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REGULAR FUNCTIONS TRANSVERSAL AT INFINITY

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Abstract. We generalize and complete some of Maxim's recent results on Alexander invariants of a polynomial transversal to the hyperplane at infinity. Roughly speaking, and surprisingly, such a polynomial behaves, both topologically and algebraically (e.g., in terms of the variation of MHS on the cohomology of its smooth fibers), like a homogeneous polynomial.

1. Introduction and the main results. In the last twenty years there has been an ever increasing interest in the topology and geometry of polynomial functions with a certain good behavior at infinity, see for instance [2, 13, 26, 27, 29, 30, 34]. In particular, the point of view of constructible sheaves was useful, see [6]. An interesting problem in this area is to understand the Alexander invariants of the complements to affine hypersurfaces defined by such polynomial functions. Various approaches, some algebro-geometric, using the superabundances of linear systems associated with singularities (cf. Remark 5.3 in the last section), and others, more topological, using the monodromy representation were proposed (see for instance [18, 19, 17, 9, 28]). Recently, Maxim [23] has considered a similar interplay but in a more general framework, which includes hypersurfaces with no restrictions on singularities and a new and very natural condition of good behavior at infinity, that we describe now.

Let X be a reduced hypersurface in the complex affine space $\boldsymbol{C}^{n+1}$ with $n \geq 1$, given by an equation $f = 0$. We say that the polynomial function $f : \boldsymbol{C}^{n+1} \to \boldsymbol{C}$ (or the affine hypersurface X) is ∞-transversal if the closure V of X in the corresponding complex projective space $\boldsymbol{P}^{n+1}$ is transversal in the stratified sense to the hyperplane at infinity $H = \boldsymbol{P}^{n+1} \setminus \boldsymbol{C}^{n+1}$. Consider the affine complement $M_X = \boldsymbol{C}^{n+1} \setminus X$, and denote by M_X^c its *infinite cyclic covering* corresponding to the kernel of the homomorphism

$$f_\sharp : \pi_1(M_X) \to \pi_1(\boldsymbol{C}^*) = \boldsymbol{Z}$$

induced by f and sending a class of a loop into its linking number with X.

Then, for any positive integer k, the homology group $H_k(M_X^c, K)$, regarded as a module over the principal ideal domain $\Lambda_K = K[t, t^{-1}]$ with $K = \boldsymbol{Q}$ or $K = \boldsymbol{C}$, is called the k-th Alexander module of the hypersurface X, see [18, 9]. When this module is torsion, we denote by $\Delta_k(t)$ the corresponding k-th Alexander polynomial of X (i.e., the Λ_K-order of $H_k(M_X^c, K)$).

With this notation, one of the main results in [23] can be stated as follows.

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THEOREM 1.1. *Assume that $f : \boldsymbol{C}^{n+1} \to \boldsymbol{C}$ is ∞-transversal. Then, for $k < n+1$, the Alexander modules $H_k(M_X^c, K)$ of the hypersurface X are torsion semisimple Λ_K-modules which are annihilated by $t^d - 1$.*

Since M_X^c is an $(n+1)$-dimensional CW complex, one has $H_k(M_X^c, K) = 0$ for $k > n+1$, while $H_{n+1}(M_X^c, K)$ is free. In this sense, the above result is optimal. To get a flavor of the second main result in [23] describing the relationship between the orders of the Alexander modules and the singularities of X, see Proposition 3.3 below.

Now we describe the more general setting of our paper. Let $W' = W_0' \cup \cdots \cup W_m'$ be a hypersurface arrangement in $\boldsymbol{P}^N$ for $N > 1$. Let d_j denote the degree of W_j' and let $g_j = 0$ be a reduced defining equation for W_j' in $\boldsymbol{P}^N$. Let $Z \subset \boldsymbol{P}^N$ be a smooth complete intersection of dimension $n+1 > 1$ which is not contained in W', and let $W_j = W_j' \cap Z$ for $j = 0, \dots, m$ be the corresponding hypersurface in Z considered as subscheme defined by the principal ideal generated by g_j. Let $W = W_0 \cup \cdots \cup W_m$ denote the corresponding hypersurface arrangement in Z. We assume troughout in this paper that the following hold.

(H1) All the hypersurfaces W_j are distinct, reduced and irreducible; moreover W_0 is smooth.

(H2) The hypersurface W_0 is transverse in the stratified sense to $V = W_1 \cup \cdots \cup W_m$, i.e., if $\mathcal{S}$ is a Whitney regular stratification of V, then W_0 is transverse to any stratum $S \in \mathcal{S}$.

The complement $U = Z \setminus W_0$ is a smooth affine variety. We consider the hypersurface $X = U \cap V$ in U and its complement $M_X = U \setminus X$. Note that $M_X = M_W$, where $M_W = Z \setminus W$. We use both notations, each one being related to the point of view (affine or projective) that we wish to emphasize.

Recall that the construction of the Alexander modules and polynomials was generalized in an obvious way in [9] to the case when $\boldsymbol{C}^{n+1}$ is replaced by a smooth affine variety U. The first result is new even in the special situation considered in [23].

THEOREM 1.2. *Assume that d_0 divides the sum $\sum_{j=1}^m d_j$, say $dd_0 = \sum_{j=1}^m d_j$. Then one has the following.*

(i) *The function $f : U \to \boldsymbol{C}$ given by*

$$f(x) = \frac{g_1(x) \cdots g_m(x)}{g_0(x)^d}$$

is a well-defined regular function on U whose generic fiber F is connected.

(ii) *The restriction $f^* : M_X \to \boldsymbol{C}^*$ of f outside the zero fiber X has only isolated singularities. The affine variety U has the homotopy type of a space obtained from X by adding a number μ of n-cells, μ being equal to the sum of the Milnor numbers of the singularities of f^*.*

Note that we need the connectedness of F, since this is one of the general assumptions made in [9]. The second claim shows that a mapping transversal at infinity behaves like an M_0-tame polynomial, see [7] for the definition and the properties of M_0-tame polynomials. These two classes of mappings are, however, distinct, e.g., the defining equation of an essential affine hyperplane arrangement is always M_0-tame, but the transversality at infinity may well fail for it.

The next result says roughly that an ∞-transversal polynomial behaves as a homogeneous polynomial up-to (co)homology of degree $n-1$. In these degrees, the determination of the Alexander polynomial of X in U is reduced to the simpler problem of computing a monodromy operator.

COROLLARY 1.3. *With the assumption in Theorem 1.2, the following hold.*

(i) *Let $\iota : \boldsymbol{C}^* \to \boldsymbol{C}$ be the inclusion. Then, $R^0 f_* \boldsymbol{Q}_U = \boldsymbol{Q}_{\boldsymbol{C}}$ and, for each $0 < k < n$, there is a $\boldsymbol{Q}$-local system $\mathcal{L}_k$ on $\boldsymbol{C}^*$ such that*

$$R^k f_* \boldsymbol{Q}_U = \iota_! \mathcal{L}_k .$$

In particular, for each $0 < k < n$, the monodromy operators of f at the origin T_0^k and at infinity T_∞^k acting on $H^k(F, \boldsymbol{Q})$ coincide, and the above local system $\mathcal{L}_k$ is precisely the local system corresponding to this automorphism of $H^k(F, \boldsymbol{Q})$.

(ii) *There is a natural morphism $H^k(M_W^c, \boldsymbol{Q}) \to H^k(F, \boldsymbol{Q})$ which is an isomorphism for $k < n$ and a monomorphism for $k = n$, and which is compatible with the obvious actions. In particular, the associated characteristic polynomial*

$$\det(t\mathrm{Id} - T_0^k) = \det(t\mathrm{Id} - T_\infty^k)$$

coincides to the k-th Alexander polynomial $\Delta_k(X)(t)$ of X in U for $k < n$, and $\Delta_n(X)(t)$ divides the G.C.D.$(\det(t\mathrm{Id} - T_\infty^n), \det(t\mathrm{Id} - T_0^n))$.

The next result can be regarded as being similar to some results in [3], [20] and [11]. Indeed, in all these results, control over the singularities of W along just one of its irreducible components (in our case along W_0) implies that certain local systems on the complement M_W are non-resonant. See [6, p. 218] for a discussion in the case of hyperplane arrangements.

THEOREM 1.4. *Let $g = g_0 \cdots g_m = 0$ be the equation of the hypersurface arrangement W in Z and let $F(g)$ be the corresponding global Milnor fiber given by $g = 1$ in the cone CZ over Z. Then*

$$H^j(F(g), \boldsymbol{Q}) = H^j(M_W, \boldsymbol{Q})$$

for all $j < n+1$. In other words, the action of the monodromy on $H^j(F(g), \boldsymbol{Q})$ is trivial for all $j < n+1$.

The main result of the present paper is the following extension of Maxim's result stated in 1.1 to our more general setting described above.

THEOREM 1.5. *Assume that d_0 divides the sum $\sum_{j=1}^m d_j$, say $dd_0 = \sum_{j=1}^m d_j$. Then the following hold.*

(i) *The Alexander modules $H_k(M_X^c, \boldsymbol{Q})$ of the hypersurface X in U are torsion semi-simple $\Lambda_{\boldsymbol{Q}}$-modules which are annihilated by $t^d - 1$ for $k < n+1$.*

(ii) *For $k < n+1$, the Alexander module $H^k(M_X^c, \boldsymbol{Q})$ of the hypersurface X in U has a canonical mixed Hodge structure, compatible with the action of $\Lambda_{\boldsymbol{Q}}$, i.e., the multiplication by*

$t : H^k(M_X^c, \boldsymbol{Q}) \to H^k(M_X^c, \boldsymbol{Q})$ is a MHS isomorphism. Moreover, there is an epimorphism of MHS $p_d^* : H^k(M_X^d, \boldsymbol{Q}) \to H^k(M_X^c, \boldsymbol{Q})$, where M_X^d is the d-cyclic covering of M_X and $p_d : M_X^c \to M_X^d$ is the induced infinite cyclic covering.

Dually, for $k < n + 1$, the Alexander module $H_k(M_X^c, \boldsymbol{Q})$ of the hypersurface X in U has a canonical mixed Hodge structure, which is compatible with the natural embedding of $H_k(M_X^c, \boldsymbol{Q})$ as a subspace in the homology $H_k(M_X^d, \boldsymbol{Q})$.

The proof of the second claim in the above theorem, given in the last section, yields also the following consequence, stating that our regular function f behaves like a homogeneous polynomial.

COROLLARY 1.6. *With the above assumptions, the MHS on the cohomology $H^k(F_s, \boldsymbol{Q})$ of a smooth fiber F_s of f is independent of s for $k < n$. In this range, the isomorphism $H^k(M_X^c, \boldsymbol{Q}) \to H^k(F_s, \boldsymbol{Q})$ given by Corollary* 1.3 (ii) *is an isomorphism of MHS.*

MHS on Alexander invariants have already been considered in the case of hypersurfaces with isolated singularities in [19] (the case of plane curves is considered also in [17]). The above relation of this MHS to the one on the cohomology groups $H^k(F_s, \boldsymbol{Q})$ is new. Notice that Corollary 1 in [17], combined with the main result in [8] and Theorem 2.10 (ii) in [9], yields the following.

COROLLARY 1.7. *Let $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ be a polynomial function such that $X = f^{-1}(0)$ is reduced and connected. Then the action of t on $H_1(M_X^c, \boldsymbol{Q})$ is semisimple.*

No example seems to be known where the action of t on some $H_k(M_X^c, \boldsymbol{Q})$ is not semisimple. On the other hand, it is easy to find examples, even for $f : \boldsymbol{C}^2 \to \boldsymbol{C}$, where the monodromy at infinity operator T_1^∞ is not semisimple, see Example 5.1 below.

Note that, although in some important cases (see for instance [18]) the Hurewicz theorem gives the identification $H_n(M_X^c, \boldsymbol{Z}) = \pi_n(M_X)$, the existence of a mixed Hodge structure on the latter cannot be deduced for example from [25], since in [25] is considered only the situation when the action of the fundamental group on the homotopy groups is nilpotent, which in general is not the case for $\pi_n(M_X)$ and of course M_X^c is not quasi-projective in general.

The proofs we propose below use various techniques. Theorem 2.2 in Section 2 is the main topological result and is established via non-proper Morse theory as developed by Hamm [16] and Dimca-Papadima [10]. The first proof of (a special case of) the first claim in Theorem 1.5 in Section 4 is based on a version of Lefschetz hyperplane section theorem due to Goreski-MacPherson and based on stratified Morse theory.

The proofs in Section 3 are based on Theorem 4.2 in [9] (which relates Alexander modules to the cohomology of a class of rank one local systems on the complement M_W) and on a general idea of getting vanishing results via perverse sheaves (based on Artin's vanishing Theorem) introduced in [3] and developped in [6, Chapter 6].

Finally, the proofs in the last section use the existence of a Leray spectral sequence of a regular mapping in the category of mixed Hodge structures (MHS for short) for which we refer to Saito [31], [32] and [33]. To show the independence of the MHS on the Alexander module $H^k(M_X^c, \boldsymbol{Q})$ on the choice of a generic fiber of f, we use a result by Steenbrink-Zucker on the MHS on the subspace of invariant cocycles, see [35].

2. Topology of regular functions transversal at infinity. The following easy remark is used repeatedly in the sequel. The proof is left to the reader.

LEMMA 2.1. *If the hypersurface V in Z has a singular locus of positive dimension, i.e.*, $\dim V_{\text{sing}} > 0$, *and W_0 is transversal to V, then*

$$\dim V_{\text{sing}} = \dim(V_{\text{sing}} \cap W_0) + 1\,.$$

In particular, the singular locus V_{sing} cannot be contained in W_0.

Now we start to prove Theorem 1.2. In order to establish the first claim, note that the closure $\bar{F}$ of F is a general member of the pencil

$$g_1(x)\cdots g_m(x) - t\,g_0(x)^d = 0$$

of hypersurfaces in Z. As such, it is smooth outside the base locus given by

$$g_1(x)\cdots g_m(x) = g_0(x) = 0\,.$$

If $d = 1$, then for t large the above equation gives a smooth hypersurface on Z, thus a smooth complete intersection in $\boldsymbol{P}^N$ of dimension $n > 0$, and hence an irreducible variety.

For $d > 2$, a closer look shows that a singular point is located either at a point where at least two of the polynomials g_j for $1 \leq j \leq n$ vanish, or at a singular point in one of the hypersurfaces W_j for $1 \leq j \leq n$. It follows essentially by Lemma 2.1 that $\operatorname{codim}(\operatorname{Sing}(\bar{F})) \geq 3$, and hence $\bar{F}$ is irreducible in this case as well. This implies that F is connected.

The second claim is more involved. Fix a Whitney regular stratification $\mathcal{S}$ for the pair (Z, V) such that W_0 is transverse to $\mathcal{S}$. Let $\mathcal{S}'$ be the induced Whitney regular stratification of CZ, the cone over Z, whose strata are either the origin, or the pull-back of strata of $\mathcal{S}$ under the projection $p : CZ \setminus \{0\} \to Z$. Then the function $h = g_1 \cdots g_m : CZ \to \boldsymbol{C}$ is stratified by the stratifications $\mathcal{S}'$ on CZ and $\mathcal{T} = \{\boldsymbol{C}^*, \{0\}\}$ on $\boldsymbol{C}$, i.e., h maps submersively strata of $\mathcal{S}'$ onto strata of $\mathcal{T}$. Using Theorem 4.2.1 in [1], it follows that the stratification $\mathcal{S}'$ satisfies the Thom condition (a_h).

Let $F_0 = \{x \in CZ\ ;\ g_0(x) = 1\}$ be the global Milnor fiber of g_0 regarded as a function germ on the isolated complete intersection singularity $(CZ, 0)$. Since W_0 is smooth, it follows that CW_0 is also an isolated complete intersection singularity and hence F_0 has the homotopy type of a bouquet of $(n+1)$-dimensional spheres. Let $\Gamma(h, g_0)$ be the closure of the set of points $x \in (CZ \setminus CV)$ such that the differentials $d_x h$ and $d_x g_0$ are linearly dependant. Here and in the sequel we regard h and g_0 as regular functions on the cone CZ, in particular we have $\operatorname{Ker} d_x h \subset T_x CZ$ for any $x \in CZ \setminus \{0\}$. Then $\Gamma(h, g_0)$ is the *polar curve* of the pair of functions (h, g_0). To proceed, we need the following key technical result.

THEOREM 2.2. *With the notation above, the following hold.*

(i) $\dim \Gamma(h, g_0) \leq 1$.

(ii) *The set Σ_1 of the singularities of the restriction of the polynomial h to $F_0 \setminus CV$ is finite.*

(iii) *For any $t \in S^1$, the unit circle in $\boldsymbol{C}$, consider the pencil of intersections $(Z_{s,t})_{s \in \boldsymbol{C}}$ given by*

$$Z_{s,t} = CZ \cap \{g_0 = s\} \cap \{h = t\}\,.$$

Then it contains finitely many singular members, and each of them has only isolated singularities. Any intersection $Z_{0,t}$ is smooth.

(iv) *F_0 has the homotopy type of a space obtained from $F_0 \cap CV$ by adding $(n+1)$-cells. More precisely, for each critical value $b \in h(\Sigma_1)$ and each small closed disc D_b centered at b, the tube $h^{-1}(D_b)$ has the homotopy type of a space obtained from $h^{-1}(c)$ for $c \in \partial D_b$ by adding a number μ_b of $(n+1)$-cells, μ_b being equal to the sum of the Milnor numbers of the singularities of $h^{-1}(b)$.*

PROOF. Note first that $\Gamma(h, g_0)$ is $\boldsymbol{C}^*$-invariant. Hence, if $\dim \Gamma(h, g_0) \leq 1$, then $\Gamma(h, g_0)$ may be the empty set, the origin or a finite set of lines in CZ passing through the origin.

Assume that contrary to (i) one has $\dim \Gamma(h, g_0) > 1$. Then its image in Z has a positive dimension and hence there exist a curve C on Z along which the differentials $d_x h$ and $d_x g_0$ are linearly dependant. Let p be a point in the non-empty intersection $C \cap V$. It follows that the line L_p in $\boldsymbol{C}^{N+1}$ associated to p is contained in CZ and that h vanishes along this line. The chain rule implies that g_0 has a zero derivative along L_p, and hence $g_0 \mid L_p$ is constant. Since g_0 is a homogeneous polynomial and the line L_p passes through the origin, this constant is zero, i.e., g_0 vanishes along L_p. Therefore $p \in W_0 \cap V$. If p is a smooth point in V, then this contradicts already the transversality $W_0 \pitchfork V$. If not, let $S \in \mathcal{S}$ be the stratum containing p. $W_0 \pitchfork \mathcal{S}$ implies that $\dim S > 0$. Let $q \in L_p$ be any nonzero vector, and let $\gamma(t)$ be an analytic curve such that $\gamma(0) = q$ and $\gamma(t) \in \Gamma(h, g_0) \setminus CV$ for $0 < |t| < \varepsilon$. Then for $t \neq 0$, $h(\gamma(t)) \neq 0$ and hence $\operatorname{Ker} d_{\gamma(t)} h = \operatorname{Ker} d_{\gamma(t)} g_0$. Passing to the limit for $t \to 0$, we get

$$T = \lim \operatorname{Ker} d_{\gamma(t)} h = \lim \operatorname{Ker} d_{\gamma(t)} g_0 = T_q(CW_0)\,.$$

On the other hand, the Thom condition (a_h) implies that

$$T \supset T_q S' = T_q(CS)\,,$$

which yields $T_p W_0 \supset T_p S$, in contradiction to $W_0 \pitchfork \mathcal{S}$. The above argument shows that $\dim \Gamma(h, g_0) \leq 1$ and hence completes the proof of (i).

To prove (ii), just note that $d_q h \mid T_q F_0 = 0$ for some point $q \in F_0 \setminus CV$ implies $q \in \Gamma(h, g_0)$. Since any line through the origin intersects F_0 in at most d_0 points, the claim (ii) follows.

The last claim of (iii) is clear by homogeneity. The rest is based on the fact that any line through the origin intersects $g = t$ in finitely many points.

To prove (iv) we use the same approach as in the proof of Theorem 3 in [10], based on Proposition 11 in loc. cit. Namely, we start by setting $A = F_0$ and $f_1 = h$ and construct inductively the other polynomials $f_2, \dots, f_{N+1}$ to be generic homogeneous polynomials of degree d_0 as in [10, p. 485] (where generic linear forms are used for the same purpose). For more details on the non-proper Morse theory used here we refer to Hamm [16]. □

We continue now the proof of the second claim in Theorem 1.2. There is a cyclic covering $F_0 \to U$ of order d_0 which restrict to a similar covering

$$p : F_0 \setminus CV \to U \setminus X$$

satisfying $f = h \circ p$. Using this and the claim (ii) above, we see that the restriction $f^* : U \setminus X \to \boldsymbol{C}^*$ of f has only isolated singularities. Let G be the cyclic group of order d_0. Then G acts on F_0 as the monodromy group of the function g_0, i.e., the group spanned by the monodromy homeomorphism $x \mapsto \kappa \cdot x$ with $\kappa = \exp(2\pi\sqrt{-1}/d_0)$. Since $d_0|d$, the function h is G-invariant. Note that the above construction of F_0 from $F_0 \cap CV$ by adding $(n+1)$-cells was done in a G-equivariant way. This implies by passing to the G-quotients the last claim in Theorem 1.2. Alternatively, one can embed U into an affine space $\boldsymbol{C}^M$, using the Veronese mapping of degree d_0, and then use in this new affine setting Proposition 11 in [10]. This completes the proof of Theorem 1.2.

Note also that we have $\tilde{H}^k(U, \boldsymbol{Q}) = \tilde{H}^k(F_0, \boldsymbol{Q})^G = 0$ for $k < n + 1$. In particular, $\tilde{H}^k(X, \boldsymbol{Q}) = 0$ for $k < n$, i.e., X is rationally a bouquet of n-spheres. In fact, $F_0 \cap CV$ can be shown to be (homotopically) a bouquet of n-spheres and $X = F_0 \cap CV/G$.

PROOF OF COROLLARY 1.3. The first claim follows from Proposition 6.3.6 and Exercise 4.2.13 in [6] in conjunction to Theorem 2.10 v in [9]. Indeed, to get the vanishing of $(R^k f_* \boldsymbol{C}_U)_0$ one has just to write the exact sequence of the triple (U, T_0, F) and to use the fact that $\tilde{H}^k(U, \boldsymbol{C}) = 0$ for $k < n + 1$ as seen above. For the second claim, one has to use Theorem 2.10 i and Proposition 2.18 in [9]. Indeed, let D be a large disc in $\boldsymbol{C}$ containing all the critical values of $f : U \to \boldsymbol{C}$ inside. Then $\boldsymbol{C}^*$ is obtained from $E = \boldsymbol{C} \setminus D$ by filling in small discs D_b around each critical value $b \neq 0$ of f. In the same way, M_X is obtained from $E_1 = f^{-1}(E)$ by filling in the corresponding tubes $T_b = f^{-1}(D_b)$. It follows from Theorem 2.2 (iv) that the inclusion $E_1 \to M_X$ is an n-equivalence. Now the total space of restriction of the cyclic covering $M_X^c \to M_X$ to the subspace E_1 is homotopy equivalent to the generic fiber F of f, in such a way that the action of t corresponds to the monodromy at infinity. In this way we get an n-equivalence $F \to M_X^c$, inducing the isomorphisms (resp. the monomorphism) announced in Corollary 1.3 (ii).

To get the similar statement for the monodromy operator T_0, we have to build $\boldsymbol{C}^*$ from a small punctured disc D_0^* centered at the origin by filling in small discs D_b around each critical value $b \neq 0$ of f. The rest of the above argument applies word for word.

The pull-back under p of the infinite cyclic covering $M_X^c \to M_X$ is just the infinite cyclic covering $(F_0 \setminus CV)^c \to F_0 \setminus CV$, and we get an induced cyclic covering $p^c : (F_0 \setminus CV)^c \to M_X^c$ of order d_0. Moreover, the action of the deck transformation group G of this covering commutes to the action of the infinite cyclic group $\boldsymbol{Z}$, and hence we get the following isomorphism (resp. projection, embedding) of $\Lambda_{\boldsymbol{Q}}$-modules

$$H^k(M_X^c, \boldsymbol{Q}) = H_k((F_0 \setminus CV)^c, \boldsymbol{Q})_G \leftarrow H_k((F_0 \setminus CV)^c, \boldsymbol{Q}) \tag{2.1}$$

and

$$H^k(M_X^c, \boldsymbol{Q}) = H^k((F_0 \setminus CV)^c, \boldsymbol{Q})^G \rightarrow H^k((F_0 \setminus CV)^c, \boldsymbol{Q})\,. \tag{2.2}$$

3. Perverse sheaf approach. In this section we prove the following weaker version of Theorem 1.5, which is used in the proof of Theorem 1.5, see Subsection 4.2.

PROPOSITION 3.1. *Assume that d_0 divides the sum $\sum_{j=1}^m d_j$, say $dd_0 = \sum_{j=1}^m d_j$. Then the Alexander modules $H_k(M_X^c, \boldsymbol{C})$ of the hypersurface X are torsion for $k < n+1$. Moreover, let $\lambda \in \boldsymbol{C}^*$ be such that $\lambda^d \neq 1$. Then λ is not a root of the Alexander polynomials $\Delta_k(t)$ for $k < n+1$.*

The proof of this proposition we give below is close in spirit to the proofs in [23], and yields with obvious minor changes (left to the reader) a proof for our Theorem 1.4.

According to Theorem 4.2 in [9], to prove Proposition 3.1, it is enough to prove the following

PROPOSITION 3.2. *Let $\lambda \in \boldsymbol{C}^*$ be such that $\lambda^d \neq 1$, where d is the quotient of $\sum_{j=1}^m d_j$ by d_0. If $\mathcal{L}_\lambda$ denotes the corresponding local system on M_W, then $H_q(M_W, \mathcal{L}_\lambda) = 0$ for all $q \neq n+1$.*

PROOF. First we recall the construction of the rank one local system $\mathcal{L}_\lambda$. Any such local system on M_W is given by a homomorphism from $\pi_1(M_W)$ to $\boldsymbol{C}^*$. To define our local system, consider the composition

$$\pi_1(M_W) \rightarrow \pi_1(M'_W) \rightarrow H_1(M'_W) = \boldsymbol{Z}^{m+1}/(d_0, \ldots, d_m) \rightarrow \boldsymbol{C}^*,$$

where the first morphism is induced by the inclusion, the second is the passage to the abelianization and the third one is given by sending the classes $e_0, \ldots, e_m$ corresponding to the canonical basis of $\boldsymbol{Z}^{m+1}$ to $\lambda^{-d}, \lambda, \ldots, \lambda$, respectively. For the isomorphism in the middle, see for instance [5, p. 102].

It is of course enough to show the vanishing in cohomology, i.e., $H^q(M_W, \mathcal{L}_\lambda) = 0$ for all $q \neq n+1$. Let $i : M_W \rightarrow U$ and $j : U \rightarrow Z$ be the two inclusions. Then one clearly has $\mathcal{L}_\lambda[n+1] \in \mathrm{Perv}(M_W)$, the abelian category of $\boldsymbol{C}$-perverse sheaves on the variety M_W, see for details [6]. It follows that $\mathcal{F} = Ri_*(\mathcal{L}_\lambda[n+1]) \in \mathrm{Perv}(U)$, since the inclusion i is a quasi-finite affine morphism. See [6, p. 214] for a similar argument.

Our vanishing result will follow from a study of the natural morphism

$$Rj_!\mathcal{F} \rightarrow Rj_*\mathcal{F}\,.$$

Extend it to a distinguished triangle

$$Rj_!\mathcal{F} \rightarrow Rj_*\mathcal{F} \rightarrow \mathcal{G} \rightarrow\ .$$

Using the long exact sequence of hypercohomology coming from the above triangle, we see exactly as on [6, p. 214] that all we have to show is that $\boldsymbol{H}^k(Z, \mathcal{G}) = 0$ for all $k < 0$. This vanishing obviously holds if we show that $\mathcal{G} = 0$.

This in turn is equivalent to the vanishing of all the local cohomology groups of $Rj_*\mathcal{F}$, namely $H^m(M_x, \mathcal{L}_x) = 0$ for all $m \in \boldsymbol{Z}$ and for all points $x \in W_0$. Here $M_x = M_W \cap B_x$ for B_x a small open ball at x in Z and $\mathcal{L}_x$ is the restriction of the local system $\mathcal{L}_\lambda$ to M_x.

The key observation is that, as already stated above, the action of an oriented elementary loop about the hypersurface W_0 in the local systems $\mathcal{L}_\lambda$ and $\mathcal{L}_x$ corresponds to multiplication by $\nu = \lambda^{-d} \neq 1$.

There are two cases to be considered.

Case 1. If $x \in W_0 \setminus V$, then M_x is homotopy equivalent to $\boldsymbol{C}^*$ and the corresponding local system $\mathcal{L}_\nu$ on $\boldsymbol{C}^*$ is defined by multiplication by ν. Hence the claimed vanishings are obvious.

Case 2. If $x \in W_0 \cap V$, then due to the local product structure of stratified sets cut by a transversal, M_x is homotopy equivalent to a product $(B' \setminus (V \cap B')) \times \boldsymbol{C}^*$, with B' a small open ball centered at x in W_0, and the corresponding local system is an external tensor product, the second factor being exactly $\mathcal{L}_\nu$. The claimed vanishings follow then from the Künneth Theorem, see [6, 4.3.14]. □

A minor variation of this proof gives also Theorem 1.4. Indeed, let $D = \sum_{j=0}^{m} d_j$ and let α be a D-root of unity, $\alpha \neq 1$. All we have to show is that $H^q(M_W, \mathcal{L}_\alpha) = 0$ for all $q \neq n+1$, see for instance [6, 6.4.6].

The action of an oriented elementary loop about the hypersurface W_0 in the local systems $\mathcal{L}_\alpha$ and in its restrictions $\mathcal{L}_x$ as above corresponds to multiplication by $\alpha \neq 1$. Therefore the above proof works word for word.

One has also the following result, in which the bounds are weaker than those in Maxim's Theorem 4.2 in [23].

PROPOSITION 3.3. *Assume that d_0 divides the sum $\sum_{j=1}^{m} d_j$, say $dd_0 = \sum_{j=1}^{m} d_j$. Let $\lambda \in \boldsymbol{C}^*$ be such that $\lambda^d = 1$ and let σ be a non negative integer. Assume that λ is not a root of the q-th local Alexander polynomial $\Delta_q(t)_x$ of the hypersurface singularity (V, x) for any $q < n+1-\sigma$ and any point $x \in W_1$, where W_1 is an irreducible component of W different from W_0. Then λ is not a root of the global Alexander polynomials $\Delta_q(t)$ associated to X for any $q < n+1-\sigma$.*

To prove this result, we start by the following general remark.

REMARK 3.4. If S is an s-dimensional stratum in a Whitney stratification of V such that $x \in S$ and W_0 is transversal to V at x, then, due to the local product structure, the q-th reduced local Alexander polynomial $\Delta_q(t)_x$ is the same as that of the hypersurface singularity $V \cap T$ obtained by cutting the germ (V, x) by an $(n+1-s)$-dimensional transversal T. It follows that these reduced local Alexander polynomials $\Delta_q(t)_x$ are all trivial except for $q \leq n-s$. It is a standard fact that, in the local situation of a hypersurface singularity, the

Alexander polynomials can be defined either from the link or as the characteristic polynomials of the corresponding the monodromy operators. Indeed, the local Milnor fiber is homotopy equivalent to the corresponding infinite cyclic covering.

Let $i : M_W \to Z \setminus W_1$ and $j : Z \setminus W_1 \to Z$ be the two inclusions. Then one has $\mathcal{L}_\lambda[n+1] \in \mathrm{Perv}(M_W)$ and hence $\mathcal{F} = Ri_*(\mathcal{L}_\lambda[n+1]) \in \mathrm{Perv}(Z \setminus W_1)$, exactly as above. Extend now the natural morphism $Rj_!\mathcal{F} \to Rj_*\mathcal{F}$ to a distinguished triangle

$$Rj_!\mathcal{F} \to Rj_*\mathcal{F} \to \mathcal{G}\,.$$

Applying Theorem 6.4.13 in [6] to this situation, and recalling the above use of Theorem 4.2 in [9], all we have to check is that $H^m(M_x, \mathcal{L}_x) = 0$ for all points $x \in W_1$ and $m < n+1-\sigma$. For $x \in W_1 \setminus W_0$, this claim is clear by the assumptions made. The case when $x \in W_1 \cap W_0$ can be treated exactly as above, using the product structure, and the fact that the monodromy of (W_1, x) is essentially the same as that of $(W_1 \cap W_0, x)$, see our remark above.

This completes the proof of Proposition 3.3.

REMARK 3.5. Here is an alternative explanation for some of the bounds given in Theorem 4.2 in [23]. Assume that λ is a root of the Alexander polynomial $\Delta_i(t)$ for some $i < n+1$. Then it follows from Proposition 3.3 that there exist a point $x \in W_1$ and an integer $l \le i$ such that λ is a root of the local Alexander polynomial $\Delta_l(t)_x$. If $x \in S$, with S a stratum of dimension s, then by Remark 3.4, we have $l \le n - s$. This provides half of the bounds in Theorem 4.2 in [23]. The other half comes from the following remark. Since λ is a root of the Alexander polynomial $\Delta_i(t)$, it follows that $H^i(M_W, \mathcal{L}_\lambda) \neq 0$. This implies via an obvious exact sequence that $\boldsymbol{H}^{i-n-1}(W_1, \mathcal{G}) \neq 0$. Using the standard spectral sequence to compute this hypercohomology group, we get that some of the groups $H^p(W_1, \mathcal{H}^{i-n-1-p}\mathcal{G})$ are non zero. This can hold only if $p \le 2\dim(\mathrm{Supp}\,\mathcal{H}^{i-n-1-p}\mathcal{G})$. Since $\mathcal{H}^{i-n-1-p}\mathcal{G}_x = H^{i-p}(M_x, L_x)$, this yields the inequality $p = i - l \le 2s$ in Theorem 4.2 in [23].

REMARK 3.6. Let $\lambda \in \boldsymbol{C}^*$ be such that $\lambda^d = 1$, where d, the quotient of $\sum_{j=1}^m d_j$ by d_0, is assumed to be an integer. Let $\mathcal{L}_\lambda$ denotes the corresponding local system on M_W. The fact that the associated monodromy about the divisor W_0 is trivial can be restated as follows. Let $\mathcal{L}'_\lambda$ be the rank one local system on $M_V = Z \setminus V$ associated to λ. Let $j : M_W \to M_V$ be the inclusion. Then

$$\mathcal{L}_\lambda = j^{-1}\mathcal{L}'_\lambda\,.$$

Let moreover $\mathcal{L}''_\lambda$ denote the restriction to $\mathcal{L}'_\lambda$ to the smooth divisor $W_0 \setminus (V \cap W_0)$. Then we have the following Gysin-type long exact sequence

$$\begin{aligned} \cdots \to H^q(M_V, \mathcal{L}'_\lambda) \to H^q(M_W, \mathcal{L}_\lambda) \to H^{q-1}(W_0 \setminus (V \cap W_0), \mathcal{L}''_\lambda) \\ \to H^{q+1}(M_V, \mathcal{L}'_\lambda) \to \cdots \end{aligned}$$

exactly as in [6, p. 222].

Since the cohomology groups $H^*(M_V, \mathcal{L}'_\lambda)$ and $H^*(W_0 \setminus (V \cap W_0), \mathcal{L}''_\lambda)$ are usually simpler to compute than $H^*(M_W, \mathcal{L}_\lambda)$, this exact sequence can give valuable information on the latter cohomology groups.

4. Semisiplicity results. In this section we prove the first claim in our main result Theorem 1.5.

4.1. First proof (the case $W_0 = H$ is the hyperplane at infinity in $\boldsymbol{P}^{n+1}$). Let $\mathcal{U}$ be a sufficiently small tubular neighborhood of the hyperplane H at infinity. We claim the following:

(i) $\pi_i(\mathcal{U} \setminus (H \cup (V \cap \mathcal{U}))) \to \pi_i(M_X)$ is an isomorphism for $1 \le i \le n-1$.

(ii) $\pi_n(\mathcal{U} \setminus (H \cup (V \cap \mathcal{U}))) \to \pi_n(M_X)$ is surjective.

First notice that, as a consequence of transversality of V and H, we have an S^1-fibration $\mathcal{U} \setminus (H \cup (V \cap \mathcal{U})) \to H \setminus (H \cap V)$. Indeed, if $f(x_0, \dots, x_{n+1}) = 0$ is an equation of V and $x_0 = 0$ is the equation for H, then the pencil $\lambda f(x_0, \dots, x_{n+1}) + \mu f(0, x_1, \dots, x_{n+1})$ defines a deformation of V to the cone over $V \cap H$. Since V is transversal to H, this pencil contains an isotopy of $\mathcal{U} \cap V$ into the intersection of $\mathcal{U}$ with the cone.

Let Y denote the above cone in $\boldsymbol{P}^{n+1}$ over $V \cap H$. The obvious $\boldsymbol{C}^*$-bundle $\boldsymbol{P}^{n+1} \setminus (Y \cup H) \to H \setminus (H \cap V)$ is homotopy equivalent to the above S^1-bundle $\mathcal{U} \setminus (H \cup (V \cap \mathcal{U})) \to H \setminus (H \cap V)$. We can apply to both M_X and $\boldsymbol{P}^{n+1} \setminus (Y \cup H)$ the Lefschetz hyperplane section theorem for stratified spaces (cf. [15, Theorem 4.3]), using a generic hyperplane H'. Thus for $i \le n-1$ we obtain the isomorphisms:

$$\pi_i(M_X) = \pi_i(M_X \cap H') = \pi_i((\boldsymbol{P}^{n+1} \setminus (Y \cup H)) \cap H') = \pi_i(\boldsymbol{P}^{n+1} \setminus (Y \cup H))$$

(the middle isomorphism takes place since, for H' near H, both spaces are isotopic). This yields (i).

To see (ii), let us apply Lefschetz hyperplane section theorem to a hyperplane H' belonging to $\mathcal{U}$. We obtain the surjectivity of the map given by the following composition:

$$\pi_i(H' \setminus (V \cup H)) \to \pi_i(\mathcal{U} \setminus (H' \cup (V \cap \mathcal{U}))) \to \pi_i(M_X)\,.$$

Hence the right map is surjective as well.

The relations (i) and (ii) yield that M_X has the homotopy type of a complex obtained from $\mathcal{U} \setminus (H \cup (V \cap \mathcal{U}))$ by adding cells having the dimension greater than or equal to $n+1$. Hence the same is true for the infinite cyclic covers defined as in Section 1 for M_X and $\mathcal{U} \setminus (H \cup (V \cap \mathcal{U}))$, respectively. Denoting by $(\mathcal{U} \setminus (H \cup (V \cap \mathcal{U})))^c$ the infinite cyclic cover of the latter, we obtain that

$$H_i((\mathcal{U} \setminus \mathcal{U} \cap (V \cup H))^c, \boldsymbol{Q}) \to H_i(M_X^c, \boldsymbol{Q}) \tag{4.1}$$

is surjection for $i = n$ and the isomorphism for $i < n$. Since the maps above are induced by an embedding map, they are isomorphisms or surjections of $\Lambda_{\boldsymbol{Q}}$-modules.

As it was mentioned above, since V is transversal to H, the space $\mathcal{U} \setminus (\mathcal{U} \cap (V \cup H))$ is homotopy equivalent to the complement in the affine space to the cone over the projective hypersurface $V \cap H$. On the other hand, the complement in $\boldsymbol{C}^{n+1}$ to the cone over $V \cap H$ is homotopy equivalent to the complement to $V \cap S^{2n+1}$ in S^{2n+1}, where S^{2n+1} is a sphere about the vertex of the cone. The latter, by Milnor's theorem [24], is fibered over the circle. Hence the fiber of this fibration, as the Milnor fiber of any hypersurface singularity, is homotopy equivalent to the infinite cyclic cover of $S^{2n+1} \setminus V \cap S^{2n+1} \approx \boldsymbol{C}^{n+1} \setminus V$. As in Section 1, this cyclic cover is the one corresponding to the kernel of the homomorphism of the fundamental group given by the linking number. In particular, since a Milnor fiber is a finite CW-complex, $H_i(\mathcal{U} \setminus (\mathcal{U} \cap (V \cup H))^c, \boldsymbol{Q})$ is a finitely generated $\boldsymbol{Q}$-module and hence a torsion $\Lambda_{\boldsymbol{Q}}$-module. Moreover, the homology of the Milnor fiber of a cone and hence $H_i(\mathcal{U} \setminus (\mathcal{U} \cap (V \cup H))^c, \boldsymbol{C})$ is annihilated by $t^d - 1$, since the monodromy on $f(0, x_1, \ldots, x_{n+1}) = 1$ is given by multiplication of coordinates by a root of unity of degree d and hence has the order equal to d. Therefore it follows from the surjectivity of (4.1) that the same is true for $H_i(M_X^c)$. In particular, $H_i(M_X^c)$ is semisimple.

4.2. Second proof (the general case). Using Equation 2.1 and Proposition 3.1, it suffices to show that the Alexander invariant $A_k = H_k((F_0 \setminus CV)^c, \boldsymbol{Q})$ of the hypersurface $h = 0$ in the affine variety F_0 is a torsion semisimple $\Lambda_{\boldsymbol{Q}}$-module killed by $t^e - 1$ for some integer e. Indeed, once we know that t is semisimple on $H_k(M_X^c, \boldsymbol{Q})$, Proposition 3.1 implies that $t^d = 1$.

The fact that A_k is torsion follows from Theorem 2.10 v in [9] and the claim (iv) in Theorem 2.2. Moreover, Theorem 2.10 ii in [9] gives for $k \leq n$, an epimorphism of $\Lambda_{\boldsymbol{Q}}$-modules

$$H_k(F_1, \boldsymbol{Q}) \to A_k\,, \tag{4.2}$$

where F_1 is the generic fiber of $h : F_0 \to \boldsymbol{C}$ and t acts on $H_k(F_1, \boldsymbol{Q})$ via the monodromy at infinity. By definition, the monodromy at infinity of $h : F_0 \to \boldsymbol{C}$ is the monodromy of the fibration over the circle S^1_R, centered at the origin and of radius $R >> 0$, given by

$$\{x \in CZ\,;\ f_0(x) = 1,\ |h(x)| = R\} \to S^1_R\,, \quad x \mapsto h(x)\,.$$

Using a rescaling, this is the same as the fibration

$$\{x \in CZ\,;\ f_0(x) = \varepsilon,\ |h(x)| = 1\} \to S^1\,, \quad x \mapsto h(x)\,, \tag{4.3}$$

where $0 < \varepsilon << 1$.

Let $R_1 >> 0$ be such that

$$\{x \in CZ\,;\ |x| \leq R_1,\ |h(x)| = 1\} \to S^1\,, \quad x \mapsto h(x)$$

is a proper model of the Milnor fibration of $h : CZ \to \boldsymbol{C}$. This implies that all the fibers $\{h = t\}$ for $t \in S^1$ are transversal to the link $K = CZ \cap S^{2N+1}_{R_1}$.

A similar argument, involving the Milnor fibration of $h : CW_0 \to \boldsymbol{C}$ shows that all the fibers $\{h = t\}$ for $t \in S^1$ are transversal to the link $K_0 = CW_0 \cap S^{2N+1}_{R_1}$. Using the usual S^1-actions on these two links, we see that transversality for all fibers $\{h = t\}$ for $t \in S^1$ is the same as transversality for $\{h = 1\}$. But saying that $\{h = 1\}$ is transversal to K_0 is the same as saying that $Z_{0,1} \pitchfork K$. By the compactness of K, there is a $\delta > 0$ such that $Z_{s,1} \pitchfork K$

for $|s| < \delta$. Using the above S^1-actions on links, this implies that $Z_{s,1} \pitchfork K$ for $|s| < \delta$ and $t \in S^1$.

Choose δ small enough such that the open disc D_δ centered at the origin and of radius δ is disjoint from the finite set of circles $g_0(\Gamma(h, g_0) \cap h^{-1}(S^1))$. Using the relative Ehresmann Fibration Theorem, see for instance [5, p. 15], we see that the map

$$\{x \in CZ\ ;\ |x| \leq R_1,\ f_0(x) < \delta,\ |h(x)| = 1\} \to D_\delta \times S^1\,, \quad x \mapsto (g_0(x), h(x))$$

is a locally trivial fibration. It follows that the two fibrations

$$\{x \in CZ\ ;\ |x| \leq R_1,\ f_0(x) = \delta/2,\ |h(x)| = 1\} \to S^1\,, \quad x \mapsto h(x)$$

and

$$\{x \in CZ\ ;\ |x| \leq R_1,\ f_0(x) = 0,\ |h(x)| = 1\} \to S^1\,, \quad x \mapsto h(x)$$

are fiber equivalent. In particular, they have the same monodromy operators. The first of these two fibration is clearly equivalent to the monodromy at infinity fibration 4.3. The homogeneity of the second of these two fibrations implies that its monodromy operator has order $e = dd_0$. This completes the proof of the semisimplicity claim in the general case.

5. Mixed Hodge structures on Alexander invariants. This proof involves several mappings and the reader may find useful to draw them all in a diagram.

Since the mapping $f : M_X \to \boldsymbol{C}^*$ has a monodromy of order d (at least in dimensions $k < n$, see Corollary 1.3 and Theorem 1.5 (i)), it is natural to consider the base change $\phi : \boldsymbol{C}^* \to \boldsymbol{C}^*$ given by $s \mapsto s^d$. Let $f_1 : M_X^d \to \boldsymbol{C}^*$ be the pull-back of $f : M_X \to \boldsymbol{C}^*$ under ϕ, and let $\phi_1 : M_X^d \to M_X$ be the induced mapping, which is clearly a cyclic d-fold covering. It follows that the infinite cyclic covering $p_c : M_X^c \to M_X$ factors through M_X^d, i.e., there is an infinite cyclic covering $p_d : M_X^c \to M_X^d$ corresponding to the subgroup $\langle t^d \rangle$ in $\langle t \rangle$, such that $\phi_1 \circ p_d = p_c$. Since $M_X^d = M_X^c / \langle t^d \rangle$, it follows that t induces an automorphism t of M_X^d of order d.

Let $F_1 = f_1^{-1}(s)$ be a generic fiber of f_1, with $|s| >> 0$. Then ϕ_1 induces a regular homeomorphism $F_1 \to F = f^{-1}(s^d)$. Let $i : F \to M_X$ and $i_1 : F_1 \to M_X^d$ be the two inclusions. Note the i has a lifting $i_c : F \to M_X^c$, which is exactly the n-equivalence mentionned in the proof of the Corollary 1.3, commuting at the cohomology level with the actions of t and T_∞. Moreover, i has a lifting $i_d = p_d \circ i_c$ such that $i = \phi_1 \circ i_d$.

Now we consider the induced morphisms on the various cohomology groups. It follows from the general spectral sequences relating the cohomology of M_X^c and $M_X^c / \langle t^d \rangle$, see [36, p. 206], that $p_d^* : H^k(M_X^d) \to H^k(M_X^c)$ is surjective. It follows that $H^k(M_X^c)$ is isomorphic (as a $\boldsymbol{Q}$-vector space endowed to the automorphism t) via i_c^* to the sub MHS in $H^k(F)$ given by $i_d^*(H^k(M_X^d))$. Note that i_d can be realized by a regular mapping and i_d^* commutes with the actions of t and T_∞.

There is still one problem to solve, namely to show that this MHS is independent of s, unlike the MHS $H^k(F, \boldsymbol{Q})$ which depends in general on s, see the example below. To do this, note that $\phi_1^* \circ i_d^*(H^k(M_X^d)) = i_1^*(H^k(M_X^d))$ as MH substructures in $H^k(F_1, \boldsymbol{Q})$. More precisely, $i_1^*(H^k(M_X^d))$ is contained in the subspace of invariant cocycles $H^k(F_1, \boldsymbol{Q})^{\mathrm{inv}}$, where inv means invariant with respect to the monodromy of the mapping $f_2 : M_2 \to S_2$ obtained from f_1 by deleting all the singular fibers, e.g., $S_2 = \boldsymbol{C}^* \setminus C(f_1)$, where $C(f_1)$ is the finite set of critical values of f_1. We have natural morphisms of MHS

$$H^k(M_X^d) \to H^k(M_2) \to H^0(S_2, R^k f_{2,*}\boldsymbol{Q}),$$

where the first is induced by the obvious inclusion and the second comes from the Leray spectral sequence of the map f_2, see [31, 5.2.17–18], [32, 4.6.2] and [33]. Moreover, the last morphism above is surjective. On the other hand, there is an isomorphisms of MHS

$$H^0(S_2, R^k f_{2,*}\boldsymbol{Q}) \to H^k(F_1, \boldsymbol{Q})^{\mathrm{inv}},$$

showing that the latter MHS is independent of s, see [35, Prop. 4.19]. It follows that $i_1^*(H^k(M_X^d))$ has a MHS which is independent of s. By transport, we get a natural MHS on $H^k(M_X^c, \boldsymbol{Q})$, which clearly satisfies all the claims in Theorem 1.5 (ii).

EXAMPLE 5.1. For $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ given by $f(x, y) = x^3 + y^3 + xy$, let F_s denote the fiber $f^{-1}(s)$. Then the MHS on $H^1(F_s, \boldsymbol{Q})$ (for F_s smooth) depends on s. Indeed, it is easy to see that the graded piece $Gr_1^W H^1(F_s, \boldsymbol{Q})$ coincides as a Hodge structure to $H^1(C_s, \boldsymbol{Q})$, where C_s is the elliptic curve

$$x^3 + y^3 + xyz - sz^3 = 0.$$

Moreover, it is known that $H^1(C_s, \boldsymbol{Q})$ and $H^1(C_t, \boldsymbol{Q})$ are isomorphic as Hodge structures if and only if the elliptic curves C_s and C_t are isomorphic, i.e., $j(s) = j(t)$, where j is the j-invariant of an elliptic curve. This proves our claim and shows that the range in Corollary 1.6 is optimal.

For $f : \boldsymbol{C}^2 \to \boldsymbol{C}$ given by $f(x, y) = (x + y)^3 + x^2y^2$ it is known that the monodromy at infinity operator has a Jordan block of size 2 corresponding to the eigenvalue $\lambda = -1$, see [14], and this should be compared to our Corollary 1.7 above.

REMARK 5.2. A “down to earth” relation between the cohomology of M_X^c and M_X^d used above and obtained from the spectral sequence [36] can be described also using the “Milnor's exact sequence”, i.e., the cohomology sequence corresponding to the sequence of chain complexes:

$$0 \to C_*(M_X^c) \to C_*(M_X^c) \to C_*(M_X^d) \to 0.$$

This is a sequence of free $\boldsymbol{Q}[t, t^{-1}]$-modules with the left homomorphism given by multiplication by $t^d - 1$. The corresponding cohomology sequence is

$$0 \to H^i(M_X^c) \xrightarrow{\iota} H^{i+1}(M_X^d) \to H^{i+1}(M_X^c) \to 0. \tag{5.1}$$

The zeros on the left and the right in (5.1) appear because of the mentioned earlier triviality of the action of t^d on cohomology. Another way to derive (5.1) is to consider the Leray spectral sequence corresponding to the classifying map $M_X^c \to BS^1 = \boldsymbol{C}^*$ corresponding to the action of t. This spectral sequence degenerates in term E_2 and is equivalent to the sequence (5.1). A direct argument shows that the image of ι coincides with the kernel of the cup product $H^1(M_X^d) \otimes H^{i+1}(M_X^d) \to H^{i+2}(M_X^d)$ (i.e., the annihilator of $H^1(M_X^d)$) which also yields the MHS on $H^i(M_X^c)$ as a subMHS on $H^{i+1}(M_X^d)$.

REMARK 5.3. The above mixed Hodge structure plays a key role in the calculation of the first non-vanishing homotopy group of the complements to a hypersurface V in $\boldsymbol{P}^{n+1}$ with isolated singularities (cf. [19]). More precisely, in this paper for each $\kappa = \exp(2\pi k\sqrt{-1}/d)$ (with $0 \leq k \leq d-1$) and each point $P \in V \subset \boldsymbol{P}^{n+1}$ which is singular on V the ideal $\mathcal{A}_{P,\kappa}$ is associated (called there *the ideal of quasiadjunction*). These ideals glue together into a subsheaf $\mathcal{A}_\kappa \subset \mathcal{O}_{\boldsymbol{P}^{n+1}}$ of ideals having at a singular point P the stalk $\mathcal{A}_{P,\kappa}$ and $\mathcal{O}_Q$ at any other point $Q \in \boldsymbol{P}^{n+1} \setminus \mathrm{Sing}(V)$. It is shown in [19] that for the κ-eigenspace of t acting on $F^0H^n((\boldsymbol{P}^{n+1} \setminus (V \cup H))^c)$ one has

$$\dim F^0H^n((\boldsymbol{P}^{n+1} \setminus (V \cup H))^c)_\kappa = \dim H^1(\boldsymbol{P}^{n+1}, \mathcal{A}_\kappa(d-n-2-k))\,.$$

The right hand side can be viewed as the difference between actual and "expected" dimensions of the linear system of hypersurfaces of degree $d-n-2-k$, whose local equations belong to the ideals of quasiadjunction at the singular points of V. For the case of plane curves, see also [12, 22].

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