Applied Mathematics, 2015, 6, 2257-2278
Published Online December 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.614199

Scientific Research Publishing

Regular Elements of the Semigroup $B_X(D)$ Defined by Semilattices of the Class $\Sigma_2(X, 8)$ and Their Calculation Formulas

Nino Tsinaridze, Shota Makharadze, Guladi Fartenadze

Department of Mathematics, Faculty of Physics, Mathematics and Computer Sciences, Shota Rustaveli Batumi State University, Batumi, Georgia
Email: ninocinaridze@mail.ru, shota_59@mail.ru, guladi@bsu.edu.ge

Received 13 February 2015; accepted 27 December 2015; published 30 December 2015

Abstract

The paper gives description of regular elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\Sigma_2(X, 8)$, for which intersection the minimal elements is not empty. When X is a finite set, the formulas are derived, by means of which the number of regular elements of the semigroup is calculated. In this case the set of all regular elements is a subsemigroup of the semigroup $B_X(D)$ which is defined by semilattices of the class $\Sigma_2(X, 8)$.

Keywords

Semilattice, Semigroup, Binary Relation, Regular Element

1. Introduction

An element α taken from the semigroup $B_X(D)$ is called a regular element of $B_X(D)$, if in $B_X(D)$ there exists an element β such that $\alpha \circ \beta \circ \alpha = \alpha$ (see [1] [2]).

Definition 1.1. We say that a complete X-semilattice of unions D is an XI-semilattice of unions if it satisfies the following two conditions:

1) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$;
2) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty element Z of D (see ([1], Definition 1.14.2), ([2], Definition 1.14.2)).

Definition 1.2. The one-to-one mapping φ between the complete X-semilattices of unions $\phi(Q, Q)$ and D'' is called a complete isomorphism if the condition $\varphi(D_1) = \bigcup_{T' \in D_1} \varphi(T')$ is fulfilled for each nonempty sub-

How to cite this paper: Tsinaridze, N., Makharadze, S. and Fartenadze, G. (2015) Regular Elements of the Semigroup $B_X(D)$ Defined by Semilattices of the Class $\Sigma_2(X, 8)$ and Their Calculation Formulas. *Applied Mathematics*, **6**, 2257-2278.
http://dx.doi.org/10.4236/am.2015.614199

set D_1 of the semilattice D' (see ([1], Definition 6.3.2), ([2], Definition 6.3.2) or [3]).

Definition 1.3. Let α be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism φ between the complete semilattices of unions Q and D' is a complete α-isomorphism if

1) $Q = V(D,\alpha)$;

2) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3) or [3]).

Theorem 1.1. *Let R be the set of all regular elements of the semigroup $B_X(D)$. Then the following statements are true*:

1) $R(D') \cap R(D'') = \varnothing$ *for any* $D', D'' \in \Sigma_{XI}(D)$ *and* $D' \neq D''$;

2) $R = \bigcup_{D' \in \Sigma_{XI}(D)} R(D')$;

3) *If X is a finite set, then* $|R| = \sum_{D' \in \Sigma_{XI}(D)} |R(D')|$ (*see* ([1], *Theorem* 6.3.6) or ([2], *Theorem* 6.3.6) *or* [3]).

2. Result

By the symbol $\Sigma_2(X,8)$ we denote the class of all X-semilattices of unions whose every element is isomorphic to an X-semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where

$$Z_6 \subset Z_3 \subset Z_1 \subset \breve{D},\ Z_6 \subset Z_4 \subset Z_1 \subset D,\ Z_6 \subset Z_4 \subset Z_2 \subset \breve{D},$$
$$Z_7 \subset Z_4 \subset Z_1 \subset \breve{D},\ Z_7 \subset Z_4 \subset Z_2 \subset \breve{D},\ Z_7 \subset Z_5 \subset Z_1 \subset \breve{D},$$
$$Z_i \setminus Z_j \neq \varnothing,\ (i,j) \in \{(7,6),(6,7),(5,4),(4,5),(5,3),(3,5),(4,3),(3,4),(2,1),(1,2)\}$$

(see [4]).

Now assume that $D \in \Sigma_2(X,8)$. We introduce the following notation:

1) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in **Figure 1**);

2) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 in **Figure 1**);

3) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in **Figure 1**);

4) $Q_4 = \{T, T', T'', \breve{D}\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T'' \subset \breve{D}$ (see diagram 4 in **Figure 1**);

5) $Q_5 = \{T, T', T'', T' \cup T''\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 5 in **Figure 1**);

6) $Q_6 = \{T, Z_4, Z, Z', \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$ and $Z \setminus Z' \neq \varnothing$, $Z' \setminus Z \neq \varnothing$ (see diagram 6 in **Figure 1**);

7) $Q_7 = \{T, T', T'', T' \cup T'', \breve{D}\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 7 in **Figure 1**);

8) $Q_8 = \{T, T', Z_4, Z_4 \cup T', Z, \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $T \subset T'$, $Z_4 \cup T'$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T' \neq \varnothing$ and $(Z_4 \cup T') \setminus Z \neq \varnothing$, $Z \setminus (Z_4 \cup T') \neq \varnothing$ (see diagram 8 in **Figure 1**);

9) $Q_9 = \{T, T', T \cup T'\}$, where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ and $T \cap T' = \varnothing$ (see diagram 9 in **Figure 1**);

10) $Q_{10} = \{T, T', T \cup T', T''\}$, where $T, T', T'' \in D$, $(T \cup T') \subset T''$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ and $T \cap T' = \varnothing$ (see diagram 10 in **Figure 1**);

11) $Q_{11} = \{Z_7, Z_6, Z_4, Z, \breve{D}\}$, where $Z \in \{Z_2, Z_1\}$ and $Z_7 \cap Z_6 = \varnothing$ (see diagram 11 in **Figure 1**);

12) $Q_{12} = \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, where $Z_7 \cap Z_6 = \varnothing$ (see diagram 12 in **Figure 1**);

13) $Q_{13} = \{T, T', T \cup T', T'', Z\}$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$ and $T \cap T'' = \varnothing$ (see diagram 13 in **Figure 1**);

14) $Q_{14} = \{T, T', Z_4, Z, Z', \breve{D}\}$, where $T, T', Z, Z' \in D$, $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $Z_4 \subset Z' \subset \breve{D}$, $T' \subset Z \subset Z'$, $Z_4 \setminus Z \neq \varnothing$, $Z \setminus Z_4 \neq \varnothing$ and $T \cap Z = \varnothing$ (see diagram 14 in **Figure 1**);

15) $Q_{15} = \{T', T, Z_4, T'', Z, T'' \cup Z_4, \breve{D}\}$, where $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $T \subset T''$, $T'' \in \{Z_5, Z_3\}$, $Z_4 \subset Z$, $(T'' \cup Z_4) \cup Z = \breve{D}$, $T'' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T'' \neq \varnothing$, $(T'' \cup Z_4) \setminus Z \neq \varnothing$, $Z \setminus (T'' \cup Z_4) \neq \varnothing$ and $T' \cap T'' = \varnothing$ (see diagram 15 in **Figure 1**);

16) $Q_{16} = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where $Z_5 \cap Z_3 = \varnothing$ (see diagram 16 in **Figure 1**).

Denote by the symbol $\Sigma(Q_i)$ $(i = 1, 2, \cdots, 16)$ the set of all *XI*-subsemilattices of the semilattice *D* isomorphic to Q_i. Assume that $D' \in \Sigma(Q_i)$ and denote by the symbol $R(D')$ the set of all regular elements α of the semigroup $B_X(D')$, for which the semilattices $V(D, \alpha)$ and Q_i are mutually α isomorphic and $V(D, \alpha) = Q_i$.

Definition 1.4. Let the symbol $\Sigma'_{XI}(X, D)$ denote the set of all *XI*-subsemilattices of the semilattice *D*.

Let, further, $D, D' \in \Sigma'(X, D)$ and $\vartheta_{XI} \subseteq \Sigma'_{XI}(X, D) \times \Sigma'_{XI}(X, D)$. It is assumed that $D \vartheta_{XI} D'$ if and only if there exists some complete isomorphism φ between the semilattices *D* and D'. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{XI}(X, D)$.

Let the symbol $Q_i \vartheta_{XI}$ denote the ϑ_{XI}-class of equivalence of the set $\Sigma'_{XI}(X, D)$, where every element is isomorphic to the *X*-semilattice Q_i and

$$R^*(Q_i) = \bigcup_{D' \in Q_i \vartheta_{XI}} R(D')$$

(see ([1], Definition 6.3.5), ([2], Definition 6.3.5) or [5]).

Lemma 1.1. If *X* be a finite set and $|\Omega(Q)| = m_0$, then the following equalities are true:

1) $|R(Q_1)| = 1$;

2) $|R(Q_2)| = m_0 \cdot \left(2^{|T' \setminus T|} - 1\right) \cdot 2^{|X \setminus T'|}$;

3) $|R(Q_3)| = m_0 \cdot \left(2^{|T' \setminus T|} - 1\right) \cdot \left(3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}\right) \cdot 3^{|X \setminus T''|}$;

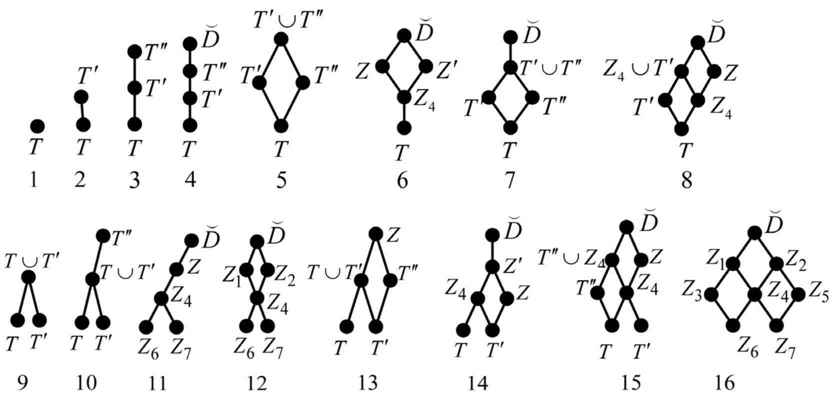

Figure 1. Diagrams of Q_i, $(i = 1, 2, 3, \cdots, 16)$.

4) $\left|R\left(Q_4\right)\right| = m_0 \cdot \left(2^{|T'\setminus T|} - 1\right) \cdot \left(3^{|T''\setminus T'|} - 2^{|T''\setminus T'|}\right) \cdot \left(4^{|\breve{D}\setminus T''|} - 3^{|\breve{D}\setminus T''|}\right) \cdot 4^{|X\setminus \breve{D}|}$;

5) $\left|R\left(Q_5\right)\right| = 2 \cdot m_0 \cdot \left(2^{|T'\setminus T''|} - 1\right) \cdot \left(2^{|T''\setminus T'|} - 1\right) \cdot 4^{|X\setminus (T'\cup T'')|}$;

6) $\left|R\left(Q_6\right)\right| = 2 \cdot m_0 \cdot \left(2^{|Z_4\setminus T|} - 1\right) \cdot 2^{|(Z\cap Z')\setminus Z_4|} \cdot \left(3^{|Z\setminus Z'|} - 2^{|Z\setminus Z'|}\right) \cdot \left(3^{|Z'\setminus Z|} - 2^{|Z'\setminus Z|}\right) \cdot 5^{|X\setminus \breve{D}|}$;

7) $\left|R\left(Q_7\right)\right| = 2 \cdot m_0 \cdot \left(2^{|T'\setminus T''|} - 1\right) \cdot \left(2^{|T''\setminus T'|} - 1\right) \cdot \left(5^{|\breve{D}\setminus (T'\cup T'')|} - 4^{|\breve{D}\setminus (T'\cup T'')|}\right) \cdot 5^{|X\setminus \breve{D}|}$;

8) $\left|R\left(Q_8\right)\right| = 2 \cdot m_0 \cdot \left(2^{|T'\setminus Z|} - 1\right) \cdot \left(2^{|Z_4\setminus T'|} - 1\right) \cdot \left(3^{|Z\setminus (Z_4\cup T')|} - 2^{|Z\setminus (Z_4\cup T')|}\right) \cdot 6^{|X\setminus \breve{D}|}$;

9) $\left|R\left(Q_9\right)\right| = 2 \cdot m_0 \cdot 3^{|X\setminus (T\cup T')|}$;

10) $\left|R\left(Q_{10}\right)\right| = 2 \cdot m_0 \cdot \left(4^{|T''\setminus (T\cup T')|} - 3^{|T''\setminus (T\cup T')|}\right) \cdot 4^{|X\setminus T''|}$;

11) $\left|R\left(Q_{11}\right)\right| = 2 \cdot m_0 \cdot \left(4^{|Z\setminus Z_4|} - 3^{|Z\setminus Z_4|}\right) \cdot \left(5^{|\breve{D}\setminus Z|} - 4^{|\breve{D}\setminus Z|}\right) \cdot 5^{|X\setminus \breve{D}|}$;

12) $\left|R\left(Q_{12}\right)\right| = 4 \cdot m_0 \cdot \left(4^{|Z_1\setminus Z_2|} - 3^{|Z_1\setminus Z_2|}\right) \cdot \left(4^{|Z_2\setminus Z_1|} - 3^{|Z_2\setminus Z_1|}\right) \cdot 6^{|X\setminus \breve{D}|}$;

13) $\left|R\left(Q_{13}\right)\right| = m_0 \cdot \left(2^{|T''\setminus (T'\cup T)|} - 1\right) \cdot 5^{|X\setminus Z|}$;

14) $\left|R\left(Q_{14}\right)\right| = m_0 \cdot \left(2^{|Z\setminus Z_4|} - 1\right) \cdot \left(6^{|\breve{D}\setminus Z'|} - 5^{|\breve{D}\setminus Z'|}\right) \cdot 6^{|X\setminus \breve{D}|}$;

15) $\left|R\left(Q_{15}\right)\right| = m_0 \cdot \left(2^{|T''\setminus Z|} - 1\right) \cdot \left(4^{|Z\setminus (T''\cup Z_4)|} - 3^{|Z\setminus (T''\cup Z_4)|}\right) \cdot 7^{|X\setminus \breve{D}|}$;

16) $\left|R\left(Q_{16}\right)\right| = 2 \cdot m_0 \cdot \left(2^{|Z_5\setminus Z_1|} - 1\right) \cdot \left(2^{|Z_3\setminus Z_2|} - 1\right) \cdot 8^{|X\setminus \breve{D}|}$.

Proof. The statements 1)-4) immediately follows from the Theorem 13.1.2 in [1], Theorem 13.1.2 in [2]; the statements 5)-7) immediately follows from the Theorem 13.3.2 in [1], Theorem 13.3.2 in [2]; the statement 8) immediately follows from the Theorem 13.7.5 in [1], Theorem 13.7.5 in [2]; the statements 9)-11) immediately follows from the Theorem 13.2.2 in [1], Theorem 13.2.2 in [2]; the statement 12) immediately follows from the Theorem 13.5.2 in [1], Theorem 13.5.2 in [2]; the statements 13), 14) immediately follows from the Theorem 13.4.2 in [1], Theorem 13.4.2 in [2], the statement 15) immediately follows from the Corollary 13.10.2 in [1] and the statement 16) immediately follows from the Theorem 2.2 in [4].

The lemma is proved.

Lemma 1.2. *Let* $D = \left\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\} \in \Sigma_2\left(X, 8\right)$ *and* $Z_7 \cap Z_6 \neq \varnothing$. *Then the following sets exhibit all XI-subsemilattices of the given semilattice D*:

1) $\{\breve{D}\}, \{Z_1\}, \{Z_2\}, \{Z_3\}, \{Z_4\}, \{Z_5\}, \{Z_6\}, \{Z_7\}$, (*see diagram* 1 *of the* ***Figure*** **1**);

2) $\{Z_7, Z_5\}, \{Z_7, Z_4\}, \{Z_7, Z_2\}, \{Z_7, Z_1\}, \{Z_7, \breve{D}\}, \{Z_6, Z_4\}, \{Z_6, Z_3\}, \{Z_6, Z_2\}, \{Z_6, Z_1\}, \{Z_6, \breve{D}\}$,
$\{Z_5, Z_2\}, \{Z_5, \breve{D}\}, \{Z_4, Z_2\}, \{Z_4, Z_1\}, \{Z_4, \breve{D}\}, \{Z_3, Z_1\}, \{Z_3, \breve{D}\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\}$,
(*see diagram* 2 *of the* ***Figure*** **1**);

3) $\{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \breve{D}\}, \{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_1, \breve{D}\}$,
$\{Z_6, Z_4, Z_2\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_4, Z_1\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_3, Z_1\}, \{Z_6, Z_3, \breve{D}\}, \{Z_6, Z_1, \breve{D}\}$,
(*see diagram* 3 *of the* ***Figure*** **1**);

4) $\{Z_7, Z_5, Z_2, \breve{D}\}, \{Z_7, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_1, \breve{D}\}$, (*see diagram* 4 *of the* **Figure 1**);

5) $\{Z_7, Z_5, Z_4, Z_2\}, \{Z_7, Z_5, Z_1, \breve{D}\}, \{Z_7, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_1\}, \{Z_6, Z_3, Z_2, \breve{D}\}, \{Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_4, Z_2, Z_1, \breve{D}\}$, (*see diagram* 5 *of the* **Figure 1**);

6) $\{Z_7, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, (*see diagram* 6 *of the* **Figure 1**);

7) $\{Z_7, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, (*see diagram* 7 *of the* **Figure 1**);

8) $\{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, (*see diagram* 8 *of the* **Figure 1**);

Proof. The statements 1)-4) immediately follows from the Theorems 11.6.1 in [1], 11.6.1 in [2] or in [5], the statements 5)-7) immediately follows from the Theorems 11.6.3 in [1], 11.6.3 in [2] or in [5] and the statement 8) immediately follows from the Theorems 11.7.2 in [1].

The lemma is proved.

Theorem 2.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α-isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the following conditions:

1) $\alpha = X \times T$, where $T \in D$;

2) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, where $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$;

3) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, where $T, T', T'' \in D$, $T \subset T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \varnothing$;

4) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D})$, where $T, T', T'' \in D$, $T \subset T' \subset T'' \subset \breve{D}$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;

5) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T''))$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \varnothing$;

6) $\alpha = (Y_T^\alpha \times T) \cup (Y_4^\alpha \times Z_4) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$, $Z \setminus Z' \neq \varnothing$, $Z' \setminus Z \neq \varnothing$, $Y_T^\alpha, Y_4^\alpha, Y_{Z'}^\alpha, Y_Z^\alpha \notin \{\varnothing\}$ and satisfies the conditions $Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cup Y_4^\alpha \supseteq \varphi(Z_4)$, $Y_T^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_T^\alpha \cup Y_4^\alpha \cup Y_{Z'}^\alpha \supseteq \varphi(Z')$, $Y_4^\alpha \cap \varphi(Z_4) \neq \varnothing$, $Y_Z^\alpha \cap \varphi(Z) \neq \varnothing$, $Y_{Z'}^\alpha \cap \varphi(Z') \neq \varnothing$;

7) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')) \cup (Y_0^\alpha \times \breve{D})$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$ $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;

8) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_4^\alpha \times Z_4) \cup (Y_{T' \cup Z_4}^\alpha \times (T' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $T \subset T'$, $Z_4 \cup T'$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T' \neq \varnothing$, $(Z_4 \cup T') \setminus Z \neq \varnothing$, $Z \setminus (Z_4 \cup T') \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_4^\alpha, Y_Z^\alpha \notin \{\varnothing\}$ and satisfies the conditions $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_4^\alpha \supseteq \varphi(Z_4)$, $Y_T^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_4^\alpha \cap \varphi(Z_4) \neq \varnothing$, $Y_Z^\alpha \cap \varphi(Z) \neq \varnothing$.

Proof. In this case, when $Z_7 \cap Z_6 \neq \varnothing$, from the Lemma 1.2 it follows that diagrams 1-8 given in **Figure 1** exhibit all diagrams of *XI*-subsemilattices of the semilattices *D*, a quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these *XI*-semilattices, may have one of the forms listed above. Then the validity of the statements 1)-4) immediately follows from the Theorem 13.1.1 in [1], Theorem 13.1.1 in [2], the statements 5)-7) immediately follows from the Theorem 13.3.1 in [1], Theorem 13.3.1 in [2] and the statement 8) immediately follows from the Theorem 13.7.1 in [1], Theorem 13.7.1 in [2].

The theorem is proved.

1) Lemm 2.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If by $R^*(Q_1)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 1) of the Theorem 2.1, then

$$\left|R^*(Q_1)\right| = 8.$$

Proof. According to the definition of the semilattice *D* we have

$$Q_1\vartheta_{XI} = \left\{\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\right\}.$$

Assume that $D_1' = \{Z_7\}, D_2' = \{Z_6\}, D_3' = \{Z_5\}, D_4' = \{Z_4\}, D_5' = \{Z_3\}, D_6' = \{Z_2\}, D_7' = \{Z_1\}, D_8' = \{\breve{D}\}$.

Then from Theorem 1.1 we obtain

$$\left|R^*(Q_1)\right| = \left|R(D_1')\right| + \left|R(D_2')\right| + \left|R(D_3')\right| + \left|R(D_4')\right| + \left|R(D_5')\right| + \left|R(D_6')\right| + \left|R(D_7')\right| + \left|R(D_8')\right|.$$

From this and by the statement 1) of Lemma 1.1 we obtain $\left|R^*(Q_1)\right| = 1+1+1+1+1+1+1+1 = 8$.

The lemma is proved.

2) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition 2) of the Theorem 2.1. In this case we have $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$. By definition of the semilattice *D* follows that

$$Q_2\theta_{XI} = \left\{\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_3\},\{Z_6,Z_2\},\{Z_6,Z_1\},\right.$$
$$\left.\{Z_6,\breve{D}\},\{Z_5,Z_2\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}\right\}.$$

If the equalities

$$D_1' = \{Z_7,\breve{D}\}, D_2' = \{Z_6,\breve{D}\}, D_3' = \{Z_4,\breve{D}\}, D_4' = \{Z_7,Z_4\}, D_5' = \{Z_7,Z_2\}, D_6' = \{Z_7,Z_1\}, D_7' = \{Z_6,Z_4\},$$
$$D_8' = \{Z_6,Z_3\}, D_9' = \{Z_6,Z_2\}, D_{10}' = \{Z_6,Z_1\}, D_{11}' = \{Z_5,Z_2\}, D_{12}' = \{Z_5,\breve{D}\}, D_{13}' = \{Z_4,Z_2\},$$
$$D_{14}' = \{Z_4,Z_1\}, D_{15}' = \{Z_7,Z_5\}, D_{16}' = \{Z_3,Z_1\}, D_{17}' = \{Z_3,\breve{D}\}, D_{18}' = \{Z_2,\breve{D}\}, D_{19}' = \{Z_1,\breve{D}\}$$

Then from Theorem 1.1 we obtain:

$$R^*(Q_2) = \bigcup_{i=1}^{19} R(D_i'). \tag{2.1}$$

Lemma 2.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then

$$\left|R^*(Q_2)\right| = \left|R(D_1')\right| + \left|R(D_2')\right| - \left|R(D_3')\right|.$$

Proof. Let $D' = \{Z, Z'\} \in Q_2\theta_{XI}$, then $Z, Z' \in D$ and $Z \subset Z'$. If $\alpha \in R(D')$ then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$ for some $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and by statement 2) of the Theorem 2.1 satisfies the conditions $Y_T^\alpha \supseteq Z$ and $Y_{T'}^\alpha \cap Z' \neq \varnothing$. Since Z_7 and Z_6 are minimal elements of the semilattice *D*, we have $Z \supseteq Z_7$ or $Z \supseteq Z_6$.

On the other hand, $\breve{D}$ is maximal elements of the semilattice *D*, therefore $\breve{D} \supseteq Z'$. Hence, in the considered case, only one of the following two conditions is fulfilled:

$$Y_T^\alpha \supseteq Z_7 \text{ and } Y_{T'}^\alpha \cap \breve{D} \neq \varnothing \text{ or } Y_T^\alpha \supseteq Z_6 \text{ and } Y_{T'}^\alpha \cap \breve{D} \neq \varnothing.$$

i.e., $\alpha \in R(D_1')$ or $\alpha \in R(D_2')$. Hence, using equality (2.1), we obtain

$$R^*(Q_2) = R(D_1') \cup R(D_2'). \tag{2.2}$$

Now, let $\alpha \in R(D_1') \cap R(D_2')$ then

$$\begin{aligned} &Y_T^\alpha \supseteq Z_7,\ Y_{T'}^\alpha \cap \breve{D} \neq \varnothing; \\ &Y_T^\alpha \supseteq Z_6,\ Y_{T'}^\alpha \cap \breve{D} \neq \varnothing. \end{aligned} \tag{2.3}$$

Of this we have that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4,\ Y_{T'}^\alpha \cap \breve{D} \neq \varnothing$, *i.e.* $\alpha \in R(D_3')$ and $R(D_1') \cap R(D_2') \subseteq R(D_3')$.

Of the other hand if $\alpha \in R(D_3')$, then $Y_T^\alpha \supseteq Z_4, Y_{T'}^\alpha \cap \breve{D} \neq \varnothing$ and the condition (2.3) is hold. Of this follows that $\alpha \in R(D_1') \cap R(D_2')$, *i.e.* $R(D_3') \subseteq R(D_1') \cap R(D_2')$. Therefore the equality

$$R(D_1') \cap R(D_2') = R(D_3') \tag{2.4}$$

is fulfilled. Now of the equalities (2.2) and (2.4) follows the following equality

$$\left|R^*(Q_2)\right| = \left|R(D_1')\right| + \left|R(D_2')\right| - \left|R(D_1') \cap R(D_2')\right| = \left|R(D_1')\right| + \left|R(D_2')\right| - \left|R(D_3')\right|.$$

The lemma is proved.

Lemma 2.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then

$$\left|R^*(Q_2)\right| = 19 \cdot \left(2^{|\breve{D} \setminus Z_7|} - 1\right) \cdot 2^{|X \setminus \breve{D}|} + 19 \cdot \left(2^{|\breve{D} \setminus Z_6|} - 1\right) \cdot 2^{|X \setminus \breve{D}|} - 19 \cdot \left(2^{|\breve{D} \setminus Z_4|} - 1\right) \cdot 2^{|X \setminus \breve{D}|}.$$

Proof: It is easy to see $\left|\Phi(Q_2, Q_2)\right| = 1$ and $\left|\Omega(Q_2)\right| = 19$, then by statement 2) of the Lemma 1.1 and by Lemma 2.2 we obtain the validity of Lemma 2.3.

The lemma is proved.

3) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 3) of the Theorem 2.1. In this case we have $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$. By definition of the semilattice D follows that

$$\begin{aligned} Q_3\vartheta_{XI} = \Big\{ &\{Z_7, Z_5, Z_2\}, \{Z_7, Z_5, \breve{D}\}, \{Z_7, Z_4, Z_2\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_4, \breve{D}\}, \{Z_7, Z_2, \breve{D}\}, \{Z_7, Z_1, \breve{D}\}, \\ &\{Z_6, Z_4, Z_2\}, \{Z_6, Z_4, Z_1\}, \{Z_6, Z_4, \breve{D}\}, \{Z_6, Z_3, Z_1\}, \{Z_6, Z_3, \breve{D}\}, \{Z_6, Z_2, \breve{D}\}, \{Z_6, Z_1, \breve{D}\}, \\ &\{Z_5, Z_2, \breve{D}\}, \{Z_4, Z_2, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}, \{Z_3, Z_1, \breve{D}\} \Big\}. \end{aligned}$$

Now if

$$\begin{aligned} &D_1' = \{Z_7, Z_5, \breve{D}\}, D_2' = \{Z_7, Z_4, \breve{D}\}, D_3' = \{Z_7, Z_2, \breve{D}\}, D_4' = \{Z_7, Z_1, \breve{D}\}, D_5' = \{Z_6, Z_4, \breve{D}\}, \\ &D_6' = \{Z_6, Z_3, \breve{D}\}, D_7' = \{Z_6, Z_2, \breve{D}\}, D_8' = \{Z_6, Z_1, \breve{D}\}, D_9' = \{Z_4, Z_2, \breve{D}\}, D_{10}' = \{Z_4, Z_1, \breve{D}\}, \\ &D_{11}' = \{Z_7, Z_5, Z_2\}, D_{12}' = \{Z_7, Z_4, Z_2\}, D_{13}' = \{Z_7, Z_4, Z_1\}, D_{14}' = \{Z_6, Z_4, Z_2\}, \\ &D_{15}' = \{Z_6, Z_4, Z_1\}, D_{16}' = \{Z_6, Z_3, Z_1\}, D_{17}' = \{Z_5, Z_2, \breve{D}\}, D_{18}' = \{Z_3, Z_1, \breve{D}\}. \end{aligned}$$

Then from Theorem 1.1 we obtain:

$$R^*(Q_3) = \bigcup_{i=1}^{18} R(D_i'). \tag{3.1}$$

Lemma 3.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then

$$\begin{aligned} \left|R^*(Q_3)\right| = &\left|R(D_1')\right| + \left|R(D_2')\right| + \left|R(D_3')\right| + \left|R(D_4')\right| + \left|R(D_5')\right| + \left|R(D_6')\right| + \left|R(D_7')\right| + \left|R(D_8')\right| \\ &- \left|R(D_9')\right| - \left|R(D_{10}')\right| - \left|R(D_1') \cap R(D_3')\right| - \left|R(D_2') \cap R(D_3')\right| - \left|R(D_2') \cap R(D_4')\right| \\ &- \left|R(D_5') \cap R(D_7')\right| - \left|R(D_5') \cap R(D_8')\right| - \left|R(D_6') \cap R(D_8')\right|. \end{aligned}$$

Proof. Let $D' = \{Z, Z', Z''\}$ $(Z \subset Z' \subset Z'')$ be arbitrary element of the set $Q_3\vartheta_{XI}$ and $\alpha \in R(D')$. Then quasinormal representation of a binary relation α has form $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)$ for some $T,T',T''\in D$, $T\subset T'\subset T''$, $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$ and by statement 3) of the Theorem 2.1 satisfies the conditions $Y_T^\alpha\supseteq Z$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z'$, $Y_{T'}^\alpha\cap Z'\neq\varnothing$ and $Y_{T''}^\alpha\cap Z''\neq\varnothing$. By definition of the semilattice D we have $Z\supseteq Z_7$ or $Z\supseteq Z_6$ and $\breve{D}\supseteq Z''$. Of this and by the conditions $Y_T^\alpha\supseteq Z$, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z'$, $Y_{T'}^\alpha\cap Z'\neq\varnothing$, $Y_{T''}^\alpha\cap Z''\neq\varnothing$ we have:

$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z',\ Y_{T'}^\alpha\cap Z'\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing \quad\text{or}$$
$$Y_T^\alpha\supseteq Z_6,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z',\ Y_{T'}^\alpha\cap Z'\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing$$

i.e. $\alpha\in R\left(D_1''\right)$ or $\alpha\in R\left(D_2''\right)$, where $D_1''=\left\{Z_7,Z',\breve{D}\right\}$ and $D_2''=\left\{Z_6,Z',\breve{D}\right\}$. Hence, using equality (3.1), we obtain

$$R^*\left(Q_3\right)=\bigcup_{i=1}^{8}R\left(D_i'\right). \tag{3.2}$$

Now we show that the following equalities are true:

$$\begin{aligned}
&R(D_1')\cap R(D_4')=\varnothing,\ R(D_1')\cap R(D_5')=\varnothing,\ R(D_1')\cap R(D_6')=\varnothing,\ R(D_1')\cap R(D_8')=\varnothing,\\
&R(D_2')\cap R(D_5')=\varnothing,\ R(D_2')\cap R(D_7')=\varnothing,\ R(D_2')\cap R(D_8')=\varnothing,\ R(D_3')\cap R(D_4')=\varnothing,\\
&R(D_3')\cap R(D_5')=\varnothing,\ R(D_3')\cap R(D_6')=\varnothing,\ R(D_3')\cap R(D_8')=\varnothing,\ R(D_4')\cap R(D_5')=\varnothing,\\
&R(D_4')\cap R(D_7')=\varnothing,\ R(D_6')\cap R(D_7')=\varnothing,\ R(D_7')\cap R(D_8')=\varnothing.\\
&R(D_1')\cap R(D_2')=R(D_1')\cap R(D_2')\cap R(D_3'),\ R(D_1')\cap R(D_7')=R(D_1')\cap R(D_3')\cap R(D_7'),\\
&R(D_2')\cap R(D_6')=R(D_2')\cap R(D_4')\cap R(D_6')=R(D_2')\cap R(D_6')\cap R(D_8'),\\
&R(D_3')\cap R(D_7')=R(D_9'),\ R(D_4')\cap R(D_6')=R(D_4')\cap R(D_6')\cap R(D_8'),\\
&R(D_5')\cap R(D_6')=R(D_5')\cap R(D_6')\cap R(D_8'),\ R(D_4')\cap R(D_8')=R(D_{10}').
\end{aligned} \tag{3.3}$$

For this we consider the following case.

a) If $\alpha\in R\left(D_1'\right)\cap R\left(D_4'\right)$, then

$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_5,\ Y_{T'}^\alpha\cap Z_5\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_1,\ Y_{T'}^\alpha\cap Z_1\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing.$$

It follows that $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_1\cup Z_5=\breve{D}$ and $\left(Y_T^\alpha\cup Y_{T'}^\alpha\right)\cap Y_{T''}^\alpha\supseteq\breve{D}\cap Y_{T''}^\alpha\neq\varnothing$. But the inequality $\left(Y_T^\alpha\cup Y_{T'}^\alpha\right)\cap Y_{T''}^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R\left(D_1'\right)\cap R\left(D_4'\right)=\varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1')\cap R(D_6')=\varnothing,\ R(D_1')\cap R(D_8')=\varnothing,\ R(D_3')\cap R(D_4')=\varnothing,\ R(D_3')\cap R(D_6')=\varnothing,$$
$$R(D_3')\cap R(D_8')=\varnothing,\ R(D_4')\cap R(D_7')=\varnothing,\ R(D_6')\cap R(D_7')=\varnothing,\ R(D_7')\cap R(D_8')=\varnothing.$$

b) If $\alpha\in R\left(D_1'\right)\cap R\left(D_5'\right)$, then

$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_5,\ Y_{T'}^\alpha\cap Z_5\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing,$$
$$Y_T^\alpha\supseteq Z_6,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_4,\ Y_{T'}^\alpha\cap Z_4\neq\varnothing,\ Y_{T''}^\alpha\cap\breve{D}\neq\varnothing$$

It follows that $Y_T^\alpha\supseteq Z_6\cup Z_7=Z_4$ and $Y_T^\alpha\cap Y_{T'}^\alpha\supseteq Z_4\cap Y_{T'}^\alpha\neq\varnothing$. But the inequality $Y_T^\alpha\cap Y_{T'}^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R\left(D_1'\right)\cap R\left(D_5'\right)=\varnothing$ is true.

The similar way we can show that the following equalities are hold:

$$R(D_2')\cap R(D_5')=\varnothing,\ R(D_2')\cap R(D_7')=\varnothing,\ R(D_2')\cap R(D_8')=\varnothing,$$
$$R(D_3')\cap R(D_5')=\varnothing,\ R(D_4')\cap R(D_5')=\varnothing.$$

c) If $\alpha \in R(D_3') \cap R(D_7')$, then

$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_6,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing. \tag{3.4}$$

It follows that

$$Y_T^\alpha \supseteq Z_6 \cup Z_7 = Z_4,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing, \tag{3.5}$$

i.e., $\alpha \in R(D_9')$. So, the inclusion $R(D_3') \cap R(D_7') \subseteq R(D_9')$ is hold.

Of the other hand, if $\alpha \in R(D_9')$, then the conditions (3.4) and (3.5) are fulfilled, *i.e.* $\alpha \in R(D_3') \cap R(D_7')$ and $R(D_9') \subseteq R(D_3') \cap R(D_7')$. Therefore, the equality $R(D_9') = R(D_3') \cap R(D_7')$ is true.

The similar way we can show that the following equality is hold: $R(D_{10}') = R(D_4') \cap R(D_8')$.

d) If $\alpha \in R(D_1') \cap R(D_2') \cap R(D_3')$, then

$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5,\ Y_{T'}^\alpha \cap Z_5 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4,\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing, \tag{3.6}$$
$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing.$$

It follows that

$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_5 \neq \varnothing, Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing \tag{3.7}$$

i.e., $\alpha \in R(D_1') \cap R(D_2')$. So, the inclusion $R(D_1') \cap R(D_2') \cap R(D_3') \subseteq R(D_1') \cap R(D_2')$ is hold.

Of the other hand, if $\alpha \in R(D_1') \cap R(D_2')$, then the conditions (3.6) and (3.7) are fulfilled, *i.e.*, $\alpha \in R(D_1') \cap R(D_2') \cap R(D_3')$ and $R(D_1') \cap R(D_2') \subseteq R(D_1') \cap R(D_2') \cap R(D_3')$. Therefore, the equality $R(D_1') \cap R(D_2') \cap R(D_3') = R(D_1') \cap R(D_2')$ is true.

The similar way we can show that the following equalities are hold:

$$R(D_1') \cap R(D_7') = R(D_1') \cap R(D_3') \cap R(D_7'),$$
$$R(D_2') \cap R(D_6') = R(D_2') \cap R(D_4') \cap R(D_6') = R(D_2') \cap R(D_6') \cap R(D_8'),$$
$$R(D_4') \cap R(D_6') = R(D_4') \cap R(D_6') \cap R(D_8'),$$
$$R(D_5') \cap R(D_6') = R(D_5') \cap R(D_6') \cap R(D_8').$$

We have that all equalities of (3.3) are true. Now, by the equalities of (3.2) and (3.3) we obtain the validity of Lemma 3.1.

The lemma is proved.

Lemma 3.2. Let $D' = \{Y, Y', \breve{D}\}$, $D'' = \{Y_1, Y_1', \breve{D}\}$, where $Y, Y', Y_1, Y_1' \in D$ $Y_1 \supseteq Y$ and $Y_1' \supseteq Y'$. If quasi-normal representation of binary relation α of the semigroup $B_X(D)$ has a form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$ for some $T, T' \in D$, $T \subset T' \subset \breve{D}$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then $\alpha \in R(D') \cap R(D'')$ iff

$$Y_T^\alpha \supseteq Y_1,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1',\ Y_{T'}^\alpha \cap Y' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

Proof. If $\alpha \in R(D') \cap R(D'')$, then by statement 3) of the Theorem 2.1 we have

$$Y_T^\alpha \supseteq Y,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y',\ Y_{T'}^\alpha \cap Y' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_T^\alpha \supseteq Y_1,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1',\ Y_{T'}^\alpha \cap Y_1' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing. \tag{3.8}$$

Of the last condition we have

$$Y_T^\alpha \supseteq Y_1,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1',\ Y_{T'}^\alpha \cap Y' \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing, \tag{3.9}$$

since $Y_1 \supseteq Y$ and $Y_1' \supseteq Y'$ by assumption.

Of the other hand, if the conditions of (3.9) are hold, then also hold the conditions of (3.8), *i.e.* $\alpha \in R(D') \cap R(D'')$.

The lemma is proved.

Lemma 3.3. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_7\cap Z_6\neq\varnothing$. If X is a finite set, then the following equalities are hold:

$$\left|R(D_1')\cap R(D_3')\right|=18\cdot 2^{|Z_2\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_2')\cap R(D_3')\right|=18\cdot 2^{|Z_2\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_2')\cap R(D_4')\right|=18\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_5')\cap R(D_7')\right|=18\cdot 2^{|Z_2\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_5')\cap R(D_8')\right|=18\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_6')\cap R(D_8')\right|=18\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}.$$

Proof. Let $D'=\{Y,Y',\breve{D}\},D''=\{Y_1,Y_1',\breve{D}\}\in\{D_1',D_2',\cdots,D_8'\}$, where $Y_1\supseteq Y$ and $Y_1'\supseteq Y'$. Assume that $\alpha\in R(D')\cap R(D'')$ and a quasinormal representation of a regular binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_0^\alpha\times\breve{D})$ for some $T,T'\in D$, $T\subset T'\subset\breve{D}$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then by statement c) of the Theorem 3.1.1, we have

$$Y_T^\alpha\supseteq Y_1,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y_1',\ Y_{T'}^\alpha\cap Y'\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing \tag{3.10}$$

Let f_α is a mapping of the set X in the semilattice D satisfying the conditions $f_\alpha(t)=t\alpha$ for all $t\in X$. $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets Y_1, $Y_1'\setminus Y_1$, $\breve{D}\setminus Y_1'$, $X\setminus\breve{D}$ respectively. It is clear, that the intersection disjoint elements of the set $\{Y_1,\ Y_1'\setminus Y_1,\ \breve{D}\setminus Y_1',\ X\setminus\breve{D}\}$ is empty set, and $Y_1\cup(Y_1'\setminus Y_1)\cup(\breve{D}\setminus Y_1')\cup(X\setminus\breve{D})=X$.

We are going to find properties of the maps $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$.

1) $t\in Y_1$. Then by the properties (3.10) we have $Y_1\subseteq Y_T^\alpha$, *i.e.*, $t\in Y_T^\alpha$ and $t\alpha=T$ by definition of the set Y_T^α. Therefore $f_{0\alpha}(t)=T$ for all $t\in Y_1$.

2) $t\in Y_1'\setminus Y_1$. Then by the properties (3.10) we have $Y_1'\setminus Y_1\subseteq Y_1'\subseteq Y_T^\alpha\cup Y_{T'}^\alpha$, *i.e.*, $t\in Y_T^\alpha\cup Y_{T'}^\alpha$ and $t\alpha\in\{T,T'\}$ by definition of the sets Y_T^α and $Y_{T'}^\alpha$. Therefore $f_{1\alpha}(t)\in\{T,T'\}$ for all $t\in Y_1'\setminus Y_1$.

Preposition we have that $Y_{T'}^\alpha\cap Y'\neq\varnothing$, *i.e.* $t'\alpha=T'$ for some $t'\in Y'$. If $t'\in Y_1$, then $t'\in Y_1\subseteq Y_T^\alpha$. Therefore $t'\alpha=T$. That is contradict of the equality $t'\alpha=T'$, while $T\neq T'$ by definition of the semilattice D. Therefore $f_{1\alpha}(t')=T'$ for some $t'\in Y'\setminus Y_1$.

3) $t\in\breve{D}\setminus Y_1'$. Then by properties (3.10) we have $\breve{D}\setminus Y_1'\subseteq\breve{D}\subseteq Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_0^\alpha=X$, *i.e.*, $t\in Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_0^\alpha$ and $t\alpha\in\{T,T',\breve{D}\}$ by definition of the sets Y_T^α, $Y_{T'}^\alpha$ and Y_0^α. Therefore $f_{3\alpha}(t)\in\{T,T',\breve{D}\}$ for all $t\in\breve{D}\setminus Y_1'$.

Preposition we have that $Y_0^\alpha\cap\breve{D}\neq\varnothing$, *i.e.* $t''\alpha=\breve{D}$ for some $t''\in\breve{D}$. If $t''\in Y_1'$. Then $t''\in Y_1'\subseteq Y_T^\alpha\cup Y_{T'}^\alpha$. Therefore $t''\alpha\in\{T,T'\}$ by definition of the set Y_T^α and $Y_{T'}^\alpha$. We have contradict of the equality $t''\alpha=T''$. Therefore $f_{3\alpha}(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y_1'$.

4) $t\in X\setminus\breve{D}$. Then by definition quasinormal representation binary relation α and by property (3.10) we have $t\in X\setminus\breve{D}\subseteq X=Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_0^\alpha$, *i.e.* $t\alpha\in\{T,T',\breve{D}\}$ by definition of the sets $Y_T^\alpha,Y_{T'}^\alpha$ and Y_0^α. Therefore $f_{4\alpha}(t)\in\{T,T',\breve{D}\}$ for all $t\in X\setminus\breve{D}$.

Therefore for every binary relation $\alpha\in R(D')\cap R(D'')$ exist ordered system $(f_{0\alpha},f_{1\alpha},f_{2\alpha},f_{3\alpha})$. It is obvious that for disjoint binary relations exist disjoint ordered systems.

Now, let $f_0:Y_1\to\{T\}$, $f_1:Y_1'\setminus Y_1\to\{T,T'\}$, $f_2:\breve{D}\setminus Y_1'\to\{T,T',\breve{D}\}$, $f_3:X\setminus\breve{D}\to\{T,T',\breve{D}\}$ are such mappings, which satisfying the conditions:

5) $f_0(t)=T$ for all $t\in Y_1$;

6) $f_1(t)\in\{T,T'\}$ for all $t\in Y_1'\setminus Y_1$ and $f_1(t')=T'$ for some $t'\in Y'\setminus Y_1$;

7) $f_2(t)\in\{T,T',\breve{D}\}$ for all $t\in\breve{D}\setminus Y_1'$ and $f_2(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y_1'$;

8) $f_3(t)\in\{T,T',\breve{D}\}$ for all $t\in X\setminus\breve{D}$.

Now we define a map f of a set X in the semilattice D, which satisfies the condition:

$$f(t)=\begin{cases} f_0(t), \text{ if } t\in Y_1, \\ f_1(t), \text{ if } t\in Y_1'\setminus Y_1, \\ f_2(t), \text{ if } t\in \breve{D}\setminus Y_1', \\ f_3(t), \text{ if } t\in X\setminus \breve{D}. \end{cases}$$

Let $\beta=\bigcup_{x\in X}(\{x\}\times f(x))$, $Y_T^\beta=\{t \mid t\beta=T\}$, $Y_{T'}^\beta=\{t \mid t\beta=T'\}$ and $Y_0^\beta=\{t \mid t\beta=\breve{D}\}$. Then binary relation β can be representation by form $\beta=(Y_T^\beta\times T)\cup(Y_{T'}^\beta\times T')\cup(Y_0^\beta\times\breve{D})$ and satisfying the conditions:

$$Y_T^\beta\supseteq Y_1,\ Y_T^\beta\cup Y_{T'}^\beta\supseteq Y_1',\ Y_{T'}^\beta\cap Y'\neq\varnothing,\ Y_0^\beta\cap\breve{D}\neq\varnothing$$

(By suppose $f_1(t')=T'$ for some $t'\in Y'\setminus Y_1$ and $f_2(t'')=\breve{D}$ for some $t''\in\breve{D}\setminus Y_1'$), *i.e.*, by lemma 2.5 we have that $\beta\in R(D')\cap R(D'')$.

Therefore for every binary relation $\alpha\in R(D')\cap R(D'')$ and ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$ exist one to one mapping.

By ([1], Theorem 1.18.2) the number of the mappings $f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}$ are respectively:

$$1,\ 2^{|Y_1'\setminus(Y'\cup Y_1)|}\cdot\left(2^{|Y'\setminus Y_1|}-1\right),\ 3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|},\ 3^{|X\setminus\breve{D}|}.$$

Note that the number $2^{|Y_1'\setminus(Y'\cup Y_1)|}\cdot\left(2^{|Y'\setminus Y_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|}\right)\cdot 3^{|X\setminus\breve{D}|}$ does not depend on choice of chains $T\subset T'\subset T''$ $(T,T',T''\in D)$ of the semilattice *D*. Sins the number of such different chains of the semilattice *D* is equal to 18, for arbitrary $T,T',T''\in D$ where $T\subset T'\subset T''$, the number of regular elements of the set $R(D')\cap R(D'')$ is equal to

$$\left|R(D')\cap R(D'')\right|=18\cdot 2^{|Y_1'\setminus(Y'\cup Y_1)|}\cdot\left(2^{|Y'\setminus Y_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|}\right)\cdot 3^{|X\setminus\breve{D}|}.$$

Note that the number $2^{|Y_1'\setminus(Y'\cup Y_1)|}\cdot\left(2^{|Y'\setminus Y_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|}\right)\cdot 3^{|X\setminus\breve{D}|}$ does not depend on choice of chains $T\subset T'\subset T''$ $(T,T',T''\in D)$ of the semilattice *D*. Since the number of such different chains of the semilattice *D* is equal to 18, for arbitrary $T,T',T''\in D$ where $T\subset T'\subset T''$, the number of regular elements of the set $R(D')\cap R(D'')$ is equal to $\left|R(D')\cap R(D'')\right|=18\cdot 2^{|Y_1'\setminus(Y'\cup Y_1)|}\cdot\left(2^{|Y'\setminus Y_1|}-1\right)\cdot\left(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|}\right)\cdot 3^{|X\setminus\breve{D}|}$. Therefore, we obtain the validity of Lemma 3.3.

The lemma is proved.

Lemma 3.4. Let *X* be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_7\cap Z_6\neq\varnothing$. If by $R^*(Q_3)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 3) of the Theorem 2.1, then

$$\begin{aligned}
\left|R^*(Q_3)\right| &= 18\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|}\right)\cdot 3^{|X\setminus\breve{D}|}+18\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+18\cdot\left(2^{|Z_2\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}+18\cdot\left(2^{|Z_1\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+18\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|}\right)\cdot 3^{|X\setminus\breve{D}|}+18\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&+18\cdot\left(2^{|Z_2\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}+18\cdot\left(2^{|Z_1\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&-18\cdot\left(2^{|Z_2\setminus Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}-18\cdot\left(2^{|Z_1\setminus Z_4|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&-18\cdot 2^{|Z_2\setminus Z_5|}\cdot\left(2^{|Z_5\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}-18\cdot 2^{|Z_2\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&-18\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}-18\cdot 2^{|Z_2\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|}\right)\cdot 3^{|X\setminus\breve{D}|}\\
&-18\cdot 2^{|Z_1\setminus Z_4|}\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}-18\cdot 2^{|Z_1\setminus Z_3|}\cdot\left(2^{|Z_3\setminus Z_6|}-1\right)\cdot\left(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|}\right)\cdot 3^{|X\setminus\breve{D}|}.
\end{aligned}$$

Proof: It is easy to see $\left|\Phi\left(Q_3,Q_3\right)\right|=1$ and $\left|\Omega\left(Q_3\right)\right|=18$, then by statement 3) of the Lemma 1.1, by Lemma 3.1 and by Lemma 3.3 we obtain the validity of Lemma 3.4.

The lemma is proved.

4) Now let binary relation α of the semigroup $B_X\left(D\right)$ satisfying the condition 4) of the Theorem 2.1. In this case we have $Q_4=\left\{T,T',T'',\breve{D}\right\}$, where $T,T',T''\in D$ and $T\subset T'\subset T''$. By definition of the semilattice D follows that

$$Q_4\vartheta_{XI}=\left\{\left\{Z_7,Z_5,Z_2,\breve{D}\right\}\left\{Z_7,Z_4,Z_2,\breve{D}\right\},\left\{Z_7,Z_4,Z_1,\breve{D}\right\}\right.$$
$$\left.\left\{Z_6,Z_4,Z_2,\breve{D}\right\},\left\{Z_6,Z_3,Z_1,\breve{D}\right\},\left\{Z_6,Z_4,Z_1,\breve{D}\right\}\right\}$$

Now if

$$D_1'=\left\{Z_7,Z_5,Z_2,\breve{D}\right\},D_2'=\left\{Z_7,Z_4,Z_2,\breve{D}\right\},D_3'=\left\{Z_7,Z_4,Z_1,\breve{D}\right\}$$
$$D_4'=\left\{Z_6,Z_4,Z_2,\breve{D}\right\},D_5'=\left\{Z_6,Z_3,Z_1,\breve{D}\right\},D_6'=\left\{Z_6,Z_4,Z_1,\breve{D}\right\}$$

Then from Theorem 1.1 we obtain

$$R^*\left(Q_4\right)=\bigcup_{i=1}^{6}R\left(D_i'\right). \tag{4.1}$$

Lemma 4.1. Let X be a finite set, $D=\left\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_2\left(X,8\right)$ and $Z_7\cap Z_6\neq\varnothing$. If by $R^*\left(Q_4\right)$ denoted all regular elements of the semigroup $B_X\left(D\right)$ satisfying the condition 4) of the Theorem 2.1, then

$$\begin{aligned}\left|R^*\left(Q_4\right)\right|=\;&6\cdot\left(2^{\left|Z_5\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_5\right|}-2^{\left|Z_2\setminus Z_5\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_2\right|}-3^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}\\
&+6\cdot\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_4\right|}-2^{\left|Z_2\setminus Z_4\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_2\right|}-3^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}\\
&+6\cdot\left(2^{\left|Z_4\setminus Z_7\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_4\right|}-2^{\left|Z_1\setminus Z_4\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_1\right|}-3^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}\\
&+6\cdot\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_2\setminus Z_4\right|}-2^{\left|Z_2\setminus Z_4\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_2\right|}-3^{\left|\breve{D}\setminus Z_2\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}\\
&+6\cdot\left(2^{\left|Z_4\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_4\right|}-2^{\left|Z_1\setminus Z_4\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_1\right|}-3^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}\\
&+6\cdot\left(2^{\left|Z_3\setminus Z_6\right|}-1\right)\cdot\left(3^{\left|Z_1\setminus Z_3\right|}-2^{\left|Z_1\setminus Z_3\right|}\right)\cdot\left(4^{\left|\breve{D}\setminus Z_1\right|}-3^{\left|\breve{D}\setminus Z_1\right|}\right)\cdot 4^{\left|X\setminus\breve{D}\right|}.\end{aligned}$$

Proof. First we show that the following equalities are hold:

$$\begin{aligned}&R\left(D_1'\right)\cap R\left(D_2'\right)=\varnothing,\ R\left(D_1'\right)\cap R\left(D_3'\right)=\varnothing,\ R\left(D_1'\right)\cap R\left(D_4'\right)=\varnothing,\\
&R\left(D_1'\right)\cap R\left(D_5'\right)=\varnothing,\ R\left(D_1'\right)\cap R\left(D_6'\right)=\varnothing,\ R\left(D_2'\right)\cap R\left(D_3'\right)=\varnothing,\\
&R\left(D_2'\right)\cap R\left(D_4'\right)=\varnothing,R\left(D_2'\right)\cap R\left(D_5'\right)=\varnothing,R\left(D_2'\right)\cap R\left(D_6'\right)=\varnothing,\\
&R\left(D_3'\right)\cap R\left(D_4'\right)=\varnothing,R\left(D_3'\right)\cap R\left(D_5'\right)=\varnothing,R\left(D_3'\right)\cap R\left(D_6'\right)=\varnothing,\\
&R\left(D_4'\right)\cap R\left(D_5'\right)=\varnothing,R\left(D_4'\right)\cap R\left(D_6'\right)=\varnothing,R\left(D_5'\right)\cap R\left(D_6'\right)=\varnothing.\end{aligned} \tag{4.2}$$

For this we consider the following case.

a) Let $\alpha\in R\left(D_1'\right)\cap R\left(D_2'\right)$. If a quasinormal representation of a regular binary relation α has the form $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$ for some $T,T',T''\in D$, $T\subset T'\subset T''$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\left\{\varnothing\right\}$. Then by statement 4) of the Theorem 2.1, we have

$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_5,\ Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq Z_2,\ Y_{T'}^\alpha\cap Z_5\neq\varnothing,\ Y_{T''}^\alpha\cap Z_2\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing;$$
$$Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq Z_2,\ Y_{T'}^\alpha\cap Z_4\neq\varnothing,\ Y_{T''}^\alpha\cap Z_2\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing;$$

It follows that $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5 \cup Z_4 = Z_2$ and $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \supseteq Z_2 \cap Y_{T''}^\alpha \neq \varnothing$. But the inequality $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \neq \varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1') \cap R(D_2') = \varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1') \cap R(D_4') = \varnothing,\ R(D_2') \cap R(D_4') = \varnothing,\ R(D_3') \cap R(D_5') = \varnothing,$$
$$R(D_3') \cap R(D_6') = \varnothing,\ R(D_5') \cap R(D_6') = \varnothing,$$

b) Let $\alpha \in R(D_1') \cap R(D_3')$ and a quasinormal representation of a regular binary relation α has the form $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T''}^\alpha \times T''\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$ for some $T, T', T'' \in D$, $T \subset T' \subset T''$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$. Then by statement 4) of the Theorem 2.1, we have

$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5,\ Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_5 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_2 \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_T^\alpha \supseteq Z_7,\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4,\ Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq Z_1,\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_1 \neq \varnothing,\ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

It follows that $Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq Z_1 \cup Z_2 = \breve{D}$ and $\left(Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha\right) \cap Y_0^\alpha \supseteq \breve{D} \cap Y_0^\alpha \neq \varnothing$. But the inequality $\left(Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha\right) \cap Y_0^\alpha \neq \varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1') \cap R(D_3') = \varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1') \cap R(D_5') = \varnothing,\ R(D_1') \cap R(D_6') = \varnothing,\ R(D_2') \cap R(D_3') = \varnothing,$$
$$R(D_2') \cap R(D_5') = \varnothing,\ R(D_2') \cap R(D_6') = \varnothing, R(D_3') \cap R(D_4') = \varnothing,$$
$$R(D_4') \cap R(D_5') = \varnothing, R(D_4') \cap R(D_6') = \varnothing$$

By equalities (4.1) and (4.2) follows, that

$$\left|R^*(Q_4)\right| = \left|R(D_1')\right| + \left|R(D_2')\right| + \left|R(D_3')\right| + \left|R(D_4')\right| + \left|R(D_5')\right| + \left|R(D_6')\right|.$$

It is easy to see $\left|\Phi(Q_4, Q_4)\right| = 1$ and $\left|\Omega(Q_4)\right| = 6$, of the last equalities and by statement 4) of the Lemma 1.1 we obtain the validity of Lemma 4.1.

The lemma is proved.

5) Now let binary relation α of the semigroup $B_X(D)$ satisfying the condition 5) of the Theorem 2.1. In this case we have $Q_5 = \{T, T', T'', T' \cup T''\}$, where $T, T', T'' \in D$ and $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$ and $T'' \setminus T' \neq \varnothing$. By definition of the semilattice *D* follows that

$$Q_5\vartheta_{XI} = \big\{\{Z_7, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_2\}$$
$$\{Z_6, Z_4, Z_3, Z_1\}, \{Z_6, Z_2, Z_1, \breve{D}\}, \{Z_4, Z_2, Z_1, \breve{D}\}\big\}$$

Now if

$$D_1' = \{Z_7, Z_2, Z_1, \breve{D}\},\ D_2' = \{Z_7, Z_1, Z_2, \breve{D}\},\ D_3' = \{Z_7, Z_5, Z_1, \breve{D}\},$$
$$D_4' = \{Z_7, Z_1, Z_5, \breve{D}\},\ D_5' = \{Z_6, Z_3, Z_2, \breve{D}\},\ D_6' = \{Z_6, Z_2, Z_3, \breve{D}\},$$
$$D_7' = \{Z_7, Z_5, Z_4, Z_2\},\ D_8' = \{Z_7, Z_4, Z_5, Z_2\},\ D_9' = \{Z_6, Z_4, Z_3, Z_1\},$$
$$D_{10}' = \{Z_6, Z_3, Z_4, Z_1\},\ D_{11}' = \{Z_6, Z_2, Z_1, \breve{D}\},\ D_{12}' = \{Z_6, Z_1, Z_2, \breve{D}\},$$
$$D_{13}' = \{Z_4, Z_2, Z_1, \breve{D}\},\ D_{14}' = \{Z_4, Z_1, Z_2, \breve{D}\}.$$

Then from Theorem 1.1 we obtain

$$R^*(Q_5) = \bigcup_{i=1}^{14} R(D_i'). \tag{5.1}$$

Lemma 5.1. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_7\cap Z_6\neq\varnothing$. If by $R^*(Q_5)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 5) of the Theorem 2.1, then

$$\begin{aligned}\left|R^*(Q_5)\right|=&\left|R(D_1')\right|+\left|R(D_2')\right|+\left|R(D_3')\right|+\left|R(D_4')\right|+\left|R(D_5')\right|+\left|R(D_6')\right|+\left|R(D_7')\right|\\&+\left|R(D_8')\right|+\left|R(D_9')\right|+\left|R(D_{10}')\right|-\left|R(D_1')\cap R(D_3')\right|-\left|R(D_1')\cap R(D_6')\right|\\&-\left|R(D_2')\cap R(D_4')\right|-\left|R(D_2')\cap R(D_5')\right|-\left|R(D_3')\cap R(D_7')\right|\\&-\left|R(D_4')\cap R(D_8')\right|-\left|R(D_5')\cap R(D_{10}')\right|-\left|R(D_6')\cap R(D_9')\right|.\end{aligned}$$

Proof. Let $\tilde{D}=\{Z,Z',Z'',Z'\cup Z''\}$ be arbitrary element of the set $\{D_1',D_2',\cdots,D_{14}'\}$ and $\alpha\in R(D')$. Then quasinormal representation binary relation α of the semigroup $B_X(D)$ has a form

$$\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right),$$

where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and by statement 5) of the Theorem 2.1 satisfies the following conditions:

$$Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z',\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z'',\ Y_{T'}^\alpha\cap Z'\neq\varnothing\ \text{ and }\ Y_{T''}^\alpha\cap Z''\neq\varnothing.$$

Of this we have that the inclusions $R(D_1')=R(D_{11}')=R(D_{13}')$, $R(D_2')=R(D_{12}')=R(D_{14}')$ are fulfilled. Therefore, of the equality (5.1) follows, that

$$R^*(Q_5)=\bigcup_{i=1}^{10}R(D_i'). \tag{5.2}$$

Now we show that the following equalities are hold:

$$\begin{aligned}&R(D_1')\cap R(D_2')=\varnothing,\ R(D_1')\cap R(D_4')=\varnothing,\ R(D_1')\cap R(D_5')=\varnothing,\ R(D_1')\cap R(D_7')=\varnothing,\\&R(D_1')\cap R(D_8')=\varnothing,\ R(D_1')\cap R(D_{10}')=\varnothing,\ R(D_2')\cap R(D_3')=\varnothing,\ R(D_2')\cap R(D_6')=\varnothing,\\&R(D_2')\cap R(D_7')=\varnothing,\ R(D_2')\cap R(D_9')=\varnothing,\ R(D_2')\cap R(D_{10}')=\varnothing,\ R(D_3')\cap R(D_4')=\varnothing,\\&R(D_3')\cap R(D_5')=\varnothing,\ R(D_3')\cap R(D_8')=\varnothing,\ R(D_3')\cap R(D_{10}')=\varnothing,\ R(D_4')\cap R(D_6')=\varnothing,\\&R(D_4')\cap R(D_7')=\varnothing,\ R(D_4')\cap R(D_9')=\varnothing,\ R(D_5')\cap R(D_6')=\varnothing,\ R(D_5')\cap R(D_7')=\varnothing,\\&R(D_5')\cap R(D_9')=\varnothing,\ R(D_6')\cap R(D_8')=\varnothing,\ R(D_6')\cap R(D_{10}')=\varnothing,\ R(D_7')\cap R(D_8')=\varnothing,\\&R(D_7')\cap R(D_9')=\varnothing,\ R(D_7')\cap R(D_{10}')=\varnothing,\ R(D_8')\cap R(D_9')=\varnothing,\ R(D_8')\cap R(D_{10}')=\varnothing,\\&R(D_9')\cap R(D_{10}')=\varnothing.\\&R(D_1')\cap R(D_9')=R(D_1')\cap R(D_6')\cap R(D_9'),\ R(D_2')\cap R(D_8')=R(D_2')\cap R(D_4')\cap R(D_8'),\\&R(D_3')\cap R(D_6')=R(D_1')\cap R(D_3')\cap R(D_6'),\\&R(D_3')\cap R(D_9')=R(D_1')\cap R(D_3')\cap R(D_9')=R(D_1')\cap R(D_3')\cap R(D_6')\cap R(D_9')\\&\qquad=R(D_3')\cap R(D_6')\cap R(D_9'),\\&R(D_4')\cap R(D_5')=R(D_2')\cap R(D_4')\cap R(D_5'),\ R(D_4')\cap R(D_{10}')=R(D_4')\cap R(D_5')\cap R(D_{10}'),\\&R(D_5')\cap R(D_8')=R(D_2')\cap R(D_5')\cap R(D_8')=R(D_4')\cap R(D_5')\cap R(D_8')\\&\qquad=R(D_2')\cap R(D_4')\cap R(D_5')\cap R(D_8'),\\&R(D_6')\cap R(D_7')=R(D_3')\cap R(D_6')\cap R(D_7').\end{aligned} \tag{5.3}$$

a) Let $\alpha\in R(D_1')\cap R(D_2')$. Then quasinormal representation binary relation α of the semigroup $B_X(D)$ has a form $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)$, where $T,T',T''\in D$, $T\subset T'$, $T\subset T''$,

$T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ and by statement 5) of the Theorem 2.1 satisfies the following conditions:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_1,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_1 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_2,\ Y_{T'}^\alpha \cap Z_1 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_2 \neq \varnothing$$

Of this conditions follows that $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1 \cup Z_2 = \breve{D}$, then $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \supseteq \breve{D} \cap Y_{T''}^\alpha \supseteq Z_1 \cap Y_{T''}^\alpha \neq \varnothing$. But the inequality $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \neq \varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1') \cap R(D_2') = \varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1') \cap R(D_4') = \varnothing,\ R(D_1') \cap R(D_5') = \varnothing,\ R(D_1') \cap R(D_{10}') = \varnothing,$$
$$R(D_2') \cap R(D_3') = \varnothing,\ R(D_2') \cap R(D_6') = \varnothing,\ R(D_2') \cap R(D_7') = \varnothing,$$
$$R(D_3') \cap R(D_4') = \varnothing,\ R(D_3') \cap R(D_5') = \varnothing,\ R(D_3') \cap R(D_{10}') = \varnothing,$$
$$R(D_4') \cap R(D_6') = \varnothing,\ R(D_4') \cap R(D_7') = \varnothing,\ R(D_4') \cap R(D_9') = \varnothing,$$
$$R(D_5') \cap R(D_6') = \varnothing,\ R(D_5') \cap R(D_7') = \varnothing,\ R(D_6') \cap R(D_8') = \varnothing,$$
$$R(D_7') \cap R(D_{10}') = \varnothing,\ R(D_8') \cap R(D_9') = \varnothing.$$

b) Let $\alpha \in R(D_1') \cap R(D_7')$. Then quasinormal representation binary relation α of the semigroup $B_X(D)$ has a form $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T''}^\alpha \times T''\right) \cup \left(Y_{T' \cup T''}^\alpha \times (T' \cup T'')\right)$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ and by statement 5) of the Theorem 2.1 satisfies the following conditions:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_1,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_1 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_4,\ Y_{T'}^\alpha \cap Z_5 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_4 \neq \varnothing.$$

Of this conditions follows that $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5 \cup Z_2 = Z_2$, then $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \supseteq Z_2 \cap Y_{T''}^\alpha \neq \varnothing$. But the inequality $\left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap Y_{T''}^\alpha \neq \varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1') \cap R(D_7') = \varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1') \cap R(D_8') = \varnothing,\ R(D_2') \cap R(D_9') = \varnothing,\ R(D_2') \cap R(D_{10}') = \varnothing,$$
$$R(D_3') \cap R(D_8') = \varnothing,\ R(D_5') \cap R(D_9') = \varnothing,\ R(D_6') \cap R(D_{10}') = \varnothing,$$
$$R(D_7') \cap R(D_8') = \varnothing,\ R(D_7') \cap R(D_9') = \varnothing,\ R(D_8') \cap R(D_{10}') = \varnothing,$$
$$R(D_9') \cap R(D_{10}') = \varnothing.$$

c) If $\alpha \in R(D_1') \cap R(D_6') \cap R(D_9')$, then

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_1,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_1 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_3,\ Y_{T'}^\alpha \cap Z_2 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_3 \neq \varnothing, \tag{5.4}$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_3,\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_3 \neq \varnothing$$

It follows that

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_2,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_1,\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ Y_{T''}^\alpha \cap Z_3 \neq \varnothing, \tag{5.5}$$

i.e., $\alpha \in R(D_1') \cap R(D_9')$. So, the inclusion $R(D_1') \cap R(D_2') \cap R(D_3') \subseteq R(D_1') \cap R(D_2')$ is hold.

Of the other hand, if $\alpha \in R(D_1') \cap R(D_9')$, then the conditions (5.4) and (5.5) are fulfilled, *i.e.*, $R(D_1') \cap R(D_9') \subseteq R(D_1') \cap R(D_6') \cap R(D_9')$. Therefore, the equality $R(D_1') \cap R(D_9') = R(D_1') \cap R(D_6') \cap R(D_9')$ is true.

The similar way we can show that the following equalities are hold:

$$R(D_2')\cap R(D_8')=R(D_2')\cap R(D_4')\cap R(D_8'),\ R(D_3')\cap R(D_6')=R(D_1')\cap R(D_3')\cap R(D_6'),$$
$$R(D_3')\cap R(D_9')=R(D_1')\cap R(D_3')\cap R(D_9')=R(D_1')\cap R(D_3')\cap R(D_6')\cap R(D_9')$$
$$=R(D_3')\cap R(D_6')\cap R(D_9'),$$
$$R(D_4')\cap R(D_5')=R(D_2')\cap R(D_4')\cap R(D_5'),\ R(D_4')\cap R(D_{10}')=R(D_4')\cap R(D_5')\cap R(D_{10}'),$$
$$R(D_5')\cap R(D_8')=R(D_2')\cap R(D_5')\cap R(D_8')=R(D_4')\cap R(D_5')\cap R(D_8')$$
$$=R(D_2')\cap R(D_4')\cap R(D_5')\cap R(D_8'),$$
$$R(D_6')\cap R(D_7')=R(D_3')\cap R(D_6')\cap R(D_7').$$

Now by equalities (5.2) and (5.3) we obtain the validity of Lemma 5.1.

The lemma is proved.

Lemma 5.2. Let $D'=\{\tilde{Z},Z,Z',Z\cup Z'\}$ and $D''=\{\tilde{Y},Y,Y',Y\cup Y'\}$ are arbitrary elements of the set $\{D_1',D_2',D_3',D_4',D_5',D_6',D_7',D_8',D_9',D_{10}'\}$, where $D'\neq D''$, $Z\supseteq Y$ and $Z'\supseteq Y'$. If quasinormal representation of binary relation α of the semigroup $B_X(D)$ has a form

$$\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right),$$

for some $T,T',T''\in D$, $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$, then $\alpha\in R(D')\cap R(D'')$ iff $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z',\ Y_{T'}^\alpha\cap Y\neq\varnothing,\ Y_{T''}^\alpha\cap Y'\neq\varnothing$.

Proof. If $\alpha\in R(D')\cap R(D'')$, then we have

$$\begin{aligned}&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z',\ Y_{T'}^\alpha\cap Z\neq\varnothing,\ Y_{T''}^\alpha\cap Z'\neq\varnothing;\\&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Y,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Y',\ Y_{T'}^\alpha\cap Y\neq\varnothing,\ Y_{T''}^\alpha\cap Y'\neq\varnothing.\end{aligned}\tag{5.6}$$

Of the last condition we have

$$Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z',\ Y_{T'}^\alpha\cap Y\neq\varnothing,\ Y_{T''}^\alpha\cap Y'\neq\varnothing,\tag{5.7}$$

since $Z\supseteq Y$ and $Z'\supseteq Y'$ by supposition.

Of the other hand, if the conditions of (5.7) are hold, then, also hold the conditions of (5.6) *i.e.* $\alpha\in R(D')\cap R(D'')$.

The lemma is proved.

Lemma 5.3. Let X be a finite set, $D'=\{\tilde{Z},Z,Z',Z\cup Z'\}$ and $D''=\{\tilde{Y},Y,Y',Y\cup Y'\}$ are arbitrary elements of the set $\{D_1',D_2',D_3',D_4',D_5',D_6',D_7',D_8',D_9',D_{10}'\}$, where $D'\neq D''$, $Z\supseteq Y$ and $Z'\supseteq Y'$. Then the following equalities are hold:

$$\left|R(D_1')\cap R(D_3')\right|=7\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_1')\cap R(D_6')\right|=7\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_2')\cap R(D_4')\right|=7\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_2')\cap R(D_5')\right|=7\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_3')\cap R(D_7')\right|=7\cdot 2^{|Z_5\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_2|}\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_4')\cap R(D_8')\right|=7\cdot 2^{|Z_1\setminus Z_2|}\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|Z_5\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_5')\cap R(D_{10}')\right|=7\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$
$$\left|R(D_6')\cap R(D_9')\right|=7\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}$$

Proof. Let $D'=\{\tilde{Z},Z,Z',Z\cup Z'\}$ and $D''=\{\tilde{Y},Y,Y',Y\cup Y'\}$ are arbitrary elements of the set

$\{D_1', D_2', D_3', D_4', D_5', D_6', D_7', D_8', D_9', D_{10}'\}$, where $D' \neq D''$, $Z \supseteq Y$ and $Z' \supseteq Y'$. If $\alpha \in R(D') \cap R(D'')$, then quasinormal representation of a binary relation α of semigroup $B_X(D)$ has a form

$$\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_{T'}^\alpha \times T'\right) \cup \left(Y_{T''}^\alpha \times T''\right) \cup \left(Y_{T' \cup T''}^\alpha \times (T' \cup T'')\right)$$

for some $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, then by statement 5) of the Theorem 2.1, we have

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z,\ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z',\ Y_{T'}^\alpha \cap Y \neq \varnothing,\ Y_{T''}^\alpha \cap Y' \neq \varnothing. \tag{5.8}$$

Let f_α is a mapping of the set X in the semilattice D satisfying the conditions $f_\alpha(t) = t\alpha$ for all $t \in X$. $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets $Z \cap Z'$, $Z \setminus Z'$, $Z' \setminus Z$, $X \setminus (Z \cup Z')$ respectively. It is clear, that the intersection disjoint elements of the set $\{Z \cap Z',\ Z \setminus Z',\ Z' \setminus Z,\ X \setminus (Z \cup Z')\}$ is empty set and $(Z \cap Z') \cup (Z \setminus Z') \cup (Z' \setminus Z) \cup (X \setminus (Z \cup Z')) = X$.

We are going to find properties of the maps $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$.

1) $t \in Z \cap Z'$. Then by the properties (5.8) we have $Z \cap Z' \subseteq \left(Y_T^\alpha \cup Y_{T'}^\alpha\right) \cap \left(Y_T^\alpha \cup Y_{T''}^\alpha\right) = Y_T^\alpha$, since $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z$ and $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z'$. *i.e.*, $t \in Y_T^\alpha$ and $t\alpha = T$ by definition of the set Y_T^α. Therefore $f_{0\alpha}(t) = T$ for all $t \in Z \cap Z'$.

2) $t \in Z \setminus Z'$. Then by the properties (5.8) we have $Z \setminus Z' \subseteq Z \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T'}^\alpha$ and $t\alpha \in \{T, T'\}$ by definition of the set Y_T^α and $Y_{T'}^\alpha$. Therefore $f_{1\alpha}(t) \in \{T, T'\}$ for all $t \in Z \setminus Z'$.

Preposition we have that $Y_{T'}^\alpha \cap Y \neq \varnothing$, *i.e.* $t'\alpha = T'$ for some $t' \in Y$. Then $t' \in Z$ sense $Y \subseteq Z$. If $t' \in Z'$, then $t' \in Z' \subseteq Y_T^\alpha \cup Y_{T''}^\alpha$. Therefore $t'\alpha \in \{T, T''\}$. That is contradiction of the equality $t'\alpha = T'$, while $T' \neq T$ and $T' \neq T''$ by definition of the semilattice *D*.

Therefore $f_{1\alpha}(t') = T'$ for some $t' \in Y \setminus Z'$.

3) $t \in Z' \setminus Z$. Then by the properties (5.8) we have $Z' \setminus Z \subseteq Z' \subseteq Y_T^\alpha \cup Y_{T''}^\alpha$, *i.e.*, $t \in Y_T^\alpha \cup Y_{T''}^\alpha$ and $t\alpha \in \{T, T''\}$ by definition of the set Y_T^α and $Y_{T''}^\alpha$. Therefore $f_{2\alpha}(t) \in \{T, T''\}$ for all $t \in Z' \setminus Z$.

Preposition we have that $Y_{T''}^\alpha \cap Y' \neq \varnothing$, *i.e.* $t''\alpha = T''$ for some $t'' \in Y'$. Then $t'' \in Z'$ sense $Y' \subseteq Z'$. If $t'' \in Z$ then $t'' \in Z \subseteq Y_T^\alpha \cup Y_{T'}^\alpha$. Therefore $t''\alpha \in \{T, T'\}$. That is contradiction of the equality $t''\alpha = T''$, while $T \neq T''$ and $T' \neq T''$ by definition of the semilattice *D*. Therefore $f_{2\alpha}(t'') = T''$ for some $t'' \in Y' \setminus Z$.

4) $t \in X \setminus (Z \cup Z')$. Then by definition quasinormal representation binary relation α and by property (5.8) we have $X \setminus (Z \cup Z') \subseteq X = Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \cup Y_{T' \cup T''}^\alpha$, *i.e.* $t\alpha \in \{T, T', T'', T' \cup T''\}$ by definition of the sets Y_T^α, $Y_{T'}^\alpha$, $Y_{T''}^\alpha$ and $Y_{T' \cup T''}^\alpha$. Therefore $f_{6\alpha}(t) \in \{T, T', T'', T' \cup T''\}$ for all $t \in X \setminus (Z \cup Z')$.

Therefore for every binary relation $\alpha \in R(D') \cap R(D'')$ exist ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$. It is obvious that for disjoint binary relations exist disjoint ordered systems.

Now let $f_0 : Z \cap Z' \to \{T\}$, $f_1 : Z \setminus Z' \to \{T, T'\}$, $f_2 : Z' \setminus Z \to \{T, T''\}$, $f_3 : X \setminus (Z \cup Z') \to \{T, T', T'', T' \cup T''\}$ are such mappings, which satisfying the conditions:

5) $f_0(t) = T$ for all $t \in Z \cap Z'$;

6) $f_1(t) \in \{T, T'\}$ for all $t \in Z \setminus Z'$ and $f_1(t_1') = T'$ for some $t_1' \in Y \setminus Z'$;

7) $f_2(t) \in \{T, T''\}$ for all $t \in Z' \setminus Z$ and $f_2(t_2') = T''$ for some $t_2' \in Y' \setminus Z$;

8) $f_{3\alpha}(t) \in \{T, T', T'', T' \cup T''\}$ for all $t \in X \setminus (Z \cup Z')$.

Now we define a map *f* of a set *X* in the semilattice *D*, which satisfies the condition:

$$f(t) = \begin{cases} f_0(t), & \text{if } t \in Z \cap Z', \\ f_1(t), & \text{if } t \in Z \setminus Z', \\ f_2(t), & \text{if } t \in Z' \setminus Z, \\ f_3(t), & \text{if } t \in X \setminus (Z \cup Z'). \end{cases}$$

Now let $\beta = \bigcup_{x \in X} (\{x\} \times f(x))$, $Y_T^\beta = \{t \mid t\beta = T\}$, $Y_{T'}^\beta = \{t \mid t\beta = T'\}$, $Y_{T''}^\beta = \{t \mid t\beta = T''\}$ and $Y_{T' \cup T''}^\beta = \{t \mid t\beta = T' \cup T''\}$. Then binary relation β can be representation by form

$$\beta = \left(Y_T^\beta \times T\right) \cup \left(Y_{T'}^\beta \times T'\right) \cup \left(Y_{T''}^\beta \times T''\right) \cup \left(Y_{T' \cup T''}^\beta \times (T' \cup T'')\right)$$

and satisfying the conditions:

$$Y_T^{\beta}\cup Y_{T'}^{\beta}\supseteq Z,\ Y_T^{\beta}\cup Y_{T''}^{\beta}\supseteq Z',\ Y_{T'}^{\beta}\cap Y\neq\varnothing,\ Y_{T''}^{\beta}\cap Y'\neq\varnothing.$$

(By suppose $f_1(t_1')=T'$ for some $t_1'\in Z\setminus Y'$ and $f_2(t_2')=T''$ for some $t_2'\in Y'\setminus Z$), *i.e.*, by Lemma 2.10 we have that $\beta\in R(D')\cap R(D'')$.

Therefore for every binary relation $\alpha\in R(D')\cap R(D'')$ and ordered system $(f_{0\alpha},f_{1\alpha},f_{2\alpha},f_{3\alpha})$ exist one to one mapping.

By ([1], Theorem 1.18.2) the number of the mappings $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ $(\alpha\in R(D')\cap R(D''))$ are respectively:

$$1,\ 2^{|Z\setminus(Y\cup Z')|}\cdot\left(2^{|Y\setminus Z'|}-1\right),\ 2^{|Z'\setminus(Y'\cup Z)|}\cdot\left(2^{|Y'\setminus Z|}-1\right),\ 4^{|X\setminus(Z\cup Z')|}.$$

Note that the number $2^{|Z\setminus(Y\cup Z')|}\cdot\left(2^{|Y\setminus Z'|}-1\right)\cdot 2^{|Z'\setminus(Y'\cup Z)|}\cdot\left(2^{|Y'\setminus Z|}-1\right)\cdot 4^{|X\setminus(Z\cup Z')|}$ does not depend on choice of elements $T,T',T''\in D$ of the semilattice D, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$ and $T''\setminus T'\neq\varnothing$. Since the number of such different elements of the semilattice D are equal to 7, the number of regular elements of the set $R(D')\cap R(D'')$ is equal to $\left|R(D')\cap R(D'')\right|=7\cdot 2^{|Z\setminus(Y\cup Z')|}\cdot\left(2^{|Y\setminus Z'|}-1\right)\cdot 2^{|Z'\setminus(Y'\cup Z)|}\cdot\left(2^{|Y'\setminus Z|}-1\right)\cdot 4^{|X\setminus(Z\cup Z')|}$.

The lemma is proved.

Lemma 5.4. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. If by $R^*(Q_5)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 5) of the Theorem 2.1, then

$$\begin{aligned}\left|R^*(Q_5)\right|&=14\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+14\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_1\setminus Z_5|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&+14\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_2\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}+14\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus Z_2|}\\&+14\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus Z_1|}\\&-7\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-7\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&-7\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_1\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-7\cdot 2^{|Z_1\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus\breve{D}|}\cdot\left(2^{|Z_2\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&-7\cdot 2^{|Z_5\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 2^{|Z_1\setminus Z_2|}\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-7\cdot 2^{|Z_1\setminus Z_2|}\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot 2^{|Z_5\setminus\breve{D}|}\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\\&-7\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 4^{|X\setminus\breve{D}|}-7\cdot 2^{|Z_2\setminus Z_1|}\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot 2^{|Z_3\setminus\breve{D}|}\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot 4^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. It is easy to see $\left|\Phi(Q_5,Q_5)\right|=2$ and $\left|\Omega(Q_5)\right|=7$, then by statement 5) of the Lemma 1.1, by Lemma 5.1 and by Lemma 5.3 we obtain the validity of Lemma 5.4.

The lemma is proved.

6) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 6) of the Theorem 2.1. In this case we have $Q_6=\{T,Z_4,Z,Z',\breve{D}\}$, where $T,Z,Z'\in D$, $T\subset Z_4\subset Z\subset\breve{D}$, $T\subset Z_4\subset Z'\subset\breve{D}$, $Z\setminus Z'\neq\varnothing$ and $Z'\setminus Z\neq\varnothing$. By definition of the semilattice D follows that

$$Q_6\theta_{XI}=\left\{\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}\right\}.$$

If $D_1'=\{Z_7,Z_4,Z_2,Z_1,\breve{D}\}$, $D_2'=\{Z_7,Z_4,Z_1,Z_2,\breve{D}\}$, $D_3'=\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}$, $D_4'=\{Z_6,Z_4,Z_1,Z_2,\breve{D}\}$, then from Theorem 1.1 we obtain

$$R^*(Q_6)=R(D_1')\cup R(D_2')\cup R(D_3')\cup R(D_4').\qquad(6.1)$$

Lemma 6.1. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. If by $R^*(Q_6)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 6) of the Theorem 2.1, then

$$\left|R^*(Q_6)\right| = 2\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}$$
$$+2\cdot\left(2^{|Z_4\setminus Z_7|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|}$$
$$+2\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 5^{|X\setminus \breve{D}|}$$
$$+2\cdot\left(2^{|Z_4\setminus Z_6|}-1\right)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot\left(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|}\right)\cdot 5^{|X\setminus \breve{D}|}.$$

Proof. First we show that the following equalities are hold:

$$\begin{aligned} &R(D_1')\cap R(D_2')=\varnothing,\ R(D_1')\cap R(D_3')=\varnothing,\ R(D_1')\cap R(D_4')=\varnothing,\\ &R(D_2')\cap R(D_3')=\varnothing,\ R(D_2')\cap R(D_4')=\varnothing, R(D_3')\cap R(D_4')=\varnothing. \end{aligned} \tag{6.2}$$

For this we consider the following case.

a) Let $\alpha\in R(D_1')\cap R(D_2')$. If a quasinormal representation of a regular binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_4^\alpha\times Z_4)\cup(Y_Z^\alpha\times Z)\cup(Y_{Z'}^\alpha\times Z')\cup(Y_0^\alpha\times\breve{D})$ for some $T,Z,Z'\in D$, $T\subset Z_4\subset Z\subset\breve{D}$, $T\subset Z_4\subset Z'\subset\breve{D}$, $Z\setminus Z'\neq\varnothing$, $Z'\setminus Z\neq\varnothing$ and $Y_T^\alpha,Y_Z^\alpha,Y_{Z'}^\alpha\notin\{\varnothing\}$. Then by statement 6) of the Theorem 2.1, we have

$$\begin{aligned} &Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z_2,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z_1,\\ &Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_{Z'}^\alpha\cap Z_2\neq\varnothing,\ Y_Z^\alpha\cap Z_1\neq\varnothing,\\ &Y_T^\alpha\supseteq Z_7,\ Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z_1,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z_2,\\ &Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_{Z'}^\alpha\cap Z_1\neq\varnothing,\ Y_Z^\alpha\cap Z_2\neq\varnothing, \end{aligned}$$

It follows that $Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z_2\cup Z_1=\breve{D}$ and $(Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha)\cap Y_Z^\alpha\supseteq\breve{D}\cap Y_Z^\alpha\supseteq Z_1\cap Y_Z^\alpha\neq\varnothing$. But the inequality $(Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha)\cap Y_Z^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1')\cap R(D_2')=\varnothing$ is hold.

The similar way we can show that the following equality is hold: $R(D_3')\cap R(D_4')=\varnothing.$

b) Let $\alpha\in R(D_1')\cap R(D_3')$ and a quasinormal representation of a regular binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_4^\alpha\times Z_4)\cup(Y_Z^\alpha\times Z)\cup(Y_{Z'}^\alpha\times Z')\cup(Y_0^\alpha\times\breve{D})$ for some $T,Z,Z'\in D$ and $T\subset Z_4\subset Z\subset\breve{D}$, $T\subset Z_4\subset Z'\subset\breve{D}$, $Z\setminus Z'\neq\varnothing$, $Z'\setminus Z\neq\varnothing$ and $Y_T^\alpha,Y_Z^\alpha,Y_{Z'}^\alpha\notin\{\varnothing\}$. Then by statement 6) of the Theorem 2.1, we have

$$\begin{aligned} &Y_T^\alpha\supseteq Z_7,\ Y_Z^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z_2,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z_1,\\ &Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_{Z'}^\alpha\cap Z_2\neq\varnothing,\ Y_Z^\alpha\cap Z_1\neq\varnothing,\\ &Y_T^\alpha\supseteq Z_6,\ Y_Z^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_{Z'}^\alpha\supseteq Z_2,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z_1,\\ &Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_{Z'}^\alpha\cap Z_2\neq\varnothing,\ Y_Z^\alpha\cap Z_1\neq\varnothing, \end{aligned}$$

It follows that $Y_T^\alpha\supseteq Z_7\cup Z_6=Z_4$ and $Y_T^\alpha\cap Y_4^\alpha\supseteq Z_4\cap Y_4^\alpha\neq\varnothing$. But the inequality $Y_T^\alpha\cap Y_4^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1')\cap R(D_3')=\varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1')\cap R(D_4')=\varnothing,\ R(D_2')\cap R(D_3')=\varnothing,\ R(D_2')\cap R(D_4')=\varnothing.$$

By equalities (6.1) and (6.2) follows that $\left|R^*(Q_6)\right|=\left|R(D_1')\right|+\left|R(D_2')\right|+\left|R(D_3')\right|+\left|R(D_4')\right|$.

It is easy to see $\left|\Phi(Q_6,Q_6)\right|=2$ and $\left|\Omega(Q_6)\right|=2$, then by statement 6) of the Lemma 1.1 we obtain validity of Lemma 6.1.

The lemma is proved.

7) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 7) of the Theorem 2.1. In this case we have $Q_7=\{T,T',T'',T'\cup T'',\breve{D}\}$, where $T,T',T''\in D$, $T\subset T'\subset\breve{D}$, $T\subset T''\subset\breve{D}$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$. By definition of the semilattice D follows that

$$Q_7\vartheta_{XI}=\{\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}\}.$$

If $D_1'=\{Z_7,Z_5,Z_4,Z_2,\breve{D}\}$, $D_2'=\{Z_7,Z_4,Z_5,Z_2,\breve{D}\}$, $D_3'=\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$, $D_4'=\{Z_6,Z_3,Z_4,Z_1,\breve{D}\}$, then from the Theorem 1.1 we obtain

$$R^*(Q_7)=R(D_1')\cup R(D_2')\cup R(D_3')\cup R(D_4'). \tag{7.1}$$

Lemma 7.1. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. If by $R^*(Q_7)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 7) of the Theorem 2.1, then

$$\begin{aligned}|R^*(Q_7)|&=2\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+2\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(2^{|Z_5\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+2\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}\\&+2\cdot\left(2^{|Z_3\setminus Z_4|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|}\right)\cdot 5^{|X\setminus\breve{D}|}.\end{aligned}$$

Proof. First we show that the following equalities are hold:

$$\begin{aligned}&R(D_1')\cap R(D_2')=\varnothing,\ R(D_1')\cap R(D_3')=\varnothing,\ R(D_1')\cap R(D_4')=\varnothing,\\&R(D_2')\cap R(D_3')=\varnothing,\ R(D_2')\cap R(D_4')=\varnothing,R(D_3')\cap R(D_4')=\varnothing.\end{aligned} \tag{7.2}$$

For this we consider the following case.

a) Let $\alpha\in R(D_1')\cap R(D_4')$. If a quasinormal representation of a regular binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))\cup(Y_0^\alpha\times\breve{D})$ for some $T,T',T''\in D$ and $T\subset T'\subset\breve{D}$, $T\subset T''\subset\breve{D}$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$. Then by statement 7) of the theorem 2.1, we have

$$\begin{aligned}&Y_T^\alpha\supseteq Z_5\cap Z_4\supseteq Z_7,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z_5,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_4,\ Y_{T''}^\alpha\cap Z_5\neq\varnothing,\ Y_{T'}^\alpha\cap Z_4\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing.\\&Y_T^\alpha\supseteq Z_4\cap Z_3\supseteq Z_6,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z_3,\ Y_Z^\alpha\cup Y_{T'}^\alpha\supseteq Z_4,\ Y_{T''}^\alpha\cap Z_3\neq\varnothing,\ Y_{T'}^\alpha\cap Z_4\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing\end{aligned}$$

It follows that $Y_T^\alpha\supseteq Z_7\cup Z_6=Z_4$ and $Y_T^\alpha\cap Y_{T'}^\alpha\supseteq Z_4\cap Y_{T'}^\alpha\neq\varnothing$. But the inequality $Y_T^\alpha\cap Y_{T'}^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1')\cap R(D_4')=\varnothing$ is hold.

The similar way we can show that the following equality is hold: $R(D_2')\cap R(D_3')=\varnothing$.

b) Let $\alpha\in R(D_1')\cap R(D_2')$. If a quasinormal representation of a regular binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T'\cup T''}^\alpha\times(T'\cup T''))\cup(Y_0^\alpha\times\breve{D})$ for some $T,T',T''\in D$ and $T\subset T'\subset\breve{D}$, $T\subset T''\subset\breve{D}$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha\notin\{\varnothing\}$. Then by statement 7) of the theorem 2.1, we have

$$\begin{aligned}&Y_T^\alpha\supseteq Z_5\cap Z_4,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z_5,\ Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_4,\ Y_{T''}^\alpha\cap Z_5\neq\varnothing,\ Y_{T'}^\alpha\cap Z_4\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing.\\&Y_T^\alpha\supseteq Z_4\cap Z_5,\ Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z_4,\ Y_Z^\alpha\cup Y_{T'}^\alpha\supseteq Z_5,\ Y_{T''}^\alpha\cap Z_4\neq\varnothing,\ Y_{T'}^\alpha\cap Z_5\neq\varnothing,\ Y_0^\alpha\cap\breve{D}\neq\varnothing\end{aligned}$$

It follows that $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq Z_5\cup Z_4=Z_2$ and $(Y_T^\alpha\cup Y_{T''}^\alpha)\cap Y_{T'}^\alpha\supseteq Z_2\cap Y_{T'}^\alpha\supseteq Z_5\cap Y_{T'}^\alpha\neq\varnothing$. But the inequality $(Y_T^\alpha\cup Y_{T''}^\alpha)\cap Y_{T'}^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1')\cap R(D_2')=\varnothing$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D_1')\cap R(D_3')=\varnothing,\ R(D_2')\cap R(D_4')=\varnothing,\ R(D_3')\cap R(D_4')=\varnothing.$$

By equalities (7.1) and (7.2) follows that $\left|R^*(Q_7)\right|=\left|R(D_1')\right|+\left|R(D_2')\right|+\left|R(D_3')\right|+\left|R(D_4')\right|$.

It is easy to see $\left|\Phi(Q_7,Q_7)\right|=2$ and $\left|\Omega(Q_7)\right|=2$ then by statement 7) of the Lemma 1.1 we obtain validity of Lemma 7.1.

The lemma is proved.

8) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 8) of the Theorem 2.1. In this case we have $Q_8=\{T,T',Z_4,Z_4\cup T',Z,\breve{D}\}$. By definition of the semilattice D follows that

$$Q_8\vartheta_{XI}=\left\{\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\right\}.$$

If $D_1'=\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$, $D_2'=\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, then from Theorem 1.1 we obtain

$$R^*(Q_8)=R(D_1')\cup R(D_2') \tag{8.1}$$

Lemma 8.1. Let X be a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. If by $R^*(Q_8)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 8) of the Theorem 2.1, then

$$\begin{aligned}R^*(Q_8)&=2\cdot\left(2^{|Z_3\setminus Z_2|}-1\right)\cdot\left(2^{|Z_4\setminus Z_3|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\\&+2\cdot\left(2^{|Z_5\setminus Z_1|}-1\right)\cdot\left(2^{|Z_4\setminus Z_5|}-1\right)\cdot\left(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|}\right)\cdot 6^{|X\setminus\breve{D}|}\end{aligned}$$

Proof. First we show that the following equalities are hold:

$$R(D_1')\cap R(D_2')=\varnothing. \tag{8.2}$$

Let $\alpha\in R(D_1')\cap R(D_2')$. If a quasinormal representation of a regular binary relation α has the form

$$\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_{T'\cup Z_4}^\alpha\times(T'\cup Z_4)\right)\cup\left(Y_Z^\alpha\times Z\right)\cup\left(Y_0^\alpha\times\breve{D}\right),$$

where $T\in\{Z_7,Z_6\}$, $T'\in\{Z_5,Z_3\}$, $Z_4\cup T'$, $Z\in\{Z_2,Z_1\}$, $Z_4\cup T'\neq Z$, $T\subset T'$, $T'\setminus Z_4\neq\varnothing$, $Z_4\setminus T'\neq\varnothing$, $(Z_4\cup T')\setminus Z\neq\varnothing$, $Z\setminus(Z_4\cup T')\neq\varnothing$ and $Y_T^\alpha,Y_{T'}^\alpha,Y_4^\alpha,Y_Z^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then by statement 8) of the Theorem 2.1, we have

$$\begin{aligned}&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_5,\ Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_Z^\alpha\supseteq Z_2,\\&Y_{T'}^\alpha\cap Z_5\neq\varnothing,\ Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_Z^\alpha\cap Z_2\neq\varnothing.\\&Y_T^\alpha\cup Y_{T'}^\alpha\supseteq Z_3,\ Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4,\ Y_T^\alpha\cup Y_4^\alpha\cup Y_{T'\cup Z_4}^\alpha\supseteq Z_1,\\&Y_{T'}^\alpha\cap Z_3\neq\varnothing,\ Y_4^\alpha\cap Z_4\neq\varnothing,\ Y_{T'\cup Z_4}^\alpha\cap Z_1\neq\varnothing.\end{aligned}$$

It follows that $Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$ and $\left(Y_T^\alpha\cap Y_4^\alpha\right)\cap Y_Z^\alpha\supseteq Z_4\cap Y_{T'}^\alpha\supseteq Z_2\cap Y_{T'}^\alpha\neq\varnothing$. But the inequality $\left(Y_T^\alpha\cap Y_4^\alpha\right)\cap Y_Z^\alpha\neq\varnothing$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D_1')\cap R(D_2')=\varnothing$ is hold.

By equalities (8.1) and (8.2) follows that $\left|R^*(Q_8)\right|=\left|R(D_1')\right|+\left|R(D_2')\right|$.

It is easy to see $\left|\Phi(Q_8,Q_8)\right|=1$ and $\left|\Omega(Q_8)\right|=1$, then by statement 8) of the Lemma 1.1 we obtain validity of Lemma 8.1.

The lemma is proved.

Let X be a finite set and $Z_7\cap Z_6\neq\varnothing$ and us assume that

$$r_1=\left|R^*(Q_1)\right|+\left|R^*(Q_2)\right|+\left|R^*(Q_3)\right|+\left|R^*(Q_4)\right|+\left|R^*(Q_5)\right|+\left|R^*(Q_6)\right|+\left|R^*(Q_7)\right|+\left|R^*(Q_8)\right|.$$

Theorem 2.2. Let X is a finite set, $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. If R_D is a set of all regular elements of the semigroup $B_X(D)$, then $\left|R_D\right|=r_1$.

Proof. This Theorem immediately follows from the Theorem 2.1.

The theorem is proved.

I was seen in ([6], Theorem 2) that if α and β are regular elements of $B_X(D)$ then $V(D,\alpha\circ\beta)$ is an *XI*-subsemilattice of *D*. Therefore $\alpha\circ\beta$ is regular element of $B_X(D)$.

Theorem 2.3. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_2(X,8)$ and $Z_6\cap Z_7\neq\varnothing$. The set of all regular elements is a subsemigroup of the semigroup $B_X(D)$ which is defined by semilattices of the class $\Sigma_2(X,8)$.

Proof. This Theorem immediately follows from the Theorem 2 in [6].

The theorem is proved.

References

[1] Diasamidze, Ya. and Makharadze, Sh. (2013) Complete Semigroups of Binary Relations. Monograph, Kriter, Turkey, 1-520 p.

[2] Diasamidze, Ya. and Makharadze, Sh. (2010) Complete Semigroups of Binary Relations. Monograph. M., Sputnik+, 657 p. (In Russian)

[3] Diasamidze, Ya. (2009) The Properties of Right Units of Semigroups Belonging to Some Classes of Complete Semigroups of Binary Relations. *Proceedings of A. Razmadze Mathematical Institute*, **150**, 51-70.

[4] Tsinaridze, N. and Makharadze, Sh. (2015) Regular Elements of the Complete Semigroups $B_X(D)$ of Binary Relations of the Class $\Sigma_2(X,8)$. *Applied Mathematics*, **6**, 447-455.

[5] Diasamidze, Ya., Makharadze, Sh. and Diasamidze, Il. (2008) Idempotents and Regular Elements of Complete Semigroups of Binary Relations. *Journal of Mathematical Sciences*, **153**, 481-499.
http://dx.doi.org/10.1007/s10958-008-9132-1

[6] Diasamidze, Ya. and Bakuridze, Al. (2015) On Some Properties of Regular Elements of Complete Semigroups Defined by Semilaices of the Class $\Sigma_4(X,8)$. *International Journal of Engineering Science and Innovative Technology* (*IJESIT*), **4**, 8-15.