]) is the Pumplün–Röhrl connection, i.e., the polarity between $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P}$ $(=H(\mathscr{X})$ in [5]) and $(\mathscr{X}\text{-}\boldsymbol{Ob})\,\boldsymbol{P}^{\mathrm{op}}$ $(=S(\mathscr{X})$ in [5]) induced by the "separating relation" $S \subseteq \mathscr{X}\text{-}\boldsymbol{Mor} \times \mathscr{X}\text{-}\boldsymbol{Ob}$. cf. Definition 1.00 and Proposition 1.01(1). The Galois connection ς was constructed under the name (S,R) in [5]. ς^* $(=R)$ maps $F \in \mathscr{M}\text{-}\boldsymbol{iCL}$ to the class of F-dense $\mathscr{X}$-morphisms. To define ς_* $(=S)$, for a class $\mathscr{N} \subseteq \mathscr{X}\text{-}\boldsymbol{Mor}$ declare all elements of $\mathscr{M}$ that are right-orthogonal to every element of $\mathscr{N}$ to be $\mathscr{N}$-closed. The $\mathscr{N}\varsigma_*$-closure of $m \in \mathscr{M}$ then is the smallest $\mathscr{N}$-closed subobject of mV that is greater than or equal to m. Finally, κ $(=(D,K)$ in [5]) goes back to [6]. κ_* $(=D)$ maps $F \in \mathscr{M}\text{-}\boldsymbol{iCL}$ to the class of F**-separated** objects, i.e., those objects X that are separated from mF for every $m \in \mathscr{M}$ (cf. Proposition 1.01(1)), while κ^* $(= K)$ maps $\mathscr{Y} \subseteq \mathscr{X}\text{-}\boldsymbol{Ob}$ to the Salbany closure operator induced by $\mathscr{Y}$. The Salbany closure $m^{\mathscr{Y}\kappa^*}$ of $m \in \mathscr{M}$ is defined as the **dominion of** m **relative to** $\mathscr{Y}$, i.e., as the intersection of all equalizers of parallel pairs $\langle r, s \rangle$ of $\mathscr{X}$-morphisms with codomain belonging to $\mathscr{Y}$ and the property that $m\,;r = m\,;s$. However, in this section we prefer to view $m\mathscr{Y}\kappa^*$ as a supremum in W/m of all those pairs $\langle n, p \rangle$ with the property that if m equalizes any parallel pair $\langle r, s \rangle$ with codomain in $\mathscr{Y}$, so does p. Hence

- *$\mathscr{Y}\kappa^*$ is the largest idempotent closure operator with respect to which all objects in $\mathscr{Y}$ are separated.*

Our program hence boils down to answering the following question: What kinds of conditions (of the type that prescribe certain sheaves, or separated objects, or closed morphisms, etc.) can we impose such that a largest closure operator (possibly with additional properties like idempotency or $\mathscr{Z}$-modality) exists that satisfies these conditions?

In order to provide some of the answers, we now specialize the Galois connections ρ and τ of Section 3 to polarities induced by (suitable restrictions of) the orthogonality relation $\perp$ and modifications thereof. In 1.00 we defined $\perp$ as a relation between collections $\boldsymbol{L}$ and $\boldsymbol{R}$. On the left side we want to restrict $\boldsymbol{L}$ to $\mathscr{M} \diamond \mathscr{M}$, or to $\mathscr{M} \diamond \mathscr{X}\text{-}\boldsymbol{Iso} \cong \mathscr{M}$ (or in case of the Pumplün-Röhrl connection to $\mathscr{X}\text{-}\boldsymbol{Mor} \diamond \mathscr{X}\text{-}\boldsymbol{Iso} \cong \mathscr{X}\text{-}\boldsymbol{Mor}$). On the right side it will be necessary to distinguish the subcollections $\boldsymbol{R}_{\mathrm{m}}$ and $\boldsymbol{R}^{\mathrm{i}}$ of $\boldsymbol{R}$ that consist of those pairs $\langle r, \boldsymbol{r} \rangle$ that have a monosource in the second component, resp. an isomorphism in the first component, and their intersection $\boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}}$. The inclusions

$$\boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}} \hookrightarrow \boldsymbol{R}_{\mathrm{m}} \hookrightarrow \boldsymbol{R} \quad \text{and} \quad \boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}} \hookrightarrow \boldsymbol{R}^{\mathrm{i}} \hookrightarrow \boldsymbol{R}$$

via their graphs induce axialities

$$\boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P} \xrightarrow{\varphi\ }\!\!\bullet\ \boldsymbol{R}_{\mathrm{m}}\boldsymbol{P} \xrightarrow{\psi\ }\!\!\bullet\ \boldsymbol{R}\boldsymbol{P} \quad \text{and} \quad \boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P} \xrightarrow{\iota\ }\!\!\bullet\ \boldsymbol{R}^{\mathrm{i}}\boldsymbol{P} \xrightarrow{\vartheta\ }\!\!\bullet\ \boldsymbol{R}\boldsymbol{P}$$

whose left adjoint parts are the inclusions of the respective power collections, and that satisfy $\varphi\,;\psi = \iota\,;\vartheta$.

Unfortunately, still further restrictions on the subcollections of $\boldsymbol{R}$ we can select as conditions for a closure operator to satisfy seem to be necessary. Recall the collection $\bar{\boldsymbol{E}}$ of sinks that naturally appeared in the proof of Proposition 2.10. We denote by $\bar{\boldsymbol{R}}$ the subcollection of all pairs in $\boldsymbol{R}$ hat are right orthogonal to every sink in $\bar{\boldsymbol{E}}$. For the intersections with $\boldsymbol{R}_{\mathrm{m}}$, $\boldsymbol{R}^{\mathrm{i}}$, and $\boldsymbol{R}^{\mathrm{i}}_{\mathrm{m}}$ we write $\bar{\boldsymbol{R}}_{\mathrm{m}}$, $\bar{\boldsymbol{R}}^{\mathrm{i}}$, and $\bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{mono}}$, respectively. Finally $\bar{\varphi}$, $\bar{\psi}$, $\bar{\iota}$ and $\bar{\vartheta}$ denote the corresponding variants of φ, ψ, ι, and ϑ above.

4.00 DEFINITION (cf. Definition 2.05(2))

The polarities from $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}$ to $\boldsymbol{RP}^{\mathrm{op}}$ and $\bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}}$ induced by $\perp_{\mathcal{Z}}$ restrict to Galois connection $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\rho_{\mathcal{Z}}} \bullet\, \boldsymbol{RP}^{\mathrm{op}}$ and $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\bar{\rho}_{\mathcal{Z}}} \bullet\, \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}}$, respectively. Similarly we obtain Galois connections $\mathscr{M}\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\tau_{\mathcal{Z}}} \bullet\, \boldsymbol{RP}^{\mathrm{op}}$ and $\mathscr{M}\boldsymbol{P}_{\mathcal{Z}} \xrightarrow{\bar{\tau}_{\mathcal{Z}}} \bullet\, \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}}$.

Clearly, the Galois connections $\bar{\rho}_{\mathcal{Z}}$ and $\bar{\tau}_{\mathcal{Z}}$, $\mathcal{Z} \in \mathscr{C}$, constitute natural transformations $(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} \xrightarrow{\bar{\rho}} \bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}$ and $\mathscr{M}\boldsymbol{P} \xrightarrow{\bar{\tau}} \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}}$ with constant codomain (cf. Proposition 2.11(1)). We now address the question as to whether or not ρ and τ factor through $\dot{\omega}$ and $\dot{\Delta}$, respectively. A closure operator F is said to **satisfy** $\boldsymbol{J} \subseteq \boldsymbol{R}$ if $mF \perp \langle r, \boldsymbol{r} \rangle$ for each $m \in \mathscr{M}$ and every $\langle r, \boldsymbol{r} \rangle \in \boldsymbol{J}$.

4.01 THEOREM

Suppose that $\bar{\boldsymbol{E}}$ is stable under pullbacks, and interpret $\bar{\varphi}^{\mathrm{op}}$, $\bar{\psi}^{\mathrm{op}}$, $\bar{\iota}^{\mathrm{op}}$, and $\bar{\vartheta}^{\mathrm{op}}$ as constant natural transformations between constant functors defined on $\mathscr{C}$.

(0) *There exist natural transformations $\boldsymbol{CL} \xrightarrow{\mu} \bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}$ and $\boldsymbol{iCL} \xrightarrow{\hat{\mu}} \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}$ such that both parts of the following diagram commute*

$$\begin{tikzcd}
& \bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}} \arrow[r, "\bar{\varphi}^{\mathrm{op}}"] & \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}} \\
(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} \arrow[ur, "\bar{\rho}"] \arrow[dr, "\dot{\omega}"'] & & \\
& \boldsymbol{CL} \arrow[uu, "\mu"] \arrow[r, "\dot{\nabla}"'] & \boldsymbol{iCL} \arrow[uu, "\hat{\mu}"']
\end{tikzcd} \tag{4-01}$$

More specifically, for any $\mathcal{Z} \in \mathscr{C}$ and for any subclass $\boldsymbol{J}$ of $\bar{\boldsymbol{R}}_{\mathrm{m}}$ there exists a largest $\mathcal{Z}$-modal closure operator F that satisfies $\boldsymbol{J}$; in fact $F(\dot{\omega}_{\mathcal{Z}})^* = \boldsymbol{J}(\bar{\rho}_{\mathcal{Z}})^*$. If $\boldsymbol{J} \subseteq \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}$, then F is even idempotent.

(1) *There exist natural transformations $\boldsymbol{wCL} \xrightarrow{\nu} \bar{\boldsymbol{R}}$ and $\boldsymbol{iwCL} \xrightarrow{\hat{\nu}} \bar{\boldsymbol{R}}^{\mathrm{i}}$ such that both parts of the following diagram commute*

$$\begin{tikzcd}
& \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}} \arrow[r, "\bar{\psi}^{\mathrm{op}}"] & \bar{\boldsymbol{R}}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{op}} \\
\mathscr{M}\boldsymbol{P} \arrow[ur, "\bar{\tau}"] \arrow[dr, "\dot{\Delta}"'] & & \\
& \boldsymbol{wCL} \arrow[uu, "\nu"] \arrow[r, "\dot{\epsilon}"'] & \boldsymbol{iwCL} \arrow[uu, "\hat{\nu}"']
\end{tikzcd} \tag{4-02}$$

More specifically, for any $\mathcal{Z} \in \mathscr{C}$ and for any subclass $\boldsymbol{J}$ of $\bar{\boldsymbol{R}}$ there exists a largest weakly hereditary $\mathcal{Z}$-modal closure operator G that satisfies $\boldsymbol{J}$; in fact $G(\dot{\Delta}_{\mathcal{Z}})^* = \boldsymbol{J}(\bar{\tau}_{\mathcal{Z}})^*$. If $\boldsymbol{J} \subseteq \bar{\boldsymbol{R}}^{\mathrm{i}}$, then G is even idempotent.

(2) *Every cell of the following diagrams commutes*

$$
\begin{tikzcd}
\mathscr{M}\boldsymbol{P} \arrow[rr, "\bar{\tau}"] \arrow[dr, "\dot{\Delta}"] \arrow[ddd, "\gamma"'] & & \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}} \arrow[ddd, "\bar{\vartheta}^{\mathrm{op}}"] \\
& \boldsymbol{wCL} \arrow[ur, "\nu"'] \arrow[d, "\ddot{\Delta}"'] & \\
& \boldsymbol{CL} \arrow[dr, "\mu"] & \\
(\mathscr{M} \diamond \mathscr{M})\,\boldsymbol{P} \arrow[ur, "\dot{\omega}"] \arrow[rr, "\bar{\rho}"'] & & \bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}} \arrow[rrr, "\bar{\psi}^{\mathrm{op}}"] \arrow[ddd, "\bar{\vartheta}^{\mathrm{op}}"'] & & & \bar{\boldsymbol{R}}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{op}} \arrow[ddd, "\bar{\iota}^{\mathrm{op}}"] \\
& \boldsymbol{wCL} \arrow[ul, "\nu"'] \arrow[r, "\dot{\epsilon}"] \arrow[d, "\ddot{\Delta}"'] & \boldsymbol{iwCL} \arrow[ur, "\hat{\nu}"] \arrow[d, "\ddot{\epsilon}"] & \\
& \boldsymbol{CL} \arrow[r, "\dot{\nabla}"'] \arrow[dl, "\mu"] & \boldsymbol{iCL} \arrow[dr, "\hat{\mu}"'] & \\
\bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}} \arrow[rrr, "\bar{\varphi}^{\mathrm{op}}"'] & & & \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}
\end{tikzcd}
\tag{4-03}
$$

Proof:

(0) Fix $\mathcal{Z} \in \mathscr{C}$. First we establish the existence of a Galois connection $\boldsymbol{CL}_{\mathcal{Z}} \xrightarrow{\mu_{\mathcal{Z}}}\!\!\bullet\ \bar{\boldsymbol{R}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}$ with $\bar{\rho}_{\mathcal{Z}} = \dot{\omega}_{\mathcal{Z}} \,;\, \mu_{\mathcal{Z}}$. Since $(\bar{\rho}_{\mathcal{Z}})^*$ maps unions to intersections, and every $\boldsymbol{H} \subseteq \bar{\boldsymbol{R}}_{\mathrm{m}}$ is a union of singletons, we only need to check whether $C = \{\langle h, \boldsymbol{h}\rangle\}(\bar{\rho}_{\mathcal{Z}})^*$ satisfies conditions (C0), (C1), and (C2) of Remark 3.05 for each fixed $\langle h, \boldsymbol{h}\rangle \in \bar{\boldsymbol{R}}_{\mathrm{m}}$.

If $\mathcal{Z} = \mathcal{I} = \mathscr{M}\text{-}\boldsymbol{Iso}$, then the hypotheses are trivially satisfied. Conditions (C0) and (C1) easily follow from the fact that $\boldsymbol{h}$ is right-orthogonal to all sinks in $\boldsymbol{E}_2$ and in $\boldsymbol{E}_{\mathscr{M}}$, respectively. Moreover, since $\boldsymbol{h}$ is a monosource, we also get (C2).

For general $\mathcal{Z}$ conditions (C0) – (C2) can be derived from the corresponding conditions in the case that $\mathcal{Z} = \mathcal{I}$ by using Proposition 2.10. For (C0) this is possible because W satisfies the Beck-Chevalley condition relative to $\mathcal{Z}$. For (C1) use the fact that $\langle f, g\rangle^{\leftarrow}$ preserves suprema for each $\langle f, g\rangle \in \mathcal{Z}$. And for (C2) notice that W-inverse image functors preserve order.

Since C is $\mathcal{Z}$-stable by construction, we now can apply Proposition 3.04(0) to get that $\mu_{\mathcal{Z}} = \langle (\dot{\omega}_{\mathcal{Z}})^* \,;\, (\bar{\rho}_{\mathcal{Z}})_*, (\bar{\rho}_{\mathcal{Z}})^* \,;\, (\dot{\omega}_{\mathcal{Z}})_*\rangle$ is a Galois connection and satisfies $\bar{\rho}_{\mathcal{Z}} = \dot{\omega}_{\mathcal{Z}} \,;\, \mu_{\mathcal{Z}}$.

If h is an isomorphism, then condition (C3) of Proposition 3.07 is easily verified. Hence $\mu_{\mathcal{Z}} \,;\, (\bar{\varphi}^{\mathrm{op}})_{\mathcal{Z}}$ factors through $\boldsymbol{CL}_{\mathcal{Z}} \xrightarrow{\dot{\nabla}_{\mathcal{Z}}}\!\!\bullet\ \boldsymbol{iCL}_{\mathcal{Z}}$ by means of a Galois connection $\boldsymbol{iCL}_{\mathcal{Z}} \xrightarrow{\hat{\mu}_{\mathcal{Z}}}\!\!\bullet\ \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}}$.

(1) The proof proceeds in a similar fashion to that of part (0), except that now the class $\{\langle h, \boldsymbol{h}\rangle\}(\bar{\tau}_{\mathcal{Z}})^*$ only needs to satisfy conditions ($\bar{\mathrm{C}}$0) and ($\bar{\mathrm{C}}$1) of Theorem 3.06. Since there is no counterpart to condition (C2), $\boldsymbol{h}$ need not be a monosource as in part (0).

(2) All inner cells of the left diagram except the right trapezoid are known to commute. Its commutativity is easily established by considering the right adjoint part and by using the construction of weakly hereditary cores given in Theorem 1.12 of [11].

In the right diagram again the only cell that needs attention is the right trapezoid. The same type of argument as in before works. $\square$

Since $\mathscr{M} \diamond \mathscr{M} \subseteq \bar{\boldsymbol{R}}_{\mathrm{m}}$, one can always prescribe relatively closed pairs in $\mathscr{M} \diamond \mathscr{M}$, hence in particular closed elements of $\mathscr{M}$. Whether or not we can prescribe, e.g., separated objects, may depend upon additional properties of $\bar{\boldsymbol{E}}$ or of $\mathscr{X}$. Proposition 1.01(2) yields

4.02 COROLLARY

Suppose $\bar{\boldsymbol{E}}$ is stable under pullbacks. If $\bar{\boldsymbol{E}}$ consists of epi-sinks, then for any $\mathcal{Z} \in \mathscr{C}$ and for any class $\mathscr{Y}$ of $\mathscr{X}$-objects

$$\{ X \xleftarrow{id_X} X \xrightarrow{id_X} X \mid X \in \mathscr{Y} \}(\tau_{\mathcal{Z}})^*$$

is the class of dense $\mathscr{M}$-elements for the largest $\mathcal{Z}$-modal closure operator F with respect to which all objects in $\mathscr{Y}$ are separated. Moreover, F is idempotent. $\square$

In order to derive the factorization of the Pumplün-Röhrl connection displayed in Diagram (4-00) from this result, identify $\mathscr{X}$-$\boldsymbol{Ob}$ with $\boldsymbol{H} = \{ X \xleftarrow{id_X} X \xrightarrow{id_X} X \mid X \in \mathscr{X}\text{-}\boldsymbol{Ob}\}$. Let S be the **separatedness** relation of Pumplün and Röhrl (cf. 1.01(1)), i.e., the restriction of $\perp$ to $\mathscr{X}$-$\boldsymbol{Mor} \times \boldsymbol{H}$, and let $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P} \xrightarrow{\bar{\sigma}_{\mathcal{I}}}\!\bullet\ \boldsymbol{H}\boldsymbol{P}^{\mathrm{op}}$ be the induced polarity. We need to define $\bar{\sigma}_{\mathcal{Z}}$ for arbitrary $\mathcal{Z} \in \mathscr{C}$ and link the resulting natural transformation $\bar{\sigma}$ between the constant functors $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P}$ and $\boldsymbol{H}\boldsymbol{P}^{\mathrm{op}}$ with $\mathscr{M}\boldsymbol{P} \xrightarrow{\bar{\tau}} \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}}$.

4.03 DEFINITION

Consider the relation $Q \subseteq \mathscr{X}\text{-}\boldsymbol{Mor} \times \mathscr{M}$ defined by

$$\langle f, m \rangle \in Q \iff \exists_{e \in \boldsymbol{E}}\ f = e \,;\, m$$

This induces an axiality $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P} \xrightarrow{\delta_{\mathcal{I}}}\!\bullet\ \mathscr{M}\boldsymbol{P}_{\mathcal{I}}$ (cf. Definition 1.04). For $\mathcal{Z} \in \mathscr{C}$ define $\delta_{\mathcal{Z}} := \delta_{\mathcal{I}} \,;\, \boldsymbol{P}_{\mathcal{I},\mathcal{Z}}$ to obtain a natural transformation δ from the constant functor $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P}$ to $\mathscr{M}\boldsymbol{P}$.

4.04 PROPOSITION

Suppose that $\bar{\boldsymbol{E}}$ *is stable under pullbacks and consists of epi-sinks. Then the class* $\boldsymbol{H} = \{\, X \xleftarrow{id_X} X \xrightarrow{id_X} X \mid X \in \mathscr{X}\text{-}\boldsymbol{Ob} \}$ *is contained in* $\bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}$ *, and this inclusion induces an axiality* $\boldsymbol{HP} \xrightarrow{\pi}\!\bullet\, \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}$ *. Viewed as a constant natural transformation, its dual can be used to define a natural transformation* $(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P} \xrightarrow{\bar{\sigma}} \boldsymbol{HP}^{\mathrm{op}}$ *via the diagram*

$$\begin{tikzcd}
(\mathscr{X}\text{-}\boldsymbol{Mor})\,\boldsymbol{P} \arrow[r, "\delta"] & \mathscr{M}\boldsymbol{P} \arrow[rrr, "\bar{\tau}\,;\,\psi^{\mathrm{op}}"] \arrow[dr, "\dot{\Delta}"] \arrow[ddd, "\gamma"'] & & & \bar{\boldsymbol{R}}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{op}} \arrow[ddd, "\iota^{\mathrm{op}}"] & \\
& & \boldsymbol{wCL} \arrow[r, "\dot{\epsilon}"] \arrow[d, "\ddot{\Delta}"'] & \boldsymbol{iwCL} \arrow[ur, "\hat{\nu}"] \arrow[d, "\ddot{\epsilon}"] & & \\
& & \boldsymbol{CL} \arrow[r, "\dot{\nabla}"'] & \boldsymbol{iCL} \arrow[dr, "\hat{\mu}"] & & \\
& (\mathscr{M}\diamond\mathscr{M})\,\boldsymbol{P} \arrow[ur, "\dot{\omega}"'] \arrow[rrr, "\bar{\rho}\,;\,\varphi^{\mathrm{op}}"'] & & & \bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}\boldsymbol{P}^{\mathrm{op}} \arrow[r, "\pi^{\mathrm{op}}"] & \boldsymbol{HP}^{\mathrm{op}}
\end{tikzcd} \tag{4-04}$$

If singleton $\boldsymbol{E}$*-sinks are epis as well, then* $\bar{\sigma}_{\mathcal{I}}$ *is the Pumplün–Röhrl connection of Diagram (4-00), and up to isomorphism* $(\pi^{\mathrm{op}}_{\mathcal{I}})^*\,;\,(\hat{\mu}_{\mathcal{I}})^*$ *agrees with the Salbany construction* κ^* *.*

Proof:

$\bar{\boldsymbol{E}}$ needs to consist of epi-sinks for $\boldsymbol{H}$ to be contained in $\bar{\boldsymbol{R}}^{\mathrm{i}}_{\mathrm{m}}$, and thus for π to be well-defined. Diagram (4-04) then is just a combination of the Diagrams (4-03)

If singleton $\boldsymbol{E}$-sinks are epis as well, then the left-orthogonality of an $\mathscr{X}$-morphism f to a 2-source $X \xleftarrow{id_X} X \xrightarrow{id_X} X$ is equivalent to to the left-orthogonality of the $\mathscr{M}$-component of the $\langle \boldsymbol{E}, \mathscr{M}\rangle$-factorization of f to $X \xleftarrow{id_X} X \xrightarrow{id_X} X$. This establishes $\bar{\sigma}_{\mathcal{I}}$ as the Pumplün–Röhrl connection.

By Theorem 4.01(0), $(\hat{\mu}_{\mathcal{I}})_*\,;\,(\pi^{\mathrm{op}}_{\mathcal{I}})_*$ maps $F \in \boldsymbol{iCL}_{\mathcal{I}}$ to those elements of $\boldsymbol{H}$ that are right-orthogonal to every relatively F-dense pair, and thus in particular are F-separated when viewed as $\mathscr{X}$-objects. Conversely, if $\langle n, p\rangle \in \mathscr{M}\diamond\mathscr{M}$ is relatively F-dense and if X is F-separated, then for $m = n\,;p$ we have $\langle n,p\rangle \perp mF \perp (X \xleftarrow{id_X} X \xrightarrow{id_X} X)$, which since n is mono implies $\langle n,p\rangle \perp (X \xleftarrow{id_X} X \xrightarrow{id_X} X)$. Hence $(\hat{\mu}_{\mathcal{I}})_*\,;\,(\pi^{\mathrm{op}}_{\mathcal{I}})_*$ essentially maps $F \in \boldsymbol{iCL}_{\mathcal{I}}$ to the class of all F-separated objects. Up to isomorphism, this identifies the right adjoint $(\pi^{\mathrm{op}}_{\mathcal{I}})^*\,;\,(\hat{\mu}_{\mathcal{I}})^*$ as the Salbany construction. $\square$

The hypotheses that $\bar{\boldsymbol{E}}$ is pullback-stable or consists of epi-sinks are frequently satisfied. Of particular importance is the case that $\mathscr{X}$ is finitely complete, $\mathscr{M}$ consists of all monos and $\boldsymbol{E}$ is stable under pullbacks. Such categories $\mathscr{X}$ are called **familially regular** in [14]. In this setting the sinks in $\boldsymbol{E}$ actually are colimit sinks, a property we will exploit in the following section.

5 APPLICATIONS OF GROTHENDIECK-TYPE

Our results so far do not cover the case of sheaves in a Grothendieck topos: in general empty sources are not right-orthogonal to the sinks in $\bar{\boldsymbol{E}}$. As we have seen above, whenever $\bar{\boldsymbol{E}}$ satisfies stronger conditions then the corresponding collection $\bar{\boldsymbol{R}}$ of pairs that are right-orthogonal to each sink in $\bar{\boldsymbol{E}}$ is less restricted. E.g., when each sink in $\bar{\boldsymbol{E}}$ is epi, then $\bar{\boldsymbol{R}}$ contains all the 2-sources of the form $X \xleftarrow{id_X} X \xrightarrow{id_X} X$ for $X \in \mathscr{X}\text{-}\boldsymbol{Ob}$, cf. Corollary 4.02. In order to address the issue of pairs in $\boldsymbol{R}$ that do not belong to $\bar{\boldsymbol{R}}$ our strategy below will be to impose further restrictions on $\bar{\boldsymbol{E}}$: not only will we require that the sinks in $\bar{\boldsymbol{E}}$ be epi-sinks, but in fact we want them to be colimit sinks of a special type, called *effective* sinks.

5.00 DEFINITION (cf. [14])

The **kernel** of a 2-sink $\bullet \xrightarrow{u} \bullet \xleftarrow{v} \bullet$ in a category $\mathscr{X}$ is the diagram that consists of all those 2-sources $\bullet \xleftarrow{a} \bullet \xrightarrow{b} \bullet$ that satisfy $a\,;u = b\,;v$. The **reduced kernel** of $\bullet \xrightarrow{u} \bullet \xleftarrow{v} \bullet$ coincides with the kernel, if the 2-sink has no pullback, and consists of any pullback of $\bullet \xrightarrow{u} \bullet \xleftarrow{v} \bullet$ otherwise. The (**reduced**) **kernel** of a sink $\boldsymbol{w}$ is the disjoint union of all (reduced) kernels of all 2-sinks contained in $\boldsymbol{w}$. A sink $\boldsymbol{w}$ is called **effective**, provided that it is a colimit of its kernel (or equivalently, of its reduced kernel).

It is well-known that all epi-sinks in a Grothendieck topos are effective. The following example shows that despite this property the techniques of Section 4 cannot be used to construct a largest weakly hereditary closure operator that has a given class of pre-sheaves as sheaves.

5.01 EXAMPLE

Consider the following small category $\mathscr{A}$

$$\begin{tikzcd}
H \arrow[dr, "h"] & & J \arrow[dl, "j"'] \arrow[dd, "f"] \\
& L & \\
I \arrow[ur, "i"'] & & K \arrow[ul, "k"]
\end{tikzcd} \tag{5-00}$$

with $j \neq f\,;k$. Identify $\mathscr{A}$ with the appropriate subcategory of representable functors in the functor category $\mathscr{X} = [\mathcal{A}, \boldsymbol{Set}]$. In $\mathscr{X}$ define E to be the coproduct of H and I, F to be the coproduct of H , I, and J, and G to be the colimit of the diagram spanned by all $\mathscr{A}$-objects except L. These colimits induce monomorphisms $E \xrightarrow{n} F \xrightarrow{p} G$. We now attempt to construct a largest weakly hereditary closure operator for which the pre-sheaf L will be a sheaf by forming the class A of all monos in $\mathscr{X}$ that are left-orthogonal to $\langle L, \emptyset \rangle$.

The only morphism from E to L is induced by the 2-sink $H \xrightarrow{h} L \xleftarrow{i} I$. Since there is only one natural sink with codomain L for the diagram that defines G, there also is only one $\mathscr{X}$-morphism from G to L. Thus $m = n$; p is left-orthogonal to $\langle L, \emptyset \rangle$, but neither n nor p has this property. Hence A cannot be the class of dense elements for any closure operator on $\mathscr{M}$. Moreover, the closure operator $A(\dot{\Delta}_{\mathcal{I}})_*$ cannot have L as a sheaf. $\square$

Condition ($\bar{\text{C}}$0) fails in this example, since $\langle p, id \rangle$ is a W-direct image of $\langle m, id \rangle$. If n in addition to m were left-orthogonal to $\langle Z, \emptyset \rangle$, we could conclude that p has this property as well. Hence it seems reasonable to restrict our attention to closure operators that are at least hereditary. However, to take advantage of the effectiveness of the sinks in $\bar{\boldsymbol{E}}$, it seems unavoidable to actually require all cartesian $\mathscr{M}$-morphisms to belong to $\mathcal{Z}$. So we are looking at closure operators that are at least modal.

Compare the next result to the instance of Theorem 4.01(1) at $\mathcal{C} = \{ \langle f, g \rangle \in \mathscr{M}\text{-}\boldsymbol{Mor} \mid \langle f, g \rangle \text{ is cartesian} \}$.

5.02 THEOREM

Suppose that W satisfies the Beck-Chevalley condition relative to $\mathscr{C}$, and that $\langle f, g \rangle^{\leftarrow}$ preserves suprema for every cartesian $\mathscr{M}$-morphism $\langle f, g \rangle$. Further assume that all sinks in $\bar{\boldsymbol{E}}$ are effective. Then there exists a Galois connection $\boldsymbol{iwCL}_{\mathcal{C}} \xrightarrow{\lambda} \boldsymbol{R}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{op}}$ such that the following diagram commutes

$$\begin{tikzcd}
 & \bar{\boldsymbol{R}}\boldsymbol{P}^{\mathrm{op}} \arrow[r, "\bar{\psi}^{\mathrm{op}}"] & \bar{\boldsymbol{R}}^{\mathrm{i}}\boldsymbol{P}^{\mathrm{op}} \\
\mathscr{M}\boldsymbol{P}_{\mathcal{C}} \arrow[ur, "\tau_{\mathcal{C}}"] \arrow[dr, "\dot{\Delta}_{\mathcal{C}}"'] & & \\
 & \boldsymbol{CL}_{\mathcal{C}} = \boldsymbol{wCL}_{\mathcal{C}} \arrow[r, "\dot{\epsilon}_{\mathcal{C}}"'] & \boldsymbol{iwCL}_{\mathcal{C}} = \boldsymbol{iCL}_{\mathcal{C}} \arrow[uu, "\lambda"']
\end{tikzcd} \tag{5-01}$$

More specifically, for every subclass $\boldsymbol{J}$ of $\boldsymbol{R}^{\mathrm{i}}$ there exists a largest idempotent modal closure operator F that satisfies $\boldsymbol{J}$; in fact $F(\dot{\Delta}_{\mathcal{C}})^* = \boldsymbol{J}(\tau_{\mathcal{C}})^*$.

Proof:

The only difference between this proof and the proof that the outer part of Diagram (4-02) commutes at $\mathcal{C}$ is that we can no longer rely on the fact that a given source $\boldsymbol{h}$ is right-orthogonal to every sink in $\bar{\boldsymbol{E}}$.

Set $A = \{\langle id, \boldsymbol{h} \rangle\}(\tau_{\mathcal{C}})^*$. This class is easily seen to be closed under composition, i.e., condition ($\bar{\text{C}}$3) is satisfied. To establish ($\bar{\text{C}}$0), i.e., that $A(\gamma_{\mathcal{C}})_*$ is closed under W-direct images, consider an $\mathscr{M}$-morphism $q \xrightarrow{\langle f,d\rangle} s$ with $q \in A$ and $\bullet \xrightarrow{d} \bullet \xleftarrow{s} \bullet$ in $\bar{\boldsymbol{E}}$. Moreover, consider an $\mathscr{X}$-morphism x and an $\mathscr{X}$-source $\boldsymbol{z}$ such that

$$x ; \boldsymbol{h} = s ; \boldsymbol{z}$$

We need to find a unique $\mathscr{X}$-morphism y that satisfies

$$x = s ; y \qquad \text{and} \qquad y ; \boldsymbol{h} = \boldsymbol{z} \tag{5-02}$$

Since $q \in A$, there exists a unique w such that

$$f ; x = q ; w \qquad \text{and} \qquad w ; \boldsymbol{h} = d ; \boldsymbol{z}$$

The reduced kernel of the sink $\langle d, s\rangle \in \bar{\boldsymbol{E}}$ consists of the pullback $\bullet \xleftarrow{s'} \bullet \xrightarrow{d'} \bullet$ of $\bullet \xrightarrow{d} \bullet \xleftarrow{s} \bullet$, the pullback of s along itself (which since s is mono, consists of two identities and hence does not contribute to the diagram that has $\langle d, s\rangle$ as a colimit), and all sources $\bullet \xleftarrow{a} \bullet \xrightarrow{b} \bullet$ that satisfy $a ; d = b ; d$. We first establish that

$$d' ; x = s' ; w \tag{5-03}$$

Form the pullback $\bullet \xleftarrow{s''} \bullet \xrightarrow{q'} \bullet$ of $\bullet \xrightarrow{q} \bullet \xleftarrow{s'} \bullet$. Since $q' \xrightarrow{\langle s'', s'\rangle} q$ is cartesian and A is $\mathscr{C}$-stable, q' belongs to A as well, i.e., q' is left-orthogonal to $\langle id, \boldsymbol{h}\rangle$. We have

$$d' ; x ; \boldsymbol{h} = d' ; s ; \boldsymbol{z} = s' ; d ; \boldsymbol{z} = s' ; w ; \boldsymbol{h} \tag{5-04}$$

Let k be the unique morphism (induced by the pullback) that satisfies $k ; s' = q$ and $k ; d' = f$. Since s' is mono, it follows that $q' = s'' ; k$. Hence

$$q' ; d' ; x = s'' ; k ; d' ; x = s'' ; f ; x = s'' ; q ; w = s'' ; k ; s' ; w = q' ; s' ; w \tag{5-05}$$

Together, (5-04), (5-05), and uniqueness in the definition of orthogonality establish (5-03).

Next we shall show that $a ; d = b ; d$ implies

$$a ; w = b ; w \tag{5-06}$$

Form the pullbacks $\bullet \xleftarrow{\tilde{a}} \bullet \xrightarrow{j} \bullet$ and $\bullet \xleftarrow{\tilde{b}} \bullet \xrightarrow{k} \bullet$ of $\bullet \xrightarrow{q} \bullet \xleftarrow{a} \bullet$ and $\bullet \xrightarrow{q} \bullet \xleftarrow{b} \bullet$, respectively. As above we conclude that j and k are left-orthogonal to $\langle id, \boldsymbol{h}\rangle$. However, this alone is not sufficient to establish (5-06). Let $\bullet \xleftarrow{k'} \bullet \xrightarrow{j'} \bullet$ be a pullback of $\bullet \xrightarrow{j} \bullet \xleftarrow{k} \bullet$. Since both j' and k' are left-orthogonal to $\langle id, \boldsymbol{h}\rangle$, so by ($\bar{\text{C}}$3) is the composite $j' ; k = h = k' ; j$. Therefore $a ; w$ is the unique morphism g that satisfies $k' ; \tilde{a} ; f ; x = h ; g$ and $g ; \boldsymbol{h} = a ; d ; \boldsymbol{z}$. Similarly, $b ; w$ is the unique morphism g that satisfies $j' ; \tilde{b} ; f ; x = h ; g$ and $g ; \boldsymbol{h} = b ; d ; \boldsymbol{z}$. We observe that

$$k' ; \tilde{a} ; f ; s = k' ; \tilde{a} ; q ; d = j' ; \tilde{b} ; q ; d = j' ; \tilde{b} ; f ; s$$

Since s is mono, this implies $k' ; \tilde{a} ; f = j' ; \tilde{b} ; f$, and thus in particular $k' ; \tilde{a} ; f ; x = j' ; \tilde{b} ; f ; x$. Since $a ; d ; \boldsymbol{z} = b ; d ; \boldsymbol{z}$ as well, it follows that $a ; w = b ; w$. Hence x in conjunction with w yields a natural sink for the reduced kernel of $\bullet \xrightarrow{d} \bullet \xleftarrow{s} \bullet$. By the effectiveness of $\langle d, s \rangle$ there exists a unique y for which (5-02) holds.

To establish ($\bar{\mathrm{C}}$1), for $m \in \mathscr{M}$ set $(A(\gamma_{\mathcal{C}})_* \cap W/m) \sup_{\ll} = \langle n, p \rangle$, and consider an $\mathscr{X}$-morphism x and an $\mathscr{X}$-source $\boldsymbol{z}$ such that $x ; \boldsymbol{h} = m ; \boldsymbol{z}$. Whenever $\langle q, r \rangle$ and $\langle s, t \rangle$ belong to $A(\gamma_{\mathcal{C}})_* \cap W/m$ there exist morphisms v and w such that

$$q ; v = x = s ; w \qquad \text{and} \qquad r ; \boldsymbol{z} = v ; \boldsymbol{h} \quad \text{and} \quad t ; \boldsymbol{z} = w ; \boldsymbol{h}$$

Moreover, r and t factor through p by means of morphisms c and d, respectively. Now form the pullback $\bullet \xleftarrow{a} \bullet \xrightarrow{b} \bullet$ of $\bullet \xrightarrow{c} \bullet \xleftarrow{d} \bullet$. We need to show that

$$a ; v = b ; w \tag{5-07}$$

Let l be the unique morphism that satisfies $l ; a = q$ and $l ; b = s$. If $\bullet \xleftarrow{a'} \bullet \xrightarrow{q'} \bullet$ is a pullback of $\bullet \xrightarrow{q} \bullet \xleftarrow{a} \bullet$, since a is mono it follows that $a' ; l = q'$. Therefore

$$q' ; a ; v = a' ; q ; v = a' ; x = a' ; s ; w = a' ; l ; b ; w = q' ; b ; w$$

Since we also know that

$$b ; w ; \boldsymbol{h} = b ; t ; \boldsymbol{z} = b ; d ; p ; \boldsymbol{z} = a ; c ; p ; \boldsymbol{z} = a ; r ; \boldsymbol{z} = a ; v ; \boldsymbol{h}$$

the left-orthogonality of q' to $\langle id, \boldsymbol{h} \rangle$ implies (5-07). Now the effectiveness of the supremum sink that induces $\langle n, p \rangle$ yields the desired unique y with

$$x = n ; y \qquad \text{and} \qquad y ; \boldsymbol{h} = p ; \boldsymbol{z}$$ □

5.03 COROLLARY

For any category $\mathscr{X}$ that satisfies the hypotheses of Theorem 5.02 every collection $\mathscr{Y}$ of objects (= full subcategory) induces an idempotent modal closure operator (unique up to isomorphism) for which all objects in $\mathscr{Y}$ are sheaves. □

5.04 REMARK

In view of the proof it seems unlikely, however, that a result like Theorem 5.02 can be extended to other collections $\mathscr{Z} \in \mathscr{C}$, e.g., to idempotent hereditary closure operators, or in the case that $\mathscr{X}$ has binary products to idempotent $\mathscr{P}$-modal closure operators, where $\mathscr{P}$ consists of all projections of the form $id_X \times n \longrightarrow n$ in $\mathscr{M}$, cf. [4], Section 2. (This latter type of closure operators is interesting, since for a cartesian closed category $\mathscr{X}$ the sheaves of a $\mathscr{P}$-modal closure operator form a cartesian closed subcategory.) But so far counterexamples similar to Example 5.01 have proved to be elusive.

It would be useful to be able to characterize those categories $\mathscr{X}$ and those closure operators F for which the F-sheaves form a reflective subcategory.

REFERENCES

[0] Adámek, J., Herrlich, H., and Strecker, G. E. *Abstract and Concrete Categories*. John Wiley, New York, 1990.

[1] Börger, R. *Kategorielle Beschreibungen von Zusammenhangsbegriffen*. PhD thesis, Fernuniversität Hagen, 1981.

[2] Castellini, G. Closure operators, monomorphisms and epimorphisms in categories of groups. *Cahiers Topologie Géom. Différentielle Catégoriques 27*, 2 (1986), 151–167.

[3] Castellini, G., Koslowski, J., and Strecker, G. E. Closure operators and polarities. In *Proceedings of the 1991 Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work* (Madison, WI, June 1991), A. R. Todd, Ed., New York Academy of Science. To appear 1993.

[4] Castellini, G., Koslowski, J., and Strecker, G. E. Hereditary and modal closure operators. In *Category Theory 1991, Proceedings of the Summer International Meeting* (Montreal, Canada, June 1991), R. A. G. Seely, Ed., no. 13 in C.M.S. Conference Proceedings, American Mathematical Society, 1992, pp. 111–132.

[5] Castellini, G., Koslowski, J., and Strecker, G. E. A factorization of the Pumplün-Röhrl connection. *Topology Appl. 44* (1992), 69–76.

[6] Castellini, G., and Strecker, G. E. Global closure operators vs. subcategories. *Quaestiones Math. 13* (1990), 417–424.

[7] Dikranjan, D., and Giuli, E. Closure operators I. *Topology Appl. 27* (1987), 129–143.

[8] Dikranjan, D., Giuli, E., and Tholen, W. Closure operators II. In *Categorical Topology* (Prague, 1988), J. Adámek and S. MacLane, Eds., World Scientific, Singapore, 1989, pp. 297–335.

[9] Erné, M., Koslowski, J., Melton, A., and Strecker, G. E. A primer on Galois connections. In *Proceedings of the 1991 Summer Conference on Topology and Applications in Honor of Mary Ellen Rudin and Her Work* (Madison, WI, June 1991), A. R. Todd, Ed., New York Academy of Science. To appear 1993.

[10] Isbell, J. R. Epimorphisms and dominions. In *Proceedings of the Conference on Categorical Algebra, La Jolla 1965* (La Jolla, 1965), S. Eilenberg, D. Harrison, S. MacLane, and H. Röhrl, Eds., Springer-Verlag, Berlin – New York, 1966, pp. 232–246.

[11] Koslowski, J. Closure operators with prescribed properties. In *Category Theory and its Applications* (Louvain-la-Neuve, 1987), F. Borceux, Ed., no. 1248 in Lecture Notes in Mathematics, Springer-Verlag, Berlin – New York, 1988, pp. 208–220.

[12] Pumplün, D., and Röhrl, H. Separated totally convex spaces. *Manuscripta Math. 50* (1985), 145–183.

[13] Salbany, S. Reflective subcategories and closure operators. In *Categorical Topology* (Mannheim, 1975), E. Binz and H. Herrlich, Eds., no. 540 in Lecture Notes in Mathematics, Springer-Verlag, Berlin – New York, 1976, pp. 548–565.

[14] Street, R. The family approach to total cocompleteness and toposes. *Trans. Amer. Math. Soc. 284* (1984), 355–369.

[15] Tholen, W. Factorizations, localizations and the orthogonal subcategory problem. *Math. Nachr. 114* (1983), 63–85.

[16] Tholen, W. Prereflections and reflections. *Comm. Algebra 14* (1986), 717–740.

Department of Mathematics, University of Puerto Rico, Mayagüez, PR 00681
Institut für Mathematik, Universität Hannover, 30167 Hannover, Germany
Department of Mathematics, Kansas State University, Manhattan, KS 66506
E-mail addresses: g_castellini@upr1.upr.clu.edu,
koslowj@math.ksu.edu or aberne@dhvrrzn1.uni-hannover.d400.de
strecker@math.ksu.edu