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Regular fractional iteration of convex functions

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Abstract. The existence of a unique C^1 solution φ of equation (1) is proved under the condition that $f\colon I \to I$ is convex or concave and of class C^1 in I, $0 < f(x) < x$ in I^*, and $f'(x) > 0$ in I. Here $I = [0, a]$ or $[0, a)$, $0 < a \leqslant \infty$, and $I^* = I \setminus \{0\}$.

1. Let $I = [0, a]$ or $[0, a)$, $0 < a \leqslant \infty$, be a real interval, and write $I^* = I \setminus \{0\}$. Further, let $f\colon I \to I$ be a function of class C^1 in I such that $0 < f(x) < x$ in I^* and $f'(x) > 0$ in I. Put $s = f'(0) \in (0, 1]$.

Several authors (cf. [1], [3], [10], [11], [13], [14], [15]) have studied the C^1 solutions $\varphi\colon I \to I$ of the functional equation

$$\varphi^N(x) = f(x), \tag{1}$$

where $N \geqslant 2$ is a positive integer, and φ^N denotes the N-th iterate of the function φ. The solutions of (1) may be regarded as iterates of the fractional order $1/N$ of the function f.

In [10] one can find the first indication that the C^1 solution of equation (1) might be unique. The uniqueness was then proved, under various additional hypotheses, in [1], [13], [14] and [15]. (Cf. also [3], where the author applies equation (1) to a problem in astronomy.) On the other hand, as has been shown in [11], in the case where $s = 0$, C^1 solutions of equation (1) are, in general, not unique.

The existence of a unique C^1 solution to (1) has been proved in [13] under the additional hypothesis that the function f fulfils the condition

$$f'(x) = s + O(x^\delta), \quad x \to 0^+, \tag{2}$$

if $s \in (0, 1)$, resp.

$$f'(x) = 1 - b(m+1)x^m + O(x^{m+\delta}), \quad x \to 0^+, \tag{3}$$

if $s = 1$, where m, b and δ are positive constants. In [15] M. C. Zdun proved that the unique C^1 solution of (1) exists whenever the function f is convex or concave and $s \in (0, 1)$. However, his proof cannot be adopted to the case where $s = 1$.

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Let us note that the convexity condition can be fulfilled although condition (2) resp. (3) is not. This may be seen from the example of the function

$$f(x) = s\int_0^x \left(1+\frac{1}{\log t}\right)dt \tag{4}$$

(cf. [6], [9]), which fulfils all the conditions imposed on f at the beginning of this section provided $a < e^{-1}$, is concave, but does not fulfil (2) resp. (3). Therefore the result of Zdun is of a considerable interest.

In the present paper we give another, simpler proof of Zdun's result, which can be also applied to the case $s = 1$, not covered in [15].

2. In the present section we assume that the function f has the following properties.

(H) $f\colon I \to I$ is of class C^1 and convex or concave in I, $0 < f(x) < x$ in I^*, and $f'(x) > 0$ in I.

Fix an $x_0 \in I^*$. If $s \in (0, 1)$, then for every $x \in I$ there exists the limit

$$\sigma(x) = \lim_{n\to\infty} f^n(x)/f^n(x_0), \tag{5}$$

and the function $\sigma\colon I \to \boldsymbol{R}$ is convex or concave (just like f), and satisfies the Schröder equation

$$\sigma[f(x)] = s\sigma(x) \tag{6}$$

for all $x \in I$ (cf. [7] and [9], Theorem 6.8). Similarly, if $s = 1$, then for every $x \in I^*$ there exists the limit

$$a(x) = \lim_{n\to\infty} \frac{f^n(x)-f^n(x_0)}{f^{n+1}(x_0)-f^n(x_0)}, \tag{7}$$

and the function $a\colon I^* \to \boldsymbol{R}$ is convex and satisfies the Abel equation

$$a[f(x)] = a(x)+1 \tag{8}$$

for all $x \in I^*$ (cf. [8] and [9], Theorem 7.5).

The relevant properties of the functions σ and a are described in the following

LEMMA. *Let the function f fulfil hypothesis* (H). *If $s \in (0, 1)$, then the function σ given by* (5) *is of class C^1 in I^*, and $\sigma'(x) > 0$ in I^*. If $s = 1$, then the function a given by* (7) *is of class C^1 in I^*, and $a'(x) < 0$ in I^*.*

Proof. If $s \in (0, 1)$, then the function σ exists and is convex or concave in I. Thus at every point $x \in (0, a)$ there exist the right derivative $\sigma'_+(x)$ and the left derivative $\sigma'_-(x)$, and the functions $\sigma'_+\colon (0, a) \to \boldsymbol{R}$

and $\sigma'_-: (0, a) \to \mathbf{R}$ are monotonic and both satisfy the functional equation

$$\sigma'[f(x)] = \frac{s}{f'(x)} \sigma'(x) \tag{9}$$

in $(0, a)$. Moreover, σ'_+ and σ'_- may differ at most at denumerably many points.

Suppose that $\sigma'_+(x^*) = 0$ for an $x^* \in (0, a)$. Then, by (9), $\sigma'_+[f(x^*)] = 0$, and by induction $\sigma'_+[f^n(x^*)] = 0$ for $n = 1, 2, \ldots$ Since the sequence $f^n(x^*)$ decreases to zero ([9], Theorem 0.4) and the function σ'_+ is monotonic, this implies that σ'_+ vanishes and σ is constant in $(0, x^*)$. But this is impossible in view of (6). Consequently,

$$\sigma'_+(x) \neq 0 \quad \text{for } x \in (0, a). \tag{10}$$

Since $\lim_{x\to 0+} s/f'(x) = 1$, monotonic solutions of equation (9) are determined uniquely up to a multiplicative constant ([2] and [9], Theorem 5.4; cf. also [5], [6]). Consequently, there exists a constant k such that

$$\sigma'_-(x) = k\sigma'_+(x) \quad \text{for } x \in (0, a).$$

However, σ'_+ and σ'_- coincide at infinitely many points. Thus $k = 1$ and $\sigma'_+(x) = \sigma'_-(x) = \sigma'(x)$ for all $x \in (0, a)$, which implies that σ' is continuous in $(0, a)$. If $a \in I$, then $f(a) < a$ in virtue of (H). Hence, by (9),

$$\lim_{x\to a-} \sigma'(x) = \lim_{x\to a-} \frac{f'(x)}{s} \sigma'[f(x)] = \frac{f'(a)}{s} \sigma'[f(a)]$$

exists, is finite and different from zero. Since the function σ is continuous at $f(a) \in (0, a)$, and f is continuous at a, it follows from equation (6) that σ is continuous at a. Consequently σ' exists, is continuous and different from zero in I^*.

Since, for every n, the function f^n is increasing and the constant $f^n(x_0)$ is positive, the function σ is non-decreasing in I. Thus, by (10), $\sigma'(x) > 0$ in I^*.

If $s = 1$, then the function f must be concave. The function α exists and is convex in I^*. The right derivative $\alpha'_+: (0, a) \to \mathbf{R}$ and the left derivative $\alpha'_-: (0, a) \to \mathbf{R}$ exist, are non-decreasing, and satisfy the equation

$$\alpha'[f(x)] = \frac{1}{f'(x)} \alpha'(x). \tag{11}$$

Moreover, similarly as in the case of σ, $\alpha'_+(x) \neq 0$ for $x \in (0, a)$. We have $\alpha'_-(x) = k\alpha'_+(x)$ for $x \in (0, a)$, since α'_+ and α'_- are monotonic solutions of (11), which implies that $\alpha'_+(x) = \alpha'_-(x) = \alpha'(x)$ for all $x \in (0, a)$. If $a \in I$, then

$$\lim_{x\to a-} \alpha'(x) = \lim_{x\to a-} f'(x)\alpha'[f(x)] = f'(a)\alpha'[f(a)]$$

exists, is finite and different from zero. Consequently α is of class C^1 in I^* and $\alpha'(x) \neq 0$ in I^*.

Since, for every n, the function f^n is increasing and the constant $f^{n+1}(x_0)-f^n(x_0)$ is negative, the function α is non-increasing in I^*. Thus $\alpha'(x) < 0$ in I^*.

The result of the present paper is contained in the following

THEOREM. *Let the function f fulfil hypothesis* (H). *Then equation* (1) *has a unique C^1 solution φ: $I \to I$. This solution is given by*

$$\varphi(x) = \sigma^{-1}\big(s^{1/N}\sigma(x)\big) \tag{12}$$

if $s \in (0,1)$, or by

$$\varphi(x) = \begin{cases} \alpha^{-1}\left(\alpha(x)+\dfrac{1}{N}\right) & \textit{for } x \in I^*, \\ 0 & \textit{for } x = 0, \end{cases} \tag{13}$$

if $s = 1$, where the functions σ and α are given by (5) *and* (7), *respectively.*

Proof. It is easily seen that the function φ is well defined in I by (12) or (13), satisfies equation (1) in I, and is of class C^1 in I^*. For the proof of the existence it remains to show that

$$\lim_{x\to 0^+} \varphi'(x) = s^{1/N}, \tag{14}$$

and that φ is continuous at $x = 0$.

Let $s \in (0,1)$. The function σ' is monotonic, say, non-decreasing (if σ' is non-increasing, the proof is analogous). By (12) we have

$$f(x) \leqslant \varphi(x) \leqslant x, \tag{15}$$

whence

$$\sigma'[f(x)] \leqslant \sigma'[\varphi(x)] \leqslant \sigma'(x) \tag{16}$$

for $x \in I^*$. Again by (12) we have $\varphi'(x) = s^{1/N}\sigma'(x)/\sigma'[\varphi(x)]$, whence by (16) and (9)

$$s^{1/N} \leqslant \varphi'(x) \leqslant \frac{s^{1/N}f'(x)}{s}$$

for $x \in I^*$, and (14) follows. Relation (15) implies that

$$\lim_{x\to 0^+} \varphi(x) = 0, \tag{17}$$

whereas by (5) $\sigma(0) = 0$, whence we get in view of (12) $\varphi(0) = 0$. Thus φ is continuous at zero.

Now let $s = 1$. Then the function α' is non-decreasing and (13) implies (15) for $x \in I^*$, whence

$$\alpha'[f(x)] \leqslant \alpha'[\varphi(x)] \leqslant \alpha'(x) \tag{18}$$

for $x \in I^*$. Since, by (13), $\varphi'(x) = a'(x)/a'[\varphi(x)]$, (18) and (11) imply that

$$f'(x) \leqslant \varphi'(x) \leqslant 1$$

for $x \in I^*$, and (14) follows. Again (15) implies (17), whence the continuity of φ at zero results in view of (13).

The proof of uniqueness is based on the ideas developed in [12] and does not differ from that given by Zdun [15] in the case $s \in (0, 1)$. Therefore we are going to prove here the uniqueness of φ only in the case $s = 1$.

It has been proved in [10] that if $\varphi\colon I \to I$ is a C^1 solution of equation (1), then it must satisfy the differential equation

$$\varphi' = G(x, \varphi), \tag{19}$$

where

$$G(x, y) = \prod_{n=0}^{\infty} \frac{f'[f^n(x)]}{f'[f^n(y)]}.$$

On the other hand, it has been proved in [2] (cf. also [9], Theorem 5.4) that if $a'\colon I^* \to \mathbf{R}$ is a monotonic solution of equation (11), then

$$a'(x) = c \prod_{n=0}^{\infty} \frac{f'[f^n(x)]}{f'[f^n(x_0)]}$$

with a suitable real constant c. Hence we get $G(x, y) = a'(x)/a'(y)$ for $x, y \in I^*$, and equation (19) becomes

$$a'[\varphi(x)]\varphi'(x) = a'(x) \tag{20}$$

for $x \in I^*$. Equation (20) can be easily integrated, and yields

$$a[\varphi(x)] = a(x) + C,$$

whence

$$\varphi(x) = a^{-1}[a(x) + C] \tag{21}$$

for $x \in I^*$. Inserting (21) into equation (1) we get $C = 1/N$. Consequently,

$$f(x) < \varphi(x) < x \quad \text{for } x \in I^*,$$

and $\varphi(0) = 0$ follows by the continuity of φ. Thus necessarily φ must be given by formula (13), which proves the uniqueness and completes the proof of the theorem.

Actually, in the above theorem it is enough to assume that the function f is convex or concave only in a neighbourhood $[0, b) \subset I$ of $x = 0$. Then the above argument yields the existence of a unique C^1 solution $\varphi\colon [0, b) \to [0, b)$ of equation (1), and this solution can be uniquely extended onto the whole interval I to a C^1 solution $\varphi\colon I \to I$ of (1) (cf. [10]).

It is worthwhile to note that the solution given in the above theorem need not be convex or concave (cf. [4], [10]); but if equation (1) has a convex or concave solution $\varphi\colon I \to I$, then the latter is unique and is identical with the C^1 solution given by formula (12) or (13) (cf. [12]). Recently Zdun [16] has proved that if the function f (fulfilling hypothesis (H)) is concave and so is its derivative f', then the (unique) C^1 solution φ of (1) is also concave.

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