Internat. J. Math. & Math. Sci.
VOL. 13 NO. 4 (1990) 651-660

REGULAR EIGENVALUE PROBLEM WITH EIGENPARAMETER CONTAINED IN THE EQUATION AND THE BOUNDARY CONDITIONS

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(Received May 8, 1989 and in revised form September 25, 1989)

ABSTRACT. The purpose of this paper is to establish the expansion theorem for a regular right-definite eigenvalue problem for the Laplace operator in R^n, $(n \geq 2)$ with an eigenvalue parameter λ contained in the equation and the Robin boundary conditions on two "parts" of a smooth boundary of a simply connected bounded domain.

KEY WORDS AND PHRASES. An expansion theorem, a regular right-definite eigenvalue problem, an eigenparameter in Robin boundary conditions, a simply connected bounded domain with a smooth boundary.
1980 AMS subject classification code (1985). 65NXX, 65N25.

1. INTRODUCTION.

Regular right-definite eigenvalue problems for ordinary differential equations with eigenvalue parameter in the boundary conditions have been studied by Fulton [1], Hinton [2], Ibrahim [3], Schneider [4], Walter [5], Zayed and Ibrahim [6], Zayed [7] and many others, while in the present paper we shall study regular right-definite eigenvalue problems for partial differential equations with eigenvalue parameter in Robin boundary conditions.

The object of this paper is to prove the expansion theorem for the following problem:

Let $\Omega \subseteq R^n$, $(n \geq 2)$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the partial differential equation

$$\tau u := \frac{1}{r}(-\Delta_n u) = \lambda u \quad \text{in} \quad \Omega , \tag{1.1}$$

together with the Robin boundary conditions

$$u_\nu + h_1(\underset{\sim}{x})u = \lambda u \quad \text{on} \quad \Gamma , \tag{1.2}$$

and

$$u_\nu + h_2(\underset{\sim}{x})u = \lambda u \qquad \text{on} \quad \partial\Omega\setminus\Gamma \tag{1.3}$$

where we assume throughout that

(i) $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n, $(n \geq 2)$.

(ii) $u_\nu = \sum_{i=1}^{n} u_{x_i}(\underset{\sim}{x})\nu_i(\underset{\sim}{x})$ denotes differentiation of $u(\underset{\sim}{x})$ along the outward unit normal $\nu(\underset{\sim}{x}) = (\nu_1(\underset{\sim}{x}),\ldots,\nu_n(\underset{\sim}{x}))$ to the boundary $\partial\Omega$, where $\underset{\sim}{x} = (x_1,\ldots,x_n)$ is a generic point in the Euclidean space R^n.

(iii) The weight function $r(\underset{\sim}{x})$ is a real-valued positive function with $r\varepsilon C^\alpha(\bar{\Omega})$, $\bar{\Omega} = \Omega \cup \partial\Omega$ where $C^\alpha(\bar{\Omega})$ is the space of all Hölder continuous functions with exponent α, $0 < \alpha < 1$ which are defined on $\bar{\Omega}$, while $C^{k+\alpha}(\bar{\Omega})$ denotes the space of all functions in $C^k(\bar{\Omega})$ whose derivatives are Hölder continuous with exponent α.

(iv) $h_1(\underset{\sim}{x})$, $(\underset{\sim}{x}\varepsilon\Gamma)$ and $h_2(\underset{\sim}{x})$, $(\underset{\sim}{x}\varepsilon\partial\Omega\setminus\Gamma)$ are non-negative real functions, where Γ is a part of the boundary $\partial\Omega$ while $\partial\Omega\setminus\Gamma$ is the remaining part of $\partial\Omega$.

(v) λ is a complex number.

If $\lambda = 0$, $h_1(\underset{\sim}{x}) = -\mu$, $h_2(\underset{\sim}{x}) = 0$, then problem (1.1)-(1.3) reduces to

$$\Delta_n u = 0 \qquad \text{in} \quad \Omega, \tag{1.4}$$

$$u_\nu = \mu u \qquad \text{on} \quad \Gamma, \tag{1.5}$$

$$u_\nu = 0 \qquad \text{on} \quad \partial\Omega\setminus\Gamma \quad , \tag{1.6}$$

wherein μ is an eigenvalue parameter. The eigenvalue problem (1.4)-(1.6) is called a "Steklov problem", which has been studied by Canavati and Minzoni [8], Odhnoff [9] and many others. Odhnoff's approach is to give problem (1.4)-(1.6) an operator-theoretic formulation by associating with it a semi-bounded self-adjoint extension operator A and to obtain a direct expansion theorem by using the spectral resolution of A. Moreover, Odhnoff proved that there exists a complete set of generalized eigenfunctions of every self-adjoint extension operator A. Canavati and Minzoni have associated with problem (1.4)-(1.6) a self-adjoint operator L which has compact resolvent and they have shown that the spectrum of L consists of a sequence $\{\lambda_j\}$ of non-negative eigenvalues such that $\lambda_j \to 0$ as $j \to \infty$. Furthermore, they have derived an eigenfunction expansion by using a suitable Green's function.

Recently, Ibrahim [3] has discussed the eigenvalue equation (1.1) together with the Robin boundary condition

$$u_\nu + h(\underset{\sim}{x})u = \lambda u \qquad \text{on} \quad \partial\Omega \quad , \tag{1.7}$$

where $h(\underset{\sim}{x})$ is a non-negative real function on the whole boundary $\partial\Omega$. Ibrahim's approach is to give the regular right-definite eigenvalue problem (1.1) and (1.7) an operator-theoretic formulation by associating with it a self-adjoint operator A with compact resolvent in a suitable Hilbert space H and he has shown that the spectrum of A consists of an unbounded sequence of eigenvalues $\{\lambda_j\}$ such that $\lambda_j \to \infty$ as $j \to \infty$ and also that the corresponding eigenfunctions of A form a complete fundamental system in H.

In this paper, our approach is to find a suitable Hilbert space H and an essentially self-adjoint operator A with compact resolvent defined in H in such a way that problem (1.1)-(1.3) can be considered as an eigenvalue problem of this operator.

2. HILBERT SPACE FORMULATION.

Let $L_r^2(\Omega)$, $L^2(\Gamma)$ and $L^2(\partial\Omega\setminus\Gamma)$ be three complex Hilbert spaces of Lebesgue measurable functions $f(\underset{\sim}{x})$ in Ω, on Γ and on $\partial\Omega\setminus\Gamma$ respectively, satisfying

(i) $\int_\Omega r(\underset{\sim}{x})|f(\underset{\sim}{x})|^2\, d\underset{\sim}{x} < \infty$,

(ii) $\int_\Gamma |f(\underset{\sim}{x})|^2\, ds_1 < \infty$,

and

(iii) $\int_{\partial\Omega\setminus\Gamma} |f(\underset{\sim}{x})|^2\, ds_2 < \infty$.

DEFINITION 2.1. We define a Hilbert space H of three-component vectors by

$$H = L_r^2(\Omega) \oplus L^2(\Gamma) \oplus L^2(\partial\Omega\setminus\Gamma); \tag{2.1}$$

with inner product

$$\langle f,g\rangle = \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x})\overline{g_1(\underset{\sim}{x})}\,d\underset{\sim}{x} + \int_\Gamma f_2(\underset{\sim}{x})\overline{g_2(\underset{\sim}{x})}\,dS_1 + \int_{\partial\Omega\setminus\Gamma} f_3(\underset{\sim}{x})\overline{g_3(\underset{\sim}{x})}\,dS_2, \tag{2.2}$$

and norm

$$||f||^2 = \int_\Omega r(\underset{\sim}{x})|f_1(\underset{\sim}{x})|^2\, d\underset{\sim}{x} + \int_\Gamma |f_2(\underset{\sim}{x})|^2\, dS_1 + \int_{\partial\Omega\setminus\Gamma} |f_3(\underset{\sim}{x})|^2\, dS_2, \tag{2.3}$$

for each $f = (f_1,f_2,f_3)$ and $g = (g_1,g_2,g_3)$ in H, where $d\underset{\sim}{x} = dx_1 \ldots dx_n$ is the volume element corresponding to Ω while dS_1 and dS_2 are the surface elements corresponding to Γ and $\partial\Omega\setminus\Gamma$ respectively.

DEFINITION 2.2. Let H_1 be a set of all those elements f satisfying

$$f\varepsilon C^1(\bar{\Omega})\cap C^2(\Omega) \quad \text{and} \quad \Delta_n f\varepsilon L_r^2(\Omega).$$

We define a linear operator A: $D(A)\rightarrow H$ by

$$Af = (\tau f_1,\ f_{1\nu} + h_1(\underset{\sim}{x})f_1,\ f_{1\nu} + h_2(\underset{\sim}{x})f_1) \tag{2.4}$$

for each $f = (f_1,f_2,f_3)$ in D(A), in which the domain D(A) of A is defined as follows:

$$D(A) = \{(f|_\Omega,\ f|_\Gamma,\ f|_{\partial\Omega\setminus\Gamma})\varepsilon H : f\varepsilon H_1\}$$

where $f|_\Omega$, $f|_\Gamma$ and $f|_{\partial\Omega\setminus\Gamma}$ are restrictions of f on Ω, on Γ and on $\partial\Omega\setminus\Gamma$ respectively.

REMARK 2.1. The parameter λ is an eigenvalue and f_1 is a corresponding eigenfunction of problem (1.1)-(1.3) if and only if

$$f = (f_1,f_2,f_3)\varepsilon D(A) \quad \text{and} \quad Af = \lambda f. \tag{2.5}$$

Therefore, the eigenvalues and the eigenfunctions of problem (1.1)-(1.3) are equivalent to the eigenvalues and the eigenfunctions of operator A in H.

REMARK 2.2. D(A) is a dense subset of H with respect to the inner product (2.2).

LEMMA 2.1. The linear operator A in H is symmetric.

PROOF. Let $f = (f_1,f_2,f_3)$ and $g = (g_1,g_2,g_3)$ be any two elements in D(A), then

$$\langle Af,g\rangle = -\int_\Omega \{\Delta_n f_1(\underset{\sim}{x})\}\bar{g}_1(\underset{\sim}{x})\,d\underset{\sim}{x} + \int_\Gamma \{f_{1\nu}(\underset{\sim}{x}) + h_1(\underset{\sim}{x})f_1(\underset{\sim}{x})\}\bar{g}_2(\underset{\sim}{x})\,dS_1 +$$

$$+ \int_{\partial\Omega\setminus\Gamma} \{f_{1\nu}(\underset{\sim}{x}) + h_2(\underset{\sim}{x})f_1(\underset{\sim}{x})\}\bar{g}_3(\underset{\sim}{x})\,dS_2. \tag{2.6}$$

Making use of first Green's formula [10, p. 50] in (2.6), we obtain

$$\langle Af,g\rangle = \int_{\Omega}(\text{grad } f_1, \text{ grad } g_1)d\underset{\sim}{x} + \int_{\Gamma} f_1(\underset{\sim}{x})h_1(\underset{\sim}{x})\overline{g_1(\underset{\sim}{x})}dS_1 + \\ + \int_{\partial\Omega\setminus\Gamma} f_1(\underset{\sim}{x})h_2(\underset{\sim}{x})\overline{g_1(\underset{\sim}{x})}dS_2. \tag{2.7}$$

where

$$(\text{grad } f_1, \text{ grad } g_1) = \sum_{i=1}^{n} f_{1x_i}(\underset{\sim}{x})\overline{g_{1x_i}(\underset{\sim}{x})} \quad \text{for } \underset{\sim}{x}\in\Omega .$$

Applying a similar argument, it follows that

$$\langle f,Ag\rangle = \int_{\Omega}(\text{grad } f_1, \text{ grad } g_1)d\underset{\sim}{x} + \int_{\Gamma} f_1(\underset{\sim}{x})h_1(\underset{\sim}{x})\overline{g_1(\underset{\sim}{x})}dS_1 + \\ + \int_{\partial\Omega\setminus\Gamma} f_1(\underset{\sim}{x})h_2(\underset{\sim}{x})\overline{g_1(\underset{\sim}{x})}dS_2. \tag{2.8}$$

From (2.7) and (2.8) we find that

$$\langle Af,g\rangle = \langle f,Ag\rangle. \tag{2.9}$$

Therefore A is a symmetric linear operator in H.

LEMMA 2.2. Let $f = (f_1,f_2,f_3)\in C^1(\overline{\Omega})$ be a complex-valued function. Then

$$\int_{\Omega}|f_1(\underset{\sim}{x})|^2 \, d\underset{\sim}{x} \le 16\mu^2 \int_{\Omega} |\text{grad } f_1(\underset{\sim}{x})|^2 \, d\underset{\sim}{x} + 2\mu \int_{\Gamma} |f_2(\underset{\sim}{x})|^2 \, dS_1 + \\ + 2\mu \int_{\partial\Omega\setminus\Gamma} |f_3(\underset{\sim}{x})|^2 \, dS_2 \tag{2.10}$$

where

$$\mu = \sup\{|x_1| : \underset{\sim}{x} = (x_1, \ldots, x_n)\in\Omega\} .$$

PROOF. Since $|f_1(\underset{\sim}{x})|$ is a real-valued function and $|f_1(\underset{\sim}{x})|\in C^1(\overline{\Omega})$, then by using Theorem 2 in [10, p. 67], we have

$$\int_{\Omega} |f_1(\underset{\sim}{x})|^2 \, d\underset{\sim}{x} \le 4\mu^2 \int_{\Omega} \sum_{i=1}^{n} \{|f_1(\underset{\sim}{x})|_{x_i}\}^2 d\underset{\sim}{x} + 2\mu \int_{\Gamma} |f_2(\underset{\sim}{x})|^2 \, dS_1 + \\ + 2\mu \int_{\partial\Omega\setminus\Gamma} |f_3(\underset{\sim}{x})|^2 \, dS_2. \tag{2.11}$$

Substituting the inequality

$$\{|f_1(\underset{\sim}{x})|_{x_i}\}^2 \le 4\{|f_{1x_i}(\underset{\sim}{x})|\}^2 , \qquad \underset{\sim}{x}\in\Omega,$$

into (2.11) we arrive at (2.10).

REMARK 2.3. Since A in H is symmetric, then it has only real eigenvalues.

3. THE BOUNDEDNESS.

We shall show that the linear operator A in H is bounded from below, unbounded from above and strictly positive.

LEMMA 3.1. The linear operator A in H is bounded from below.

PROOF. Let $f = (f_1,f_2,f_3)$ be any element in D(A). We have

$$\langle Af,f\rangle = - \int_{\Omega} \{\Delta_n f_1(\underset{\sim}{x})\}\overline{f_1(\underset{\sim}{x})}d\underset{\sim}{x} + \int_{\Gamma} \{f_{1\nu}(\underset{\sim}{x}) + h_1(\underset{\sim}{x})f_1(\underset{\sim}{x})\}\overline{f_2(\underset{\sim}{x})}dS_1 + \\ + \int_{\partial\Omega\setminus\Gamma} \{f_{1\nu}(\underset{\sim}{x}) + h_2(\underset{\sim}{x})f_1(\underset{\sim}{x})\}\overline{f_3(\underset{\sim}{x})}dS_2. \tag{3.1}$$

By using the first Green's formula, (3.1) becomes

$$\langle Af,f\rangle = \int_{\Omega} |\text{grad } f_1(\underset{\sim}{x})|^2 \, d\underset{\sim}{x} + \int_{\Gamma} h_1(\underset{\sim}{x})|f_2(\underset{\sim}{x})|^2 \, dS_1 + \int_{\partial\Omega\setminus\Gamma} h_2(\underset{\sim}{x})|f_3(\underset{\sim}{x})|^2 \, dS_2 \tag{3.2}$$

With $\beta = \max\{16\mu^2, 2\mu, 2\mu\}$, Lemma 2.2. gives the inequality

$$\frac{1}{\beta}\int_{\Omega}|f_1(\underset{\sim}{x})|^2\,d\underset{\sim}{x}-\int_{\Gamma}|f_2(\underset{\sim}{x})|^2\,dS_1-\int_{\partial\Omega\setminus\Gamma}|f_3(\underset{\sim}{x})|^2\,dS_2\leq\int_{\Omega}|\text{grad }f_1(\underset{\sim}{x})|^2\,d\underset{\sim}{x}. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\langle Af,f\rangle\geq\frac{1}{\beta}\int_{\Omega}\frac{1}{r(\underset{\sim}{x})}\,r(\underset{\sim}{x})|f_1(\underset{\sim}{x})|^2\,d\underset{\sim}{x}+\int_{\Gamma}\{h_1(\underset{\sim}{x})-1\}|f_2(\underset{\sim}{x})|^2\,dS_1+$$

$$+\int_{\partial\Omega\setminus\Gamma}\{h_2(\underset{\sim}{x})-1\}|f_3(\underset{\sim}{x})|^2\,dS_2\geq C_o||f||^2, \tag{3.4}$$

where

$$C_o=\min\{\frac{1}{\beta}\inf_{\underset{\sim}{x}\in\Omega}\frac{1}{r(\underset{\sim}{x})},\ \inf_{x\in\Gamma}[h_1(\underset{\sim}{x})-1],\ \inf_{\underset{\sim}{x}\in\partial\Omega\setminus\Gamma}[h_2(\underset{\sim}{x})-1]\}. \tag{3.5}$$

This proves that the linear operator A in H is bounded from below.

REMARK 3.1.

(i) Since $r(\underset{\sim}{x}) > 0$ for $x\in\Omega$, and if $h_1(\underset{\sim}{x}) > 1$ for $\underset{\sim}{x}\in\Gamma$ and if $h_2(\underset{\sim}{x}) > 1$ for $\underset{\sim}{x}\in\partial\Omega\setminus\Gamma$ then $C_o > 0$ and consequently the linear operator A in H is strictly positive. We assume these conditions on h_1 (x) and h_2 (x) for the remainder of the paper.

(ii) Since A in H is strictly positive, then $\lambda = 0$ is not an eigenvalue of A in H.

LEMMA 3.2. The linear operator A in H is unbounded from above.

PROOF. Let χ $(\underset{\sim}{x})$ be a test function with the compact support on $\bar{\Omega}$ and define a sequence of this test function in D(A) by

$$\chi_N(\underset{\sim}{x}) = \chi(N\underset{\sim}{x}), \qquad \underset{\sim}{x}\in\bar{\Omega}, \quad N = 1,2,\ \ldots$$

By using the same argument of Lemma 3.1, we find that

$$\langle A\chi_N,\chi_N\rangle \geq C_1N^4||\chi_N||^2 \tag{3.6}$$

where C_1 is a positive constant.

Taking the limit as $N\to\infty$ in (3.6), we obtain

$$\lim_{N\to\infty}\langle A\chi_N,\chi_N\rangle = \infty. \tag{3.7}$$

In other words, A is unbounded from above.

REMARK 3.2.

(i) Since A in H is bounded from below, then the set of all eigenvalues of A is also bounded from below by the constant C_o defined by (3.5).

(ii) Since A in H is unbounded from above, then the set of all eigenvalues is too.

DEFINITION 3.1. The linear operator A in H is said to be essentially self-adjoint if

(i) A in H is symmetric

(ii) (A + iE)D(A) and (A - iE)D(A) are dense in H, where E is the identity operator and $i = \sqrt{-1}$ (see [10, p. 172]).

REMARK 3.3. Since A in H is symmetric, then $\pm i$ cannot be an eigenvalue of A.

LEMMA 3.3. The linear operator A in H is essentially self-adjoint.

PROOF. We must prove that $(A \pm iE)D(A)$ is dense in H. Suppose the contrary; first of all, suppose that $(A + iE)D(A)$ is not dense in H. Then there exists a non-zero element $\underset{\sim}{0} \neq f = (f_1,f_2,f_3) \in H$ such that

$$<f,(A + iE)g> = 0, \qquad \forall\ g = (g_1,g_2,g_3) \in D(A).$$

By using the same argument of Lemma 2.1, we find that

$$<(A - iE)f,g> = 0, \qquad \forall\ g \in C^1(\bar{\Omega}) \cap C^2(\Omega),$$

which means that $(A - iE)f = 0$ and consequently $Af = if$.

Since $f \in H$, it follows that $Af \in H$. Thus $f \in D(A)$ and since $f \neq \underset{\sim}{0}$, then $+i$ must be an eigenvalue of A. This contradicts the fact that A in H is symmetric.

Similarly, we can show that $(A - iE)D(A)$ is dense in H.

4. THE RESOLVENT OPERATOR.

Since $\lambda = 0$ is not an eigenvalue of the linear operator A in H, then the inverse operator A^{-1} of A exists in H. To study the operator A^{-1} it is convenient to give an explicit formula for it in terms of the Robin's function $R(\underset{\sim}{x},\underset{\sim}{y})$ for the Laplacian Δ_n on Ω.

Here it is difficult to characterize $D(A^{-1}) = R(A)$, the range of A, exactly. In any case, it is not true that

$$D(A^{-1}) = \{(f|_\Omega,\ f|_\Gamma,\ f|_{\partial\Omega\setminus\Gamma}) \in H : f \in C^0(\bar{\Omega})\};$$

because for such an f we cannot in general find $u = (u_1,u_2,u_3) \in D(A)$ with $Au = f$. Hence we define A^{-1} in H by

$$D(A^{-1}) = \{(f|_\Omega,\ f|_\Gamma,\ f|_{\partial\Omega\setminus\Gamma}) \in H : f \in C^\alpha(\bar{\Omega})\}\ , \tag{4.1}$$

and

$$A^{-1} : D(A^{-1}) \to H,$$

$$A^{-1}f = \Big(\int_\Omega R(\underset{\sim}{x},\underset{\sim}{y})f_1(\underset{\sim}{y})r(\underset{\sim}{y})d\underset{\sim}{y},\quad \int_\Gamma R(\underset{\sim}{x},\underset{\sim}{y})f_2(\underset{\sim}{y})dS_1,\quad \int_{\partial\Omega\setminus\Gamma} R(\underset{\sim}{x},\underset{\sim}{y})f_3(\underset{\sim}{y})dS_2\Big), \tag{4.2}$$

for each $f = (f_1,f_2,f_3) \in D(A^{-1})$.

REMARK 4.1.

(i) $D(A^{-1})$ is dense in H.

(ii) A^{-1} is a linear operator in H.

REMARK 4.2.

The Robin's function $R(\underset{\sim}{x},\underset{\sim}{y})$ for fixed $\underset{\sim}{x} \in \bar{\Omega}$ is a fundamental solution of $\underset{\sim}{y}$ with respect to Ω (see [10],[11]), i.e.,

$$R(\underset{\sim}{x},\underset{\sim}{y}) = S(\underset{\sim}{x},\underset{\sim}{y}) + K(\underset{\sim}{x},\underset{\sim}{y}) \tag{4.3}$$

where $S(\underset{\sim}{x},\underset{\sim}{y})$ is a singularity function defined as follows:

$$S(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} \dfrac{1}{(n-2)\omega_n}|\underset{\sim}{x}-\underset{\sim}{y}|^{2-n} & \text{for } n > 2, \\ -\dfrac{1}{2\pi}\log|\underset{\sim}{x} - \underset{\sim}{y}| & \text{for } n = 2, \end{cases} \tag{4.4}$$

which is the solution of the equation $\Delta_n u = 0$ for $\underset{\sim}{x} \neq \underset{\sim}{y}$, where ω_n denotes the surface of the unit ball in R^n, while $K(\underset{\sim}{x},\underset{\sim}{y})$ is a regular function satisfying the following:

$$K(\underset{\sim}{x},\underset{\sim}{y})\varepsilon C^1(\bar{\Omega})\bigcap C^2(\Omega),$$
$$\Delta_n K(\underset{\sim}{x},\underset{\sim}{y}) = 0 \quad \text{in } \Omega,$$
$$K_\nu(\underset{\sim}{x},\underset{\sim}{y}) + h_1(\underset{\sim}{y})K(\underset{\sim}{x},\underset{\sim}{y}) = -\{S_\nu(\underset{\sim}{x},\underset{\sim}{y}) + h_1(\underset{\sim}{y})S(\underset{\sim}{x},\underset{\sim}{y})\} \quad \text{on } \Gamma,$$
and
$$K_\nu(\underset{\sim}{x},\underset{\sim}{y}) + h_2(\underset{\sim}{y})K(\underset{\sim}{x},\underset{\sim}{y}) = -\{S_\nu(\underset{\sim}{x},\underset{\sim}{y}) + h_2(\underset{\sim}{y})S(\underset{\sim}{x},\underset{\sim}{y})\} \quad \text{on } \partial\Omega\setminus\Gamma.$$

DEFINITION 4.1. We define the linear operators B_1, B_2, B_3 as follows:

(i) $D(B_1) = \{u\varepsilon L_r^2(\Omega):u\varepsilon C^o(\bar{\Omega})\}$,
$$B_1 u = \int_\Omega R(\underset{\sim}{x},\underset{\sim}{y})u(\underset{\sim}{y})r(\underset{\sim}{y})d\underset{\sim}{y},$$
for each $u\varepsilon D(B_1)$.

(ii) $D(B_2) = \{u\varepsilon L^2(\Gamma):u\varepsilon C^o(\bar{\Omega})\}$,
$$B_2 u = \int_\Gamma R(\underset{\sim}{x},\underset{\sim}{y})u(\underset{\sim}{y})dS_1,$$
for each $u\varepsilon D(B_2)$.

(iii) $D(B_3) = \{u\varepsilon L^2(\partial\Omega\setminus\Gamma) :u\varepsilon C^o(\bar{\Omega})\}$,
$$B_3 u = \int_{\partial\Omega\setminus\Gamma} R(\underset{\sim}{x},\underset{\sim}{y})u(\underset{\sim}{y})dS_2,$$
for each $u\varepsilon D(B_3)$.

REMARK 4.3.

(i) With reference to [10, p. 128] we conclude that the linear operators B_1,B_2,B_3 are compact in $L_r^2(\Omega)$, $L^2(\Gamma)$, $L^2(\partial\Omega\setminus\Gamma)$ respectively. Consequently, formula (4.2) shows that A^{-1} is also compact.

(ii) From Lemmas 2.1, 3.1, 3.2 and theorem 3 in [10, p. 60], we deduce that the set of all eigenvalues of A, counted according to multiplicity, forms an increasing sequence
$$0 < C_o \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty \quad \text{as} \quad j \to \infty \;.$$

(iii) Since A in H is symmetric, then A^{-1} in H is also symmetric.

(iv) Since $D(A^{-1}) \neq H$, then only the closure of A^{-1} is self-adjoint.

(v) On using theorem 3 in [10, p. 30] we deduce that the density of D(A) in H gives us the completeness of the orthonormal system of the eigenfunctions Φ_1,Φ_2,Φ_3, ... of the operator A.

5. AN EXPANSION THEOREM.

We now arrive at the problem of expanding an arbitrary function $f\varepsilon H$ in terms of the eigenfunctions $\{\Phi_j\}_{j=1}^\infty$ of the operator A.

The results of our investigations are summarized in the following theorem:

THEOREM 5.1. The spectrum of A consists of an unbounded sequence of real eigenvalues of finite multiplicity without accumulation point in $(-\infty,\infty)$. Denoting them by
$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$
and the corresponding eigenfunctions by Φ_1,Φ_2,Φ_3, ..., we have $\{\Phi_j\}_{j=1}^\infty$ forms a complete fundamental system in H and for every $f\varepsilon H$ we have the expansion formula
$$f = \sum_{j=1}^\infty \langle f,\Phi_j\rangle\Phi_j \tag{5.1}$$
in the sense of strong convergence in H.

The above theorem has some interesting corollaries for particular choices of the function $f \varepsilon H$.

COROLLARY 5.1. If $f = (f_1, f_2, 0) \varepsilon H$, $f_1 \varepsilon L^2_r(\Omega)$ and $f_2 \varepsilon L^2(\Gamma)$ then we have

$$f_1(\underset{\sim}{x}) = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 \} \Phi_{j1}(\underset{\sim}{x}),$$

$$f_2(\underset{\sim}{x}) = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 \} \Phi_{j2}(\underset{\sim}{x}),$$

and

$$0 = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 \} \Phi_{j3}(\underset{\sim}{x}).$$

COROLLARY 5.2. If $f = (f_1, 0, f_3) \varepsilon H$, $f_1 \varepsilon L^2_r(\Omega)$ and $f_3 \varepsilon L^2(\partial\Omega \setminus \Gamma)$ then we have

$$f_1(\underset{\sim}{x}) = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j1}(\underset{\sim}{x}),$$

$$0 = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j2}(\underset{\sim}{x}),$$

and

$$f_3 = \sum_{j=1}^{\infty} \{ \int_\Omega r(\underset{\sim}{x}) f_1(\underset{\sim}{x}) \Phi_{j1}(\underset{\sim}{x}) d\underset{\sim}{x} + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j3}(\underset{\sim}{x}).$$

COROLLARY 5.3. If $f = (0, f_2, f_3) \varepsilon H$, $f_2 \varepsilon L^2(\Gamma)$ and $f_3 \varepsilon L^2(\partial\Omega \setminus \Gamma)$ then we have

$$0 = \sum_{j=1}^{\infty} \{ \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j1}(\underset{\sim}{x}),$$

$$f_2(\underset{\sim}{x}) = \sum_{j=1}^{\infty} \{ \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j2}(\underset{\sim}{x}),$$

and

$$f_3(\underset{\sim}{x}) = \sum_{j=1}^{\infty} \{ \int_\Gamma f_2(\underset{\sim}{x}) \Phi_{j2}(\underset{\sim}{x}) dS_1 + \int_{\partial\Omega \setminus \Gamma} f_3(\underset{\sim}{x}) \Phi_{j3}(\underset{\sim}{x}) dS_2 \} \Phi_{j3}(\underset{\sim}{x}).$$

ACKNOWLEDGEMENT. The authors would like to express their sincere thanks to the referee for his interesting suggestions and comments.

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