

# Regular completions of Cauchy spaces via function algebras

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A regular completion with the universal property is obtained for each member of a certain class of Cauchy spaces by embedding the Cauchy space in a complete function algebra with the continuous convergence structure.

## 1. Preliminaries

A Cauchy space  $(X, \mathcal{C})$  is a set  $X$  equipped with a Cauchy structure  $\mathcal{C}$ ; that is,  $\mathcal{C}$  is a set of filters on  $X$  (called "Cauchy filters") subject to the following conditions:

- (C<sub>1</sub>) the ultrafilter  $\dot{x}$  containing  $\{x\}$  is in  $\mathcal{C}$  for all  $x$  in  $X$ ;
- (C<sub>2</sub>)  $F \in \mathcal{C}$  and  $G \geq F$  implies  $G \in \mathcal{C}$ ;
- (C<sub>3</sub>) if  $F, G \in \mathcal{C}$  and  $F \vee G \neq 0$  (meaning each set in  $F$  intersects each set in  $G$ ), then  $F \cap G \in \mathcal{C}$ .

A Cauchy structure  $\mathcal{C}$  generates a convergence structure on  $X$  defined as follows:  $F \rightarrow x$  in  $X$  if  $\dot{x} \cap F \in \mathcal{C}$ . Keller [5] showed that a set of filters on  $X$  satisfying (C<sub>1</sub>), (C<sub>2</sub>), and (C<sub>3</sub>) is precisely the set of Cauchy filters for some uniform convergence structure (see [3]) on  $X$ . A definitive study of Cauchy spaces has been made by Ramaley and Wyler ([8], [9]).

A Cauchy space  $(X, \mathcal{C})$  is *complete* if each Cauchy filter is *convergent*. Each convergence space which admits a uniform convergence

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structure (see [5] for a characterization of such spaces) can be regarded as a complete Cauchy space if the Cauchy structure is taken to be the set of all Cauchy filters. A function  $f : (X, \mathcal{C}) \rightarrow (X_1, \mathcal{C}_1)$  is *Cauchy-continuous* if  $F \in \mathcal{C}$  implies  $f(F) \in \mathcal{C}_1$ . The pair  $((X_1, \mathcal{C}_1), j)$  is a *completion* of a Cauchy space  $(X, \mathcal{C})$  if  $(X_1, \mathcal{C}_1)$  is a complete Cauchy space, and  $j : (X, \mathcal{C}) \rightarrow (X_1, \mathcal{C}_1)$  is a dense Cauchy embedding. If, in addition,  $F \rightarrow x$  in  $X$  implies the existence of a filter  $G \in \mathcal{C}$  such that  $jG \rightarrow x$  in  $X$  and  $F \geq \text{cl}_X(jG)$ , the completion is said to be *strict*.

Reed [10], showed that each completion of a Cauchy space  $(X, \mathcal{C})$  induces a completion of each uniform convergence space  $(X, \mathcal{J})$  for which  $\mathcal{C}$  is the set of  $\mathcal{J}$ -Cauchy filters. A Cauchy space  $(X, \mathcal{C})$  is *regular* if  $\text{cl}_X F \in \mathcal{C}$  whenever  $F \in \mathcal{C}$ . It is shown in [6] that a strict, regular, Hausdorff completion of a Cauchy space, if it exists, is unique up to Cauchy-homeomorphism.

The Cauchy filters in a Cauchy space  $(X, \mathcal{C})$  are partitioned into *Cauchy equivalence classes* by the following equivalence relation:  $F \sim G$  if and only if  $F \cap G \in \mathcal{C}$ . Reed [10] used the set  $\{[F] : F \in \mathcal{C}\}$  of these equivalence classes to construct a family of completions for any Cauchy space. In particular, she showed that every Hausdorff Cauchy space (that is, Hausdorff in its underlying convergence structure) has a strict Hausdorff completion; we will make use of this fact later.

A Cauchy space  $(X, \mathcal{C})$  is said to be *pseudo-topological* if a filter  $F$  is in  $\mathcal{C}$  whenever all ultrafilters finer than  $F$  are in the same Cauchy equivalence class relative to  $\mathcal{C}$ . A convergence space is *pseudo-topological* if a filter converges to  $x$  whenever each refining ultrafilter converges to  $x$ . Note that a pseudo-topological convergence space can be regarded as a complete pseudo-topological Cauchy space.

Let  $X$  be a convergence space,  $C(X)$  the  $R$ -algebra of all continuous real valued functions on  $X$ . The *continuous convergence structure*  $c$  on  $X$  is the coarsest relative to which the function  $w : C(X) \times X \rightarrow R$ , defined by  $w(f, x) = f(x)$ , is continuous. The resulting convergence function space  $C_c(X)$  has been studied extensively

(see, for instance, [1], [2], [4], [11], and [12]). The natural map  $i : X \rightarrow C_c C_c(X)$  is defined by  $i(x) = F_x$ , where  $F_x(f) = f(x)$ , all  $f \in C(X)$ . Binz [1], has shown that  $i(X) = \text{hom}_{C_c}(X)$ , the set of all continuous algebra homomorphisms from  $C_c(X)$  into  $R$  which preserve the multiplicative identity. The space  $X$  is said to be *c-embedded* if the function  $i$  is an embedding. In order to state a characterization of *c-embedded* spaces, we need a few preliminary definitions.

Starting with a space  $X$ , let  $\omega X$  denote the finest completely regular topological space coarser than  $X$  on the same underlying set;  $\omega X$  is called the completely regular modification of  $X$ . The fact that  $C(X) = C(\omega X)$  is useful and will be used without explicit mention in what follows. We will use the symbol " $\text{cl}_X$ " to represent the closure operator for any space  $X$ .  $X$  is said to be  *$\omega$ -regular* if  $\text{cl}_{\omega X} F \rightarrow x$  in  $X$  whenever  $F \rightarrow x$  in  $X$ . The first proposition is established in [12] and, implicitly, in [11].

**PROPOSITION 1.1.** *A space  $X$  is c-embedded iff  $X$  is pseudo-topological, Hausdorff, and  $\omega$ -regular.*

Binz [2], has shown that the space  $C_c(X)$  is always *c-embedded*.

Next, let  $(X, C)$  be a Cauchy space, and let  $C^\wedge(X, C)$  be the  $R$ -algebra of all real-valued Cauchy-continuous functions on  $(X, C)$ . Let  $C^\wedge$  denote the *continuous Cauchy structure* on  $C^\wedge(X, C)$ , defined as follows: a filter  $\Phi$  on  $C^\wedge(X, C)$  is in  $C^\wedge$  iff  $\Phi(F)$  is  $R$ -Cauchy (that is,  $R$ -convergent) whenever  $F \in C$ . If one defines the notion of product Cauchy structure in the natural way, then there is no difficulty in establishing the following proposition; we omit the details.

**PROPOSITION 1.2.** *For any Cauchy space  $(X, C)$ ,  $C^\wedge$  is the coarsest Cauchy structure on  $C^\wedge(X, C)$  relative to which the function  $w : C^\wedge(X, C) \times (X, C) \rightarrow R$ , defined by  $w(f, x) = f(x)$ , is Cauchy-continuous.*

**PROPOSITION 1.3.** *For any Cauchy space  $(X, C)$ , the Cauchy function space  $(C^\wedge(X, C), C^\wedge)$  is complete.*

**Proof.** Let  $\Phi \in C^\wedge$ , and let  $f : X \rightarrow R$  be defined by

$f(x) = \lim \Phi(\dot{x})$ , all  $x \in X$ .

To show that  $f \in C^\wedge(X)$ , let  $F \in \mathcal{C}$ ; then  $\Phi(F) \rightarrow a$ , for some  $a \in R$ . Let  $W$  be a closed neighborhood of  $a$  in  $R$ ; then there is  $A \in \Phi$  and  $F \in \mathcal{F}$  such that  $A(F) \subset W$ . If  $x \in F$ , then  $A(x) \subset W$ , and so  $f(x) \in \text{cl}_R A(x) \subset W$ . Thus  $f(F) \subset W$ , and  $f(F) \rightarrow a$  in  $R$ .

To show that  $\Phi \rightarrow f$ , it must be shown that  $\Phi \cap \dot{f} \in C^\wedge$ . If  $F \subset X$ ,  $A \in \Phi$ , and  $x \in F$ , then  $A(F) \in \Phi(\dot{x})$ , and so  $f(F) \subset \text{cl}_R A(F)$ . Thus, if  $F \in \mathcal{C}$ ,  $\dot{f}(F) \supset \text{cl}_R \Phi(F)$ , and hence  $\dot{f}(F)$  and  $\Phi(F)$  must converge to the same point in  $R$ .

A complete Cauchy space is merely a convergence space in which the convergent filters are designated to be Cauchy. Thus the Cauchy structure  $C^\wedge$  on the function space is somewhat redundant and will usually be deleted. Also, it is convenient to shorten the notation from  $C^\wedge(X, \mathcal{C})$  to  $C^\wedge(X)$  if there is no danger of confusing which Cauchy structure on  $X$  is being considered.

The next proposition follows by inspection of the respective definitions.

**PROPOSITION 1.4.** *If  $(X, \mathcal{C})$  is a complete Cauchy space, then  $C^\wedge(X)$  coincides with  $C_{\mathcal{C}}(X)$ .*

Reed [10], proved that each Hausdorff Cauchy space  $(X, \mathcal{C})$  has a strict, Hausdorff completion  $((X_1, \mathcal{C}_1), j)$  with the *regular extension property*. The latter term means that each Cauchy-continuous function from  $(X, \mathcal{C})$  into a Hausdorff, regular, complete Cauchy space  $(X_2, \mathcal{C}_2)$  has a Cauchy-continuous extension  $f_1 : (X_1, \mathcal{C}_1) \rightarrow (X_2, \mathcal{C}_2)$ . The existence of such a completion (but not the details of the construction) are needed for the following proposition.

**PROPOSITION 1.5.** *Let  $(X, \mathcal{C})$  be a Hausdorff Cauchy space, and let  $((X_1, \mathcal{C}_1), j)$  be a strict, Hausdorff completion with the regular extension property. Then  $C^\wedge(X)$  and  $C^\wedge(X_1)$  are both isomorphic and homeomorphic.*

**Proof.** Let  $\Psi : C^\wedge(X) \rightarrow C^\wedge(X_1)$  be defined by  $\Psi(f) = f_1$ , where  $f_1$

is the extension from  $X$  to  $X_1$  whose existence is derived from the regular extension property. The uniqueness of  $f_1$  is guaranteed by the Hausdorff property, and so  $\Psi$  is well defined. It is a trivial matter to verify that  $\Psi$  is an algebra isomorphism.

To show that  $\Psi$  is Cauchy continuous, let  $\Phi$  be  $C^\wedge(X)$ -Cauchy and let  $A \in C_1$ . Since  $X_1$  is complete,  $A$  converges in  $X_1$ , and by the strictness of the completion, there is  $F \in C$  such that  $A \geq \text{cl}_{X_1} jF$ . It then follows easily that  $(\Psi\Phi)(A) \geq \text{cl}_R \Phi(F)$ . But  $\Phi(F)$  is  $R$ -convergent, by definition of the Cauchy structure on  $C^\wedge(X)$ , and, since  $R$  is regular,  $(\Psi\Phi)(A)$  is likewise  $R$ -convergent. Thus  $\Psi\Phi$  is  $C^\wedge(X_1)$ -Cauchy.

To show that  $\Psi^{-1}$  is continuous, let  $\Lambda$  be a  $C^\wedge(X_1)$ -Cauchy filter, and let  $F \in C$ . Then  $(\Psi^{-1}\Lambda)(F) \geq (jF)$ , and it follows that  $\Psi^{-1}\Lambda$  is  $C^\wedge(X)$ -Cauchy.

**COROLLARY 1.6.** *For any Hausdorff Cauchy space  $(X, C)$ , there is a convergence space  $X_1$  such that  $C^\wedge(X)$  is isomorphic and homeomorphic to  $C_c(X_1)$ . Consequently,  $C^\wedge(X)$  is a Hausdorff,  $\omega$ -regular, pseudo-topological convergence space.*

*Proof.* These statements follow immediately from Proposition 1.1, the remark following Proposition 1.1, Proposition 1.4, and Proposition 1.5.

It follows from the preceding results that, for any Cauchy space  $(X, C)$ ,  $C^\wedge C^\wedge(X)$  is equal to  $C_c C^\wedge(X)$  and is equivalent, in the sense of Corollary 1.6, to  $C_c C_c(X_1)$ , where  $(X_1, C_1)$  is the completion of Proposition 1.5. A Cauchy space  $(X, C)$  is said to be  $c^\wedge$ -embedded if the function  $i : (X, C) \rightarrow C^\wedge C^\wedge(X)$ , where  $i(x)(f) = f(x)$  for all  $f$  in  $C^\wedge(X)$ , is a Cauchy embedding. In order to obtain an internal characterization of  $c^\wedge$ -embedded spaces, we will introduce several new Cauchy space concepts.

For any Cauchy space  $(X, C)$ , let  $\mu X$  be the set  $X$  with the weak topology induced by the set of functions  $C^\wedge(X)$ . Since  $C^\wedge(X) \subset C(X)$ , it follows that  $\mu X \leq \omega X$ , and  $\mu X = \omega X$  if  $(X, C)$  is complete.  $(X, C)$  is

said to be  $\mu$ -regular if  $\text{cl}_{\mu X} F \in \mathcal{C}$  whenever  $F \in \mathcal{C}$ . It is clear that  $\mu$ -regular implies  $\omega$ -regular implies regular.

A Cauchy space  $(X, \mathcal{C})$  is said to be *Cauchy-separated* if  $X$  is Hausdorff and, whenever  $F, G \in \mathcal{C}$  and  $F \cap G \notin \mathcal{C}$ , there is  $f \in C^{\wedge}(X)$  such that  $\lim f(F) \neq \lim f(G)$  in  $R$ . In other words,  $(X, \mathcal{C})$  is Cauchy-separated iff there are enough functions in  $C^{\wedge}(X)$  to separate Cauchy equivalence classes. Since distinct points in a Hausdorff Cauchy space correspond to distinct equivalence classes, it follows that  $\mu X$  is Hausdorff whenever  $(X, \mathcal{C})$  is Cauchy-separated.

**PROPOSITION 1.7.** *Let  $(X, \mathcal{C})$  be a Cauchy space,  $A$  a  $\mu X$ -closed subset of  $X$ , and  $x \in X - A$ . Then there is a function  $f$  in  $C^{\wedge}(X)$  such that  $f(x) = 0$  and  $f(A) = 1$ .*

*Proof.* Let  $J$  be the uniformity on  $X$  generated by sets of the form  $J(\varepsilon, f) = \{(x, y) : |f(x) - f(y)| < \varepsilon\}$ , where  $\varepsilon > 0$  and  $f \in C^{\wedge}(X, \mathcal{C})$ . Note that the  $J$ -Cauchy filters are precisely those filters  $F$  such that  $f \in C^{\wedge}(X, \mathcal{C})$  implies that  $f(F)$  is Cauchy on  $R$ . Thus, if  $\mathcal{C}_J$  is the set of  $J$ -Cauchy filters,  $\mathcal{C} \subset \mathcal{C}_J$ . It also follows from this construction that  $\mathcal{C}_J$  is an admissible Cauchy structure for  $\mu X$ .

Let  $x$  and  $A$  be as stated in the proposition, and let  $(X_1, J_1)$  be a complete uniform space containing  $(X, J)$ . Then there is an  $X_1$ -closed set  $B \subset X_1$  such that  $A = X_1 \cap B$ . Since  $X_1$  is a completely regular topological space, there is  $g \in C(X_1)$  such that  $g(x) = 0$  and  $g(B) = 1$ . If  $f$  is the restriction of  $g$  to  $X$ , then  $f \in C^{\wedge}(X, \mathcal{C}_J)$ ,  $f(x) = 0$ , and  $f(A) = 1$ . Since  $\mathcal{C} \subset \mathcal{C}_J$ ,  $C^{\wedge}(X, \mathcal{C}_J) \subset C^{\wedge}(X, \mathcal{C})$ , and the proof is complete.

**PROPOSITION 1.8.** *A Cauchy space  $(X, \mathcal{C})$  which has an  $\omega$ -regular, Hausdorff completion is  $\mu$ -regular and Cauchy-separated.*

*Proof.* Let  $((X_1, \mathcal{C}_1), j)$  be a Hausdorff,  $\omega$ -regular completion of  $(X, \mathcal{C})$ . We will first show that  $(X, \mathcal{C})$  is  $\mu$ -regular. Let  $F \in \mathcal{C}$ ; then  $\text{cl}_{\omega X_1}(jF) \in \mathcal{C}_1$ . If  $A \subset X$ , then

$$cl_{\omega X_1}(jA) = \bigcap \left\{ f^{-1}(B) : B \text{ closed subset of } R, jA \subset f^{-1}(B), f \in C(X_1) \right\},$$

by Proposition 1.7, and

$$cl_{\mu X}A = \bigcap \{ f^{-1}(B) : B \text{ closed subset of } R, A \subset f^{-1}(B), f \in C(X) \}.$$

If  $f \in C(X_1)$ , then the restriction of  $f$  to  $X$  is in  $C(X)$ . Thus

$$j(cl_{\mu X}A) \subset cl_{\omega X_1}(jA), \text{ and it follows that } cl_{\mu X}F \geq j^{-1}\left( cl_{\omega X_1}(jF) \right).$$

Thus  $cl_{\mu X}F \in C$ , and  $(X, C)$  is  $\mu$ -regular.

To show that  $(X, C)$  is Cauchy separated, let  $F, G \in C$  and  $F \cap G \notin C$ . Then  $jF$  and  $jG$  are in distinct Cauchy equivalence classes in  $C_1$ , and so  $jF \rightarrow a$  in  $X_1$ ,  $jG \rightarrow b$  in  $X_1$ , and  $a \neq b$ . Since  $X_1$  is  $\omega$ -Hausdorff, there is a function  $f$  in  $C(X_1)$  which separates the points  $a$  and  $b$ . If  $g = f \circ j$ , then  $g \in C(X)$ , and  $limfF \neq limfG$ .

## 2. The natural completion

In this section, we will characterize the  $c^\wedge$ -embedded Cauchy spaces both internally and externally; the external characterization leads to what we call the *natural completion*. The natural completion is an  $\omega$ -regular, Hausdorff, pseudo-topological completion obtained by embedding  $(X, C)$  into  $C^\wedge C^\wedge(X)$  under the natural injection  $i$ .

Let  $hom^\wedge C^\wedge(X)$  denote the subspace of  $C^\wedge C^\wedge(X)$  consisting of all continuous algebra homomorphisms from  $C^\wedge(X)$  into  $R$  which preserve the multiplicative identity.  $hom^\wedge C^\wedge(X)$  coincides with  $hom_{\mathcal{C}}^\wedge C^\wedge(X)$ , and by Corollary 1.6,  $hom^\wedge C^\wedge(X)$  is an  $\omega$ -regular, Hausdorff, pseudo-topological convergence space, since all of these properties are inherited from  $C_{\mathcal{C}}^\wedge C^\wedge(X)$ . We will assume that  $hom^\wedge C^\wedge(X)$  is equipped with the Cauchy structure inherited from  $C^\wedge C^\wedge(X)$ .

**PROPOSITION 2.1.** *Let  $(X, C)$  be a Hausdorff Cauchy space,  $i : (X, C) \rightarrow C^\wedge C^\wedge(X)$  the natural function.*

- (a)  $i(X)$  is a dense subset of  $hom^\wedge C^\wedge(X)$ .
- (b)  $i$  is Cauchy-continuous.

(c) If  $(X, C)$  is Cauchy-separated, then  $i$  is one-to-one.

Proof. Let  $((X_1, C_1), j)$  be a strict, Hausdorff completion of  $(X, C)$  (see remarks preceding Proposition 1.5). Consider the following commutative diagram:

$$\begin{array}{ccc}
 (X, C) & \xrightarrow{i} & C^\wedge C^\wedge(X) \\
 j \downarrow & & \uparrow \Gamma \\
 (X_1, C_1) & \xrightarrow{i_1} \text{hom}_C C_C(X_1) \xrightarrow{id} & C^\wedge C^\wedge(X_1)
 \end{array}$$

In this diagram,  $id$  is the identity map and  $\Gamma$  the homeomorphism and algebra isomorphism derived from the homeomorphism  $\Psi : C^\wedge(X) \rightarrow C^\wedge(X_1)$  of Proposition 1.5. By the work of Binz [1],  $i_1$  is always continuous and onto  $\text{hom}_C C_C(X_1)$ . Thus  $i$  is a composition of Cauchy-continuous functions (recall that all the spaces in the above diagram except  $(X, C)$  are complete Cauchy spaces, and continuous functions between complete Cauchy spaces are Cauchy continuous).  $jX$  is dense in  $X_1$ , and therefore  $i_1 jX$  is dense in  $\text{hom}_C C_C(X_1)$ . Since  $\Gamma$  is an algebra isomorphism,  $\Gamma$  carries  $\text{hom}_C C_C(X_1)$  onto  $\text{hom}^\wedge C^\wedge(X)$ , and (a) is thus established.

Finally, assume that  $(X, C)$  is Cauchy-separated, and let  $i(x) = i(y)$ . This implies  $f(x) = f(y)$ , all  $f \in C^\wedge(X)$ . But  $\mu X$  has the weak topology induced by  $C^\wedge(X)$ , and  $x = y$  follows from the fact that  $\mu X$  is Hausdorff whenever  $(X, C)$  is Cauchy-separated (see the remark preceding Proposition 1.7).

Let  $(X, C)$  be a Cauchy space. A collection  $B$  of subsets of  $X$  will be called a *Cauchy covering system* if each member of  $C$  contains a member of  $B$ . This definition is the obvious extension to Cauchy spaces of the notion of "covering system" defined for convergence spaces in [4].

Starting with a Cauchy system  $B$  for  $(X, C)$ , let  $U$  be the uniformity on the set  $C^\wedge(X)$  with basic sets of the form  $T(B_1, \dots, B_n, \epsilon) = \{(f, g) : |f(x) - g(x)| < \epsilon \text{ for all } x, y \in \cup B_i\} \cup \Delta$ , where  $\Delta$  is the diagonal in  $C^\wedge(X) \times C^\wedge(X)$ ,  $\{B_1, \dots, B_n\}$  is an arbitrary finite subcollection of  $B$ , and  $\epsilon > 0$ . Let  $\bar{0}$  be the zero



function in  $C^\wedge(X)$ , and  $N(\bar{0})$  the neighborhood filter at  $\bar{0}$  in the uniform topology on  $C^\wedge(X)$ .

LEMMA 2.2. *In the notation of the preceding paragraph,  $N(\bar{0}) \in C^\wedge$ , the continuous Cauchy structure on  $C^\wedge(X)$ .*

Proof. Let  $H \in C$ ; then  $H$  contains a set  $B \in \mathcal{B}$ , and, for any  $\varepsilon > 0$ ,  $(T(B, \varepsilon)(\bar{0}))(B) \subset (-\varepsilon, \varepsilon)$ .

THEOREM 2.3. *The following statements about a Cauchy space  $(X, C)$  are equivalent:*

- (a)  $(X, C)$  is  $C^\wedge$ -embedded;
- (b)  $(\text{hom}^\wedge C^\wedge(X), i)$  is a completion of  $(X, C)$ ;
- (c)  $(X, C)$  has an  $\omega$ -regular, Hausdorff, pseudo-topological completion;
- (d)  $(X, C)$  is  $\mu$ -regular, Cauchy-separated, and pseudo-topological.

Proof. (a) implies (b). By Proposition 2.1,  $i(X)$  is a dense subset of  $\text{hom}^\wedge C^\wedge(X)$ .  $\text{hom}^\wedge C^\wedge(X)$  is a closed subspace of the complete Cauchy space  $C^\wedge C^\wedge(X)$ , so  $\text{hom}^\wedge C^\wedge(X)$  is also a complete Cauchy space.

(b) implies (c). We noted in the second paragraph of Section 2 that  $\text{hom}^\wedge C^\wedge(X)$  is  $\omega$ -regular, Hausdorff, and pseudo-topological.

(c) implies (d). See Proposition 1.8.

(d) implies (a). The function  $i : (X, C) \rightarrow \text{hom}^\wedge C^\wedge(X)$  is Cauchy-continuous and one-to-one by Proposition 2.1. Thus it remains only to show that if  $G$  is a filter on  $X$  such that  $iG$  is convergent in  $\text{hom}^\wedge C^\wedge(X)$ , then  $G \in C$ . Since  $(X, C)$  is pseudo-topological, it is sufficient to show:

- (1) each ultrafilter finer than  $G$  is in  $C$ ;
- (2) all ultrafilters finer than  $G$  are in the same equivalence class.

To establish (1), assume that  $F$  is an ultrafilter,  $F \geq G$ , and  $F \notin C$ . Then  $i(F)$  converges in  $\text{hom}^\wedge C^\wedge(X)$ . Since  $F \notin C$ , and  $(X, C)$  is  $\mu$ -regular,  $F \not\leq \text{cl}_{\mu X} H$  for all  $H \in C$ . Thus, for each  $H \in C$ , there

is  $H_H$  in  $H$ , but not in  $F$ , such that  $H_H$  is  $\mu X$ -closed. The set  $\mathcal{B} = \{H_H : H \in \mathcal{C}\}$  is a Cauchy covering system for  $(X, \mathcal{C})$ . Let  $U$  be the uniformity on the set  $\mathcal{C}^\wedge(X)$  derived from  $\mathcal{B}$ , as in the paragraph preceding Lemma 2.2, and let  $N(\bar{0})$  be the uniform neighborhood filter at the constant function  $\bar{0}$ . By Lemma 2.2,  $N(\bar{0}) \in \mathcal{C}^\wedge$ . Since  $i(F)$  is  $\text{hom}^\wedge \mathcal{C}^\wedge(X)$ -Cauchy,  $i(F)(N(\bar{0})) \rightarrow 0$ ; which implies that  $(N(\bar{0}))(F) \rightarrow 0$  in  $R$ .

Let  $W$  be a neighborhood of 0 in  $R$  not containing 1. Then there are sets  $B_1, \dots, B_n$  in  $\mathcal{B}$ ,  $\varepsilon > 0$ , and  $F \in \mathcal{F}$  such that  $(T(B_1, \dots, B_n, \varepsilon))(F) \subset W$ . Since  $F$  is an ultrafilter and  $B_i \notin F$  for  $i = 1, \dots, n$ ,  $F$  is not a subset of  $B_0 = \cup\{B_i : i = 1, \dots, n\}$ . Let  $x \in F - B_0$ . Since  $B_0$  is  $\mu X$ -closed, it follows from Proposition 1.7 that there is a function  $f \in \mathcal{C}^\wedge(X)$  such that  $f(x) = 1$  and  $f(B_0) = 0$ . By this construction,  $f \in T(B_1, \dots, B_n, \varepsilon)$ , but  $f(x) \notin W$  for  $x \in F$ , a contradiction. This contradiction establishes that  $F \in \mathcal{C}$ , and assertion (1) is proved.

To establish (2), suppose that  $F$  and  $H$  are ultrafilters finer than  $G$ ; thus  $i(F \cap H) \geq i(G)$ . Let  $f \in \mathcal{C}^\wedge(x)$ . Then  $(i(F \cap H))(f)$  is equal to  $f(F) \cap f(H)$ , and so the latter filter is Cauchy in  $R$ . Thus  $f(F)$  and  $f(H)$  converge to the same point in  $R$ , and this is true for all  $f \in \mathcal{C}^\wedge(X)$ . Thus, since  $(X, \mathcal{C})$  is Cauchy-separated,  $F \cap G \in \mathcal{C}$ . This completes the proof of Theorem 2.3.

If  $(X, \mathcal{C})$  is a  $\mathcal{C}^\wedge$ -embedded Cauchy space, we will refer to  $\text{hom}^\wedge \mathcal{C}^\wedge(X)$  as the *natural completion* of  $(X, \mathcal{C})$ . As our concluding result in this paper, we will prove that the natural completion of a  $\mathcal{C}^\wedge$ -embedded space has the universal property in the class of all  $\mathcal{C}^\wedge$ -embedded Cauchy spaces.

Let  $\phi : (X, \mathcal{C}) \rightarrow (X_1, \mathcal{C}_1)$  be a Cauchy continuous function from one  $\mathcal{C}^\wedge$ -embedded Cauchy space into another. Then it is a routine matter to show that the function  $\phi_1 : \mathcal{C}^\wedge(X_1) \rightarrow \mathcal{C}^\wedge(X)$ , defined by  $\phi_1(f) = f\phi$  for all  $f \in \mathcal{C}^\wedge(X_1)$ , is a Cauchy-continuous algebra homomorphism.

**THEOREM 2.4.** *If  $(X, C)$  is a  $c^\wedge$ -embedded Cauchy space,  $(X_1, C_1)$  a complete  $c^\wedge$ -embedded Cauchy space, and  $\phi : (X, C) \rightarrow (X_1, C_1)$  is Cauchy-continuous, then  $\phi$  has a unique Cauchy-continuous extension  $\psi : \text{hom}^\wedge C(X) \rightarrow (X_1, C_1)$ .*

*Proof.* Define the natural function  $\phi_2 : C^\wedge C(X) \rightarrow C^\wedge C(X_1)$  in the manner described in the preceding paragraph. Let  $\sigma$  be the restriction of  $\phi_2$  to  $\text{hom}^\wedge C(X)$ . Since  $\phi_2$  is a Cauchy-continuous homomorphism,  $\sigma : \text{hom}^\wedge C(X) \rightarrow \text{hom}^\wedge C(X_1)$  is Cauchy-continuous. Since  $(X_1, C_1)$  is complete,  $(X_1, C_1)$  is Cauchy-homeomorphic to  $\text{hom}^\wedge C(X_1)$ . Let  $\psi = i_1^{-1} \sigma$ , where  $i_1$  is the natural embedding from  $X_1$  onto  $\text{hom}^\wedge C(X_1)$ . It is clear from this construction that  $\psi$  is an extension of  $\phi$  (in the sense that  $\phi = \psi i$ ) and also that this extension is unique.

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