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Regular A-optimal spring balance weighing designs with correlated errors

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Abstract

The problems linked with an A-optimal spring balance weighing design with correlated errors are discussed. The topic is focus on the determining the lowest bound of the trace of inverse information matrix in a special class of design matrices. The constructing method of the optimal design, based on the incidence matrices of balanced incomplete block designs, is presented.

Keywords: A-optimal design, balanced incomplete block design, spring balance weighing design.

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1. Introduction

Consider the linear model

(1.1) $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$,

where

(a) $\mathbf{y}$ is an $n \times 1$ random vector of the observations,
(b) $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0, 1)$, where $\mathbf{\Phi}_{n\times p}(0, 1)$ denotes the class of $n \times p$ matrices $\mathbf{X} = (x_{ij})$ of known elements $x_{ij} = 1$ or 0 according as in the ith weighing operation the jth object is placed on the pan or not. Any matrix $\mathbf{X}$ belonging to the class $\mathbf{\Phi}_{n\times p}(0, 1)$ is called the design matrix of the spring balance weighing design.
(c) $\mathbf{w}$ is a $p \times 1$ vector of unknown weights of objects,
(d) $\mathbf{e}$ is an $n \times 1$ random vector of errors for that $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ denotes the $n \times 1$ vector with zero elements everywhere, $\mathbf{G}$ is a known positive definite matrix.

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For the estimation of $\mathbf{w}$ we use the normal equations $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\mathbf{w} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$. Any spring balance weighing design is singular or nonsingular, depending on whether $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. Since $\mathbf{G}$ is a known positive definite matrix then $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if $\mathbf{X}$ has a full column rank. However, if $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then the generalized least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}} = \left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

There are several problems concerning to the optimality criteria of experimental designs. The best general references here are books [14] and [11]. The study results of determining the optimal weighing designs are shown in many papers, see for instance [12]. The standard work on A-, D- and E-optimality is the paper [5]. The deliberation related to A-optimal criterion for $\mathbf{G} = \mathbf{I}_n$ is presented in many papers. In [8] the robustness optimal designs are considered, whereas in [3] the problem of adding additionally weighing operation is presented. For a recent account on the theory of weighing designs, for $\mathbf{G}$ being any positive definite diagonal matrix, we refer the reader to [4].
The problems of determining of the regular D-optimal designs are included in several papers: in [10] some infinite families of D-optimal matrices based on Hadamard matrices are considered, however in [7] the deliberation on D-optimal designs under correlated structure of errors is presented. The construction of optimal design for eight objects is given in [9], while D-optimal weighing designs with autoregressive errors in [6]. Moreover, weighing designs as 2^n factorial designs were presented in [1] and [2].

2. The main result

In this paper, we emphasize a special interest of the existence conditions for A-optimal criterion. For given matrix $\mathbf{G}$, the problem is to determine such matrix $\mathbf{X}$ that $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ takes the minimal value over all possible matrices in $\mathbf{\Phi}_{n\times p}(0,1)$.

2.1. Definition. For given variance matrix of errors $\sigma^2\mathbf{G}$, any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ is A-optimal if $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal. Moreover, if $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ attains the lower bound then $\mathbf{X}$ is called regular A-optimal.

It's worth underlining that for given variance matrix of errors $\sigma^2\mathbf{G}$ and in any class $\mathbf{\Phi}_{n\times p}(0,1)$ A-optimal spring balance weighing design exists always, whereas regular A-optimal design may exist.

In order to determine the lower bound of $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ the following theorems will be required.

2.2. Theorem. *Let* $\mathbf{M}$ *be any positive definite* $p \times p$ *matrix and* $\mathbf{\Pi}$ *be the set of all* $p \times p$ *permutation matrices. The average of* $\mathbf{M}$ *over all elements of* $\mathbf{\Pi}$*, i.e.* $\bar{\mathbf{M}} = \frac{1}{p!}\sum_{\mathbf{P}\in\mathbf{\Pi}}\mathbf{P}^{'}\mathbf{M}\mathbf{P}$ *and*

$$\bar{\mathbf{M}} = \frac{p\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}\mathbf{I}_p + \frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p - \mathrm{tr}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}. \tag{2.1}$$

Besides, $\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\bar{\mathbf{M}})$ *and* $\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p = \mathbf{1}_p^{'}\bar{\mathbf{M}}\mathbf{1}_p$.

Proof. Let us consider $p!$ elements of the set of all $p \times p$ permutation matrices $\mathbf{\Pi}$. When we put all matrices into $\sum_{\mathbf{P} \in \mathbf{\Pi}} \mathbf{P}^{'}\mathbf{M}\mathbf{P}$ an easy computation makes it obvious that

$$\bar{\mathbf{M}} = \frac{1}{p!} \begin{bmatrix} (p-1)!\,\mathrm{tr}(\mathbf{M}) & (p-2)!\,Q(\mathbf{M}) & \dots & (p-2)!\,Q(\mathbf{M}) \\ (p-2)!\,Q(\mathbf{M}) & (p-1)!\,\mathrm{tr}(\mathbf{M}) & \dots & (p-2)!\,Q(\mathbf{M}) \\ \dots & \dots & \dots & \dots \\ (p-2)!\,Q(\mathbf{M}) & (p-2)!\,Q(\mathbf{M}) & \dots & (p-1)!\,\mathrm{tr}(\mathbf{M}) \end{bmatrix},$$

where $Q(\mathbf{M})$ denotes the sum of all offdiagonal elements. Because $\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p = \mathrm{tr}(\mathbf{M}) + Q(\mathbf{M})$ we obtain 2.1. Moreover, the form the matrix $\bar{\mathbf{M}}$ indicates that it has two eigenvalues $\mu_1 = \frac{p\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}$ with the multiplicity $p-1$ and $\mu_2 = \frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p}$ with the multiplicity 1.

□

2.3. Theorem. *Let t_1 be the eigenvalue with the multiplicity $p-1$, t_2 be the eigenvalue with the multiplicity 1 of any positive definite $p \times p$ matrix $\mathbf{M}$ and let q_1 be the eigenvalue with the multiplicity $p-1$ and q_2 be the eigenvalue with the multiplicity 1 of the matrix $\bar{\mathbf{M}}$. If $(p-1)t_1 + t_2 = (p-1)q_1 + q_2$, $t_1 \leq t_2$, $q_1 \leq q_2$, $t_1 \leq q_1$ then $\mathrm{tr}(\mathbf{M}^{-1}) \geq \mathrm{tr}(\bar{\mathbf{M}}^{-1})$. The equality is satisfied if and only if the eigenvalues of matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ are the same.*

Proof. $\mathrm{tr}(\mathbf{M}^{-1}) - \mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{p-1}{t_1} + \frac{1}{t_2} - \frac{p-1}{q_1} - \frac{1}{q_2} = \frac{(p-1)t_2q_1q_2 - (p-1)t_1t_2q_2 + t_1q_1q_2 - t_1t_2q_1}{t_1t_2q_1q_2}$. Because $(p-1)q_1 = (p-1)t_1 + t_2 - q_2$ then $\mathrm{tr}(\mathbf{M}^{-1}) - \mathrm{tr}(\bar{\mathbf{M}}^{-1}) = \frac{(t_2-q_2)(t_2q_2-t_1q_1)}{t_1t_2q_1q_2}$. We observe $\frac{t_2}{t_1} \geq 1$, $\frac{q_1}{q_2} \leq 1$. Thus $t_2q_2 - t_1q_1 \geq 0$. Finally $\mathrm{tr}(\mathbf{M}^{-1}) \geq \mathrm{tr}(\bar{\mathbf{M}}^{-1})$. It is obvious the equality is satisfied if and only if the eigenvalues of the matrices $\mathbf{M}$ and $\bar{\mathbf{M}}$ are equal. □

To aim at a target determining the regular A-optimal design let us consider the class of all design matrices of the spring balance weighing design $\mathbf{\Phi}_{n\times p}(0,1)$. For positive definite matrix $\mathbf{G}$ and any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ we take $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$. Let $m_1,\ m_2,\ ...\ m_p$, $m_1 \leq m_2 \leq ... \leq m_p$ be the eigenvalus of the matrix $\mathbf{M}^{-1}$. Then $\mathrm{tr}(\mathbf{M}^{-1}) = m_1 + m_2 + ... + m_p \geq pm_1$. The minimum of $\mathrm{tr}(\mathbf{M}^{-1})$ is attained if $m_1 = m_2 = ... = m_p$ and m_1 attains the minimal value. The equality is fulfilled if and only if $\mathbf{M}^{-1}$ is proportional to identity matrix. Such form of the matrix $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is not interesting from the point of view of experiment as in each measurement only one object is included. Therefore, let $m_1 = m_2 = ... = m_{p-1} \leq m_p$ and $\mathrm{tr}(\mathbf{M}^{-1}) = (p-1)m_1 + m_p$ and its minimum is attained if and only if m_1 and m_p are minimal. So, we consider the matrix $\mathbf{M}$ with two different eigenvalues, only.

Here, we consider the subclass of the spring balance weighing designs in the following form

$$\mathbf{\Omega}^{\xi}_{n\times p}(0,\ 1) =$$

$$\left\{\mathbf{X}:\ \mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,\ 1),\ \mathbf{X}\mathbf{1}_p = \xi\mathbf{1}_n,\ \mathbf{X}^{'}\mathbf{1}_n = \tfrac{n\xi}{p}\mathbf{1}_p,\ \tfrac{n\xi}{p} \in \mathrm{N},\ \xi \leq p\right\}.$$

Moreover, from now on until the end of the paper we consider $\mathbf{G}$ to be of the form

$$\mathbf{G} = g\left[(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^{'}\right],\ \ g > 0,\ \ \frac{-1}{n-1} < \rho < 1. \tag{2.2}$$

Condition on the values of g and ρ is equivalent to the matrix $\mathbf{G}$ being positive definite. When the variance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix of the form 2.2 then we say that the errors are equally correlated and they have the same variances. Let note, $\mathbf{G}^{-1} = \frac{1}{g(1-\rho)}\left[\mathbf{I}_n - \frac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}_n^{'}\right]$. Next let us consider

$$\mathbf{M}=\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}=\frac{1}{g(1-\rho)}\left[\mathbf{X}^{'}\mathbf{X}-\frac{\rho}{1+\rho(n-1)}\mathbf{X}^{'}\mathbf{1}_n\mathbf{1}_n^{'}\mathbf{X}\right].$$

We will denote by s the the number of elements equal to 1 in any row of the design matrix $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,\ 1)$. It is evident that $\mathrm{tr}(\mathbf{M})=\frac{ns}{g(1-\rho)}\left[1-\frac{ns\rho}{p(1+\rho(n-1))}\right]$ and $\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p=\frac{ns^2}{g(1+\rho(n-1))}$. From the above considerations and Theorem 2.2, eigenvalues of $\bar{\mathbf{M}}$ are $\mu_1=\frac{ns(p-s)}{p(p-1)g(1-\rho)}$ and $\mu_2=\frac{ns^2}{pg(1+\rho(n-1))}$. Thus the matrix $\bar{\mathbf{M}}^{-1}$ has also two eigenvalues $\frac{1}{\mu_1}$ with the multiplicity $p-1$ and $\frac{1}{\mu_2}$ with the multiplicity 1. Then $\mathrm{tr}(\bar{\mathbf{M}}^{-1})=\frac{p-1}{\mu_1}+\frac{1}{\mu_2}$. Furthermore, to determine A-optimal spring balance weighing design, we need to find the smallest value of $\mathrm{tr}(\bar{\mathbf{M}}^{-1})$. The $\mathrm{tr}(\bar{\mathbf{M}}^{-1})$ attains the lowest bound when $\frac{p-1}{\mu_1}$ and $\frac{1}{\mu_2}$ are minimized. We have

$$\mathrm{tr}(\bar{\mathbf{M}}^{-1})=\frac{pg}{n}\phi(s), \tag{2.3}$$

where $\phi(s)=\frac{(p-1)^2(1-\rho)}{s(p-s)}+\frac{1+\rho(n-1)}{s^2}$, $s=1,2,...,p-1$.

2.4. Theorem. *Let p be even. In any nonsingular spring balance weighing design $\mathbf{X}\in\mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$*

(i) if $\rho\in\left(\frac{-1}{n-1},\ P_1\right)$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right)\geq\frac{4g}{np}\left(1+\rho(n-1)+(p-1)^2(1-\rho)\right), \tag{2.4}$$

the equality in 2.4 is satisfied if and only if $\mathbf{X}\mathbf{1}_p=\frac{p}{2}\mathbf{1}_n$,

(ii) if $\rho\in(P_a,\ P_{a+1})$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right)\geq\frac{4pg}{n(p+2a)}\left(\frac{1+\rho(n-1)}{p+2a}+\frac{(p-1)^2(1-\rho)}{p-2a}\right), \tag{2.5}$$

the equality in 2.5 is satisfied if and only if $\mathbf{X}\mathbf{1}_p=\frac{p+2a}{2}\mathbf{1}_n$,

(iii) if $\rho=P_a$ then

$$\mathrm{tr}\left(\mathbf{M}^{-1}\right)\geq\frac{n(p-1)^2\left((p+2a-2)(2a-1)+(p+2a-1)(p-2a+2)\right)}{(p+2a)\left(n(p+2a-1)(p-2a)(p-2a+2)+L(a)\right)} \tag{2.6}$$

the equality in 2.6 is satisfied if and only if $\mathbf{X}\mathbf{1}_p=\frac{p+2a-2}{2}\mathbf{1}_n$ *or* $\mathbf{X}\mathbf{1}_p=\frac{p+2a}{2}\mathbf{1}_n$,

where $P_a=\frac{L(a)}{n(p+2a-1)(p-2a)(p-2a+2)+L(a)}$, $L(a)=(p-1)^2(2a-1)(p+2a-2)(p+2a)-(p+2a-1)(p-2a)(p-2a+2)$, $a=1,2,...,\frac{p-2}{2}$.

Proof. Based on the delibarations given above, we will consider the matrix $\mathbf{M}$ with two eigenvalues. Theorem 2.3 implies $\mathrm{tr}(\mathbf{M}^{-1})\geq\mathrm{tr}(\bar{\mathbf{M}}^{-1})$. Thus we have to asses the equality 2.3. For given $n,\ p,\ \rho$ and g, 2.3 is the function of s. Furthermore, to determine A-optimal spring balance weighing design, we need to find s for which $\phi(s)$ takes the smallest value. Because $s=1,2,...,p-1$, then we should investigate the sequence $\phi(1),\ \phi(2),...,\phi(p-1)$. Therefore we study the difference

$$\phi(s)-\phi(s+1)=\frac{(2s+1)(1+\rho(n-1))}{s^2(s+1)^2}+\frac{(p-2s-1)(p-1)^2(1-\rho)}{s(s+1)(p-s-1)(p-s)}. \tag{2.7}$$

For $s = 1, 2, ..., \frac{p-2}{2}$ and any $n,\ p,\ \rho$, we have $\phi(s) \geq \phi(s+1)$. Thus, we investigate the sequence for $s = \frac{p-2}{2}+a,\ a = 1, 2, ..., \frac{p-2}{2}$. We denote $P_a = \frac{L(a)}{n(p+2a-1)(p-2a)(p-2a+2)+L(a)}$, $L(a) = (p-1)^2(2a-1)(p+2a-2)(p+2a) - (p+2a-1)(p-2a)(p-2a+2)$. Next, let us consider the interval $\rho \in \left(\frac{-1}{n-1},\ P_1\right)$. If $s < \frac{p}{2}$ then $\phi(s) \geq \phi(s+1)$, if $s > \frac{p}{2}$, then $\phi(s) \leq \phi(s+1)$. The smallest value of 2.3 is attained if $s = \frac{p}{2}$ and then we obtain (i). Thus, we study $\rho \in (P_a,\ P_{a+1})$. If $s < \frac{p+2a}{2}$, then $\phi(s) \geq \phi(s+1)$. The inequality $s > \frac{p+2a}{2}$ implies $\phi(s) \leq \phi(s+1)$. The smallest value of 2.3 is attained for $s = \frac{p+2a}{2}$, thus (ii). If $\rho = P_a$, then $\phi(s) = \phi(s+1)$ and for $s = \frac{p+2a-2}{2}$ or $s = \frac{p+2a}{2}$, we receive (iii). □

2.5. Theorem. *Let p be even. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular A-optimal*

(i) *for fixed $\rho \in \left(\frac{-1}{n-1},\ P_1\right)$ if and only if $\mathbf{X1}_p = \frac{p}{2}\mathbf{1}_n$,*

(ii) *for fixed $\rho \in (P_a,\ P_{a+1})$ if and only if $\mathbf{X1}_p = \frac{p+2a}{2}\mathbf{1}_n$,*

(iii) *for fixed $\rho = P_a$ if and only if $\mathbf{X1}_p = \frac{p+2a-2}{2}\mathbf{1}_n$ or $\mathbf{X1}_p = \frac{p+2a}{2}\mathbf{1}_n$,*

where $a = 1, 2, ..., \frac{p-2}{2}$.

Proof. Any spring balance weighing design is regular A-optimal if and only if the equalities in 2.4-2.6 hold, i.e. if and only if the design matrix $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ is given as above. □

2.6. Theorem. *Let p be even. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular A-optimal*

(i) *for fixed $\rho \in \left(\frac{-1}{n-1},\ P_1\right)$ if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{np}{4(p-1)}\mathbf{I}_p + \frac{n(p-2)}{4(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \frac{\rho n^2}{4(1+\rho(n-1))}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

(ii) *for fixed $\rho \in (P_a,\ P_{a+1})$ if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a)(p-2a)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a)(p-2a-2)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} + \phi_a\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

(iii) *for fixed $\rho = P_a$ if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a)(p-2a)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a)(p-2a-2)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} + \phi_a\mathbf{1}_p\mathbf{1}_p^{'}\right] \quad \textit{or}$$

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+2)(p-2a-2)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+2)(p-2a-4)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} + \phi_{a+1}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

where $\phi_a = \frac{n(p+2a)(4ap(1-\rho)-\rho n(p(p-1)-2a(p+1)))}{4p^2(p-1)(1+\rho(n-1))}$, $a = 1, 2, ..., \frac{p-2}{2}$.

Proof. From Theorem 2.3, we obtain $\mathrm{tr}(\mathbf{M}^{-1}) = \mathrm{tr}(\bar{\mathbf{M}}^{-1})$ if and only if the eigenvalues of $\mathbf{M}$ and $\bar{\mathbf{M}}$ are equal. Hence for $\mathbf{G}$ in the form 2.2 and $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ the best design for which minimum of $\mathrm{tr}(\mathbf{M}^{-1})$ is attained if the $\bar{\mathbf{M}} = \mathbf{M}$ one. Thus to prove this Theorem it is worthy to notice that from 2.1 we have $\bar{\mathbf{M}} = \frac{p\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)}\mathbf{I}_p + \frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p - \mathrm{tr}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}$. Moreover, taking $s = \frac{p+2a}{2}$ we obtain $\frac{p\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)} = \frac{n(p+2a)(p-2a)}{4p(p-1)g(1-\rho)}$ and $\frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p - \mathrm{tr}(\mathbf{M})}{p(p-1)} = \frac{1}{g(1-\rho)}\left(\frac{n(p+2a)(p-2a-2)}{4p(p-1)} + \frac{n(p+2a)(4ap(1-\rho)-\rho n(p(p-1)-2a(p+1)))}{4p^2(p-1)(1+\rho(n-1))}\right)$, thus (ii). For $a = 0$ we obtain (i). The above consideration and the condition (iii) of Theorem 2.5 imply formulas given in (iii). □

2.7. Corollary. *In the special case, $g = 1$ and $\rho = 0$, the Condition (i) of Theorem 2.6 is equivalent to equality given in* [5]. *If additionally, $a = 0$ then the condition (ii) of Theorem 2.6 is the same as given in* [5] *one.*

2.8. Theorem. *Let p be odd. In any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$

(i) if $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg}{n(p+1)^2}\left(1+\rho(n-1)+(p^2-1)(1-\rho)\right), \tag{2.8}$$

the equality in 2.8 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p+1}{2}\mathbf{1}_n$,

(ii) if $\rho \in (R_a,\ R_{a+1})$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg}{n(p+2a+1)}\left(\frac{1+\rho(n-1)}{p+2a+1} + \frac{(p-1)^2(1-\rho)}{p-2a-1}\right), \tag{2.9}$$

the equality in 2.9 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

(iii) if $\rho = R_a$ then

$$\operatorname{tr}\left(\mathbf{M}^{-1}\right) \geq \frac{4pg(p-1)^2\left(2a(p+2a+1)+(p+2a)(p-2a-1)\right)}{(p+2a-1)\left(n(p+2a)(p-2a+1)(p-2a-1)+N(a)\right)} \tag{2.10}$$

the equality in 2.10 is satisfied if and only if $\mathbf{X}\mathbf{1}_p = \frac{p+2a-1}{2}\mathbf{1}_n$ *or* $\mathbf{X}\mathbf{1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

where $R_a = \frac{N(a)}{n(p+2a)(p-2a+1)(p-2a-1)+N(a)}$, $N(a) = 2(p-1)^2a(p+2a-1)(p+2a+1) - (p+2a)(p-2a+1)(p-2a-1)$, $a = 1,2,...,\frac{p-3}{2}$.

Proof. The proof of Theorem is similar to that given in Theorem 2.4. Since, we will give the most important steps, only. For $s = 1,2,...,\frac{p+1}{2}$, $\phi(s) \geq \phi(s+1)$, for any $n,\ p,\ \rho$. Thus, we investigate the sequence for $s = \frac{p}{2}+a$, $a = 1,2,...,\frac{p-3}{2}$. We denote $R_a = \frac{N(a)}{n(p+2a)(p-2a+1)(p-2a-1)+N(a)}$, $N(a) = 2(p-1)^2a(p+2a-1)(p+2a+1)-(p+2a)(p-2a+1)(p-2a-1)$, $a = 1,2,...,\frac{p-3}{2}$. Next, let us consider the interval $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$. If $s < \frac{p+1}{2}$ then $\phi(s) \geq \phi(s+1)$, if $s > \frac{p+1}{2}$, then $\phi(s) \leq \phi(s+1)$. The smallest value of 2.8 is attained if $s = \frac{p+1}{2}$. When we put $s = \frac{p+1}{2}$ in 2.3 we obtain (i). Now, we study $\rho \in (R_a,\ R_{a+1})$. If $s < \frac{p+2a+1}{2}$, then $\phi(s) \geq \phi(s+1)$. If $s > \frac{p+2a+1}{2}$, then $\phi(s) \leq \phi(s+1)$. The smallest value of 2.3 is attained for $s = \frac{p+2a+1}{2}$, thus (ii). If $\rho = R_a$, then $\phi(s) = \phi(s+1)$ and for $s = \frac{p+2a-1}{2}$ or $s = \frac{p+2a+1}{2}$, we receive (iii). □

2.9. Theorem. *Let p be odd. Any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *is regular A-optimal*

(i) *for fixed* $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ *if and only if* $\mathbf{X}\mathbf{1}_p = \frac{p+1}{2}\mathbf{1}_n$,
(ii) *for* $\rho \in (R_a,\ R_{a+1})$ *if and only if* $\mathbf{X}\mathbf{1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,
(iii) *for fixed* $\rho = R_a$ *if and only if* $\mathbf{X}\mathbf{1}_p = \frac{p+2a-1}{2}\mathbf{1}_n$ *or* $\mathbf{X}\mathbf{1}_p = \frac{p+2a+1}{2}\mathbf{1}_n$,

where $a = 1,2,...,\frac{p-3}{2}$.

Proof. According to the investigation given above, a spring balance weighing design is regular A-optimal if and only if the equalities in 2.8-2.10 are satisfied, i.e. if and only if the design matrix $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ is given as in Theorem 2.8. □

2.10. Theorem. *Let p be odd. Any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{n\times p}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *is regular A-optimal*

(i) *for fixed* $\rho \in \left(\frac{-1}{n-1},\ R_1\right)$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+1)}{4p}\mathbf{I}_p + \frac{n(p+1)}{4p}\mathbf{1}_p\mathbf{1}_p^{'} - \frac{\rho n^2(p+1)^2}{4p^2(1+\rho(n-1))}\mathbf{1}_p\mathbf{1}_p^{'}\right]$$

(ii) *for* $\rho \in (R_a,\ R_{a+1})$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_a\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

(iii) *for* $\rho = R_a$ *if and only if*

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_a\mathbf{1}_p\mathbf{1}_p^{'}\right] \quad or$$

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\frac{n(p+2a+3)(p-2a-3)}{4p(p-1)}\mathbf{I}_p + \frac{n(p+2a+3)(p-2a-3)}{4p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'} - \psi_{a+1}\mathbf{1}_p\mathbf{1}_p^{'}\right],$$

where $\psi_a = \frac{n(p+2a+1)(\rho n(p^2-1)-4ap(1-\rho)-2an\rho(p+1))}{4p^2(p-1)(1+\rho(n-1))}$, $a = 1, 2, ..., \frac{p-3}{2}$.

Proof. The proof is similar to given in Theorem 2.6 one. It is sufficient to show that taking $s = \frac{p+2a+1}{2}$ we obtain $\frac{p\mathrm{tr}(\mathbf{M})-\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p}{p(p-1)} = \frac{n(p+2a+1)(p-2a-1)}{4p(p-1)g(1-\rho)}$ and $\frac{\mathbf{1}_p^{'}\mathbf{M}\mathbf{1}_p-\mathrm{tr}(\mathbf{M})}{p(p-1)} = \frac{1}{g(1-\rho)}\left(\frac{n(p+2a+1)(p-2a-1)}{4p(p-1)} - \frac{n(p+2a+1)(\rho n(p^2-1)-4ap(1-\rho)-2an\rho(p+1)))}{4p^2(p-1)(1+\rho(n-1))}\right)$. Thus (ii). For $a = 0$ we obtain (i). Moreover, the above considerations and the condition (iii) of Theorem 2.9 imply the formulas presented in (iii). □

2.11. Corollary. *In the special case,* $g = 1$ *and* $\rho = 0$*, the Condition (i) of Theorem 2.10 is equivalent to equality given in* [5]*. If additionally,* $a = 0$ *then (ii) of Theorem 2.10 is the same as given in* [5] *one.*

3. Examples

Take into the consideration $\mathbf{X} = \mathbf{N}^{'}$, where $\mathbf{N}$ is the incidence matrix of balanced incomplete block design with the parameters $v,\ b,\ r,\ k,\ \lambda$, see [13]. To simplify the notation it is customary to write v instead of p and b instead of n. It is obvious that we are not able to give the construction of regular A-optimal spring balance weighing design for any combination of p, n and ρ. With the results obtained until now we can establish the following corollaries which indicate the series of the parameters of balanced incomplete block designs. Based on that incidence matrices we form the design matrices of regular A-optimal designs for an appropriate ρ.

3.1. Corollary. *Let* v *be even. If exists the balanced incomplete block design with the parameters* v, $b = v(v-1)$, $r = 0.5(v-1)(v+2a-2)$, $k = 0.5(v+2a-2)$, $\lambda = 0.25(v+2a-2)(v+2a-4)$, $a = 1, 2, ..., \frac{v-2}{2}$, *given by the incidence matrix* $\mathbf{N}$ *then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{v(v-1)\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$ *is regular A-optimal spring balance weighing with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ P_1\right]$ *or* $\rho \in [P_a,\ P_{a+1})$.

3.2. Corollary. *Let* v *be even. If exists the balanced incomplete block design with the parameters* $v = 2(t+1)$, $b = 2(2t+1)$, $r = 2t+1$, $k = t+1$, $\lambda = t$, $t = 1, 2, ...,$ *given by incidence matrix* $\mathbf{N}$*, then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{2(2t+1)\times 2(t+1)}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{4t+1},\ \frac{2t^3+5t^2+3t+1}{6t^3+13t^2+6t+1}\right]$.

3.3. Corollary. *Let* v *be even. Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{b\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}^{'}$*, where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* v, $b = \begin{pmatrix} v \\ 0.5(v+2a-2) \end{pmatrix}$, $r = \begin{pmatrix} v-1 \\ 0.5(v+2a-4) \end{pmatrix}$, $k = \frac{v+2a-2}{2}$, $\lambda = \begin{pmatrix} v-2 \\ 0.5(v+2a-6) \end{pmatrix}$, $a = 1, 2, ..., \frac{v-2}{2}$, *is regular A-optimal spring balance weighing design with the variance* *matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ P_1\right]$ *or* $\rho \in [P_a,\ P_{a+1})$, *where* $\binom{\eta}{\tau}$ *denotes binomial coefficient.*

3.4. Corollary. *Let* v *be odd. If exists the balanced incomplete block design with the parameters* $v,\ b = 0.5v(v-1),\ r = 0.25(v-1)(v+2a-1),\ k = 0.5(v+2a-1),\ \lambda = 0.125(v+2a-1)(v+2a-3),\ a = 1,2,...,\frac{v-3}{2}$, *given by the incidence matrix* $\mathbf{N}$, *then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{0.5v(v-1)\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ R_1\right]$ *or* $\rho \in [R_a,\ R_{a+1})$.

3.5. Corollary. *Let* v *be odd. If exists the balanced incomplete block design with the parameters* $v = 2t+1,\ b = 2(2t+1),\ r = 2(t+1),\ k = t+1,\ \lambda = t+1,\ t = 2,3,...,$ *given by the incidence matrix* $\mathbf{N}$, *then any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{2(2t+1)\times(2t+1)}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{4t+1},\ \frac{8t^3+22t^2+15t+3}{16t^3+30t^2+5t-3}\right]$.

3.6. Corollary. *Let* v *be odd. Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{b\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$, *where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* $v,\ b = \binom{v}{0.5(v+2a-1)},\ r = \binom{v-1}{0.5(v+2a-3)},\ k = \frac{v+2a-1}{2},\ \lambda = \binom{v-2}{0.5(v+2a-5)}$, $a = 1,2,...,\frac{v-1}{2}$, *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{n-1},\ R_1\right]$ *or* $\rho \in [R_a,\ R_{a+1})$.

3.7. Corollary. *Any* $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{v\times v}(0,1)$ *in the form* $\mathbf{X} = \mathbf{N}'$, *where* $\mathbf{N}$ *is the incidence matrix of balanced incomplete block design with the parameters* $v = b,\ r = k = v-1,\ \lambda = v-2,\ v = 3,4,...,$ *is regular A-optimal spring balance weighing design with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left[\frac{v^4-8v^3+24v^2-34v+19}{(v-1)(v^3-7v^2+17v-13)},\ 1\right)$.

3.8. Example. Let $\mathbf{X} \in \mathbf{\Omega}^{\xi}_{30\times 6}(0,1)$ and let for $\mathbf{G},\ g > 0,\ \rho \in (-0.034,\ 1),\ \xi \le 6$.

(i) If $\rho \in (-0.034,\ 0.170)$ then $\mathbf{X} = \mathbf{N}_1'$,
(ii) if $\rho \in (0.170,\ 0.733)$ then $\mathbf{X} = \mathbf{N}_2'$,
(iii) if $\rho \in (0.733,\ 1)$ then $\mathbf{X} = \mathbf{N}_3'$,
(iv) if $\rho = 0.170$ then $\mathbf{X} = \mathbf{N}_h',\ h = 1,2$,
(v) if $\rho = 0.733$ then $\mathbf{X} = \mathbf{N}_h',\ h = 2,3$,

is regular A-optimal spring balance weighing design, where $\mathbf{N}_h,\ h = 1,2,3$, is the incidence matrix of the balanced incomplete block design with parameters $v = 6,\ b_1 = 30,\ r_1 = 15,\ k_1 = 3,\ \lambda_1 = 6,\ v = 6,\ b_2 = 30,\ r_2 = 20,\ k_2 = 4,\ \lambda_2 = 12,\ v = 6,\ b_3 = 30,\ r_3 = 25,\ k_3 = 5,\ \lambda_3 = 20$, respectively.

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