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k-REGULAR MAPPINGS OF 2^n-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT. A map $f\colon X \to R^n$ is said to be k-regular if whenever $x_1, \dots, x_k$ are distinct points of X, then $f(x_1), \dots, f(x_k)$ are linearly independent. Using configuration spaces and homological methods, it is shown that there does not exist a k-regular map from R^n into $R^{n(k-\alpha(k))+\alpha(k)-1}$ where $\alpha(k)$ denotes the number of ones in the dyadic expansion of k and n is a power of 2.

A continuous map $f\colon X \to R^n$ is said to be k-regular if whenever $x_1, \dots, x_k$ are distinct elements of X, then $f(x_1), \dots, f(x_k)$ are linearly independent. The study of k-regular maps is prompted by the theory of Čebyšev approximation. The reader is referred to [**12**, pp. 237–242] for the relationship between these two concepts.

Results on existence and nonexistence of k-regular maps can be found in [**1**], [**2**], [**4**]–[**7**]. In [**4**], David Handel and Fred Cohen, using algebraic-topological methods, obtained a nonexistence theorem about k-regular mappings of the plane. The object of the present paper is to generalize their result to k-regular mappings of R^n where n is a power of 2. We obtain an improvement upon the following result, for the case n a power of 2.

THEOREM 1 (BOLTJANSKIĬ-RYŠKOV-ŠAŠKIN). *If a $2k$-regular map of R^n into R^n exists, then $N \geqslant (n+1)k$.*

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THEOREM 2. *There does not exist a k-regular map of R^n into $n(k-\alpha(k)) + \alpha(k) - 1$ dimensional Euclidean space where $\alpha(k)$ denotes the number of ones in the dyadic expansion of k and n is a power of 2.*

In the proof we utilize algebraic-topological properties of the configuration space of R^n, denoted $F(R^n, k)$, is the subspace of $(R^n)^k$ consisting of ordered k-tuples of distinct points in R^n. The symmetric group Σ_k acts freely on $F(R^n, k)$ and orthogonally on R^k by permuting factors. Let $P_{n,k}$ denote the k-plane bundle $F(R^n, k) \times_{\Sigma_k} R^k \to F(R^n, k)/\Sigma_k$.

LEMMA 3. *If n and k are powers of 2, then $\bar{\omega}_{(n-1)(k-1)}(P_{n,k}) \neq 0$.*

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PROOF. All homology and cohomology groups are with $\mathbf{Z}_2$ coefficients. Let $\mathcal{C}_n$ denote the little n-cubes operad with associated monad C_n as constructed by J. P. May in [8]. By [8, Theorem 4.8], $F(R^n, k)$ is Σ_k-equivariantly homotopy equivalent to $\mathcal{C}_n(k)$. So, as in [4] replace $F(R^n, k)/\Sigma_k$ by $\mathcal{C}_n(k)/\Sigma_k$ in $P_{n,k}$.

The following composition is a classifying map for $P_{n,k}$;

$$\mathcal{C}_n(k)/\Sigma_k \xrightarrow{\sigma_k} \mathcal{C}_\infty(k)/\Sigma_k \cong B\Sigma_k \xrightarrow{\rho} BO(k)$$

where ρ is induced from the regular representation $\Sigma_k \to O(k)$ and σ_k is the direct limit of $\sigma_{m,k}$ where $\sigma_{m,k}$ is given in [8, p. 31].

Let [1] denote the element of $H_0(S^0)$ determined by the nonbase point of S^0. By [3, §3], $H_*(C_n S^0)$ is given in terms of the Dyer-Lashof operations on [1].

Suppose $k = 2^r$, then $I = (2^{r-1}(n-1), \ldots, 2(n-1), n-1)$ is an admissible sequence of degree $(n-1)(k-1)$ and excess $n-1$. By [3, §§1,4], $Q^I[1]$ is an element of $H_*(C_n S^0)$ and by filtration arguments given there it follows that $Q^I[1] \in H_*(\mathcal{C}_n(k)/\Sigma_k)$. Since σ_k is the restriction of a map of $\mathcal{C}_n$-spaces, we have

$$\sigma_{k*}(Q^I[1]) = Q^I[1].$$

Thus it suffices to show that $\langle \rho^*\bar{\omega}_{(n-1)(k-1)}, Q^I[1]\rangle \neq 0$, where $\langle\ ,\ \rangle$ denotes the Kronecker index and $\bar{\omega}_{(n-1)(k-1)}$ is the $(n-1)(k-1)$-universal dual Stiefel-Whitney class.

As a first step we now show

$$\langle \rho^*\bar{\omega}_{(n-1)(k-1)}, Q^I[1]\rangle = \langle \rho^*\omega_{k-1}^{n-1}, Q^I[1]\rangle.$$

By [10, p. 220], $\rho^*\bar{\omega}_{(n-1)(k-1)} = \rho^*(\Sigma_j c_j \omega_1^{j_1} \cdots \omega_{k-1}^{j_{k-1}})$ where j runs over all $(j_1, \ldots, j_{k-1})$ with $j_i \geqslant 0$, $\Sigma_{i=1}^{k-1} ij_i = (n-1)(k-1)$ and $c_j = (j_1 + \cdots + j_{k-1})!/j_1! \cdots j_{k-1}!$, $\rho^*\omega_k = 0$ since every k-plane bundle with structural group Σ_k admits a nowhere zero section.

Suppose $1 \leqslant i < k-1$ is odd, $k = 2^r$, $u \in H^*B\Sigma_k$, $L = (s_1, \ldots, s_r)$ and $\psi(Q^L[1]) = \Sigma Q^A[1] \otimes Q^B[1]$ where ψ is the diagonal map in homology. See [9, p. 6]. Thus

$$\left\langle \rho^*\omega_i u, Q^L[1]\right\rangle = \sum \langle \rho^*\omega_i, Q^A[1]\rangle\langle \rho^*u, Q^B[1]\rangle = 0.$$

This follows since for each A, the length of A is r and $Q^A = \Sigma Q^M$ where each M is admissible and of the same degree and length as A, and since we may assume the degree of each A is i. Hence each A has positive excess and by [11, Theorem 4.7], $\langle \rho^*\omega_i, Q^A[1]\rangle = 0$.

Thus

$$\left\langle \rho^*\bar{\omega}_{(n-1)(k-1)}, Q^I[1]\right\rangle = \sum_j \left\langle c_j \rho^*\omega_2^{j_2}\omega_4^{j_4} \cdots \omega_{k-2}^{j_{k-2}}\omega_{k-1}^{j_{k-1}}, Q^I[1]\right\rangle$$

where $j = (j_2, j_4, \ldots, j_{k-2}, j_{k-1})$ and $2j_2 + 4j_4 + \cdots + (k-2)j_{k-2} + (k-1)j_{k-1} = (n-1)(k-1)$.

For such j we now show $c_j \equiv 0 \pmod 2$ if $j_{k-1} \neq n-1$. Since j_{k-1} is odd, $j_{k-1} = 2^m S - 1$, S odd. Suppose $j_{k-1} \neq n-1$. Thus j_{k-1} can be written as $2^0 + 2^1 + \cdots + 2^{m-1}$ + other distinct powers of 2. Let $2 \leqslant i \leqslant k-2$ be even. If $j_i \neq 0$, then j_i can be written as 2^p + other distinct powers of 2 where p is minimal in this expression of j_i. By writing c_j as a product of binomial coefficients, $c_j = y(j_i + j_{k-1})!/j_i!j_{k-1}!$, for some integer y. Thus if $p < m-1$, then $c_j \equiv 0 \pmod 2$. We may thus assume that j_i is divisible by 2^m for i even. So we have;

$$2j_2 + \cdots + (k-2)j_{k-2} = 2^m(k-1)(n'-s) \tag{1}$$

where $n' = n/2^m$ is even. Since the L.H.S. of (1) is divisible by 2^{m+1} and the R.H.S. of (1) is not divisible by 2^{m+1} we have a contradiction. Hence

$$\langle \rho^* \bar{\omega}_{(n-1)(k-1)}, Q^I[1] \rangle = \langle \rho^* \omega_{k-1}^{n-1}, Q^I[1] \rangle.$$

We now show that $\langle \rho^* \omega_{k-1}^m, Q^{mJ}[1] \rangle = 1$ where $mJ = (2^{r-1}m, \ldots, 2m, m)$, $m \geqslant 1$, $k = 2^r$, concluding the proof of the lemma. It is easily seen, by induction on r, that $1J = J$ is the only admissible sequence of length r and degree $2^r - 1$ such that $Q^J \neq 0$. Next, using induction and the internal Cartan formula [**9**, p. 6], we see that if L is an admissible sequence of length r and degree $2^r - 1$, then $Q^L = 0$. By [**11**, Theorem 4.7], the diagonal Cartan formula [**9**, p. 6] and induction on m, $\langle \rho^* w_{k-1}^m, Q^{mJ}[1] \rangle = 1$.

In [**4**], the following result is proved.

THEOREM 4 (HANDEL-COHEN). *If a k-regular map of X into R^N exists, then $F(X, k) \times_{\Sigma_k} R^k \to F(X, k)/\Sigma_k$ admits an N-k-plane inverse.*

PROOF OF THEOREM 2. By Theorem 4, it suffices to show $\bar{\omega}_{(n-1)(k-\alpha(k))}(P_{n,k}) \neq 0$. Write $k = \sum_{i=1}^{\alpha(k)} j(i)$ where $j(i) = 2^{m(i)}$, $m(1) < m(2) < \cdots < m(\alpha(k))$. We have a map of k-plane bundles

$$f\colon P_{n,j(1)} \times \cdots \times P_{n,j(\alpha(k))} \to P_{n,k}$$

as follows: Choose pairwise disjoint open discs $E_1, \ldots, E_{\alpha(k)}$ in R^n. Then we can regard $P_{n,j(i)}$ as $F(E_i, j(i)) \times_{\Sigma_{j(i)}} R^{j(i)} \to F(E_i, j(i))/\Sigma_{j(i)}$. Define f by

$$f\big((x_1, v_1), \ldots, (x_{\alpha(k)}, v_{\alpha(k)})\big) = (x_1, \ldots, x_{\alpha(k)}; v_1, \ldots, v_{\alpha(k)}),$$
$$(x_i, v_i) \in F(E_i, j(i)) \times_{\Sigma_{j(i)}} R^{j(i)}.$$

Thus

$$f^* \bar{\omega}_{(n-1)(k-\alpha(k))}(P_{n,k}) = \bar{\omega}_{(n-1)(k-\alpha(k))}\big(P_{n,j(1)} \times \cdots \times P_{n,j(\alpha(k))}\big),$$

which has as a nonzero component, by Lemma 3,

$$\bar{\omega}_{(n-1)(j(1)-1)}\big(P_{n,j(1)}\big) \times \cdots \times \bar{\omega}_{(n-1)(j(\alpha(k))-1)}\big(P_{n,j(\alpha(k))}\big).$$

Thus $f^* \bar{\omega}_{(n-1)(k-\alpha(k))}(P_{n,k}) \neq 0$.

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