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REGULAR FUNCTIONS OF COMPLEX QUATERNIONIC VARIABLE

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1. INTRODUCTION

The development of the quaternionic analysis started only recently and – in comparison with the complex analysis – the theory is certainly still underdeveloped with its best years to come. It took a long time to find a suitable generalization of the $C - R$ equation to have a nice, distinguished class of "regular functions". The school of Fueter [2, 3, 4] in the thirties made the first and substantial step in the building of quaternionic analysis. Its up-to-date summary can be found in [6]. Regular functions – as mappings of R^4 into R^4 – are real-analytic, hence it is certainly worth while to complexify the situation and to use the full power of complex analysis for the investigation of the properties of the class of regular functions. At the same time, through this complexification, the connection with the equations of mathematical physics on the complex Minkowski space can be established (see [7, 5]).

In the present paper the theory of analytic spaces (as described in [1]) is used for the investigation of properties of the holomorphic extension of the Fueter equation. The main result proved in the paper is the fact that the zero sets of (complex) Fueter functions are "null surfaces" (in the sense used e.g. in general relativity), i.e. (roughly speaking), the "gradients" at the points of the surface lie in the null cone N.

A definition of Fueter-meromorphic functions is suggested and it is shown that the poles of such functions have similar properties as zero sets of Fueter functions.

As to the contents of this paper, after introducing the basic notation, a short summary of the results is given at the end of Section 1. Section 2 contains a review of the necessary facts from the theory of analytic spaces. The properties of orthogonal cones, on which the proofs are based, are described in Section 3. Section 4 contains the main theorem on zero sets of regular functions, while in Section 5 similar facts are proved for poles of meromorphic functions.

Notation. We denote the four-dimensional complex associative algebra of complex quaternions by CH, its identity by $i_0 = 1$, and we regard C as a subset of CH by identifying $c \in C$ with $ci_0 \in CH$. Then we have a direct sum decomposition $CH = C \oplus \oplus P$, where P is the oriented three-dimensional complex vector space having the quaternionic units i_1, i_2, i_3, as a basis. With the usual notation for three-dimensional vectors, the product of two elements of CH is given by

$$vw = (v_0, \mathbf{v})(w_0, \mathbf{w}) = (v_0 w_0 - \mathbf{v} \cdot \mathbf{w},\ v_0 \mathbf{w} + w_0 \mathbf{v} + \mathbf{v} \times \mathbf{w}).$$

A complex quaternion can be written as $v = \sum_{\alpha=0}^{3} v_\alpha i_\alpha$, $v_\alpha \in C$ and we define:

$$v^+ = v_0 - i_1 v_1 - i_2 v_2 - i_3 v_3 ,$$
$$\bar{v} = \bar{v}_0 + i_1 \bar{v}_1 + i_2 \bar{v}_2 + i_3 \bar{v}_3 ,$$
$$N(v) = vv^+ = \sum_\alpha v_\alpha^2 ,$$

$N = \{v \in CH;$ there exists $w \in CH$ such that $vw = 0\}$. Then $v^{-1} = v^+/(vv^+)$ for $v \in CH$, $N(v) \neq 0$ and $N = \{v \in CH;\ N(v) = 0\}$. A complex quaternionic function can be written as $f = \sum_\beta f_\beta i_\beta$ where $f_\beta : CH \to C$ are complex valued functions. Define

$$\frac{\partial}{\partial z_\alpha} f = \sum_\beta i_\beta \frac{\partial}{\partial z_\alpha} f_\beta .$$

Definition 1.1. Let B be an open subset of CH. Let $D = \sum_\alpha i_\alpha (\partial/\partial z_\alpha)$. Then a function $f: B \to CH$ is *left regular* (or *right regular*) in B iff:

1. f is holomorphic,
2. $Df \equiv 0$ (or $fD \equiv 0$) in B.

The set of left regular functions in B will be denoted by $F^-(B)$. The set of germs of left regular functions at a point $q \in CH$ will be denoted by F_q^-.

We are going to summarize now the basic results of this paper. Let $V \subset C^n$ be an analytic cone (see Definition 2.6) and V^- its regular points. In Section 3 we define an orthogonal cone $V^\perp = \operatorname{clos} \bigcup_{p \notin V^-} C(V, p)^\perp$, where $C(V, p)$ is the tangent space to V at the point $p \in V^-$ and $C(V, p)^\perp = \{q \in C^n,\ \sum d_i \bar{q}_i = 0$ for all $d \in C(V, p)\}$. We shall show that $V^\perp$ is an analytic cone and $(V^\perp)^\perp = V$.

Definition 1.2. The cone V is N-orthogonal (or N-ω-orthogonal) iff V is an analytic cone and $V = Q^\perp$ (or $V = Q_\omega^\perp$, see Definition 3.1) for some analytic cone $Q \subset N = \{v \in CH,\ \sum_\alpha v_\alpha^2 = 0\}$.

Definition 1.3. Let B be an open set in CH. Then $V \subset B$ is an F^- *zero set in* B iff for all $b \in B$ there exist $U(b) \subset B$, $f \in F^-(U(b))$ such that $V \cap U(b) = f^{-1}(0)$.

An F^- zero set in B is clearly an analytic set in B. Let $V = \bigcup_{\alpha=0}^{3} V_\alpha$ be the splitting of V by dimension (see Section 2). In Section 4 we show that $C(V_3, q)$ is N-orthogonal and N-ω-orthogonal for all $q \in V$.

Definition 1.4. Let $B \subset CH$ be an open set in CH and $H \subset B$ an open and dense subset of B. The couple (H, f) is F^-*meromorphic function in B* iff:

(i) For all $p \in B$ there exist $U(p)$ and $\Phi, \Psi: U(p) \to CH$, Φ, Ψ holomorphic, $\Psi(U(p)) \not\subset N$ such that $f(x) = \Phi(x)/\Psi(x)$ for all $x \in U(p) \cap H$, $\Psi(x) \notin N$;

(ii) f is a left regular function on H (regularity condition).

In Section 5 we investigate the pole set P_f of an F^- meromorphic functions. We show that $C(P_f, q)$ is an N-orthogonal and N-ω-orthogonal analytic cone for all $p \in P_f$ except some "bad points".

2. PRELIMINARIES

In this section we recall, for the convenience of the reader, some basic facts from the theory of analytic sets.

A subset V of C^n is *analytic near p*, iff there is a neighbourhood U of p and holomorphic functions $f_1, \ldots, f_s$ in U such that $V \cap U$ is the zero set of these functions. A subset $V \subset C^n$ is *locally analytic*, iff it is analytic near each of its points. Let H be open in C^n and suppose $V \subset C^n$. Then *V is analytic in H* iff V is analytic near each point of H. A point p of an analytic set V is a *simple (regular) point of V* iff V is an analytic manifold near p. Otherwise p is a *singular point of V*. Let V^- denote the set of simple points of V and $V^\times$ the set of singular points of V. A subset W of a locally analytic set V is an *analytic subset of V* iff W is analytic near each point of V.

Let f be a germ of a holomorphic function at a point $p \in C^n$. Then $f = \sum_I a_I(z - p)^I$, $I = (i_1, \ldots, i_n) \in N^n$ near p.

Definition 2.1. The *initial polynomial* of f at p is the polynomial $f_p^* = \sum_{|I|=m} a_I(z - p)^I$ where $m = \min_I \{|I|, a_I \neq 0\}$ and $|I| = \sum_{j=1}^n i_j$.

Definition 2.2. Let V be locally analytic in C^n and $p \in V$. The *tangent cone of V at p* is the set $C(V, p) = \{v \in C^n; \exists p_i \in V, \exists a_i \in C, p_i \to p, a_i(p_i - p) \to v\}$.

If $p \in V^-$ (p is a regular point), then this definition is equivalent to the usual definition of the tangent space in terms of differentiable curves [1, 7-3C].

Theorem 2.3 [1, 7-4D]. *Denote by $I(V, p)$ the set of germs of holomorphic functions at p which vanish on V. Then $C(V, p)$ is the zero set of the set of all initial polynomials f_p^* of the germs $f \in I(V, p)$.*

Theorem 2.4 [1, 7-4A]. *If f is holomorphic in C^n near p and has Z for its zero set, then $C(Z, p)$ is the zero set of f_p^*.*

The *dimension of V* (dim V) is the largest dimension of any simple point. V is said to have a *constant dimension r* ($\dim V \equiv r$) iff V^- is an analytic manifold of dimension r. A nonvoid locally analytic set $V \subset C^n$ is *reducible* iff there are analytic subsets V_1 and V_2 if V such that $V = V_1 \cup V_2$, $V \neq V_1$, $V \neq V_2$. Otherwise, V is *irreducible*. An analytic subset W of a locally analytic set V is an *irreducible component of* V iff it is a maximal irreducible analytic subset.

Theorem 2.5 [1, 3-1G]. *Let V be a locally analytic set in C^n, let $M_1, M_2, \ldots$ be the connected pieces of V^- (the number of them is clearly finite or denumerable), and set $\overline{M}_i = V$-clos M_i. Then $V = \bigcup_i \overline{M}_i$. Further:*

(a) *Each $\overline{M}_i$ is an irreducible component of V;* $\dim \overline{M}_i = \dim M_i$.

(b) *The $\overline{M}_i$'s form a locally finite set of sets in V.*

(c) *If the numbers $\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots$ are all distinct then $V' = \bigcup_i \overline{M}_{\lambda_i}$, $V'' = \bigcup_i \overline{M}_{\mu_i}$ are analytic subsets of V and* $\dim (V' \cap W'') < \dim V'$.

The *splitting of a locally analytic set V by dimension* is the decomposition $V = \bigcup_{j=1}^{r} V_j$ where $r = \dim V$ and $V_j = \bigcup_{\dim M_i = j} \overline{M}_i$.

Definition 2.6. A set $V \subset C^n$ is an *analytic cone* iff it is analytic in C^n and is the union of a set of lines through 0.

If $V \subset C^n$ is an analytic cone then V is the zero set of a finite set of homogeneous polynomials in C^n [1, 5-9E]. The tangent cone is an example of an analytic cone.

Lemma 2.6. *Let $V \subset C^n$ be a cone and let V be analytic in a neighbourhood of* 0, *then V is an analytic cone.*

Proof. Define $F_c \colon C^n \to C^n$, $z \mapsto cz$, $c \in C$. F_c is the biholomorphism and maps V onto V. Let $V \cap U(0)$ be an analytic subset of $U(0)$ and consider a point $p \in C^n$. Then we can find such a number $c \in C$ that $F_c(p) \in U(0)$. Hence V is analytic in $F_c^{-1}(U(0))$, $p \in F_c^{-1}(U(0))$ and V is analytic in p.

Theorem 2.7 [1, 3-7A]. *Let $V \subset C^n$ be a locally analytic set, and suppose* $\dim V \equiv$ $\equiv p$. *Then for each $p \in V$ there is a neighbourhood U of p and holomorphic vector functions $w_1, \ldots, w_m$ in U such that*

(a) *$w_i(x) = 0$ for all $i = 1, \ldots, m$ if $x \in V^\times \cap U$,*

(b) *if $x \in V^- \cap U$, then $w_i(x) \in C(V, x)$ for all i, and the functions* $w_i(x)$ span $C(V, x)$.

The following theorem is an extension of the Remmert proper mapping theorem.

Definition 2.8 [1, 7-11A]. Let X abd Y be locally compact spaces, and let $f : X \to Y$ be continuous. Then f is *semiproper* into Y iff for each compact set $Q \subset Y$ there is a compact (perhaps void) set $K \subset X$ such that $f(K) = f(X) \cap Q$.

Theorem 2.9 [1, 7-11B]. *Let f be a semiproper holomorphic mapping of an analytic space V into an analytic space W. Then $f(V)$ is analytic in W and* $\dim_q f(V) = \sup_{f(p)=q} \operatorname{srnk}_p f$, $q \in f(p)$.

The zero set of a holomorphic function $f \neq 0$ in an open and connected set $H \subset C^n$ is an analytic set in H of the constant dimension $n - 1$. Conversely, such an analytic set is, at least locally, the zero set of a single holomorphic function.

Theorem 2.10 [1, 2-10D]. *Suppose that $V \subset C^n$ is a locally analytic set,* $\dim V \equiv \equiv n - 1$ *and $p \in V$. Then there exist a neighbourhood $U(p)$ and a holomorphic function ω in $U(p)$ such that*

(a) $V \cap U(p) = \omega^{-1}(0)$.

(b) *If g is a holomorphic function in $U(p)$ such that its zero set contains $V \cap U(p)$, then $g = h\omega$ for some holomorphic functions h in $U(p)$.*

The function ω is called the *characteristic function* of V at p.

Lemma 2.11. *Let $V \subset C^n$ be an analytic cone of the constant dimension $n - 1$. Then there exists a homogeneous polynomial ω such that*

(a) $V = \omega^{-1}(0)$.

(b) *If p is a holomorphic function and $p \mid V = 0$, then there exists a holomorphic function h such that $p = h\omega$.*

(c) *If ω_1 and ω_2 satisfy* (a) *and* (b), *then there exists $c \in C$ such that $\omega_1 = c\omega_2$.*

Proof. By Theorem 2.10 there exists a holomorphic function f that satisfies (a) and (b) in some neighbourhood $U(0)$. First we prove that $f_0^* = f$. We can write $f = = \sum p_n$, where p_n are homogeneous polynomials. Take $z \in V \cap U(0)$ and $t \in C$ arbitrary but such that $tz \in V \cap U(0)$. Then $0 = f(z) = f(tz) = \sum t^n p_n(z)$. Hence $p_n(z) = 0$ for all $n \in N$ and $z \in V$; particularly, f_0^* has this property. By Theorem 2.10 (b), $f_0^* = hf$ in some neighbourhood $U(0)$ and the required equality follows. Set $\omega = f_0^*$. To prove that p/ω is holomorphic in $z \in C$ we can find a biholomorphism $F_c : C^n \to C^n$, $z \mapsto (1/c)\, z$ such that $(1/c)\, z \in U(0)$. Then $p(z)/\omega(z) = p(F_c(z')) : : \omega(F_c(z')) = (1/c)^m\, (p(F_c(z'))/\omega(z'))$, $z' \in U(0)$. We can now use Theorem 2.10 (b) in $U(0)$. To prove (c) it is sufficient to apply (b) both to ω_1 and ω_2.

Lemma 2.12. (i) *Let $h : C^n \to C$ be holomorphic at $p \in C^n$ and let $g(z) = \overline{h(\bar{z})}$. Then g is holomorphic at $\bar{p} \in C^n$.*

(ii) *If $V \subset C^n$ is analytic near p, then $\bar{V} = \{q = (q_1, \ldots, q_n); \bar{q} = (\bar{q}_1, \ldots, \bar{q}_n) \in V\}$ is analytic near $\bar{p}$ and $C(\bar{V}, \bar{p}) = \overline{C(V, p)}$.*

Proof. In some neighbourhood of p we can write $g(z) = \overline{h(\bar{z})} = \overline{\sum a_I(\bar{z} - p)^I} = = \sum \bar{a}_I(z - \bar{p})^I$ and the holomorphicity of g follows. If V is the zero set at p of $h_1, \ldots$ $\ldots, h_n$, then $\bar{V}$ is the zero set at $\bar{p}$ of $g_i(z) = \overline{h_i(\bar{z})}$, for $g_i(z) = 0$ iff $h_i(\bar{z}) = 0$. As

$C(\bar{V}, \bar{p}) = \{v;$ there exist $p_i \in V$ and $a_i \in C$ such that $(\bar{p}_i - \bar{p})\, a_i \to v\} = \overline{C(V, p)}$, the lemma is proved.

3. THE ORTHOGONAL CONE

Let us consider the Hermitian scalar product $(u, v) = \sum_{i=1}^{n} u_i \bar{v}_i$, $u, v \in C^n$, and the regular bilinear form $\omega(u, v) = \sum u_i v_i$ in C^n. Let $T \subset C^n$ be a vector subspace of C^n. The orthogonal subspace (ω-orthogonal subspace) is defined to be the set $T^\perp = \{z \in C^n;\ (T, z) = 0\}$ $(T^\perp_\omega = \{z;\ \omega(T, z) = 0\})$. Clearly we have $(T^\perp)^\perp = T$ and $(T^\perp_\omega)^\perp_\omega = T$. Our main aim in this section is to define the orthogonal analytic cone (see Definition 3.1) that has similar properties (Theorem 3.8).

Definition 3.1. Let $V \subset C^n$ be an analytic cone. The *orthogonal cone to* V is the cone $V^\perp = \operatorname{clos}(\bigcup_{p \in V^-} C(V, p)^\perp)$. The *$\omega$-orthogonal cone to* V is the cone $V^\perp_\omega = \operatorname{clos}(\bigcup_{p \in V^-} C(V, p)^\perp_\omega)$.

Remark 3.2. *Let* $V \subset C^n$ *be an analytic cone of a constant dimension. Then* $V^\perp = \bar{V}^\perp_\omega$, $V^\perp = (\bar{V})^\perp_\omega$, $(\bar{V})^\perp = V^\perp_\omega$.

Proof follows immediately from Lemma 2.12 (ii) and from the special case of the subspace V.

Denote

$$p_1 : C^{2n} \to C^n, \quad p_1 : (z, z') \mapsto z,$$
$$p_2 : C^{2n} \to C^n, \quad p_2 : (z, z') \mapsto z'.$$

Definition 3.3. Let V be an analytic cone. The cone $\hat{V} = \operatorname{clos}(\bigcup_{p \in V^-} p \times C(V, p)^\perp_\omega) \subset C^{2n}$ is called the *blanket of* V.

Theorem 3.4. *Let* $V \subset C^n$ *be an analytic cone. Then:*

(i) $\hat{V}$ *is an analytic cone and* $\dim \hat{V} \equiv n$.

(ii) $V = p_1(\hat{V})$ *and* $p_1^{-1}(p) = p \times C(V, p)^\perp_\omega$ *for all* $p \in V^-$.

(iii) $p_2 : \hat{V} \to C^n$ *is a semiproper mapping and* $p_2(\hat{V}) = V^\perp_\omega$.

(iv) $V^\perp_\omega$ *and* $V^\perp$ *are analytic cones.*

(v) *If* $V = \bigcup_i V_i$ *is the splitting of* V (see Theorem 2.5), *then* $\hat{V} = \bigcup \hat{V}_i$, $V^\perp_\omega = \bigcup (V_i)^\perp_\omega$, $V^\perp = \bigcup V_i^\perp$.

Proof. First we prove the theorem with the additional assumption $\dim V \equiv r$. Set $W = [\bigcup_{p \in V^-} p \times C(V, p)^\perp_\omega] \cup [\bigcup_{p \in V^\times} p \times C^n] \subset C^{2n}$. Obviously W is a cone. We want to show that W is an analytic cone. As $\dim V \equiv r$ we can apply Theorem 2.7

to V at $0 \in V$. Let $w_1, \ldots, w_m$ be holomorphic mappings satisfying the conditions (a) and (b) of the theorem in a neighbourhood $U \ni 0$. Define $g_i : U \times C^n \to C$, $g_i(z, z') = \omega(w_i(z), z')$ for $i = 1, \ldots, m$ and $W_1 = \{(z, z') \in U \times C^n, g_i(z, z') = 0, i = 1, \ldots, m\} \cap (V \times C^n)$. The set W_1 is analytic in $U \times C^n$. We have $W \cap (U \times \times C^n) = W_1$, hence by Lemma 2.6, W is an analytic cone. It follows from [1, 3-21] that $\hat{V} = \text{clos}(W \setminus (V^\times \times C^n)) = W\text{-clos}(W \setminus (V^\times \times C^n))$ is an analytic cone. If we show that $W \setminus V^\times \times C^n$ is an analytic manifold of dimension n, then $\dim \hat{V} \equiv n$. For an arbitrary point $p \in V^-$ we can choose $U(p), U(p) \cap V \subset V^-$, $w_{i_1}, \ldots, w_{i_r}$, $f_1, \ldots, f_{n-r}$ such that $w_{i_1}(q), \ldots, w_{i_r}(q)$ span $C(V, q)$ at all $q \in U(p) \cap V$ and

$$V \cap U(p) = \{z, f_1(z) = \ldots = f_{n-r}(z) = 0\} \quad \text{and} \quad \text{rnk}(f_1, \ldots, f_{n-r}) = n - r .$$

Set $f_i'(z, z') = f_i(z)$, $i = 1, \ldots, (n - r)$. The zero set of $f_1', \ldots, f_{n-r}', g_{i_1}, \ldots, g_{i_r}$ in $U(p) \times C^n$ is $\hat{V} \cap (U(p) \times C^n) = (W \setminus V^\times \times C^n) \cap (U(p) \times C^n)$ and $\text{rnk}(f_1', \ldots, f_{n-r}', g_{i_1}, \ldots, g_{i_r}) = n$ in $\hat{V} \cap (U(p) \times C^n)$. Hence $W \setminus (V^\times \times C^n)$ is an analytic manifold of dimension n. The proof of (i) is complete.

We have $V^- \times \{0\} \subset \hat{V}$, hence $V \times \{0\} \subset \hat{V}$ and $V = p_1(\hat{V})$.

To prove that $p_2 : \hat{V} \to C^n$ is a semiproper mapping it is sufficient to demonstrate the following assertion:

$$(u, v) \in \hat{V} \,\&\, c \in C \Rightarrow (cu, v) \in \hat{V} .$$

For $u \in V^-$ the implication holds as $C(V, u) = C(V, cu)$. If $u \in V^\times$, then there exist $(u_n, v_n) \to (u, v)$, $u_n \in V^-$, $(u_n, v_n) \in \hat{V}$. We know that $(cu_n, v_n) \in \hat{V}$ and $(cu_n, v_n) \to \to (cu, v)$. Hence $(cu, v) \in \hat{V}$.

As p_2 is *a* continuous mapping we have $p_2(W \setminus (V^\times \times C^n)) \subset p_2(\hat{V}) \subset \subset \text{clos}(p_2(W \setminus (V^\times \times C^n))$. By Theorem 2.9, $p_2(\hat{V})$ is analytic in C^n, hence closed. So we immediately obtain $p_2(\hat{V}) = \text{clos}(p_2(W \setminus (V^\times \times C^n))) = V_\omega^\perp$ and (iii) is proved.

Part (iv) follows from Theorem 2.9 and Lemma 2.12.

Now we can return to the general case of V. By Theorem 2.5 we can split $V = = \bigcup_{i=1}^k V_i$ where V_i are irreducible analytic cones, $\dim V_i \equiv n_i$ and $\dim (V_j \cap \bigcup_{i \neq j} V_i) < < \dim V_j$,

$$\hat{V} = \text{clos}\big(\bigcup_{p \in V^-} p \times C(V, p)_\omega^\perp\big) = \text{clos}\big(\bigcup_{i=1}^k \bigcup_{V_i^- \setminus V^\times} p \times C(V, p)_\omega^\perp\big) =$$

$$= \text{clos}\big(\bigcup_{i=1}^k \bigcup_{V_i^-} p \times C(V, p)_\omega^\perp\big) = \bigcup_{i=1}^k \hat{V}_i .$$

Similarly $V_\omega^\perp = \bigcup_i (V_i)_\omega^\perp$ and $V^\perp = \bigcup_i V_i^\perp$. We can now apply the special case of $\dim V \equiv r$ (as V_i's are irreducible and hence of a constant dimension). The proof of the theorem is complete.

Theorem 3.5. *Let V be an analytic irreducible cone. Then $\hat{V}$, $V_\omega^\perp$ and $V^\perp$ are irreducible analytic cones and* $\dim V_\omega^\perp = \dim V^\perp = \text{rnk}\, p_2 \mid \hat{V}$.

Proof. Suppose that $\hat{V}$ is not irreducible. By Theorem 2.5 and because $\hat{V}$ is an analytic cone, $\hat{V} = \bigcup_{i=1}^{k} \operatorname{clos} V_i$, where V_i are connected pieces of $\hat{V}^-$, $\operatorname{clos} V_i$ are irreducible analytic subsets of $\hat{V}$ and $\dim(\operatorname{clos} V_i) \equiv n$ (we use Theorem 3.4 (i)). Set $\hat{V} = W_1 \cup W_2$, $W_1 = \operatorname{clos} V_1$, $W_2 = \bigcup_{i \neq 1} \operatorname{clos} V_i$ and $P_i = \{p \in V^-,\ p \times C(V, p)^{\perp}_{\omega} \subset W_i \subset \hat{V}\}$ for $i = 1, 2$. Let $p \in V^-$ be an arbitrary point. Then there exists $U(p) \subset V^-$ such that $V(p) = (U(p) \times C^n) \cap \hat{V} \subset \hat{V}^-$. The set $V(p)$ is a connected neighbourhood of p, therefore $V(p) \subset W_1^-$ or $V(p) \subset W_2^-$ and P_1, P_2 are open in $\hat{V}^-$. As $W_1^- \cup W_2^- = \hat{V}^-$ we have $P_1 \cup P_2 = V^-$. By assumption we also have $W_1^- \cap W_1^- = \emptyset$. Hence $P_1 \cap P_2 = \emptyset$. But this is impossible as V^- is connected (V is irreducible).

If $V^{\perp}_{\omega} = V_1 \cup V_2$, then by Theorem 3.4, $\hat{V} = p_2^{-1}(V_1) \cup p_2^{-1}(V_2)$, a contradiction. The irreducibility of $V^{\perp}$ follows from the irreducibility of $V^{\perp}_{\omega}$. The last part of the theorem follows from 3.4(iii) and 2.9.

Proposition 3.6. *Let $V \subset C^n$ be an irreducible analytic cone and $\hat{V}$ be the blanket of V. Then there exists an analytic cone $W \subset \hat{V}$ such that:*

(i) $\dim W < \dim \hat{V}$.

(ii) *Let $W_1 = \{q \in V;\ p_1^{-1}(q) \subset W\}$ and $W_2 = \{q \in V^{\perp}_{\omega};\ p_2^{-1}(q) \subset W\}$. Then W_1 is thin in V and W_2 is thin in $V^{\perp}_{\omega}$.*

(iii) *If $(p, q) \in \hat{V} \setminus W$, then $\omega(p, C(V^{\perp}_{\omega}, q)) = 0$, $(p, C(V^{\perp}, q)) = 0$.*

Proof. By Theorem 3.4(i), $\hat{V}$ is an irreducible analytic cone of the constant dimension n. Denote $\varrho_1 = \operatorname{rnk} p_1$, $\varrho_2 = \operatorname{rnk} p_2$, where $p_1 : \hat{V} \to V$, $p_2 : \hat{V} \to V^{\perp}_{\omega}$, and $M_i = \{q \in \hat{V}^-;\ \operatorname{rnk}_q p_i = \varrho_i\}$, $i = 1, 2$. By Theorem [1, 4-7F], $\hat{V} \setminus M_1$ and $\hat{V} \setminus M_2$ are analytic sets. As $\hat{V}$ is irreducible we have $\dim(\hat{V} \setminus M_i) < \dim \hat{V}$, $i = 1, 2$. Set $W = p_1^{-1}(V^{\times}) \cup p_2^{-1}((V^{\perp}_{\omega})^{\times}) \cup (\hat{V} \setminus M_1) \cup (\hat{V} \setminus M_2)$. Clearly W is an analytic cone and $\dim W < \dim \hat{V}$.

To prove that $(V^{\perp}_{\omega})^- \setminus \operatorname{clos} W_2$ is dense in $(V^{\perp}_{\omega})^-$ it is sufficient to demonstrate that for all $q \in (V^{\perp}_{\omega})^-$ and for all open neighbourhoods $U(q) \subset (V^{\perp}_{\omega})^-$ there exists an open set $P \subset U(q)$ such that $P \cap W = \emptyset$. Denote $\tilde{U}(q) = p_2^{-1}(U(q))$. $\tilde{U}(q) \subset \hat{V}$ is open in $\hat{V}$. As W is a proper analytic subset of the irreducible analytic set $\hat{V}$ we have that $\tilde{U}(q) \setminus W$ is an open and nonvoid subset of $\hat{V}$. Choose an arbitrary open $\tilde{P} \subset \tilde{U}(q) \setminus W$. Set $P = p_2(\tilde{P})$. As $\operatorname{rnk}_u p_2 = \varrho_2$ at all $u \in \tilde{P}$ and $\dim U(q) = \varrho_2$ we conclude that P is an open subset of $V^{\perp}_{\omega}$ [1, Apendix II, 7F]. Moreover, $P \cap W_2 = \emptyset$ which completes the proof that W_2 is thin in $V^{\perp}_{\omega}$. The proof that W_1 is thin in V is quite analogous.

Take any $z = (p, q) \in \hat{V} \setminus W$ and any $v \in C(V^{\perp}_{\omega}, q)$. We shall show that $\omega(p, v) = \emptyset$. Denote by $\mathrm{d}p_1(z)$, $\mathrm{d}p_2(z)$ the differential mappings of p_1 and p_2 at the point $z \in \hat{V}$. From $z \notin W$ it follows that in some neighbourhood $U(z)$ of z we have p_1 and p_2 of maximum rank ϱ_1 and ϱ_2. Hence $\mathrm{d}p_i(z)^{-1}(0) = C(p_i^{-1}(p_i(z)), z)$, $i = 1, 2$, $\mathrm{d}p_1(z)^{-1}(0) \cap \mathrm{d}p_2(z)^{-1}(0) = \emptyset$, and $\mathrm{d}p_1(z)(C(\hat{V}, z)) = C(V, p)$, $\mathrm{d}p_2(z)(C(\hat{V}, z)) =$

$= C(V_\omega^\perp, q)$. In this situation we can find $w \in C(\hat{V}, z)$ such that $dp_2(z)(w) = v$, $0 \neq v_1 = dp_1(z)(w) \in C(V, p)$ for an arbitrary $v \in C(V_\omega^\perp, q)$. Clearly $w = (v_1, v) \subset C^{2n}$. By the definition of the tangent cone there exists a sequence $z_n = (p_n, q_n) \in \hat{V} \setminus W$ and a sequence $a_n \in C$ such that $a_n(z_n - z) \to w$. Hence $a_n(q_q - q) \to v$, $a_n(p_n - p) \to v_1$.

By Theorem 3.4 we have:

(A) $\omega(p, q) = 0$ since $p \in V^-$ and $p \in C(V, p)$;

(B) $\omega(p_n, q_n) = 0$ since $p_n \in V^-$ and $p_n \in C(V, p_n)$;

(C) $\omega(v_1, q) = 0$ since $p \in V^-$ and $v_1 \in C(V, p)$.

We can now compute: $\omega(p, v) = \omega(p, \lim a_n(q_n - q)) = \lim a_n \omega(p_n, q_n - q) =$ $=$ (by (A) and (B)) $= \lim a_n(\omega(p_n, -q) + \omega(p, q)) = \lim a_n \omega(p - p_n, q) =$ $= \omega(\lim (p - p_n) a_n, q) = \omega(v_1, q) = 0$ by (C). This completes the proof.

Theorem 3.7. *Let V be an analytic cone. Then $(V^\perp)^\perp = V$ and $(V_\omega^\perp)_\omega^\perp = V$.*

Proof. Because of Theorem 3.4(v) we can assume that V is irreducible. By the last proposition (iii) we have $V \setminus W_1 \subset (V_\omega^\perp)_\omega^\perp$ and by (ii), $V \subset (V_\omega^\perp)_\omega^\perp$. We have yet to prove the second inclusion $(V_\omega^\perp)_\omega^\perp \subset V$. Let W and W_2 be the exceptional sets as in Proposition 3.6 and rnk $p_2 = \varrho$. Choose $(p, q) \in \hat{V} \setminus W$. Since rnk $p_2 = \varrho$ on some neighbourhood $U(p, q) \subset \hat{V} \setminus W$ and $\dim \hat{V} \equiv n$ we have $\dim (p_2^{-1}(q) \cap U(p, q)) = n - \varrho$. By Theorem 3.5 we have $\dim_q V_\omega^\perp = \varrho$, therefore $\dim C(V_\omega^\perp, q)_\omega^\perp = n - \varrho$ for $q \in (V_\omega^\perp)^-$. However, by Proposition 3.6 we have $(p_2^{-1}(q) \cap U(p, q)) \subset C(V_\omega^\perp, q)_\omega^\perp \times q$. Hence we have proved that some open part of $C(V_\omega^\perp, q)_\omega^\perp \times q$ lies in $\hat{V} \cap (C^n \times q)$. As $C(V_\omega^\perp, q)_\omega^\perp \times q$ is irreducible we have $C(V_\omega^\perp, q)_\omega^\perp \times q \subset \hat{V}$. Thus we have proved that $C(V_\omega^\perp, q)_\omega^\perp \subset V$ for any $q \in V_\omega^\perp \setminus W_2$. Now we can write $(V_\omega^\perp)_\omega^\perp = \text{clos} \bigcup_{(V_\omega^\perp)^-} C(V_\omega^\perp, q)_\omega^\perp = \text{clos} \bigcup_{(V_\omega^\perp)^- \setminus W_2} C(V_\omega^\perp, q)_\omega^\perp \subset V$. Finally, $(V^\perp)^\perp = (\bar{V}_\omega^\perp)^\perp = (V_\omega^\perp)_\omega^\perp = V$, completing thus the proof of the theorem.

Corollary 3.8. *Let $V \subset C^n$ be an irreducible analytic cone and suppose that $\dim V + \dim V^\perp = n$. Then V is a linear subspace of C^n.*

Proof. Let V be of dimension r and $p \in V^-$. Then $\dim C(V, p)^\perp = n - r$ and by the assumption also $\dim V^\perp = n - r$. As $V^\perp$ is irreducible (see Theorem 3.5) we have $V^\perp = C(V, p)^\perp$. Thus (see Theorem 3.7) $V = (V^\perp)^\perp = (C(V, p)^\perp)^\perp = C(V, p)$ and the assertion is proved.

In the remaining part of the section we give a new expression of $V^\perp$ for $V \subset C^n$, $\dim V \equiv n - 1$.

Definition 3.9. Let $V \subset C^n$ be an analytic cone of constant dimension $(n - 1)$. The homogeneous polynomial ω satisfying the conditions (a) and (b) of Lemma 2.11 is called the *characteristic polynomial of V*. Denote by h_ω the mapping

$$h_\omega : C^n \to C^n, \quad z \mapsto \left(\frac{\partial}{\partial z_1}\omega(z), \ldots, \frac{\partial}{\partial z_n}\omega(z)\right).$$

Theorem 3.10. *Let $V \subset C^n$ be an irreducible analytic cone of constant dimension $(n-1)$ and ω the characteristic polynomial of V.* (i) *Assume* $\operatorname{st} \omega \geqq 2$. *Then* $V^\perp = \operatorname{clos} \overline{h_\omega(V)}$, $V_\omega^\perp = \operatorname{clos} h_\omega(V)$.

(ii) *If* $\operatorname{st} \omega = 1$, *then* $V_\omega^\perp = \{c h_\omega(0); c \in C\}$, $V^\perp = \{\overline{c h_\omega(0)}; c \in C\}$.

Proof. (i) Clearly

$$\left.\frac{\partial \omega}{\partial z_i}\right| V \not\equiv 0 \quad \text{as} \quad \operatorname{st} \frac{\partial \omega}{\partial z_i} = \operatorname{st} \omega - 1$$

and ω is the characteristic polynomial. Denote by $\boldsymbol{p}$ the complex line defined by the points $p \in C$ and $0 \in C$. We have $V_\omega^\perp = \operatorname{clos} \bigcup_{p \in V^-} C(V, p)_\omega^\perp = \operatorname{clos} \bigcup_{p \in V^- \setminus h_\omega^{-1}(\emptyset)} C(V, p)_\omega^\perp =$ $= \operatorname{clos} \bigcup_{\boldsymbol{p} \in V^- \setminus h_\omega^{-1}(0)} \bigcup_{z \in \boldsymbol{p}} C(V, z)_\omega^\perp = \operatorname{clos} \bigcup_{\boldsymbol{p} \in V^-} h_\omega(\boldsymbol{p}) = \operatorname{clos} h_\omega(V)$.

(ii) Clearly V is a linear subspace and the assertion is obvious.

4. ZERO SETS OF LEFT REGULAR FUNCTIONS

Assume that $f : CH \to CH$ is holomorphic in some neighbourhood of 0 and $f = \sum_\alpha f_\alpha \mathrm{i}_\alpha$. The functions f_α are holomorphic and in some neighbourhood of 0 we can write $f_\alpha = \sum a_{\alpha,I} z^I$, $I \in N^4$. Set $A_I = \sum_\alpha a_{\alpha,I} \mathrm{i}_\alpha$. Then $f = \sum_I A_I z^I$ is the expansion of a function f into the power series. Denote $f^* = \sum_{|I|=m} A_I z^I$ where $m = \min \{|I|, A_I \neq 0\}$ and $P_n = \sum_{|I|=n} A_I z^I$.

Proposition 4.1. *Let $f : CH \to CH$ be holomorphic in some neighbourhood of* 0 *and let $f = \sum_n P_n$ be the described expansion into homogeneous polynomials. Then $f \in F_0^-$ iff $P_n \in F_0^-$ for all $n \in N$.*

Proof. Denote $Q_{n-1} = \mathrm{D}P_n = \sum_\alpha \mathrm{i}_\alpha (\partial/\partial z_\alpha) P_n$. Q_{n-1} is a homogeneous polynomial of degree $(n-1)$ and $\mathrm{D}f = \mathrm{D} \sum_n P_n = \sum_n \mathrm{D}P_n = \sum Q_{n-1}$. Clearly $\mathrm{D}f \equiv 0$ iff $Q_{n-1} \equiv$ $\equiv 0$ for all $n \in N$ and the proposition follows.

Proposition 4.2. *Let V be a germ of F^- zero set at a point $q \in CH$ and let $V =$ $= \bigcup_{\alpha=0}^{3} V_\alpha$ be the splitting of V by dimension. Assume that $V_3 \neq 0$.*

(i) *There exists a left regular homogeneous polynomial p such that the zero set of p contains $C(V_3, q)$.*

(ii) *If* $q \in V_3^-$ *then* $C(V_3, q)^\perp \subset N$ *and* $C(V_3, q)^\perp_\omega \subset N$.

Proof. Let $f : U(q) \subset CH \to CH$ be a left regular function such that $U(q) \cap V = f^{-1}(0) \cap U(q)$. Let $\omega_1, \ldots, \omega_l$ be the characteristic functions of the irreducible components of V_3 (see Theorem 2.10). Then $V_3 = \{z, \omega(z) = \omega_1(z) \ldots \omega_l(z) = 0\}$ and there exist $k_1, \ldots, k_l$ such that $f = \omega_1^{k_1} \ldots \omega_l^{k_l} g$ where $g \not\equiv 0$ on every irreducible component of V_3. By the last proposition $f^* = (\omega_1^*)^{k_1} \ldots (\omega_l^*)^{k_l} g^*$ is a left regular homogeneous polynomial. By Theorem 2.4, $C(V_3, q) = \{z; \omega^*(z) = 0\}$. We see now that $f^* \equiv 0$ on $C(V_3, q)$ and the first part of the proposition is proved.

If $q \in V_3^-$ then V_3 is irreducible at the point q and we have $C(V_3, q) = \{z, \omega^*(z) = \sum_\alpha a_\alpha z_\alpha = 0\}$, where ω is the characteristic polynomial of V_3. By Theorem 3.10, $C(V_3, q)^\perp_\omega = \{c \sum_\alpha \mathrm{i}_\alpha (\partial/\partial z_\alpha)\, \omega(0);\ c \in C\} = \{c \sum_\alpha \mathrm{i}_\alpha a_\alpha,\ c \in C\}$. To prove the second part of our proposition it is sufficient to show that $\sum_\alpha a_\alpha \mathrm{i}_\alpha \in N$. We already know that $f = \omega^l g$ and $f^* = (\omega^*)^l g^* \in F_q^-$. Assume that $g^* = (\omega^*)^k g_1$ where $k \geqq 0$ and $g_1 \not\equiv 0$ on $C(V_3, q)$. Then $0 = Df^* = D((\omega^*)^{l+k} g_1) = (\sum_\alpha a_\alpha \mathrm{i}_\alpha)(l + k)(\omega^*)^{l+k-1} g + (\omega^*)^{l+k} \mathrm{D} g_1$. If $(l + k - 1) > 0$ then $(\omega^*)^{l+k-1} \not\equiv 0$ in CH and we can cancel it in the last equation. We obtain $0 \equiv (\sum_\alpha a_\alpha \mathrm{i}_\alpha)(l + k)\, g_1 + \omega^* \mathrm{D} g_1$. On $C(V, q)$ we have $0 \equiv (\sum_\alpha a_\alpha \mathrm{i}_\alpha)(l + k)\, g_1$. By our assumption on g_1 there exists $z \in C(V_3, q)$ such that $g_1(z) = 0$ and $(\sum_\alpha a_\alpha \mathrm{i}_\alpha) \in N$ follows. As $\bar{N} = N$, the second part of the proposition is proved.

Theorem 4.3. *Let* V *be a* F^- *zero set in* $B \subset CH$ *and let* $V = \bigcup_{\alpha=0}^{3} V_\alpha$ *be the splitting of* V *by dimension. Then* $C(V_3, q)$ *is an* N*-orthogonal and an* N*-*ω*-orthogonal cone for every* $q \in V$.

Proof. Let $q \in V_3$ be an arbitrary point. By Proposition 4.2 (i) there exists $p \in F^-(CH)$, $Z = p^{-1}(0) = \bigcup_\alpha Z_\alpha$, such that $C(V_3, q) \subset Z_3$. By Theorem 3.7 we have to prove that $C(V_3, q)^\perp_\omega \subset N$ for $C(V_3, q) = (C(V_3, q)^\perp_\omega)^\perp_\omega$. With respect to Theorem 3.4 (v) it is sufficient to prove that $Z_3^\perp \subset N$. But $Z_3^\perp = \operatorname{clos} \bigcup_{z \in Z_3^-} C(Z_3, z)^\perp_\omega \subset N$ by Proposition 4.2 (ii) and we have proved that $C(V_3, q)$ is an N-ω-orthogonal cone. The proof that $C(V_3, q)$ is an N-orthogonal cone is quite analogous.

Remark 4.4. $N^\perp = N$ and $N^\perp_\omega = N$, where $N = \{v \in CH; \sum v_\alpha^2 = 0\}$.

Proof. By Theorem 3.10,

$$N^\perp_\omega = \operatorname{clos} h_N(N) = \operatorname{clos} \left(\frac{\partial N}{\partial z_0}, \ldots, \frac{\partial N}{\partial z_3}\right)(N) = (2z_0, 2z_1, 2z_2, 2z_3)(N) = N$$

and $N^\perp = \operatorname{clos} \overline{h_N(N)} = \bar{N} = N$.

Example 4.5. Let us suppose that Q is a line in N. Then $Q = \{c \sum_\alpha a_\alpha \mathrm{i}_\alpha\}$ for some $A = \sum_\alpha a_\alpha \mathrm{i}_\alpha \in N$. We have $Q_\omega^\perp = \{z, \sum a_\alpha z_\alpha = 0\}$. Set $p_A = (\sum a_\alpha z_\alpha) A^+$. Then $p_A \in F^-(CH)$ for $\mathrm{D}p_A = AA^+ = 0$ and $Q_\omega^\perp = p_A^{-1}(0)$.

Lemma 4.6. *Denote* $N_a = \{z \in CH;\ az = 0\}$, $a \in N$. *Then* $N_a \subset N$ *and* N_a *is a linear subspace of dimension* 2.

Proof. Let $T_a : CH \to CH$, $z \mapsto az$ be a linear mapping. Clearly $T_a(CH) \subset N$, $\operatorname{Ker} T_a = N_a \subset N$. So we have $\dim T_a(CH) \leqq 2$ and $\dim \operatorname{Ker} T_a \leqq 2$. It follows that $\dim T_a(CH) = \dim \operatorname{Ker} T_a = 2$.

Example 4.7. Suppose that $a \in N$ and $B \subset N_a$ (see the last lemma), and that B is a finite set. Denote $Q = \bigcup_{b \in B} \{cb;\ c \in C\} \subset N$ and $p_{a,B}(z) = a \prod_{b \in B} (\sum b_\alpha z_\alpha)^{n_b}$ where $n_b > 0$ are arbitrary integers. Then $p_{a,B} \in F^-(CH)$ and $p_a^{-1}(0) = Q_\omega^\perp = \bigcup_{b \in B} \{z;\ \sum z_\alpha b_\alpha = 0\}$.

Proof.

$$\mathrm{D}p_{a,B} = \sum_\alpha \frac{\partial}{\partial z_\alpha} \mathrm{i}_\alpha p_{a,B} = \sum_{b \in B} (\sum b_\alpha \mathrm{i}_\alpha)\, a \Big(\prod_{b' \in B \setminus \{b\}} (\sum b'_\alpha z_\alpha)^{n_{b'}} \Big) (\sum b_\alpha z_\alpha)^{n_b - 1} = 0$$

as $ba = 0$ for all $b \in B$.

5. THE SET OF POLES OF LEFT REGULAR FUNCTIONS

A meromorphic function on an open set $B \subset C^n$ is a pair (H, f), where H is a dense open subset of B and f is a holomorphic function in H with the following property: For each $p \in B$ there is a neighbourhood U of p and holomorphic functions Φ and Ψ in U such that the zero set of Ψ is nowhere dense in U and $f = \Phi/\Psi$ in $U \cap H \setminus \Psi^{-1}(0)$ [1,6-1A]. It is shown in [1,6-1K,L] that f may be extended to be holomorphic in an open subset $H_f \subset B$ such that the complement $P_f = B \setminus H_f$ is a nowhere dense analytic set in B. Set $Z_f(a) = \{p \in B;$ there exist $z_n \in H$, $z_n \to p$, $f(z_n) \to a\}$. Then $Z_f(a)$ is an analytic set in B [1,6-1L] and $P_f = Z_f(\infty)$. In 1.4 we proposed the definition of a F^- meromorphic function (H, f) in $B \subset CH$. Set, like in the case of meromorphic functions, $Z_f(a) = \{p \in B;$ there exist $z_n \in H$, $z_n \to p$, $f(z_n) \to a\}$ for every $a \in CH$. Denote $P_f = Z_f(\infty)$, $Z_f = Z_f(0)$. In the case of a left regular function f in $B \subset CH$, Z_f is its zero set and $P_f = 0$. The following proposition gives some definitions equivalent to 1.4 (i).

Proposition 5.1. *Let* $B \subset CH$ *be an open set. Let* $H \subset B$ *be open and dense in* B. *Then the following properties of a mapping* $f : H \to CH$ *are equivalent*:

A) *For all* $p \in B$ *there exist* $U(p)$ *and holomorphic mappings* $\Phi, \Psi : U(p) \to CH$, $\Psi(U(p)) \not\subset N$ *such that* $f = \Phi/\Psi$ *in* $U(p) \cap H \setminus \Psi^{-1}(N)$.

B) *There exist meromorphic functions* f_α *such that* $f = \sum_\alpha i_\alpha f_\alpha$.

C) *For all* $p \in B$ *there exist* $U(p)$ *and holomorphic mappings* $\Phi : U(p) \to CH$, $\Psi : U(p) \to C$, $U(p) \not\subset Z_\Psi$ *such that* $f = \Phi/\Psi$ *in* $U(p) \cap H \setminus Z_\Psi$ *and* $\dim (Z_\Phi \cap \cap Z_\Psi \cap U(p)) < 3$.

Proof. A $\Rightarrow$ B. Set $f = \sum_\alpha \mathrm{i}_\alpha f_\alpha$ in H. But $f = \Phi/\Psi = \Phi \,.\, \Psi^+/(\Psi \,.\, \Psi^+)$ in $U(p) \cap \cap H \setminus \Psi^{-1}(N)$ and therefore f_α are meromorphic functions.

B $\Rightarrow$ C. For every $p \in B$ there exist $U(p)$ and holomorphic functions Φ_α, Ψ_α in $U(p)$ such that $f_\alpha = \Phi_\alpha/\Psi_\alpha$ in $U(p) \cap H \setminus Z_{\Psi_\alpha}$. Set $\Psi = \prod_\alpha \Psi_\alpha$ and $\Phi = = \sum_\alpha (i_\alpha \Phi_\alpha \prod_{\beta \neq \alpha} \Psi_\beta)$. Then $f = \Phi/\Psi$ in $U(p) \cap H \setminus Z_\Psi$. If $\dim (Z_\Phi \cap Z_\Psi \cap U(p)) = 3$ then we can divide both Φ and Ψ by the characteristic functions of $Z_\Phi \cap Z_\Psi \cap U(p)$. We repeat this process, if necessary, until the condition C) is satisfied.

C $\Rightarrow$ *A*. As we have C imbedded in CH, by identifying $C \ni c$ with $c\mathrm{i}_0 \in CH$ we have nothing to prove.

Proposition 5.2. *Let* $B \subset CH$ *be an open set, let* $H \subset B$ *be an open and dense subset and let* $f : H \to CH$ *be a* F^- *meromorphic function. Then there exists* H_f, $H \subset H_f \subset B$, *such that*:

(i) *We can extend* f *to be a left regular function in* H_f.

(ii) $B \setminus H_f = P_f$ *and* P_f *is an analytic set in* B, $\dim P_f \equiv 3$.

(iii) *Assume that* $f = \Phi/\Psi$ *is an expression of* f *in some neighbourhood* $U(q)$ *of* $q \in P_f$ *satisfying the condition* 5.1C. *Then* $P_f \cap U(q) = Z_\Psi$ *and* $(Z_f)_3 \cap U(q) = = (Z_\Phi)_3$, *where* $Z_f = \bigcup_\alpha (Z_f)_\alpha$ *and* $Z_\Phi = \bigcup_\alpha (Z_\Phi)_\alpha$ *are the splittings of* Z_f *and* Z_Φ *by dimension.*

Proof. By Proposition 5.1B there exist meromorphic functions f_α such that $f = \sum_\alpha \mathrm{i}_\alpha f_\alpha$ in H. We can extend f_α to be holomorphic in $B \setminus P_{f_\alpha}$. Set $H_f = B \setminus \bigcup_\alpha P_{f_\alpha}$. Then we can extend f to be a left regular function in H_f (we use the identity theorem for the holomorphic mapping $\mathrm{D}f$). Clearly $B \setminus H_f = \bigcup_\alpha P_{f_\alpha} = P_f$. As $\dim P_{f_\alpha} \equiv 3$ for the meromorphic function f_α, it follows at once that $\dim P_f \equiv 3$. To prove (iii) we use the fact that $\dim (Z_\Phi \cap Z_\Psi \cap U(p)) < 3$.

In what follows we want to show that the description of poles of F^- meromorphic functions is in many ways similar to the description of F^- zero sets.

Definition 5.3. Let (H_f, f) be an F^- meromorphic function in $B \subset CH$, let $q \in P_f$ and let $f = \Phi/\Psi$ be an expression of f satisfying the condition 5.1C in a neighbourhood $U(q)$. A point $q \in P_f$ is a *bad point* iff $\dim (Z_{\Phi_q^*} \cap Z_{\Psi_q^*}) = 3$, where Φ_q^*, Ψ_q^* are the initial polynomials of Φ, Ψ at the point q.

Remark 5.4. The bad points are thin in P_f as $Z_f \cap P_f$ is thin in P_f.

Proposition 5.5. *Let (H_f, f) be an F^- meromorphic function in B and assume that $q \in P_f$ is not a bad point.*

(i) *Then there exists an F^- meromorphic function (H_g, g) in CH such that $C(P_f, q) = P_g$ and g is the quotient of two homogeneous polynomials.*

(ii) *If $q \in P_f^-$ then $C(P_f, q)^\perp \subset N$ and $C(P_f, q)^\perp_\omega \subset N$.*

Proof. Following Proposition 5.2(iii) we have, in some neighbourhood of $U(q)$, $f = \Phi/\Psi$, $P_f = Z_\Psi$ and $C(P_f, q) = \{z \in CH;\ \Psi_q^*(z) = 0\}$. If we use the regularity condition on f we obtain $0 = \mathrm{D}(\Phi/\Psi) = \mathrm{D}((1/\Psi)\,\Phi) = ((\mathrm{D}\Phi)\,\Psi - (\mathrm{D}\Psi)\,\Phi)/\Psi^2 = ((\mathrm{D}\Phi_q^*)\,\Psi_q^* - (\mathrm{D}\Psi_q^*)\,\Phi_q^*$ + powers at the point q of degrees higher than st Φ_q^* + st $\Psi_q^* - 1)/\Psi^2$. As $\Psi \not\equiv 0$ in a neighbourhood of q we can multiply the last equality by Ψ^2: $0 = ((\mathrm{D}\Phi_q^*)\,\Psi_q^* - (\mathrm{D}\Psi_q^*)\,\Phi_q^*$ + powers of degrees higher than st Φ_q^* + st $\Psi_q^* - 1) = F$. Using the last equality only for the initial polynomial on the right hand side we obtain

$$(1) \qquad 0 = F_q^* = (\mathrm{D}\Phi_q^*)\,\Psi_q^* - (\mathrm{D}\Psi_q^*)\,\Phi_q^*\,.$$

Set $g = \Phi_q^*/\Psi_q^*$. Then (1) is the regularity condition for g and g is an F^- meromorphic function. As q is not a bad point, the expresion Φ_q^*/Ψ_q^* satisfies the condition 5.1C, and by 5.2(iii) we have proved (i).

Consider now $q \in P_f^-$. Denote by ω the characteristic function of P_f at the point q. Then $C(P_f, q) = \{z, \omega^*(z) = \sum_\alpha z_\alpha a_\alpha = 0\}$. We must verify, as in the proof of Proposition 4.2(ii), that $\mathrm{D}\omega^* = \sum_\alpha \mathrm{i}_\alpha a_\alpha \in N$. By 5.2(iii) we have $f = \Phi/\Psi$, $\Psi = \omega^l \delta$, where $\delta \mid P_f \not\equiv 0$. Write $\Psi^* = (\omega^*)^l\,\delta^* = (\omega^*)^{l+k}\,\delta'$, where $\delta' \mid C(P_f, q) \not\equiv 0$. Using (1) we obtain

$$0 \equiv (D\Phi_q^*)\,(\omega^*)^{l+k}\,\delta' - (l + k)\,(\mathrm{D}\omega^*)\,(\omega^*)^{l+k-1}\,\delta'\Phi_q^* - (\omega^*)^{l+k}\,(\mathrm{D}\delta')\,\Phi_q^*\,.$$

If $l + k + (-1) > 0$ then we can divide the equation by $(\omega^*)^{l+k-1}$:

$$0 = (\mathrm{D}\Phi_q^*)\,\omega^*\delta' - (l + k)\,(\mathrm{D}\omega^*)\,\delta'\Phi_q^* - \omega^*(\mathrm{D}\delta')\,\Phi_q^*\,.$$

On $C(P_f, q)$ we have

$$(2) \qquad 0 \equiv (l + k)\,(\mathrm{D}\omega^*)\,\delta'\Phi_q^*\,.$$

Since q is not a bad point and $\delta' \mid C(P_f, 0) \not\equiv 0$ we can choose $z \in C(P_f, q)$ such that $\delta'(z)\,\Phi^*(z) \neq 0$. If we apply (2) at such a point z we obtain that $(\mathrm{D}\omega^*) \in N$ and the proposition is proved.

Theorem 5.6. *Let (H_f, f) be an F^- meromorphic function in B and P_f its set of poles. Assume that $q \in P_f$ and q is not a bad point. Then $C(P_f, q)$ is an N-orthogonal and an N-ω-orthogonal cone.*

Proof. The proof is quite similar to that of Theorem 4.3. By Proposition 5.5(i) there exists an F^- meromorphic function g such that $C(P_f, g) = P_g$. Denote by S_g the set of bad points of g. Using Proposition 5.5(ii) and Remark 5.4 we have $C(P_f, q)^\perp = \text{clos} \bigcup_{z \in P_g^-} C(P_g, z)^\perp = \text{clos} \bigcup_{z \in P_g^- \setminus S_g} C(P_g, z)^\perp \subset N$. The proof is complete.

Example 5.7. Suppose that $a \in N$ and $B \subset N_a$ (see 4.6), and let B be a finite set. Denote $Q = \bigcup_{b \in B} \{cb;\ c \in C\} \subset N_a \subset N$ and $f_{a,B}(z) = a / \prod_{b \in B} (b_\alpha z_\alpha)^{n_b}$, where $n_b > 0$ are arbitrary integers. Then $f_{a,B}$ is an F^- meromorphic function and $P_{f_{a,b}} = Q_\omega^\perp = \bigcup_{b \in B} \{z;\ \sum z_\alpha b_\alpha = 0\}$.

Example 5.8. Suppose that $Q \subset N$ is an irreducible analytic cone, dim $Q = 3$. Then $Q = N$ (as N is an irreducible analytic cone) and $Q_\omega^\perp = Q^\perp = N$ (see 4.4). Set $f_N = z^+/(N(z))^2$. Then f_N is an F^- meromorphic function and $P_{f_N} = N$.

Proof.

$$Df_N = \sum_\alpha i_\alpha \frac{\partial}{\partial z_\alpha} (z^+/(N(z))^2) =$$

$$= \left(\sum_\alpha i_\alpha \left(\frac{\partial z^+}{\partial z_\alpha} \right) / (N(z))^2 \right) - (2 \sum_\alpha i_\alpha 2z_\alpha) N(z) z^+ / (N(z))^4 = 0 .$$

Example 5.9. Set $Q = N \cap \{z;\ z_0 = z_3 = 0\} = \{z;\ z_0 = z_3 = 0,\ z_1^2 + z_2^2 = 0\}$. Clearly $Q \not\subset N_a$ for any $a \in N$ and Q is the union of two lines. Then $Q_\omega^\perp = \{z \in CH,\ z_1^2 + z_2^2 = 0\}$ as $(Q_\omega^\perp)_\omega^\perp = Q$ by Theorem 3.10. Set

$$f = \frac{-i_1 z_1 - i_2 z_2}{z_1^2 + z_2^2} .$$

Then f is an F^- meromorphic function and $P_f = Q_\omega^\perp$.

Proof.

$$Df = \sum_\alpha i_\alpha \frac{\partial}{\partial z_\alpha} \left(\frac{-i_1 z_1 - i_2 z_2}{z_1^2 + z_2^2} \right) = \sum_{\alpha = 1,2} i_\alpha \frac{-i_\alpha (z_1^2 + z_2^2) - 2z_\alpha(-i_1 z_1 - i_2 z_2)}{(z_1^2 + z_2^2)^2} = 0 .$$

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