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Regular or stochastic dynamics in real analytic families of unimodal maps

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To Jacob Palis on his 60th birthday

Abstract. In this paper we prove that in any non-trivial real analytic family of quasiquadratic maps, almost any map is either regular (i.e., it has an attracting cycle) or stochastic (i.e., it has an absolutely continuous invariant measure). To this end we show that the space of analytic maps is foliated by codimension-one analytic submanifolds, "hybrid classes". This allows us to transfer the regular or stochastic property of the quadratic family to any non-trivial real analytic family.

Contents

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1. Introduction

In this paper we will consider the dynamics of unimodal maps of an interval I, i.e., smooth endomorphisms of I with a unique critical point that will be assumed to be quadratic. The simplest and most famous example of this kind is given by the real quadratic family:

$$Q_\lambda : [0, 1] \to [0, 1], \quad Q_\lambda(x) = \lambda x(1 - x),$$

where λ is a real parameter between 1 and 4. An 1976 article by R. May [May] had a big impact on the scientific community by demonstrating that this simple mathematical model exhibits a very interesting and complex dynamical behavior.

The interest in this special family grew further when Milnor and Thurston [MT] showed the *qualitative universality* of this family: any unimodal map has essentially the same dynamics as some quadratic map Q_λ. This statement becomes particularly complete if we restrict ourselves to maps with negative Schwarzian derivative: any such map is *topologically conjugate* to some quadratic map, as was shown by Guckenheimer and Misiurewicz in late 1970's. This suggests that a typical one parameter family $\{f_t\}$ of unimodal maps should have a similar qualitative dynamical evolution as the quadratic family. Discovery of Feigenbaum, Coullet and Tresser (made approximately at the same time) of *quantitative universality* of the quadratic family raised its significance even further.

In this paper we will describe a picture of an appropriate space of real analytic unimodal maps, which will give a justification for the special role of the quadratic family. We will use it to transfer some important dynamical properties from the quadratic family to any non-trivial real analytic family of *quasiquadratic maps* (a class which includes maps with negative Schwarzian derivative, see §2.8 for the precise definition).

A unimodal map is called *regular* if its critical point belongs to the basin of a hyperbolic periodic attractor and all its periodic orbits are hyperbolic. It is called *stochastic* if it has an invariant measure absolutely continuous with respect to the Lebesgue measure. Given a smooth one-parameter family $\{f_t\}$ of unimodal maps, we refer to a parameter t as regular or stochastic if the corresponding map f_t is such. The set of regular parameter values is always open. The set of stochastic parameter values has positive Lebesgue measure for an open set of families containing the quadratic family [J,BC]. In fact, in the case of the quadratic family $\{Q_\lambda\}$ much more is known:

- The set of regular parameter values λ is open and dense in the quadratic family [L4,GS2].
- The set of stochastic parameter values λ has full Lebesgue measure in the complement of the regular parameters [L7].

The former result was extended by Kozlovski [K1] to any non-trivial real analytic family $\{f_t\}$ of real analytic unimodal maps: For an open and dense set of parameter values t in such a family, the map f_t has a finite number of periodic attractors whose basin has full Lebesgue measure. One of our main theorems, Theorem B, extends the latter result to all non-trivial real analytic families $\{f_t\}$ of unimodal maps with negative Schwarzian derivative: *In such a family the map f_t is either regular or stochastic for a set of parameter values t of full Lebesgue measure.* Note that this result fits nicely to the general program of studying attractors in finite parameter families of dynamical systems (in all dimensions) formulated by Palis [Pa].

Let us now describe the picture which allows us to transfer the results from the quadratic family to other families of quasiquadratic maps. We consider an appropriate Banach space $\mathcal{B}$ of real analytic quasiquadratic maps of an interval and describe its partition into topological classes (i.e., the classes of topologically conjugate maps). One of our main results, Theorem A, states that each topological class is either an open set (in the regular case) or a codimension-one Banach submanifold. Different codimension-one classes fit together nicely giving a lamination structure in a neighborhood of any map that does not have a parabolic periodic point. This lamination is transversally quasisymmetric.

Any non-trivial real analytic one parameter family $\{f_t\} \subset \mathcal{B}$ is transverse to this lamination except possibly on a closed countable set of parameter values. Moreover, the quadratic family $\{Q_\lambda\}$ is a global transversal to this lamination. Thus, the bifurcation locus of the quadratic family (i.e., the complement of the set of regular parameters) has a *universal quasisymmetric structure*: outside a closed countable set of parameters, the bifurcation locus in any non-trivial real analytic family $\{f_t\} \subset \mathcal{B}$ is locally quasisymmetrically equivalent to that in the quadratic family. However, since quasisymmetric maps are not necessarily absolutely continuous, measure-theoretical applications of this result involve some extra work.

If we had the lamination structure in a neighborhood of parabolic maps as well, we would have a stronger result: the set of tangencies would be discrete. This would also imply the following conjecture. There is a C^∞ open and dense set of analytic families $\{f_t\} \subset \mathcal{B}$ which are transverse to all topological classes and satisfy the following property: for any nearby family $\{g_t\}$ there is a quasisymmetric homeomorphism ϕ between the parameter intervals such that f_t is quasisymmetrically conjugate to $g_{\phi(t)}$.

Problem 1.1. Is it true that codimension-one topological classes form a lamination in the space of quasiquadratic maps (near parabolic maps as well)?[1]

Though this paper is concerned with *real* unimodal maps, it is mostly based on the *complex* methods. The complex tools which are particularly important for us are the theory of holomorphic motions and the Pullback Argument, especially, its *infinitesimal* version introduced in this paper. It allows us to carry out an infinitesimal analysis of topological classes of unimodal maps which yields the above lamination structure.

[1] One can show that this fails without the quasiquadratic assumption.

After describing the dynamical and analytic background (§2), we state the main results of the paper (§3.1) and then give an outline of the main ideas of the proofs (§3.2). We encourage the reader to read this part first before plunging into the ocean of technical details. The structure of the rest of the paper containing the proofs of the main results should be clear from the Table of Contents.

Remark 1.1. We would like to draw the reader's attention to §A.6 of Appendix A which implies a short proof of Yoccoz's Rigidity Theorem.

Further development: The main results of this paper are still valid without the negative Schwarzian derivative assumption, see [Av].

The regular or stochastic dichotomy in families of unimodal maps has been recently refined, giving a better statistical description of the dynamics of typical stochastic parameters: they satisfy the "Collet-Eckmann condition", among other nice properties. Those results, first obtained in the context of the quadratic family ([AM1]), were generalized in [AM2], [Av] to non-trivial analytic families using results and methods of this paper, and then to generic smooth families.

The lamination constructed in this paper fails to be absolutely continuous in a rather dramatic way, see [AM3].

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2. Preliminaries

2.1. General terminology and notations. As usual, $\mathbb{N} = \{1, 2, \dots\}$ stands for the set of natural numbers; $\mathbb{R}$ stands for the real line; $\mathbb{C}$ stands for the complex plane, and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere.

Let $\mathbb{D}_r(x) = \{z \in \mathbb{C} : |z - x| < r\}$, $\mathbb{D}_r = \mathbb{D}_r(0)$, and let $\mathbb{D} = \mathbb{D}_1$.

$\overline{X}$ or cl X denotes the closure of a set X; int X denotes its interior.

$U \Subset V$ means that U is *compactly contained in* V, i.e., $\overline{U}$ is a compact set contained in V.

For an open set $U \subset \mathbb{C}$ and a point $z \in U$, let $U(z)$ stand for the connected component of U containing z.

For two sets X and Y in $\mathbb{C}$, let

$$\operatorname{dist}(X, Y) = \inf_{x \in X,\, y \in Y} |x - y|.$$

If S is a hyperbolic Riemann surface, we consider the hyperbolic metric dist_S in it and for $X, Y \subset S$ we define

$$\operatorname{dist}_S(X, Y) = \inf_{x,y} \operatorname{dist}_S(x, y)$$

and

$$\overline{\mathrm{dist}}_S(X, Y) = \max\{\sup_{x \in X} \mathrm{dist}_S(x, Y), \sup_{y \in Y} \mathrm{dist}_S(y, X)\},$$

which is the standard Hausdorff distance.

We will reserve notation I for the interval $[-1, 1]$. For $a > 0$ let

$$\Omega_a = \{z \in \mathbb{C} : \ \mathrm{dist}(z, I) < a\}.$$

A *topological disk* is a simply connected domain in $\mathbb{C}$; a *Jordan disk* is a topological disk bounded by a Jordan curve;

a *topological annulus* is a doubly connected domain in $\mathbb{C}$.

The Lebesgue measure of a set $X \subset \mathbb{R}$ will be denoted by $|X|$; notation $\mathrm{meas}(X)$ will be reserved for the planar Lebesgue measure of a set $X \subset \mathbb{C}$.

A set $X \subset \mathbb{C}$ is called $\mathbb{R}$-*symmetric* if it is invariant under the conjugacy $z \mapsto \bar{z}$. A function, or vector field, or differential defined on an $\mathbb{R}$-symmetric set will be called $\mathbb{R}$-symmetric if it commutes with the conjugacy. A set X is called 0-*symmetric* if it is invariant under the 0-symmetry $z \mapsto -z$.

For a bounded function, or a vector field, or a differential, $\|\cdot\|_\infty$ will denote its sup-norm.

Given a bounded open set $V \subset \mathbb{C}$, let $\mathcal{B}_V$ be the Banach space of holomorphic functions $f : V \to \mathbb{C}$ which are continuous up to the boundary endowed with the sup-norm.

The tangent space to a manifold M at a point x is denoted by $T_x M$.

Given a map $f : X \to X$ on some metric space X, f^n will denote its *iterates*, $n = 0, 1, 2, \dots$.

For $x \in X$, $\mathrm{orb}(x) \equiv \mathrm{orb}_f(x) = \{f^n x\}_{n=0}^{\infty}$ will denote the *forward orbit* or *trajectory* of x.

We will also use this notation for partially defined maps so that $\mathrm{orb}_f(x)$ consists of those points $f^n(x)$ which are well defined.

$\omega(x) \equiv \omega_f(x)$ is the *limit set* of $\mathrm{orb}(x)$:

$$\omega(x) = \bigcap_{n=0}^{\infty} \overline{\mathrm{orb}(f^n(x))}.$$

A point x is called *recurrent* if $x \in \omega(x)$.

A point x is called *fixed* if $f(x) = x$; it is called *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$; the smallest $n \in \mathbb{N}$ with this property is called the *period* of x; The orbit of a periodic point is also called a *cycle*.

id stands for the identity map, $\mathrm{id}(x) \equiv x$.

Given a set $Y \subset X$, the *first return map* to Y is defined as follows: For $y \in Y$, let $F(y) = f^{l(y)}(y)$, where $l = l(y) \in \mathbb{N}$ is the first moment when $f^l(y) \in Y$. Such a moment may or may not exist, so that the first return map is only partially defined on Y. Somewhat abusing notations we will still write $F : Y \to Y$.

The *first landing map* $L : X \to Y$ is defined as follows: For $x \in X$, let $L(x) = f^{l(x)}(x)$, where $l = l(x) \in \mathbb{N} \cup \{0\}$ is the first moment such that $f^l(x) \in Y$. (Note that $L|Y = \mathrm{id}$.) Again, this map is only partially defined on X.

Let $X \subset X'$ and $Y \subset Y'$. Two maps $f : X \to X'$ and $g : Y \to Y'$ are called *topologically conjugate* or *topologically equivalent* if there is a homeomorphism $h : X' \to Y'$ such that $h(X) = Y$ and

$$h(f(z)) = g(h(z)), \ z \in X. \tag{2.1}$$

Classes of topologically conjugate maps are called *topological classes*.

We will also say that h is *equivariant* (with respect to the actions of f and g). This terminology will also be used in the case when h is only partially defined. Then it means that (2.1) is satisfied whenever it makes sense.

Given a diffeomorphism $\phi : J \to J'$ between two real intervals, its *distortion* or *non-linearity* is defined as

$$\sup_{x,y\in J} \log \frac{|D\phi(x)|}{|D\phi(y)|}.$$

Its Schwarzian derivative is given by the formula:

$$S\phi = \frac{D^3\phi}{D\phi} - \frac{3}{2}\left(\frac{D^2\phi}{D\phi}\right)^2.$$

The condition of negative Schwarzian derivative plays an important role in one-dimensional dynamics. This condition is preserved under composition.

Let $U \subset \mathbb{C}$ be a bounded open set. We say that a holomorphic function $f : U \to \mathbb{C}$ belongs to class $A^1(U)$ if f and its derivative f' admit a continuous extension to the closure $\overline{U}$. We will use the same notations f and f' for the extensions. We supply $A^1(U)$ with the seminorm

$$\|f\|_1 = \max_{z\in\overline{U}} |f'(z)|. \tag{2.2}$$

If $f \in A^1(U)$, $f|\overline{U}$ is a homeomorphism onto its image and f' does not vanish on $\overline{U}$, we say that $f|\overline{U}$ is a diffeomorphism (onto the image).

Remark 2.1. Notice that if U is a bounded *connected* open set then $\|\cdot\|_1$ is a Banach *norm* in the subspace $\Lambda_z \subset A^1(U)$ of functions vanishing at a given point $z \in \overline{U}$.

2.2. Hyperbolic metric. A domain $D \subset \mathbb{C}$ is called *hyperbolic* if its universal covering is conformally equivalent to the unit disc. This happens if and only if $\mathbb{C} \setminus D$ consists of at least two points. Hyperbolic domains possess the *hyperbolic (or Poincaré) metric* ρ_D of constant negative curvature. This metric is obtained by pushing down the Poincaré metric $d\rho_{\mathbb{D}} = |dz|/(1 - |z|^2)|$ from the unit disc $\mathbb{D}$.

In the case of a simply connected hyperbolic domain D ("conformal disk"), $d\rho_D = p_D(z)|dz|$ is the pull-back of the $\rho_{\mathbb{D}}$ by the Riemann mapping $D \to \mathbb{D}$. In this case the density $p_D(z)$ is comparable with $\operatorname{dist}(z, \partial D)^{-1}$:

$$\frac{1}{4} \operatorname{dist}(z, \partial D)^{-1} \le p_D(z) \le \operatorname{dist}(z, \partial D)^{-1}.$$

The main virtue of the hyperbolic metric is that it is contracted under holomorphic maps (Schwarz Lemma) and hence is conformally invariant.

Consider an open interval $J \subset \mathbb{R}$. A special role in the complex dynamics of real maps belongs to the slit plane $\mathbb{C}_J = \mathbb{C} \setminus (\mathbb{R} \setminus J)$ endowed with the hyperbolic metric. By symmetry, J is a hyperbolic geodesic in $\mathbb{C}_J$. Let us consider hyperbolic r-neighborhoods consisting of points $z \in \mathbb{C}_J$ whose hyperbolic distance to J is at most r. It is easy to see that such a neighborhood is the union of two symmetric disk sectors based on J. For $\phi \in (0, \pi)$, let $D_\phi(J)$ denote such a neighborhood whose boundary curve meets J at angle ϕ. (One can easily work out the explicit relation between r and ϕ.) Note that for $\phi = \pi/2$ we obtain the round disk with diameter J.

2.3. Quasiconformal maps and Beltrami differentials. We assume that the reader is familiar with the basic theory of quasiconformal maps (see, e.g., [A,LV]). The goal of this section is to fix terminology and notations, and to state a few basic facts particularly important for this paper.

A homeomorphism $h : U \to V$ between two open sets $U, V \subset \mathbb{C}$, is called a *quasiconformal map*, or briefly a *qc* map, if it has locally integrable distributional derivatives ∂h, $\bar\partial h$, and $|\bar\partial h/\partial h| \le k < 1$ almost everywhere. As this local definition is conformally invariant, one can define qc homeomorphisms between Riemann surfaces.

One can associate to a qc map an analytic object called the *Beltrami differential* of h,

$$\mu = \frac{\bar\partial h}{\partial h}\frac{d\bar z}{dz},$$

with $\|\mu\|_\infty < 1$. (We will identify the Beltrami differential of a map $h : \mathbb{C} \to \mathbb{C}$ with the function $\bar\partial h/\partial h$.) The corresponding geometric object is a measurable family of infinitesimal ellipses (defined up to dilatation), pull-backs by Dh of the field of infinitesimal circles. The eccentricities of these ellipses are ruled by $|\mu|$, and are uniformly bounded almost everywhere, while the orientation of the ellipses is ruled by the arg μ. The *dilatation*

$$\mathrm{Dil}(h) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

of h is the essential supremum of the eccentricities of these ellipses. A qc map h is called K-qc if $\mathrm{Dil}(h) \leq K$.

Weyl's Lemma. *A 1-qc map is holomorphic.*

One of the most remarkable facts of analysis is that *any Beltrami differential with* $\|\mu\|_\infty < 1$ *(or rather a measurable field of ellipses with essentially bounded eccentricities) is locally generated by a qc map*, unique up to post-composition with a conformal map. Thus, such a Beltrami differential on a Riemann surface S induces a conformal structure quasiconformally equivalent to the original structure of S. Together with the Riemann Mapping Theorem this leads to the following result:

Measurable Riemann Mapping Theorem. *Let* μ *be a Beltrami differential on* $\overline{\mathbb{C}}$ *with* $\|\mu\|_\infty < 1$. *Then there is a qc map* $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *which solves the Beltrami equation:* $\overline{\partial} h / \partial h = \mu$. *This solution is unique if it is normalized to fix three points in* $\overline{\mathbb{C}}$. *The normalized solution* h_μ *depends holomorphically*[2] *on* μ.

A map $f : U \to V$ between domains in $\overline{\mathbb{C}}$ is called *quasiregular* if it is a composition of a holomorphic map and a qc homeomorphism. Beltrami differentials can be naturally pulled back by quasiregular maps $\mu \mapsto f^*\mu$. A Beltrami differential (defined on an open set containing $U \cup V$) is called f*-invariant* if $f^*\mu = \mu$ a.e. in U.

Assume that a quasiregular map $f : U \to V$ admits an invariant Beltrami differential μ defined on $\overline{\mathbb{C}}$. Let us solve the Beltrami equation

$$\frac{\overline{\partial} h_\lambda}{\partial h_\lambda} = \lambda\mu, \quad |\lambda| < a \equiv \frac{1}{\|\mu\|_\infty},$$

by means of qc maps $h_\lambda : \mathbb{C} \to \mathbb{C}$ fixing two given points. Then the maps $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ preserve the standard conformal structure and hence are holomorphic (by Weyl's Lemma). The dependence of f_λ on λ is also holomorphic (that is, the map $(\lambda, z) \mapsto f_\lambda(z)$, $|\lambda| < a$, $z \in h_\lambda(U)$ is holomorphic). This family of maps is called the *Beltrami disk* through f in the direction of μ. If we restrict λ to the real interval $(-a, a)$, we obtain the Beltrami *path* through f in the direction of μ.

One more fundamental property of qc maps exploited in this paper is compactness:

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[2] The space of Beltrami differentials is the unit ball in the complex Banach space $L^\infty(\overline{\mathbb{C}})$, and thus it is endowed with the natural complex structure. Holomorphic dependence of h_μ on μ is understood in the pointwise sense: for any $z \in \overline{\mathbb{C}}$, $\mu \mapsto h_\mu(z)$ is holomorphic.

First Compactness Lemma. *The space of K-qc maps $h : \mathbb{C} \to \mathbb{C}$ fixing two points is compact in the uniform topology on the Riemann sphere.*

A useful consequence is the following:

Lemma 2.1. *Let $S \subset \mathbb{C}$ be a hyperbolic domain, and let $H : S \to S$ be a K-qc map homotopic to the identity rel the boundary. Then* $\mathrm{dist}_S(x, H(x)) \le C(K)$, $x \in S$, *where* $C(K) \to 0$ *as* $K \to 1$.

Proof. If $S = \mathbb{D}$, then the situation can be normalized so that $x = 0$, and the statement follows from the First Compactness Lemma. In general, cover S by the unit disk and lift H to a qc homeomorphism of $\mathbb{D}$ homotopic to id rel boundary (see Theorem 2.2 of [EM]). □

Let $L : \mathbb{R}_+ \to \mathbb{R}_+$. A map $h : X \to Y$ between two metric spaces is called a *L-quasi-isometry* if for any $\varepsilon > 0$,

$$\mathrm{dist}(h(x), h(y)) \le \max\{L(\varepsilon)\, \mathrm{dist}(x, y), \varepsilon\}, \quad x, y \in X.$$

Quasiconformal maps are quasi-isometries with respect to the hyperbolic metric:

Lemma 2.2. *For every $K \ge 1$ there exists $L_K : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $h : S \to \tilde{S}$ is a K-qc map between two hyperbolic Riemann surfaces then h is a L_K quasi-isometry in the hyperbolic metric. Furthermore, for every $\varepsilon > 0$, $\lim_{K\to 1} L_K(\varepsilon) = 1$.*

Proof. Lifting H to the universal covering, we reduce the situation to the case when $S = \mathbb{D}$. Again, the conclusion follows from compactness argument. □

Lemma 2.3. *Given $M > m > 0$, there is a constant δ with the following property. Let $S, \tilde{S} \subset \overline{\mathbb{C}}$ be hyperbolic Riemann surfaces and $h_1, h_2 : S \to \tilde{S}$ be $(1 + \delta)$-qc maps homotopic rel boundary. Let X and Y be subsets of S. If* $\mathrm{dist}_S(X, Y) > M$ *then* $\mathrm{dist}_{\tilde{S}}(h_1(X), h_2(Y)) > m$.

Proof. Let $L = L_{1+\delta}(m)$ be the quasi-isometric constant for h_2^{-1} (see Lemma 2.2). Let $H = h_2^{-1} \circ h_1$. Assume there is a point $x \in X$ such that $\mathrm{dist}_{\tilde{S}}(h_1(x), h_2(Y)) \le m$. Applying h_2^{-1} we conclude that $\mathrm{dist}_S(H(x), Y) \le \max\{Lm, m\} < (M + m)/2$, provided δ is sufficiently small. Hence $\mathrm{dist}_S(H(x), x) > (M - m)/2$, which for sufficiently small δ contradicts Lemma 2.1. □

A qc map $h : \mathbb{C} \to \mathbb{C}$ will be called *normalized* if it fixes points -2 and 2 (this gives a little bit of space for the dynamical interval $[-1, 1]$). We will use the notation μ_h for the Beltrami differential of the qc map h.

A homeomorphism $h : X \to h(X) \subset \mathbb{C}$ *of a closed set* $X \subset \mathbb{C}$ will be called quasiconformal if it admits a qc extension to $\mathbb{C}$.

If X is a measurable set and $h : X \to \overline{\mathbb{C}}$ is a homeomorphism which admits qc extensions h_1 and h_2 to neighborhoods of X, it is easy to see that $\mu_{h_1}|X = \mu_{h_2}|X$ up to sets of zero Lebesgue measure. This allows us to define the dilatation of h, $\mathrm{Dil}(h)$ as the essential supremum over X of the eccentricities of the corresponding ellipses field (or 0 if meas $X = 0$).

A *quasidisk* (or *quasiarc*) is the image of $\mathbb{D}$ (or $[-1, 1]$) by a qc map of $\mathbb{C}$.

2.4. Quasisymmetric maps. A homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is called *quasisymmetric* (briefly: "qs") if there exists a constant $\kappa \geq 1$ such that for any $h > 0$ and any $x \in \mathbb{R}$ we have

$$\frac{1}{\kappa} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq \kappa.$$

The *dilatation* $\mathrm{Dil}(f)$ of a qs map is defined as the smallest such κ. A map f is called κ-qs if $\mathrm{Dil}(f) \leq \kappa$.

Qs maps are important because of their relation to qc maps:

Theorem 2.4 (Ahlfors-Beurling). *Let $F : \mathbb{C} \to \mathbb{C}$ be a K-qc homeomorphism preserving the real line. Then the restriction $f|\mathbb{R}$ is $\kappa(K)$-qs. Vice versa, any κ-qs homeomorphism $f : \mathbb{R} \to \mathbb{R}$ admits a $K(\kappa)$-qc extension to the complex plane.*

Quasisymmetric maps fixing 0 and 1 are Hölder continuous with an absolute constant and an exponent depending only on the dilatation. Since the dilatation is invariant under compositions with affine maps, we obtain:

Lemma 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a κ-qs homeomorphism. Then there exist constants $C > 0$ and $\delta > 0$ such that for any two nested intervals $J \subset T \subset \mathbb{R}$ we have:*

$$\frac{|f(J)|}{|f(T)|} \leq C \left(\frac{|J|}{|T|} \right)^{\delta}.$$

A map f *on a set* $X \subset \mathbb{R}$ is called quasisymmetric if f extends to a qs map of $\mathbb{R}$.

2.5. Holomorphic motions and codimension-one laminations. Given a domain $\mathcal{V}$ in a complex Banach space E with a base point $*$ and a set $X_* \subset \mathbb{C}$, a *holomorphic motion* of X_* over $\mathcal{V}$ is a family of injections $h_\lambda : X_* \to \mathbb{C}$, $\lambda \in \mathcal{V}$, such that $h_* = \mathrm{id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_*$. Let $X_\lambda = h_\lambda X_*$.

We will summarize fundamental properties of holomorphic motions which are usually referred to as the λ-*lemma*. It consists of two parts: extension of the motion and transversal quasiconformality, which will be stated separately. The first extension result was obtained in [L1], [MSS], and states that any holomorphic motion of a subset of $\overline{\mathbb{C}}$ extends to a holomorphic motion of the closure. A much more involved result was obtained by [BR], [ST]. Let $B_r \subset E$ stand for the Banach ball of radius r centered at $*$.

Extension Lemma ([BR], [ST])**.** *A holomorphic motion $h_\lambda : X_* \to X_\lambda$ of a set $X_* \subset \mathbb{C}$ over a Banach ball B_r admits an extension to a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole complex plane over the ball $B_{r/3}$.*

Remark 2.2. Assume that a complex Banach space E is supplied with an anti-linear isometric involution conj : $E \to E$. Let

$$E^{\mathbb{R}} = \{\lambda \in E \mid \operatorname{conj}(\lambda) = \lambda\}$$

and $* \in E^{\mathbb{R}}$. Let us say that a holomorphic motion of an $\mathbb{R}$-symmetric set $X \subset \mathbb{C}$ over B_r is $\mathbb{R}$-symmetric if $\overline{h_{\operatorname{conj}\lambda}(\overline{z})} = h_\lambda(z)$. Then the above Extension Lemma actually provides an $\mathbb{R}$-symmetric extension of this motion over $B_{r/3}$. This follows from the fact that the extension constructed by [BR] is canonical.

In what follows, this remark will be applied to certain spaces of holomorphic functions on some $\mathbb{R}$-symmetric domains, and $(\operatorname{conj} f)(z) = \overline{f(\overline{z})}$.

Remark 2.3. If a holomorphic motion $h_\lambda : X_* \to X_\lambda$ is defined over a *simply connected hyperbolic domain $D \subset \mathbb{C}$*, then Slodkowski's Theorem [Sl] gives the existence of an extension to $H_\lambda : \mathbb{C} \to \mathbb{C}$ defined over the full parameter space D.

Quasiconformality Lemma [MSS], [BR]**.** *Let $h_\lambda : U_* \to U_\lambda$ be a holomorphic motion of a domain $U_* \subset \mathbb{C}$ over a hyperbolic domain $D \subset \mathbb{C}$. Then the maps h_λ are $K(r)$-qc, where r is the hyperbolic distance between $*$ and λ in D. Moreover, $K(r) = 1 + O(r)$ as $r \to 0$.*

Remark 2.4. Combining the Extension Lemma and the Quasiconformality Lemma, we obtain the following statement: If $h_\lambda : X_* \to X_\lambda$ is a holomorphic motion of a subset $X_* \subset \mathbb{C}$ over a hyperbolic domain D then h_λ admits a $K(r)$-qc extension to the whole $\mathbb{C}$, where r is the hyperbolic distance between $*$ and λ.

Indeed, it is enough to work in the universal cover (notice that we do not require the extensions to fit together in a holomorphic motion). If $r \le 1/4$, a direct application of the Extension Lemma and the Quasiconformality Lemma will do. In the general case, we can cover a geodesic path linking $*$ to λ by at most $4r + 1$ balls of hyperbolic diameter bounded by $1/4$, so we just have to apply the previous case at most $4r + 1$ times. See also [BR], Theorem 1, which provides optimal bounds for the dilatation of the extensions.

A holomorphic motion $h_\lambda : X_* \to X_\lambda$ over $\mathcal{V}$ can be viewed as a complex codimension-one lamination on $\mathcal{V} \times \mathbb{C}$, whose leaves are graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in \mathcal{V}$. More generally, a codimension-one *holomorphic lamination* $\mathcal{L}$ on a complex Banach manifold $\mathcal{M}$ is a family of disjoint codimension-one Banach submanifolds of $\mathcal{M}$, called the *leaves* of the lamination that locally looks like a holomorphic motion:

- For any point $p \in \mathcal{M}$, there exists a holomorphic local chart $\Phi : \mathcal{W} \to \mathcal{V} \oplus \mathbb{C}$ (where $\mathcal{W}$ is an open neighborhood of p in $\mathcal{M}$ and $\mathcal{V}$ is an open

set in some complex Banach space) such that for any leaf L and any connected component L_0 of $L \cap \mathcal{W}$, the image $\Phi(L_0)$ is a graph of a holomorphic function $\mathcal{V} \to \mathbb{C}$.
The neighborhood $\mathcal{W}$ in the above definition is called a *flow box*, and the connected components L_0 are called *local leaves* in this flow box.

Note that by the Extension Lemma, any holomorphic lamination extends locally, near any point $p \in \mathcal{M}$, to a lamination whose leaves fill out a full Banach neighborhood of p.

A one-dimensional holomorphic submanifold of $\mathcal{M}$ which only has transversal intersections with the leaves of $\mathcal{L}$ is called a *transversal* to $\mathcal{L}$. Given two transversals X and Y within one flow box $\mathcal{W}$, we have a partially defined local holonomy map $H : X \to Y$, $H(p) = q$ iff p and q belong to the same local leaf in $\mathcal{W}$. This definition can be extended to local transversals X and Y through two points p and q on the same leaf connected by some path γ. To this end, cover the path with finitely many flow boxes $\mathcal{W}_i$ and and define the local holonomy $H : X \to Y$ as the composition of local holonomies within the $\mathcal{W}_i$.

A map $H : X \to Y$ is called *locally qc* at $p \in X$ if it admits a qc extension to some neighborhood of p. We say that a lamination $\mathcal{L}$ is *transversally quasiconformal* if the holonomy between any two transversals is locally qc. The λ-lemma implies (see e.g., [L6, Appendix 2]):

Corollary 2.6 (Transverse qc structure)**.** *Any codimension-one holomorphic lamination is transversally quasiconformal.*

A holomorphic motion $h_\lambda : \mathbb{C} \to \mathbb{C}$ will be called *normalized* if it fixes points 2 and -2.

The Measurable Riemann mapping Theorem can be interpreted in terms of holomorphic motions in the following way. Let μ_λ, $\lambda \in \mathbb{D}$, be a holomorphic family of Beltrami differentials on $\mathbb{C}$ such that $\|\mu_\lambda\|_\infty < 1$ for all $\lambda \in \mathbb{D}$, and $\mu_0 = 0$. Then there exists a unique normalized holomorphic motion h_λ of $\mathbb{C}$ based at 0 such that $\mu_{h_\lambda} = \mu_\lambda$. The converse is also true:

Theorem 2.7 (see [BR], Theorem 2)**.** *Let h_λ be a holomorphic motion of an open set of $\mathbb{C}$. Then μ_{h_λ} is a holomorphic family of Beltrami differentials.*

Holomorphic motions also enjoy the compactness properties of qc maps:

Lemma 2.8. *Let $X \subset \overline{\mathbb{C}}$ be a set containing* 3 *distinct points* $\{a, b, c\}$ *and let $\mathcal{V}$ be an open subset of a separable Banach space. Consider a holomorphic motion of X over $\mathcal{V}$ as a map from $\mathcal{V}$ to the space of continuous maps from X to $\overline{\mathbb{C}}$ endowed with the uniform metric. The space of all holomorphic motions $h_\lambda : X \to \overline{\mathbb{C}}$, $\lambda \in \mathcal{V}$, fixing $\{a, b, c\}$ is compact in the uniform topology over compacts of $\mathcal{V}$.*

For a proof, see [D1].

2.6. Infinitesimal deformations

2.6.1. Quasiconformal vector fields. A continuous vector field $v \equiv v(z)/dz$ on an open set $U \subset \overline{\mathbb{C}}$ is called *K-quasiconformal* (or briefly, K-qc) if it has locally integrable distributional partial derivatives ∂v and $\overline{\partial} v$, and $\|\overline{\partial} v\|_\infty \leq K$. A vector field is *quasiconformal* if it is K-qc for some K.

The theory of qc vector fields has many parallels with the theory of qc maps. Given $\mu \in L^\infty(\mathbb{C})$ one obtains a qc vector field α with $\overline{\partial}\alpha = \mu$. (Global) solutions to this problem are obtained from local ones, and those can be explicitly given (see [AB]) using the Cauchy transform

$$-\frac{1}{\pi}\int \frac{\mu(\zeta)}{z-\zeta} d\zeta \wedge d\overline{\zeta}.$$

The Cauchy transform also implies that local solutions have modulus of continuity $\phi(x) = -x \ln(x)$ (see [Mc2], Theorem A.10).

Two qc vector fields α and $\tilde{\alpha}$ such that $\overline{\partial}\alpha = \overline{\partial}\tilde{\alpha}$ differ by a conformal vector field (this is another instance of Weyl's Lemma). A conformal vector field on $\overline{\mathbb{C}}$ which vanishes at three given points vanishes on the whole sphere.

Second Compactness Lemma. *The space of K-qc vector fields of the Riemann sphere $\overline{\mathbb{C}}$ vanishing at three given points is compact in the topology of uniform convergence on $\overline{\mathbb{C}}$.*

(See Corollary A.11, p. 199 of [Mc2] for the proof.)

Corollary 2.9. *For any $L > 0$, there exists a $C > 0$ such that if α is a L-qc vector field on $\overline{\mathbb{C}}$ that vanishes at ∞ and on the boundary of some interval $T \subset \mathbb{R}$, then $|\alpha(z)| < C|T|$, for all $z \in T$.*

Proof. Let $A : [0, 1] \to T$ be an affine transformation and let $\beta = A^*\alpha$. Then the vector field β is L-qc and vanishes at $0, \ 1$ and ∞. By the Second Compactness Lemma, β is bounded by some universal C. Hence α is bounded by $C|T|$. □

A qc vector field will be called *normalized* if it vanishes on $\{-2, 2, \infty\}$.

A continuous vector field v *on a closed set* $X \subset \mathbb{C}$ is called *quasiconformal* if it extends to a qc vector field on $\mathbb{C}$. If a vector field v on a closed set X admits a normalized qc extension to $\overline{\mathbb{C}}$ (this is always the case when X does not intersect $\{-2, 2, \infty\}$) then we let

$$\|v\|_{\mathrm{qc}} = \inf \|\overline{\partial}\beta\|_\infty, \tag{2.3}$$

where β runs over all normalized qc extensions of v.

Notice that if α_1 and α_2 are qc vector fields that coincide in some measurable set X then $\overline{\partial}\alpha_1 = \overline{\partial}\alpha_2$ on X up to sets of zero Lebesgue measure. If we define α as the restriction of those vector fields to X, the object $\overline{\partial}\alpha \in L^\infty(X)$ is well defined.

Quasiconformal vector fields are the infinitesimal counterparts of qc maps. More precisely, they are tangent at identity to holomorphic motions.

Lemma 2.10. *Let $h_\lambda : X \to \mathbb{C}$, $\lambda \in \mathbb{D}$, be a holomorphic motion with base point* 0. *Then*

$$\alpha \equiv \frac{d}{d\lambda} h_\lambda \bigg|_{\lambda=0}$$

is a qc vector field on X. Moreover, if X is an open set,

$$\bar\partial \alpha = \frac{d}{d\lambda} \mu_{h_\lambda} \bigg|_{\lambda=0} . \tag{2.4}$$

Proof. Consider an extension of h_λ, which we still denote h_λ. By Theorem 2.7, μ_{h_λ} depends holomorphically on λ, and (2.4) follows from the proof of Lemma 19 of [AB]. □

2.6.2. Equivariant vector fields. Let $f : \Omega \to \mathbb{C}$ be a holomorphic map and let v be a holomorphic vector field on Ω. A vector field α is called *equivariant* on some set $X \subset \Omega$ (with respect to the pair (f, v)) if for any $z \in X$,

$$v(z) = \alpha(f(z)) - f'(z)\alpha(z). \tag{2.5}$$

Note that this equation can also be written in the form

$$f^*\alpha - \alpha = \frac{v}{f'}. \tag{2.6}$$

This equation tells us that α is an "infinitesimal conjugacy" between f and its "infinitesimal deformation" v. It is obtained by linearizing the following commutative diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow[\mathrm{id}+\varepsilon\alpha]{} & \Omega_\varepsilon \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+\varepsilon v} \\ \mathbb{C} & \xrightarrow[\mathrm{id}+\varepsilon\alpha]{} & \mathbb{C} \end{array} \tag{2.7}$$

Let $X \subset \Omega$ and let α be a vector field on $Y \equiv f(X)$. A vector field β is called the *lift* of α to X by (f, v) if $v = \alpha \circ f - f'\beta$. This equation is obtained by linearization of the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow[\mathrm{id}+\varepsilon\beta]{} & X_\varepsilon \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+\varepsilon v} \\ \mathbb{C} & \xrightarrow[\mathrm{id}+\varepsilon\alpha]{} & \mathbb{C} \end{array}$$

Note that if 0 is a critical point of f, then a "liftable" vector field α must necessarily satisfy condition $v(0) = \alpha(f(0))$ (this condition is also sufficient if 0 is a simple critical point, see Lemma 6.4).

Obviously, a vector field is equivariant if and only if it is equal to its lift.

Assume now that the set X is open and the vector field α is quasiconformal. Since $\beta = f^*\alpha - v/f'$ where v/f' is holomorphic,

$$\bar\partial\beta = \bar\partial(f^*(\alpha|Y)) = f^*\bar\partial(\alpha|Y),$$

where the former pullback acts on vector fields while the latter one acts on Beltrami differentials. Hence

$$\|\bar\partial\beta\|_\infty = \|f^*\bar\partial(\alpha|Y)\|_\infty = \|\bar\partial(\alpha|Y)\|_\infty,$$

i.e., *lifts preserve the qc norm of vector fields.*

2.6.3. Variational formula. Let us consider now the iteration operator $S_n : f \mapsto f^n$ acting between some spaces of (real or complex) analytic functions. Linearizing the expression $(f+\varepsilon v)^n$, we obtain (by induction) the following formula for the differential of S_n:

(2.8)
$$v^n \equiv DS_n(f)v = Df^{n-1}\circ f \sum_{k=0}^{n-1} \frac{v\circ f^k}{Df^k\circ f} = Df^n \sum_{k=0}^{n-1}(f^k)^*\left(\frac{v}{f'}\right).$$

(Though f is implicit in the notation v^n, it should not lead to a confusion.)

Note that if f_t is a one-parameter family of analytic maps such that

$$\left.\frac{d}{dt} f_t\right|_{t=0} = v$$

then

$$\left.\frac{d}{dt} f_t^n\right|_{t=0} = v^n. \tag{2.9}$$

Applying to (2.6) the iterates of f^* and summing up, we see that if α is equivariant with respect to (f, v) on $\cup_{k=0}^{n-1} f^k(X)$, then it is equivariant with respect to (f^n, v^n) on X:

$$(f^n)^*\alpha - \alpha = \sum_{k=0}^{n-1}(f^k)^*(f^*\alpha - \alpha) = \sum_{k=0}^{n-1}(f^k)^*\left(\frac{v}{f'}\right) = \frac{v^n}{Df^n}. \tag{2.10}$$

(Another way of seeing it is to linearize the "iteration" of the diagram (2.7).)

The variational formula makes it clear that if α is a bounded vector field equivariant on $\mathrm{orb}_f(x)$ and $Df^k(x) \to \infty$ then (using (2.8) and (2.10)) we have

$$\alpha(x) = -\sum_{j=0}^{\infty} \frac{v\circ f^j}{Df^{j+1}}.$$

Note finally that if β is obtained from α by n consecutive lifts by (f, v), then β is the lift of α by (f^n, v^n).

2.7. Markov maps and expanding Cantor sets.

Definition 2.1. *Consider two open sets $U \Subset \tilde{U}$ and a smooth map $f : \tilde{U} \to \mathbb{C}$. The map $f : U \to \mathbb{C}$ is called Markov if:*

(i) U is the union of finitely many Jordan disks U_i with piecewise smooth boundary and disjoint closures;
(ii) the restrictions $f|\overline{U_i}$ are diffeomorphism onto the image;
(iii) for any i and k, the curves $f(\partial U_k)$ and $f(\partial U_i)$ are either disjoint or coincide;
(iv) for any i, j, the image $f(U_i)$ is either disjoint from U_j or contains U_j (Markov property).

A Markov map will be called *strictly* Markov if the Markov property (iv) is strengthened to the strict Markov property: $fU_i \Supset U_j$.

The "Julia set" of a Markov map is defined as

$$K(f) = \{z : f^n z \in U,\ n = 0, 1, 2, \dots\}.$$

The map f and its Julia set are called *expanding* if $K(f)$ is compact and there exist constants $C > 0$ and $\rho > 1$ such that

$$|Df^n(z)v| \geq C\rho^n |v|, \quad z \in K(f),\ v \in T_z U,\ n = 0, 1, 2, \dots.$$

We will be mostly concerned with holomorphic (on $\tilde{U}$) Markov maps, though we will sometimes encounter a more general situation. It follows from the Schwarz Lemma that the Julia set of a holomorphic strictly Markov map is an expanding Cantor set.

Recall that $\mathcal{B}_V$ stands for the Banach space of holomorphic functions $f : V \to \mathbb{C}$ which are continuous up to the boundary.

Proposition 2.11. *Let $f : U \to \mathbb{C}$ be a holomorphic strictly Markov map as above. Let V be an open set such that $U \Subset V \Subset \tilde{U}$. Then there exists a holomorphic motion $h_g : \mathbb{C} \to \mathbb{C}$ over some neighborhood $\mathcal{V} \subset \mathcal{B}_V$ of f conjugating $f : U \to \mathbb{C}$ to $g : U_g \to \mathbb{C}$, where $U_g = h_g(U)$.*

Proof. We will give a proof of this well-known statement which illustrates a simplest version of the so called "pull-back argument". By property (iii) of Definition 2.1, the image $\Gamma = f(\partial U)$ is a 1-cycle, i.e., a finite union of disjoint Jordan curves. Let $U^n = f^{-n}U$. Since the restrictions $f|U_i$ are univalent, each U^n is a finite union of Jordan disks with disjoint closures. Moreover,

$$U \equiv U^0 \Supset U^1 \Supset U^2 \Supset \dots.$$

Let $\gamma = \partial U$.

Let $\gamma_g = g^{-1}\Gamma$. If g is sufficiently close to f in $\mathcal{B}_V$, then γ_g is a 1-cycle moving holomorphically with g, where the motion $h_g^0 : \gamma \to \gamma_g$ is defined by the requirement: $g(h_g^0(z)) = f(z)$, $z \in \gamma$. Moreover, γ_g bounds some domain U_g and the map $g : U_g \to \mathbb{C}$ is a strictly Markov map.

Extend h_g^0 to Γ as the identity. By the λ-lemma, this motion further extends to a holomorphic motion of the whole plane over some neighborhood $\mathcal{V} \subset \mathcal{B}_V$. Let us keep the same notation h_g^0 for the extension. By definition, h_g^0 conjugates $f : \gamma \to \Gamma$ to $g : \gamma_g \to \Gamma$.

Let $U_g^n = g^{-n}U$.

Since the restrictions $f|U_i$ are univalent, we can lift h_g^0 to U as follows:

$$g \circ h_g^1 \big| U = h_g^0 \circ f \big| U.$$

Since h_g^0 is equivariant on γ, it matches with h_g^1 on γ. Thus, we can let $h_g^1 = h_g^0$ in $\mathbb{C} \setminus U$. We obtain a holomorphic motion of the whole plane which conjugates $f : U \setminus U^1 \to \mathbb{C}$ to $g : U_g \setminus U_g^1 \to \mathbb{C}$.

Similarly, pulling this motion back to U^2, we obtain a motion h_g^2 conjugating $f : U \setminus U^2 \to \mathbb{C}$ to $g : U_g \setminus U_g^2 \to \mathbb{C}$, etc. Since the motions h_g^l, $l \geq n$, coincide with h_g^n on $\mathbb{C} \setminus U^n$, at the end we obtain a motion $h_g : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus K(g)$ conjugating $f : U \setminus K(f) \to \mathbb{C}$ to $g : U \setminus K(g) \to \mathbb{C}$. By the λ-lemma, this motion admits an extension through the Julia sets. Since the Julia sets are nowhere dense, this extension conjugates $f : U \to \mathbb{C}$ to $g : U_g \to \mathbb{C}$. $\square$

Remark 2.5. The same result is valid for (non-strictly) Markov maps for which one can construct a holomorphic motion h_g^0 of $\gamma \cup \Gamma$, provided its Julia set has empty interior. The rest of the argument carries to this more general situation without changes.

Given a smooth map $f : U \to \mathbb{C}$ and a Riemannian metric ν on $\mathbb{C}$, we say that ν is *expanded* by f on an invariant set $Q \subset U$ if there is a constant $\lambda > 1$ such that $\|Df(x)v\|_\nu \geq \lambda \|v\|_\nu$ for all $x \in Q$ and $v \in T_x\mathbb{C}$. The following useful fact is well-known:

Lemma 2.12. *Assume that a smooth map $f : U \to \mathbb{C}$ is expanding on a compact invariant set $Q \subset U$. Then there exists a Riemannian metric ν which is expanded by f on Q. If f is holomorphic then the metric ν can be selected to be conformal.*

2.8. Unimodal maps. We refer to the book of de Melo & van Strien [MS] for the general background in one-dimensional dynamics.

A smooth map $f : I \to I$ of the interval $I = [-1, 1]$ is called *unimodal* if it has a single critical point and this point is an extremum. We always assume that the critical point is located at the origin. Let $\mathcal{U}^3$ be the space of C^3 unimodal maps $f : I \to I$ with quadratic critical point, which are even (symmetric), that is $f(x) = f(-x)$, $Df(0) = 0$ and $D^2(f(0)) \neq 0$. We normalize the maps so that -1 is a fixed point and $f(1) = -1$. We endow $\mathcal{U}^3$ with the C^3 topology. If $Df(-1) < 1$, either the dynamics is trivial (-1 is the global attractor) or the map has a proper unimodal restriction. For this reason we will assume further that $Df(-1) \geq 1$.

The symmetry assumption above was introduced to simplify the proofs, but is not essential. In fact the general case reduces to the symmetric case, as we show in Appendix C.

We let $c_n = f^n(0)$, $n = 0, 1, 2, \dots$.

Basic examples of unimodal maps are given by quadratic maps

$$q_\tau : I \to I, \quad q_\tau(x) = \tau - 1 - \tau x^2, \tag{2.11}$$

where $\tau \in [1/2, 2]$ is a real parameter.

A map $f \in \mathcal{U}^3$ is *quasiquadratic* if any nearby map $g \in \mathcal{U}^3$ is topologically conjugate to some quadratic map. We denote by $\mathcal{U} \subset \mathcal{U}^3$ the space of quasiquadratic maps. By the theory of Milnor-Thurston [MT] and Guckenheimer [G], a map $f \in \mathcal{U}^3$ with negative Schwarzian derivative is quasiquadratic, so the quadratic family is contained in $\mathcal{U}$.

Let us consider a periodic point q of some period n and its cycle $\overline{q} = \{f^k(q)\}_{k=0}^{n-1}$. Let $\lambda = (Df^n)(q)$ be its *multiplier*. The point q and its cycle $\overline{q}$ are called *attracting, parabolic*, or *repelling* depending on whether $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$. A periodic point is called *superattracting* if $\lambda = 0$, which means that the cycle of this point contains 0.

The *basin of attraction* $D(\overline{q})$ of an attracting cycle $\overline{q}$ is defined as $\{x \in I : f^n(x) \to \overline{q}\}$. The basin of attraction $D(\overline{q})$ of a parabolic cycle $\overline{q}$ is defined similarly, except that the orbits landing at the cycle itself are not considered to be in the basin (this makes the basin open).

A quasiquadratic map f is called *hyperbolic* or *regular* if it has an attracting cycle $\overline{q}$. In this case the orbit of the critical point 0 converges to $\overline{q}$, hence a quasiquadratic map can have at most one attracting cycle [Fa,Si]. Moreover, if it has one then almost all orbits converge to this cycle (this follows from a result by Guckenheimer and Mañé, see [M]).

A quasiquadratic map f is called *parabolic* if it has a parabolic cycle $\overline{q}$. Similarly to the hyperbolic case, in the parabolic case the critical point 0 belongs to the basin $D(\overline{q})$ and this basin has full Lebesgue measure in I. Thus, a quasiquadratic map can have at most one parabolic cycle (and the map cannot be simultaneously hyperbolic and parabolic).

Remark 2.6. Let us denote by $D^0(\overline{q})$ the union of connected components of $D(\overline{q})$ whose closure intersects $\overline{q}$ (the *immediate basin* of $\overline{q}$).

There is a simple criteria to decide if a unimodal map $f \in \mathcal{U}^3$ is topologically equivalent to a quadratic map: any non-repelling periodic orbit must contain the critical point in its immediate basin. This is a consequence of Milnor-Thurston Theory and non-existence of wandering intervals for maps in $\mathcal{U}^3$ (see [MS], Theorem 6.2, p. 156 and Theorem 6.4, p. 162).

To prove that a map is quasiquadratic, we need a robust version of the above criteria given in the following:

Lemma 2.13. *If $f \in \mathcal{U}^3$ is topologically conjugate to a quadratic map and has no parabolic periodic orbit then f is quasiquadratic.*

Proof. We only have to show that the criteria of Remark 2.6 is valid for all g in a neighborhood of f.

By the result of Guckenheimer and Mañé (see also Proposition 2.15 below), if J is a neighborhood of the critical point then $f|I \setminus J$ is expanding. This property is persistent, so we conclude that for any J there is a neighborhood $\mathcal{V} \subset \mathcal{U}^3$ of f such that for $g \in \mathcal{V}$ all periodic orbits which do not intersect J are repelling. This implies the result if f has an attracting periodic orbit, so let us assume that all periodic orbits are repelling. In this case, the (automatically repelling) periodic cycles of f accumulate on 0.[3]

By Theorem A of [K2], there exists a symmetric interval J_f containing the critical point such that the first return map to $f(J_f)$ has negative Schwarzian derivative. Shrinking the interval J_f we may suppose that $f(J_f) = [p_f, f(0)]$ where p_f is a repelling periodic point and $\overline{p}_f \cap f(J_f) = \{p_f\}$ (this condition implies the nice condition, to be defined in the next section).

The interval J_f has a continuation J_g for $g \in \mathcal{U}^3$ near f such that $g(\partial J_g) = p_g$ is the continuation of p_f, and by the proof in [K2], it is clear that there exists a neighborhood $\mathcal{V} \subset \mathcal{U}^3$ of f such that if $g \in \mathcal{V}$ then the first return map to $g(J_g)$ still has negative Schwarzian derivative. By the argument of Singer [Si], it follows that any non-repelling periodic orbit for g which intersects J_g must contain the critical point in its immediate basin. Shrinking $\mathcal{V}$ we may assume that there exists an interval J which is contained in $\cap_{g \in \mathcal{V}} J_g$. Shrinking $\mathcal{V}$ again, we conclude (by the previous argument) that all periodic orbits contained in $I \setminus J$ are repelling, and the result follows. □

In what follows, all unimodal maps under consideration will be assumed to be quasiquadratic.

If the fixed point -1 is repelling, then f has a unique fixed point in the interior of I. This point will always be denoted by α. If the α-point is repelling, then it is orientation reversing, that is, $f'(\alpha) < 0$.

A unimodal map is called *preperiodic* if the orbit of the the critical point 0 lands at a repelling cycle. Simplest examples of preperiodic maps are provided by *Ulam-Neumann* (or *Chebyshev*) maps which are defined by the condition $f(0) = 1$ (so that the second iterate of 0 lands at the fixed point -1, which is automatically repelling in this case). For instance, the quadratic map q_2 is Chebyshev.

A unimodal map f is *renormalizable* if there exists an interval J containing the critical point and an integer $n \geq 2$ such that $f^n(J) \subset J$ and the intervals $J,\ f(J), \dots, f^{n-1}(J)$ have pairwise disjoint interior. The smallest such n is called the *renormalization period.*

Let n be the renormalization period and $J \ni 0$ be the maximal periodic interval of period n as above. This interval is bounded by a periodic point p

[3] To see this, first notice that by the No Wandering Intervals Theorem, the preimages of the critical point are dense on I. By the Intermediate Value Theorem, if $n \geq 1$ and $f^n(x) = 0$ then there exists $p \in [-x, x]$ with $f^n(p) = p$.

(of period n if $n \geq 3$ and period 1 if $n = 2$) and the symmetric point. The restriction $f^n|J$ is called the *pre-renormalization* of f.

Let $A : I \to J$ be the affine scaling mapping -1 to p. Then the map

$$R(f) \equiv A^{-1} \circ f^n \circ A : I \to I$$

is called the *renormalization* of f.

Remark 2.7. The renormalization of a quadratic map has negative Schwarzian derivative and hence is quasiquadratic. Since the renormalization operator is an open continuous map in the topology of $\mathcal{U}^3$ and respects topological equivalence, it acts on the space of quasiquadratic maps.

If the renormalization $R(f)$ is in turn renormalizable, then the map f is *twice renormalizable*. In this way we can define n *times renormalizable* maps for any $n = 1, 2, \ldots,$ including $n = \infty$.

A hyperbolic or parabolic map with an attracting/parabolic cycle of period $n > 1$ is renormalizable but at most finitely many times (the last renormalization of this map has an attracting/parabolic fixed point). Preperiodic maps are at most finitely renormalizable. In fact, infinitely renormalizable maps have a recurrent critical point.

A unimodal map is called *Yoccoz* if it is not infinitely renormalizable and has all periodic orbits repelling. A Yoccoz map with a non-recurrent critical point is called *Misiurewicz*.

2.9. Spaces of unimodal maps. Let $\mathcal{UN}$ stand for the set of Ulam-Neumann maps $f \in \mathcal{U}^3$. It is a domain in an affine hyperplane of the Banach space C^3.

Unimodal maps $f \in \mathcal{U}^3$ with negative Schwarzian derivative form an open subset of $\mathcal{U}$ containing the quadratic family $\{q_\tau : \tau \in [1/2, 2]\}$. We will be interested in the intersection of $\mathcal{U}$ with some Banach spaces of real analytic unimodal maps.

Let $a > 0$, and let $\mathcal{E}_a \subset \mathcal{B}_{\Omega_a}$ be the complex Banach space of holomorphic maps $v : \Omega_a \to \mathbb{C}$ continuous up to the boundary which are 0-symmetric (that is, $v(z) = v(-z)$) and such that $v(-1) = v(1) = 0$, endowed with the sup-norm $\|v\|_a = \|v\|_\infty$. It contains the real Banach space $\mathcal{E}_a^{\mathbb{R}}$ of "real maps" v, i.e, holomorphic maps symmetric with respect to the real line: $v(\bar{z}) = \overline{v(z)}$.

The complex affine subspace $q_2 + \mathcal{E}_a$ will be denoted as $\mathcal{A}_a$.

If $f \in \mathcal{A}_a$, we denote the postcritical set $\overline{\mathrm{orb}(f(0))}$ by O_f.

Let $\mathcal{U}_a = \mathcal{U} \cap \mathcal{A}_a$. Note that $\mathcal{U}_a$ is the union of an open set in the affine subspace $\mathcal{A}_a^{\mathbb{R}} = q_2 + \mathcal{E}_a^{\mathbb{R}}$ and a codimension-one space of Ulam-Neumann maps.

2.10. Real puzzle. Let us start with some combinatorial preparation, which will be used throughout this paper. *We assume that both fixed points of f, -1 and α, are repelling.*

A symmetric interval J containing 0 is called *nice* in the sense of Martens if the orbits of its boundary points do not intersect int J.

The *real Yoccoz puzzle* $\mathcal{P}^{\mathbb{R}}$ for a quasiquadratic map f is a collection of closed intervals P_i^n, $n \in \mathbb{N} \cup \{0\}$, called *real Yoccoz puzzle pieces* such that $P_0^0 = [-\alpha, \alpha]$ and the P_i^n, $n > 0$, are the components of $f^{-n} P_0^0$. Intervals of the Yoccoz puzzle containing the critical point are called *critical* and are labeled as P_0^n. Any critical Yoccoz puzzle piece P_0^n is nice. Moreover,

- any non-critical Yoccoz puzzle piece P_i^n is diffeomorphically mapped onto some other puzzle piece $P_{k(i)}^{n-1}$;
- any critical Yoccoz puzzle piece P_0^n, $n > 0$, is folded into the Yoccoz puzzle piece P_1^{n-1} containing the critical value c_1 in such a way that $f(\partial P_0^n) \subset \partial P_1^{n-1}$.

Take now a critical Yoccoz puzzle piece $J_0 \in \mathcal{P}^{\mathbb{R}}$ and consider the first landing map L to it. The domain of this map consists of a family $\mathcal{J}$ of disjoint Yoccoz puzzle pieces $J_i \in \mathcal{P}^{\mathbb{R}}$, $i \in \mathbb{N}$, satisfying the following properties (see [Ma]): Any J_i, $i > 0$, is diffeomorphically mapped by f onto some other interval $J_{k(i)} \in \mathcal{J}$, and there exists $n_i \in \mathbb{N}$ such that the branch $L|J_i = f^{n_i}|J_i$ diffeomorphically maps J_i onto J_0.

More generally, a similar description for the landing map applies to any nice interval J_0. In this case, we will loosely say that the collection $\{J_i\}$ (of pairwise disjoint intervals) is the real puzzle associated to J_0, and we will call the J_i real puzzle pieces.

The following statement explains the role of nice intervals.

Theorem 2.14 ([Ma])**.** *Let f be a quasiquadratic map. Let us consider two symmetric intervals $J_0 \subset T_0$ such that the orbit of $f(\partial J_0)$ does not return to* int T_0 *(in particular, J_0 is nice). Let $f^{n_i} : J_i \to J_0$ be a branch of the landing map L to J_0. Then f^{n_i} diffeomorphically maps some interval $T_i \supset J_i$ onto T_0. Moreover, the distortion of $L|J_i$ is $O(|J_0|/|T_0|)$.*

The following is a consequence of the result by Guckenheimer and Mañé.

Proposition 2.15 (see [MS], Corollary 1, p. 248)**.** *Let $f : I \to I$ be a quasiquadratic map with all periodic orbits repelling and let $L : \cup J_i \to J_0$ be the first landing map to a nice interval J_0. Then the complement $Q = I \setminus \cup$* int *J_i is an expanding set.*

Remark 2.8. The above expanding set is usually a Cantor set (for instance, if the boundary points of J_0 are not periodic). If ∂J_0 is contained in the preorbit of a periodic point q, then this Cantor set admits a real Markov partition $\{M_j^{\mathbb{R}}\}$ constructed with the help of finitely many preimages of q (see the proof of Lemma 7.10 for an explicit construction of a real Markov partition). If f is analytic, this partition can be further refined and thickened to become a (complex) strictly Markov partition $\{M_j\}$.

For a given unimodal map f, let us construct the *principal nest*

$$[-\alpha, \alpha] = T^0 \supset T^1 \supset T^2 \supset \dots \tag{2.12}$$

of real puzzle pieces. Consider the first return map $g_1 : \cup T_i^1 \to T^0$ to the interval T^0. It is defined on the union of disjoint closed intervals T_i^1 contained in T^0. Assuming the critical point 0 returns to T^0, one of these intervals contains the critical point. Call it $T^1 \equiv T_0^1$. Otherwise we stop and the principal nest consists only of T_0.

We define inductively the first return maps

$$g_n : \cup T_i^n \to T^{n-1} \tag{2.13}$$

to T^{n-1}, and we let $T^n \equiv T_0^n$ be the critical interval of the domain of g_n. If there is n such that the critical point never returns to T^n, the principal nest consists of the intervals T^0, ..., T^n. In this case f is necessarily non-renormalizable and has a non-recurrent critical point. Otherwise the principal nest is an infinite sequence of intervals. If f is non-renormalizable, the intervals T^n shrink to zero. Otherwise they shrink to the domain of the pre-renormalization of f (see [L3]).

The *scaling factors* of f are defined as follows: $\lambda_n = |T^n|/|T^{n-1}|$.

We will now reformulate Theorem 2.14 in a way convenient for further references.

Theorem 2.16. *Let f be a non-renormalizable quasiquadratic map with recurrent critical point such that $f(0) \neq 0$. Let $J_0 = T^{n+1}$, $n \in \mathbb{N}$, and let $L : \cup J_i \to J_0$ be the corresponding first landing map. Let $L|J_i = f^{n_i}$ and $\tilde{J}_i$ be the monotonicity interval of f^{n_i} containing J_i. Then $f^{n_i}(\tilde{J}_i) \supset T^n$ and hence the distortion of $L|J_i$ is $O(\lambda_{n+1})$.*

It is important to distinguish two combinatorial possibilities for the returns of the critical point: central and non-central returns. The return to level $n-1$ (and the level $n-1$ itself) is called *central* if $g_n(0) \in T^n$. Let $\{n_k - 1\}$ be the sequence of non-central levels in the principal nest. (Under the assumption that the principal nest consists of infinitely many intervals, the sequence $\{n_k - 1\}$ is infinite if and only if f is non-renormalizable.)

The following result will provide us with a big space around certain intervals of the principal nest.

Theorem 2.17 ([L3]). *Let f be a non-renormalizable quasiquadratic map with non-trivially recurrent critical point (that is, $f(0) \neq 0$). Then there exist constants $C > 0$ and $\rho \in (0, 1)$ such that*

$$\lambda_{n_k+1} \leq C\rho^k.$$

Combining the last two theorems, we see that the branches of the first landing map $L : \cup J_i \to J_0$ become almost linear if J_0 is selected sufficiently deep in the principal nest.

Remark 2.9. Theorems 2.14 and 2.17 were proven in the quoted papers for quasiquadratic maps with negative Schwarzian derivative. The general case was reduced to this one by Kozlovski [K2], who has proven that for sufficiently big n, the first return map to T^{n-1} has negative Schwarzian derivative, and this is enough to obtain good properties for g_n.

Lemma 2.18. *Let us use the notations of Theorem 2.16. There is a function $C(\lambda) \to \infty$ as $\lambda \to 0$ with the following property. For any n, there exist topological disks $U_i \supset J_i$ such that* $\mathrm{mod}(U_0 \setminus J_0) = C(\lambda_{n+1})$ *and the first landing map $L|_{J_i}$ extends to a univalent map from U_i onto U_0.*

Proof. Fix some $\phi \in (0, \pi)$ and let U_0 be the hyperbolic neighborhood $D_\phi(T^n)$ of T^n (see §2.2). □

2.11. Stochastic maps. A unimodal map $f : I \to I$ is called *stochastic* if it has an invariant measure μ which is absolutely continuous with respect to the Lebesgue measure on I (such a measure will be abbreviated as a.c.i.m.).

Existence of an a.c.i.m. is related to the rate of expansion along the orbit of the critical value c_1. It was shown by Collet & Eckmann [CE] (with a complement by Nowicki [N]) that the map is stochastic if the expansion rate is exponential:

$$Df^n(c_1) \geq C\lambda^n, \quad C > 0,\ \lambda > 1.$$

This criterion was improved by Nowicki & van Strien [NS] who replaced the exponential rate with the summability condition:

$$\sum |Df^n(c_1)|^{-1/2} < \infty. \tag{2.14}$$

Since the strongest contraction occurs near the critical point 0, one should expect that the rate of *expansion* along the critical orbit is related to the rate of *recurrence* of the critical orbit. Here is an efficient criterion of this kind:

Theorem 2.19 (Martens & Nowicki [MN]). *Let f be a non-renormalizable quasiquadratic map and let λ_n be its scaling factors. If*

$$\sum \sqrt{\lambda_n} < \infty$$

then f is stochastic (in fact, f satisfies (2.14)).

This result together with Theorem 2.17 implies the following combinatorial criterion:

Theorem 2.20. *Let f be a non-renormalizable quasiquadratic map. If all but finitely many levels in its principal nest are non-central then f is stochastic.*

2.12. Quadratic-like maps. A holomorphic map $f : U \to U'$ is called *quadratic-like* if it is a double branched covering between topological disks U, U' such that $U \Subset U'$. It has a single critical point which is assumed to be located at the origin 0, unless otherwise is stated. We will also make the following technical assumptions:

- U is symmetric with respect to the origin and f is even, i.e., $f(-z) = f(z)$.
- The domains U and U' are bounded by piecewise smooth curves.

Quadratic-like maps are considered up to affine conjugacy. We will say that a quadratic-like map f is *normalized* at 0 if it has the following expansion at 0:

$$f(z) = c + z^2 + O(z^3). \tag{2.15}$$

Note that this normalization is not the same as the normalization we use for unimodal maps.

A quadratic-like map is called *real* if the domains U and U' are $\mathbb{R}$-symmetric and f preserves the real line: $f(U \cap \mathbb{R}) \subset \mathbb{R}$.

The *filled Julia set* of a quadratic-like map is defined as the set of non-escaping point:

$$K(f) = \{z : f^n z \in U,\ n = 0, 1 \dots\}.$$

Its boundary is called the *Julia set*, $J(f) = \partial K(f)$. The sets $K(f)$ and $J(f)$ are connected if and only if the critical point itself is non-escaping: $0 \in K(f)$. Otherwise these sets are Cantor.

If f is a real quadratic-like map with connected Julia set, then $K(f) \cap \mathbb{R}$ is an interval $[-\beta, \beta]$, where β is a fixed point of f. Since f is considered up to affine conjugacy, we can normalize it so that $\beta = -1$.

The *fundamental annulus* of a quadratic-like map $f : U \to U'$ is the annulus between the domain and the range of f, $A = U' \setminus \overline{U}$.

Two quadratic-like maps f and g are called *topologically equivalent* if they are topologically conjugate in some neighborhoods of their Julia sets. They are called *hybrid equivalent* if they are conjugate by a qc map h with $\bar\partial h = 0$ a.e. on $K(f)$. Note that in the hyperbolic case (when f has an attracting cycle), the hybrid class of f consists of topologically equivalent quadratic-like maps with the same multiplier of the attracting cycle (Douady & Hubbard).

By the Douady-Hubbard Straightening Theorem [DH1], every hybrid class with connected Julia set intersects the quadratic family

$$\{P_c : z \mapsto z^2 + c\}_{c \in \mathbb{C}}$$

at a single point c of the Mandelbrot set. (Recall that the Mandelbrot set is defined as the set of parameter values $c \in \mathbb{C}$ for which the Julia set $J(P_c)$ is connected.) It follows that given a real quadratic-like map f without parabolic points, the restriction $f|I$ is quasiquadratic (this is still true when parabolic points are allowed, but requires an extra argument).

Theorem 2.21 ([L4,GS2]). *Let us consider two real non-hyperbolic quadratic-like maps f and g with connected Julia set. If f and g are topologically equivalent then they are hybrid equivalent. Thus, there exists a unique quadratic polynomial q_τ, $\tau \in [1/2, 2]$, in the topological class of f.*

Let us now consider the projectivized *real* tangent bundle $\mathcal{L} \to \mathbb{C}$ over the plane $\mathbb{C}$. An *invariant line field* on the Julia set $J(f)$ is a measurable section $X \to \mathcal{L}$ invariant under the action of f, where $X \subset J(f)$ is a measurable invariant set of positive (plane) Lebesgue measure. In other words, it is a measurable function $\theta : X \to \mathbb{R} \bmod \pi\mathbb{Z}$ such that for a.e. $z \in X$,

$$\theta(fz) = \theta(z) + \arg Df(z) \mod \pi\mathbb{Z}.$$

Associated to an invariant line field on the Julia set, there is family of f-invariant Beltrami differentials

$$\mu_\lambda(z) = \lambda e^{2i\theta(z)}, \quad |\lambda| < 1,$$

on X (extended by 0 to the whole complex plane). Hence any invariant line field generates a Beltrami disk $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$, $|\lambda| < 1$, of the map f, where h_λ is the solution of the Beltrami equation $\bar\partial h_\lambda = \mu_\lambda \partial h_\lambda$. This deformation is non-trivial on the Julia set. Thus, Theorem 2.21 yields the following result due to Yoccoz (in the finitely renormalizable case, see [H]) and McMullen [Mc1]:

Theorem 2.22. *A real quadratic-like map with connected Julia set does not have invariant line fields on the Julia set.*

2.13. Real hybrid classes. Theorem 2.21 can be carried to the class of real analytic maps:

Theorem 2.23 (see §B.3). *If two non-hyperbolic real analytic unimodal maps*
$f, g \in \mathcal{U}_a$ are topologically conjugate then they are quasisymmetrically conjugate.

(For a a stronger statement in the Yoccoz case, see Theorem B.1.)

The above discussion motivates the following definition. Two quasiquadratic maps f and g of class $\mathcal{U}_a$ are called *(real) hybrid equivalent* if they are topologically equivalent and (in the hyperbolic case) their attracting cycles have the same multiplier. (Because of Theorem 2.21, two quasiquadratic maps which have quadratic-like extensions belong to the same real hybrid class if and only if they belong to the same hybrid class as quadratic-like maps.)

We denote by $\mathcal{H}_f^{\mathbb{R}} \equiv \mathcal{H}_{f,a}^{\mathbb{R}} \subset \mathcal{U}_a$ the real hybrid class of f (for simplicity we often omit a in the notation). By Theorem 2.21, each hybrid class intersects the quadratic family $\{q_\tau\}_{\tau\in[1/2,2]}$ at a single point. Hence we can consider the *straightening* map $\chi : \mathcal{U} \to [1/2, 2]$, which associates to a quasiquadratic map $f \in \mathcal{U}$ the hybrid equivalent quadratic polynomial.

Remark 2.10. By the Milnor-Thurston theory, two quasiquadratic maps belong to the same hybrid class if and only if they have the same *kneading sequence* and the multipliers of any non-repelling cycles are the same for both maps. This is a consequence of non-existence of wandering intervals for quadratic maps, and can be easily obtained by the argument of [MS], Corollary of Theorem 6.2, p. 157.

This has the following nice consequence for the action of renormalization in the space of quasiquadratic maps. Let f and g be renormalizable of the same type. It is clear that they have the same kneading sequence if and only if their renormalizations do, so f and g belong to the same hybrid class if and only if their renormalizations do. So each renormalization operator (corresponding to some renormalization combinatorics) acts on the hybrid classes injectively.

2.14. A priori bounds. A relation between general real analytic and quadratic-like maps is provided by the renormalization: an appropriate renormalization of a real analytic map is quadratic-like. This statement is usually encoded as *a priori* bounds:

Theorem 2.24 ([LS1,LY])**.** *Let f be an infinitely renormalizable real analytic map of class $\mathfrak{U}_a$. Then some renormalization $R^n f$ admits a quadratic-like extension to the complex plane.*

Note that this property is robust: if it holds for some map $f_0 \in \mathfrak{U}_a$ then it also holds, with the same n, for nearby maps $f \in \mathfrak{U}_a$.

2.15. Parameter geometry of the quadratic family. The real quadratic family $\{q_\tau\}$ can be partitioned according to the combinatorics of the first return maps [L5]. According to this construction, the set $\mathcal{N}$ of non-renormalizable quadratic maps with both fixed points repelling[4] is covered with countable unions $\mathcal{D}^n$ of intervals Δ_i^n each of which is endowed with a family of first return maps $g_{\tau,l} : \cup T_{\tau,j}^l \to T_\tau^{l-1}$ of a certain level $l = l(n, i)$. Each interval Δ_i^n contains a subinterval Π_i^n corresponding to the central return of the critical point: $g_{\tau,l}(0) \in T_{\tau,0}^l$.

Theorem 2.25 ([L5])**.** *There exist constants $C > 0$ and $\rho \in (0, 1)$ such that*

$$\frac{|\Pi_i^n|}{|\Delta_i^n|} \leq C\rho^n. \tag{2.16}$$

Thus, the probability of the central return on level n is exponentially small in n. By the Borel-Cantelli Lemma, the probability of infinitely many central returns is equal to 0. By Theorem 2.20, almost all quadratic maps q_τ with $\tau \in \mathcal{N}$ are stochastic.

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[4] $\mathcal{N}$ will also denote the corresponding set of parameter values $\tau \in [1/2, 2]$, and the same convention applies to other sets which appear below: $\mathcal{F}$, $\mathcal{I}$, etc.

By means of the renormalization, this result extends to quadratic maps which are not infinitely renormalizable:

Theorem 2.26 ([L5]). *Almost all Yoccoz quadratic maps q_τ, $\tau \in [1/2, 2]$, are stochastic maps whose last renormalization satisfies the Martens-Nowicki criterion for existence of a.c.i.m.*

Recently, Avila and Moreira (see [AM1]) have strengthened the result above by showing that almost every Yoccoz map satisfies the Collet-Eckmann condition.

2.16. Regular or Stochastic Theorem. Let $\mathcal{I}$ stand for the set of infinitely renormalizable quadratic maps q_τ, $\tau \in [1/2, 2]$.

We say that a set $X \subset \mathbb{R}$ has *definite gaps everywhere* if there exists a $C > 0$ such that for any $x \in X$ and any $\varepsilon > 0$ there exists an interval $J \subset (x - \varepsilon, x + \varepsilon) \setminus X$ such that

$$C^{-1} \operatorname{dist}(x, J) \leq |J| \leq C \operatorname{dist}(x, J).$$

By the Lebesgue Density Theorem, such a set has zero measure. But unlike the measure zero property, the property to have definite gaps everywhere is preserved by quasisymmetric maps.

Theorem 2.27 ([L7], §4.1). *The set $\mathcal{I}$ has definite gaps everywhere and hence has zero Lebesgue measure in the parameter interval* $[1/2, 2]$.

Putting together the last two theorems, we obtain:

Theorem 2.28. *Almost any real quadratic map q_τ is either regular or stochastic.*

2.17. Invariant line fields and equivariance. In this work, the non-existence of invariant line fields on the set of non-escaping points of a holomorphic dynamical system will be a recurrent hypothesis. It will allow us to estimate the global dilatation of equivariant qc maps or vector fields in terms of their "external dilatation". The following two Lemmas will clarify this relation.

Lemma 2.29. *Let $f : U \to \mathbb{C}$, $\tilde{f} : \tilde{U} \to \mathbb{C}$ be two non-constant holomorphic maps such that there exists a qc map $h : \mathbb{C} \to \mathbb{C}$, $h(U) = \tilde{U}$, equivariant on U. Let $K(f)$ be the set of non-escaping points of f. Then*

$$\operatorname{Dil}(h) = \max\{\operatorname{Dil}(h|\mathbb{C} \setminus U), \operatorname{Dil}(h|K(f))\}.$$

Furthermore, if $\operatorname{Dil}(h|K(f)) > 1$ *then f has an invariant line field on $K(f)$.*

Proof. Equivariance implies that $f^*(\mu_h) = \mu_h$, so

$$\operatorname{Dil}(h|\mathbb{C} \setminus K(f)) = \max \operatorname{Dil}(h|f^{-n}(\mathbb{C} \setminus U)) = \operatorname{Dil}(h|\mathbb{C} \setminus U).$$

If $\operatorname{Dil}(h|K(f)) > 1$, μ_h gives an invariant line field on $K(f)$. $\square$

The same argument gives the following infinitesimal version:

Lemma 2.30. *Let $f : U \to \mathbb{C}$ be a non-constant holomorphic map and let v be a holomorphic vector field on U such that there exists a qc vector field α on the plane equivariant on U. Let $K(f)$ be the set of non-escaping points. Then $\|\overline{\partial}\alpha\|_\infty = \max\{\|\overline{\partial}\alpha|\mathbb{C} \setminus U\|_\infty, \|\overline{\partial}\alpha|K(f)\|_\infty\}$. Furthermore, if $\|\overline{\partial}\alpha|K(f)\|_\infty > 0$ then f has an invariant line field on $K(f)$.*

2.18. Some generalities about Banach spaces. Let E denote a complex Banach space.

We say that a set $\mathcal{K} \subset E \setminus \{0\}$ is a *cone* if $v \in \mathcal{K}$ implies $\lambda v \in \mathcal{K}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. We say that a codimension-one subspace F is *transverse* to $\mathcal{K}$ if $F \cap \mathcal{K} = \emptyset$.

Lemma 2.31. *Let F be a codimension-one subspace transverse to an open cone $\mathcal{K}$ and $v \in \mathcal{K}$. There exists $C > 0$ such that if $w \in E \setminus \mathcal{K}$ and $w = w_1 + \lambda v$ with $w_1 \in F$ then $|\lambda| \leq C\|w_1\|_E$.*

For a sequence of subspaces $F_n \subset E$, let

$$\limsup F_n = \mathrm{Lin}\{v \in E | \liminf\ \mathrm{dist}_E(v, F_n) = 0\}.$$

Lemma 2.32. *If the F_n are codimension-one subspaces, then* $\limsup F_n$ *is either E or a codimension-one subspace.*

Proof. Let G be any subspace of E of dimension 2. Since each F_n has codimension one, $F_n \cap G$ contains a unitary vector v_n. So there exists a subsequence converging to some unitary vector $v \in \limsup F_n$. Since $\limsup F_n$ is a subspace which intersects any 2-dimensional subspace of E, we conclude that $\limsup F_n$ is either E or a codimension-one subspace. □

3. Results and methods

3.1. Statement of the results. We are now ready to formulate the main results of this paper.

Fix some $a > 0$. Note that the affine space $\mathcal{A}_a$ has a natural involution around its real subspace $\mathcal{A}_a^{\mathbb{R}}$. A subset in $\mathcal{A}_a$ is called $\mathbb{R}$-*symmetric* if it is invariant under this involution.

Theorem A. *Every real hybrid class $\mathcal{H}_f^{\mathbb{R}}$, $f \in \mathcal{U}_a$, is an embedded codimension-one real analytic Banach submanifold of $\mathcal{U}_a$. Furthermore, the hybrid classes laminate a neighborhood of any non-parabolic map $f \in \mathcal{U}_a$. More precisely, any non-parabolic map $f \in \mathcal{U}_a$ has an $\mathbb{R}$-symmetric neighborhood $\mathcal{V}$ in the complex affine space $\mathcal{A}_a$ endowed with a codimension-one $\mathbb{R}$-symmetric holomorphic lamination such that for any $g \in \mathcal{V} \cap \mathcal{U}_a^{\mathbb{R}}$, the intersection of the leaf through g with $\mathcal{A}_a^{\mathbb{R}}$ coincides with $\mathcal{H}_g^{\mathbb{R}} \cap \mathcal{V}$.*

A real analytic one-parameter family $\{f_\lambda\}_{\lambda\in\Lambda}$ of quasiquadratic maps is called *non-trivial* if it is not contained in a single real hybrid class.

Theorem B. *Let $\{f_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{U}_a$ be a non-trivial one-parameter real analytic family of quasiquadratic maps. Then for Lebesgue almost all parameter values $\lambda \in \Lambda$, the map f_λ is either regular or stochastic.*

Theorem C. *Under the circumstances of the previous theorem, there is an open and dense set $\Lambda_0 \subset \Lambda$ of parameter values with countable complement such that the straightening map $\chi(f_t)$ is quasisymmetric on any compact interval contained in Λ_0.*

Remark 3.1. The map conjugating f_t to its straightening does not depend continuously on t (see [NPT], Theorem 3.2, p. 15, or [DH1], Proposition 15, p. 315).

3.2. Ideas of the proofs.

3.2.1. Quadratic-like maps. The entry point for this paper is the theory of quadratic-like maps $g : U \to U'$ described in §2.12. Briefly, the picture is as follows. The space $\mathcal{Q}$ of quadratic-like maps is endowed with a natural complex analytic structure based on families of Banach spaces. The connectedness locus $\mathcal{C}$ of this space is laminated by the hybrid classes $\mathcal{H}_f$ each of which is a codimension-one complex analytic submanifold in $\mathcal{Q}$. By the λ-lemma, this lamination is transversally quasiconformal (but it is not transversally smooth!). The quadratic family $\{P_c : z \mapsto z^2 + c\}_{c\in\mathbb{C}}$ is a global transversal to this lamination.

The tangent ("horizontal") space to the hybrid class $\mathcal{H}_f$ consists of holomorphic vector fields v on U, which admits representation (2.5), $v = \alpha \circ f - f'\alpha$, where α is a qc vector field on U whose $\bar\partial$-derivative vanishes a.e. on the filled Julia set $K(f)$. This equation tells us that v is an "infinitesimal hybrid deformation" of f and α is the corresponding "infinitesimal conjugacy".

A transverse ("vertical") direction to the hybrid class $\mathcal{H}_f$ can be selected as a holomorphic vector field $V^{\mathrm{tr}} \equiv V^{\mathrm{tr}}_f$ which can be represented as (2.5) in $U \setminus K(f)$, where $\alpha(z)/dz$ is a *holomorphic* vector field on $\overline{\mathbb{C}} \setminus K(f)$ vanishing at ∞. This explicit description of a vertical direction exploits essentially the "external structure" of quadratic-like maps (existence of the fundamental annulus $U' \setminus U$). Lack of this structure for general real analytic maps is the source of major difficulties addressed in this paper.

3.2.2. Horizontal space. Theorem 2.23 motivates the following definition. Assume that $f \in \mathcal{U}_a$ is not hyperbolic, and let v be a holomorphic vector field in the neighborhood Ω_a. We say that it is *horizontal* if there is a qc vector field $\alpha(z)$ on $\mathbb{C}$ satisfying (2.5) on $\mathrm{orb}(0)$. (This definition is designed in such a way that it will still make sense for complex perturbations of f.)

Clearly, the horizontal vector fields form a linear space, which will be denoted T_f. Moreover, it is easy to see (by the Implicit Function Theorem) that for simple combinatorics, like parabolic or preperiodic, the hybrid class $\mathcal{H}_f^{\mathbb{R}}$ is a real analytic codimension-one submanifold in $\mathcal{U}_a$, whose tangent space coincides with T_f. For general combinatorics, this is still true but requires much finer analysis; it will be outlined below.

3.2.3. Infinitely renormalizable case. As usual in one-dimensional dynamics, the analysis depends on the combinatorial properties of the maps under consideration. It turns out that for our problem, the infinitely renormalizable case is easier to handle. The reason is that by means of renormalization, it can be reduced to the quadratic-like case.

Take some infinitely renormalizable map $f \in \mathcal{U}_a$. By *a priori* bounds (Theorem 2.24) some renormalization $R^n f$ is a quadratic-like map, and the same is true for all (complex) $\tilde{f} \in \mathcal{A}_a$ near f. Moreover, all maps $R^n \tilde{f}$ belong to some Banach ball $\mathcal{V}$ of quadratic-like maps.

The classical Runge Theorem implies that the differential $DR^n(f)$ has a dense image in $\mathcal{V}$. By the Implicit Function Theorem, the pullback of the hybrid lamination near $g = R^n f$ in $\mathcal{V}$ is a holomorphic lamination near f in $\mathcal{A}_a$. It is easy to see that the real slices of the leaves of this lamination are local real hybrid classes.

3.2.4. Puzzle maps. To handle the non-renormalizable case (the finitely renormalizable case is completely analogous), we need to consider a special class of piecewise holomorphic maps. A *puzzle map* is defined on a disjoint union $\cup_{i\geq 0} U_i$ of topological disks U_i called "puzzle pieces" such that the "critical" piece U_0 contains 0 and is symmetric with respect to 0, while any noncritical puzzle piece U_i is univalently mapped by f onto some other puzzle piece $U_{j(i)}$ (see Fig. 1). Moreover, we require that for any i, either $f(U_0)$ contains U_i or $\overline{f(U_0)}$ is disjoint from $\overline{U_i}$ (and there are also some other technical requirements). Let U_1 denote the puzzle piece containing the critical value $c_1 = f(0)$.

The *filled Julia set* $K(f)$ of the puzzle map is defined as the set of all non-escaping (from $\cup U_i$) points. It is not necessarily compact.

Let us fix from now on a non-renormalizable unimodal map $f \in \mathcal{U}_a$ with repelling fixed points and recurrent critical point. Any such map can be restricted to a puzzle map in the following way. Select a nice critical interval $J_0 \equiv T^n$ in the principal nest (see §2.10), and let U_0 be an appropriate hyperbolic neighborhood of J_0 in the slit complex plane $\mathbb{C} \setminus (\mathbb{R} \setminus J_0)$. Consider the first landing map $L : \cup_{j\geq 0} J_j \to J_0$ to J_0, where we denote by J_1 the component containing the critical value c_1 (notice that $f^{-1}(J_1) = T^{n+1}$). Then $U_i \supset J_i$ are defined as the preimages of U_0 under L. The last requirement on the puzzle map can be ensured by selecting the interval $J_0 = T^n$ sufficiently deep in the principal nest to make the corresponding scaling factor λ_n so small (by Theorem 2.17) that $\overline{f^{-1}(U_1)} \subset U_0$ (see Lemma 5.5).

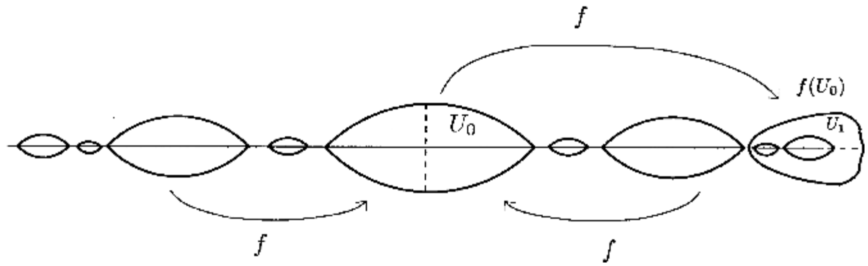

Fig. 1. Puzzle map

This puzzle structure of unimodal maps $f \in \mathcal{U}_a$ will serve as a substitute for the external structure of quadratic-like maps. A crucial property of the puzzle structure (see §5.2) is that it is *persistent* under perturbations of the map, even *complex*, and moreover *moves holomorphically* under the perturbation.

3.2.5. Key estimate. By definition, a horizontal vector field v for a non-hyperbolic puzzle map is a holomorphic vector field on $\cup U_i$ satisfying (2.5) for some qc vector field α on $\mathrm{orb}(0)$. In the hyperbolic case, we also require that the multiplier of the corresponding periodic attractor does not change "infinitesimally" along the direction v (that is, v satisfies (5.1)) .

The key estimate says that, for any map $g \in \mathcal{V}$ in some neighborhood $\mathcal{V} \ni f$ which either does not have invariant line fields on $K(g)$ or is hyperbolic, the dependence of α on v is *uniformly bounded*:

$$\|\alpha\|_{\mathrm{qc}} \leq L\|v\|_a, \quad v \in T_g. \tag{3.1}$$

(The norms $\|\cdot\|_{\mathrm{qc}}$ and $\|\cdot\|_a$ are defined in §2.6.1 and 2.9.) Recall that the no-invariant line fields assumption is satisfied for all real maps which are not hyperbolic or parabolic (see Theorem 2.22 and Lemma A.24). It is also satisfied for complex preperiodic puzzle maps for simpler reasons (see Appendix B).

The proof goes as follows. Select an $\varepsilon > 0$ so that the puzzle structure is persistent in the 2ε-neighborhood $\mathcal{V}$ of f, and let g belong to the ε-neighborhood of f. Consider a holomorphic curve $g_\lambda = g + \lambda v \in \mathcal{V}$ tangent to v, $|\lambda| < \varepsilon/\|v\|_a$. Let $h_\lambda : (\mathbb{C}, \cup U_i) \to (\mathbb{C}, \cup U_i^\lambda)$ be a holomorphic motion of the puzzle structure, and let

$$\alpha_0 = \left.\frac{dh_\lambda}{d\lambda}\right|_{\lambda=0}.$$

Since h_λ is equivariant on the boundary of the puzzle, α_0 is equivariant with respect to (g, v) on ∂U. Moreover,

$$\bar{\partial}\alpha_0 = \left.\frac{d\mu_\lambda}{d\lambda}\right|_{\lambda=0},$$

where μ_λ is the Beltrami differential of h_λ. Applying the Schwarz Lemma to the map $\lambda \mapsto \mu_\lambda$, we obtain: $\|\overline{\partial}\alpha_0\|_\infty \le \varepsilon^{-1}\|v\|_a$.

We exploit now a new tool, the "Infinitesimal Pullback Argument" (§6.2), to construct a qc vector field β equivariant with respect to (g, v) on U, coinciding with α on $\mathrm{orb}(0)$, and coinciding with α_0 on $\mathbb{C}\setminus U$. Moreover, if g has no invariant line fields on the filled Julia set, then $\|\overline{\partial}\beta\|_\infty \le \|\overline{\partial}\alpha_0\|_\infty$, and the key estimate follows with $L = \varepsilon^{-1}$.

3.2.6. Transverse direction. We are now approaching the most delicate issue, construction of a "transverse vector field", i.e., a non-horizontal vector field belonging to $T_f(\mathcal{U}_a)$. The idea is first to construct a *smooth* transverse vector field v and then to approximate it with a holomorphic one using Mergelyan's Theorem. In fact, v is going to be holomorphic on the critical value puzzle piece U_1 and vanishing on all other puzzle pieces U_i, $i \ne 1$. Moreover, v can be selected in such a way that it has a definite hyperbolic length (with respect to the hyperbolic metric of U_1) at any given point $q \in J_1 = U_1 \cap \mathbb{R}$, no matter how close this point to is ∂J_1 (it is essential here that v is required to be only C^1). It turns out that if the nice interval J_0 was originally selected sufficiently deep, then the vector field v corresponding to $q = c_1$ cannot be horizontal. Assuming otherwise, we use the infinitesimal pullback argument to construct a qc vector field α equivariant on $\mathrm{orb}(0)$ and vanishing on the boundary of the real puzzle. Let us then consider the first landing point p of the $\mathrm{orb}(q)$ in the domain $W = f^{-1}U_1 \Subset U_0$ (notice that by construction, α must vanish in $\partial W \cap \mathbb{R}$). Equation (2.5) allows us to bound from below the hyperbolic length of $\alpha(p)$ in U_0 via the hyperbolic length of v in U_1. So, the former length is also definite. Moreover, using the ideas of the proof of the Key estimate we are able to bound from above the qc norm of α uniformly.

But the hyperbolic diameter of W in U_0 is very small, provided the nice interval J_0 was selected deep enough (note that $W \cap \mathbb{R}$ is the puzzle piece of the principal nest following J_0). Hence the length of $\alpha(p)$ is very big compared with the diam W. But this contradicts Corollary 2.9, since α vanishes at the boundary points of $W \cap \mathbb{R}$ and has uniformly bounded qc norm.

Next we approximate v in the union of two appropriate sectors by polynomial vector fields v_n vanishing to the first order at the boundary point of J_1 where the sectors touch. We claim that the v_n are eventually not horizontal (and hence they represent transverse directions at f in $\mathcal{U}_a$). Assuming contrary, we prove the key estimate for the corresponding vector fields α_n. It follows the same lines as described above, though technically more involved. The key estimate allows us to pass to a limit in equation (2.5) and to conclude that v is horizontal – contradiction.

Remark 3.2. In this construction we control uniformly the $\|\cdot\|_1$ norm of v. The scaling invariance of the $\|\cdot\|_1$ norm is essential for this argument, as it is for the C^1 Closing Lemma of Pugh [Pu]. In fact, the construction of the transverse direction can be seen as the infinitesimal counterpart to the C^1 Connecting Lemma of Hayashi [Ha] (we want to close the critical orbit at the infinitesimal level). The key estimate, based heavily on complex analysis, is what allows us to promote a C^1 perturbation to an infinitesimal holomorphic perturbation.

3.2.7. Transverse cone field. Thus, the tangent space at f can be decomposed in the direct sum of the horizontal and transverse spaces, $T_f\mathcal{A}_a = T_f \oplus V_f^{\mathrm{tr}}$, where $V_f^{\mathrm{tr}} = \mathrm{Lin}\{V^{\mathrm{tr}}\}$ and V^{tr} is the transverse vector field constructed above. Since $\mathcal{A}_a$ is an affine space, we can use this decomposition as a "coordinate system" in it. Let $\theta \in (0, \pi/2)$. For a complex map $g \in \mathcal{A}_a$ near f, let us consider a tangent cone in $T_g\mathcal{A}_a$,

$$\mathcal{K}_g^\theta = \{v \in T_g\mathcal{A}_a : \|v^h\| < \tan\theta\,\|v^{\mathrm{tr}}\|\},$$

where v^h and v^{tr} are the projections of v to T_f and V_f^{tr} respectively.

If the angle θ is sufficiently small and g (possibly complex) is sufficiently close to f and does not have invariant line fields on $K(g)$, or is hyperbolic, then

$$\mathcal{K}_g^\theta \cap T_g = \emptyset. \tag{3.2}$$

Otherwise there would exist a sequence of maps $g_n \to f$, either without invariant line fields on $K(g_n)$ or hyperbolic, and a sequence of unitary horizontal vector fields $v_n \in T_{g_n}\mathcal{A}_a$ converging to V^{tr}. Let α_n be a a qc vector field equivariant with respect to (g_n, v_n) on $\mathrm{orb}_{g_n}(0)$. By (3.1) and the Second Compactness Lemma, the sequence $\{\alpha_n\}$ admits a subsequence converging to a qc vector field α equivariant on $\mathrm{orb}_f(0)$ with respect to (f, V^{tr}). It follows that the vector field V^{tr} is horizontal – contradiction.

3.2.8. Local laminations. Let $\mathcal{V} \subset \mathcal{A}_a$ be a neighborhood of f where (3.2) holds. We can select this neighborhood as a product $\mathcal{V}^h \times \Sigma^{\mathrm{tr}}$ where $\mathcal{V}^h \subset T_f$ and $\Sigma^{\mathrm{tr}} \subset V_f^{\mathrm{tr}}$ is a transverse segment. Exploiting a macroscopic version of the Key estimate, we show that for Σ^{tr} small enough, each hybrid class may intersect Σ^{tr} only at a single point (the estimate we use essentially implies that if two nearby maps g_1 and g_2 are hybrid conjugate then $g_2 - g_1$ is almost tangent to the hybrid class of g_1, so g_1 and g_2 cannot be both in the transverse segment Σ^{tr}). This implies that preperiodic and hyperbolic maps are dense in Σ^{tr} (thus, at this stage we obtain a new proof of Kozlovski's Theorem [K1]).

If $g \in \mathcal{V}$ is preperiodic or hyperbolic then the hybrid class $\mathcal{H}_g$ is a complex analytic submanifold in $\mathcal{A}_a$ whose tangent space coincides with the horizontal space T_g. By the cone transversality (3.2), this submanifold has a bounded slope in the coordinate system $T_f \oplus V_f^{\mathrm{tr}}$. This implies (together with the existence of uniform qc conjugacies for $\mathcal{H}_g \cap \mathcal{V}$, obtained using the Macroscopic Pullback Argument) that it is a graph with a bounded slope over the whole neighborhood $\mathcal{V}^h$. For these simple combinatorics (preperiodic or hyperbolic), it is easy to see that the local hybrid classes $\mathcal{H}_g \cap \mathcal{V}$ of different maps cannot intersect, so we have indeed constructed the lamination through a dense subset of Σ^{tr}. An application of the Extension Lemma promotes it to a lamination through the whole transversal Σ^{tr}.

Since we already dealt with infinitely renormalizable maps, the remaining case of the construction of local laminations is for hyperbolic maps, which is much easier. If f is hyperbolic, there is a neighborhood $\mathcal{V}^{\mathbb{R}} \subset \mathcal{U}_a$ of f consisting of hyperbolic maps. The analytic map that associates to each $g \in \mathcal{V}^{\mathbb{R}}$ the multiplier of its attracting periodic orbit is a submersion. This implies that the real hybrid classes in $\mathcal{V}^{\mathbb{R}}$ form a transversally analytic foliation.

3.2.9. Connectivity of the hybrid classes. In the Appendix B, we prove that any hybrid class $\mathcal{H}^{\mathbb{R}}_{f,a}$ of a Yoccoz map f is connected. To this end we complement the puzzle structure of f with a Markov structure in such a way that the combinatorics of this pattern depends on the combinatorics of f only, while the geometry of the pattern is definite. In other words, these patterns are qc equivalent for any two maps f and $\tilde{f} \in \mathcal{H}^{\mathbb{R}}_f$. By the pullback argument, f and $\tilde{f}$ are qc equivalent in a complex neighborhood of the interval I.

In particular, we can select $\tilde{f}$ as the quadratic polynomial q_τ in the hybrid class of f. Let h be an $\mathbb{R}$-symmetric qc map conjugating f to $\tilde{f}$ near I, and let μ be its Beltrami differential. Let $\mu_t = t\mu$, $0 \le t \le 1$, and let $h_t : \mathbb{C} \to \mathbb{C}$ be the solution of the corresponding Beltrami equation fixing 0 and 1. Then the Beltrami path $f_t = h_t \circ f \circ h_t^{-1}$ is a real analytic family of unimodal maps connecting f to $\tilde{f}$ in $\mathcal{H}^{\mathbb{R}}_{f,a'}$ for some a'.

Starting with this path f_t in $\mathcal{H}^{\mathbb{R}}_{f,a'}$ we build a two parameter family $f_{t_1,t_2} = f_{t_1} + t_2 v_{t_1}$ in $\mathcal{U}_{a'}$ using a transverse direction v_t along the path f_t. This family can be approximated by a two parameter family in $\mathcal{U}_a$, passing through f and $\tilde{f}$. By the Implicit Function Theorem, the intersection of this family with $\mathcal{H}^{\mathbb{R}}_{f,a}$ yields a path in $\mathcal{H}^{\mathbb{R}}_{f,a}$ linking f and $\tilde{f}$.

3.2.10. Regular or Stochastic Theorem. We will outline now a proof of Theorem B. Take some map $f \in \mathcal{U}_a$ which is not parabolic, and consider a real analytic family $f_t \in \mathcal{U}_a$, $f_0 = f$, transverse to the hybrid class $\mathcal{H}^{\mathbb{R}}_f$. We wish to prove that almost all maps f_t near f are either regular or stochastic. Of course, we can assume that f is not hyperbolic. Moreover, since maps with non-recurrent critical point have absolutely continuous invariant measures (Misiurewicz [Mi]), it is enough to consider the recurrent case.

Assume first that f is at most finitely renormalizable. Since the real hybrid class $\mathcal{H}^{\mathbb{R}}_f$ is connected, there is a local holonomy from the family $\{f_t\}$ to the quadratic family $\{q_\tau\}$. By Corollary 2.6, this holonomy is quasisymmetric.

Quasisymmetric maps are not in general absolutely continuous. However, by Lemma 2.5, they respect the property of exponential decay of the parapuzzle geometry (Theorem 2.25). This property implies that almost any map f_t near f, which is not hyperbolic and is at most finitely renormalizable, satisfies the Martens-Nowicki criterion for existence of a.c.i.m. Hence almost any map f_t near f which is at most finitely renormalizable is either regular or stochastic.

If f is infinitely renormalizable then some renormalization $\{g_t = R^n f_t\}$ is a quadratic-like family near $g = R^n f$ transverse to the real hybrid class $\mathcal{H}_g^{\mathbb{R}}$ (since the image of $DR^n(f)$ is dense, R^n is transversally non-singular). Since the hybrid classes form a holomorphic lamination with connected leaves in the space of quadratic-like germs, the holonomy from $\{g_t\}$ to the quadratic family $\{q_\tau\}$ is quasisymmetric. By Theorem 2.27 the set $\mathcal{I}$ of infinitely renormalizable quadratic maps has definite gaps everywhere. This property carries to the family $\{g_t\}$ by means of the quasisymmetric holonomy. Since R^n analytically maps $\{f_t\}$ to $\{g_t\}$, the same is true for the family $\{f_t\}$. By the Lebesgue Density Theorem, the set of infinitely renormalizable maps f_t near f has zero Lebesgue measure.

Consider now an arbitrary non-trivial real analytic family $\{f_t\} \subset \mathcal{U}_a$. We show in §9.1 that this family contains at most countably many tangencies with the real hybrid classes and at most countably many parabolic points. At all other points it is transverse to the hybrid classes. It follows that almost all parameter values in this family are either regular or stochastic. The set of transverse points is an open set with countable complement, so this argument also proves Theorem C.

4. Infinitely renormalizable case

4.1. Hybrid lamination in the space of quadratic-like maps

4.1.1. Banach balls of quadratic-like maps. Let U be a 0-symmetric domain, and let $\mathcal{B}_U^0$ be the space of normalized at 0 even holomorphic functions $f \in \mathcal{B}_U$. This is an affine subspace of $\mathcal{B}_U$. The ball or radius ε centered at $f \in \mathcal{B}_U^0$ is denoted as $\mathcal{B}_U^0(f, \varepsilon)$.

Each class of affinely conjugate quadratic-like maps contains a unique representative normalized at 0. Let $\mathcal{Q}$ stand for the space of normalized at 0 quadratic-like maps. The *connectedness locus* $\mathcal{C}$ is the set of quadratic-like maps $f \in \mathcal{Q}$ with connected Julia set. The *hybrid class* $\mathcal{H}_f \subset \mathcal{Q}$ is the set of quadratic-like maps $g \in \mathcal{Q}$ which are hybrid equivalent to f on some neighborhoods of the filled Julia sets.

Take some quadratic-like map $f : U \to U'$ in $\mathcal{Q}$. Consider a 0-symmetric Jordan domain $V \Subset U$ with a piecewise smooth boundary such that $V \supset K(f)$ and $fV \Supset V$. Then any normalized at 0 even holomorphic function $g \in \mathcal{B}_U^0$ which is sufficiently close to f restricts to a quadratic-like map $g : V \to V'$. Thus, we have an embedding $j_V : \mathcal{B}_U^0(f, \varepsilon) \to \mathcal{Q}$. Its image $\mathcal{Q}_V(f,\varepsilon)$ endowed with the topology induced from $\mathcal{B}_U$ will be called a *Banach ball* (centered at f of radius ε) of quadratic-like maps.

Given a set $X \subset \mathcal{Q}$, the intersection $X \cap \mathcal{Q}_V(f,\varepsilon)$ will be called a *Banach slice* of X (by the ball $\mathcal{Q}_V(f,\varepsilon)$).

4.1.2. Hybrid lamination. Below we will summarize the results of [L6] (especially §4) about the hybrid lamination in the space of quadratic-like maps.

Theorem 4.1. *Let $f : U \to U'$ be a normalized at 0 quadratic-like map with connected Julia set. Then there exists a Banach ball $\mathcal{V} = \mathcal{Q}_V(f,\varepsilon)$ such that for any $g \in \mathcal{V} \cap \mathcal{C}$, the slice of the hybrid class $\mathcal{H}_g$ by the ball $\mathcal{V}$ is a complex codimension-one analytic submanifold in $\mathcal{V}$. These submanifolds form a holomorphic lamination in $\mathcal{V}$.*

In the setting of the above theorem, let us consider a map $\tilde{f} \in \mathcal{V}$ which is hybrid equivalent to f and in the same connected component of $\mathcal{V} \cap \mathcal{H}_f$. Take two transversals $\mathcal{T} \ni f$ and $\tilde{\mathcal{T}} \ni \tilde{f}$ to $\mathcal{H}_f \cap \mathcal{V}$. Then we have a well defined local holonomy $h : \mathcal{T} \cap \mathcal{C} \to \tilde{\mathcal{T}} \cap \mathcal{C}$ along the leaves of the lamination. Moreover, by the λ-lemma, h admits a qc extension to a neighborhood of f in $\mathcal{T}$. (We will formulate this property briefly by saying that "the holonomy is locally quasiconformal" or that the "the lamination is transversally quasiconformal"). In fact, these holonomies can be extended to the whole hybrid class of f:

Proposition 4.2. *Let $f \in \mathcal{C}$ be a normalized quadratic-like map with connected Julia set. If $\tilde{f} \in \mathcal{H}_f$ is hybrid equivalent to f then it can be joined to f with a path $\{f_t\} \subset \mathcal{H}_f$ covered with finitely many Banach balls from Theorem 4.1. Thus, there is a well-defined local qc holonomy between any two transversals $\mathcal{T} \ni f$ and $\tilde{\mathcal{T}} \ni \tilde{f}$ to $\mathcal{H}_f$.*

The role of the quadratic family is partly explained by the following statement:

Proposition 4.3. *The quadratic family $\{q_\tau\}$ is transverse to the hybrid lamination (in any Banach ball from Theorem 4.1).*

The tangent space T_f to the above Banach slice of $\mathcal{H}_f$ consists of vector fields $v \in \mathcal{B}_U$ that admit representation (2.5), $v = \alpha \circ f - f'\alpha$, where α is a qc vector field on U whose $\bar\partial$-derivative vanishes a.e. on the filled Julia set $K(f)$. Such vector fields are called *horizontal*.

A transverse, *vertical* direction to the hybrid class $\mathcal{H}_f$ can be selected as a holomorphic vector field V_f^{tr} which can be represented as (2.5) in $U \setminus K(f)$, where $\alpha(z)/dz$ is a *holomorphic* vector field on $\overline{\mathbb{C}} \setminus K(f)$ vanishing at ∞. The latter condition means that near ∞, $\alpha(z) = az + h(z)$, where h is a bounded holomorphic function. If f is $\mathbb{R}$-symmetric, the transverse direction can also be chosen $\mathbb{R}$-symmetric.

4.2. From real analytic to quadratic-like. Let us consider an infinitely renormalizable quasi-quadratic map $f \in \mathcal{U}_a$. By the *a priori bounds*, there is a quadratic-like renormalization of f. This means that there exist $\mathbb{R}$-symmetric Jordan domains $U \equiv U_0, \dots, U_{n-1}, U_p \equiv U'$ such that:

- $U_i \Subset \Omega_a$, $i = 0, 1, \dots, n-1$.
- $U_0 \ni 0$ is 0-symmetric and $f : U_0 \to U_1$ is a branched double covering;
- the maps $f : U_i \to U_{i+1}$ are conformal isomorphisms, $i = 1, \dots, n-1$;
- $U' \Supset U$.

We call the map

$$P(f) \equiv f^n : U \to U'$$

a *quadratic-like pre-renormalization* of f. If $g \in \mathcal{U}_a$ is sufficiently close to f, then the restriction $g^n|U$ gives a quadratic-like pre-renormalization of g. Normalizing these maps at the origin, we obtain a renormalization $R : \mathcal{V} \to \mathcal{Q}_W^{\mathbb{R}}$, where $\mathcal{V} \subset \mathcal{U}_a$ is some neighborhood of f and $W \ni 0$ is an $\mathbb{R}$-symmetric and 0-symmetric Jordan disk obtained from U by little shrinking and rescaling.

To carry out the infinitesimal analysis of the renormalization operator, we will make use of the variational formula (2.8).

Lemma 4.4. *The renormalization operator $R : \mathcal{V} \to \mathcal{Q}_W^{\mathbb{R}}$ is real analytic.*

Proof. The pre-renormalization of f analytically depends on f, since it is a restricted iterate of f (with the differential explicitly given by (2.8)). The normalization of a function f is the rescaling by factor $f''(0)$ analytically depending on f as well. □

We want to prove that the derivative of the renormalization operator is transversally non-singular, that is, the image of $DR(f)$ contains transverse vectors to $\mathcal{H}_{R(f)}^{\mathbb{R}}$. Let us deal first with an easier case.

Lemma 4.5. *Assume $\overline{U}_i$ are disjoint,* $0 \le i \le n-1$. *Then the image of the infinitesimal renormalization $DR(f) : T_f\mathcal{A}_a^{\mathbb{R}} \to T_{R(f)}\mathcal{Q}_W^{\mathbb{R}}$ is dense.*

Proof. The operator R is a composition of the pre-renormalization operator P from $\mathcal{A}_a^{\mathbb{R}}$ to $\mathcal{B}_U^{\mathbb{R}}$ and a rescaling operator from $\mathcal{B}_U^{\mathbb{R}}$ to $\mathcal{Q}_W^{\mathbb{R}}$. It is easy to see that the derivative of the rescaling operator has dense image, so we just have to show that $DP(f) : T_f\mathcal{A}_a^{\mathbb{R}} \to T_{R(f)}\mathcal{B}_U^{\mathbb{R}}$ has dense image. Since $P(f)$ is the restriction of the iterate f^n, the differential $DP(f)v$ is given by the variational formula (2.8).

Take an even holomorphic vector field $w \in \mathcal{B}_U^{\mathbb{R}}$. Let us define a holomorphic vector field $\tilde{v}$ on $\cup U_i$ such that $DP(f)\tilde{v} = w$. First let $\tilde{v} = 0$ on $\cup_{i=1}^{n-1} U_i$. Using (2.8) with w in the left-hand side, extend this vector field to U:

$$\tilde{v}(z) = \frac{w(z)}{Df^{n-1}(f(z))} \tag{4.1}$$

(notice that in (2.8) the only non-vanishing term corresponds to $k = 0$).

Then $\tilde{v}$ is an even vector field on U, and the pre-renormalization of $\tilde{v}$ is equal to w. By the Mergelyan Polynomial Approximation Theorem (see [R], Theorem 20.5, p. 423), $\tilde{v}$ can be approximated by a polynomial vector field v with which is even and $\mathbb{R}$-symmetric. Then the pre-renormalization of v approximates w. Rescaling the domain U, we obtain the assertion. □

The above situation occurs whenever the last renormalization is not period doubling. In the doubling case, the intersections between the U_i are unavoidable, since the little Julia sets (the forward images of $J(f^{2n}|U)$, where $2n$ is the period of renormalization) intersect.

In this case, we can still select the U_i with the following properties:

- If the closure of U_i intersects the closure of U_j and $i < j$ then $j = i + n$ and $U_i \cup U_j$ is simply connected;
- $\overline{U}_0 \cap \overline{U}_n$ is contained in a small neighborhood V of a repelling periodic point q of period n;
- V is a topological disk such that $f^n|V$ is a diffeomorphism and $\overline{V} \subset f^n(V)$; hence $f^n|V$ is linearizable.

Lemma 4.6. *Under the circumstances just described, $DR(f) : T_f\mathcal{A}_a^{\mathbb{R}} \to T_{R(f)}\mathcal{Q}_W^{\mathbb{R}}$ has dense image.*

We will make use of the following lemma.

Lemma 4.7. *Let $|\lambda| > 1$, $L : s \mapsto \lambda s$, and let S be a bounded connected open set such that $0 \in S \subset L(S)$. Consider two holomorphic functions $a, b \in \mathcal{B}_S$ such that $b(0) = \lambda$. Then there exists a holomorphic function $u \in \mathcal{B}_S$ such that*

$$a(s) = u(s) + b(s)u(\lambda^{-1}s), \quad s \in S. \tag{4.2}$$

Proof. Let us first consider the case where $S = \mathbb{D}_\varepsilon$ and $\|b\|_{\mathcal{B}_S} < |\lambda|^2$. Let $a(s) = \sum_{k=0}^{\infty} a_k s^k$, $b(s) = \sum_{k=0}^{\infty} b_k s^k$. Then (4.2) has a formal solution $u(z) = u_k z^k$ whose coefficients are recursively found from the equations:

$$(\lambda + \lambda^k)u_k = a_k - \sum_{j=0}^{k-1} b_{k-j} u_j.$$

To prove convergence of the formal solution, let us consider the linear jet of u, $\hat{u}(s) = u_0 + u_1 s$, and let $\hat{a}(s) = \hat{u}(s) + b(s)\hat{u}(\lambda^{-1}s)$. A simple calculation gives $a(s) - \hat{a}(s) = O(|s^2|)$.

Hence it is enough to solve (4.2) in the space of functions u with vanishing linear jet (assuming that a is also in the same space). In terms of $w(s) = u(s)/s^2$ and $c(s) = a(s)/s^2$, (4.2) assumes the form

$$c(s) = w(s) + \lambda^{-2} b(s) w(\lambda^{-1}s) \equiv (\mathrm{id} + T)w(s),$$

where $T : \mathcal{B}_S \to \mathcal{B}_S$ is a contracting linear operator (since $\|T\| = \lambda^{-2}\|b\| < 1$). Hence $\mathrm{id} + T$ is invertible, and the conclusion follows.

In the general case, we first notice that for ε sufficiently small we still have $a, b \in \mathcal{B}_{\mathbb{D}_\varepsilon}$ and $\|b\|_{\mathcal{B}_{\mathbb{D}_\varepsilon}} < |\lambda|^2$. By the previous consideration, we obtain $u \in \mathcal{B}_{\mathbb{D}_\varepsilon}$ satisfying (4.2) for $s \in \mathbb{D}_\varepsilon$. Then we find a $k \geq 0$ such that $L^{-k}(S) \subset \mathbb{D}_\varepsilon$ and use the functional equation k times to extend u to S. $\square$

Proof of Lemma 4.6. As before, we just have to show that the image of $DP(f)$ is dense.

Arguing as in Lemma 4.5, let $w \in \mathcal{B}_U^{\mathbb{R}}$ be an arbitrary even polynomial vector field (note that even polynomial vector fields are dense in $\mathcal{B}_U^{\mathbb{R}}$). We claim that there exists a holomorphic vector field v_{lin} in $f^n(V)$ such that

(4.3)
$$w(z) = Df^{n-1}(f^{n+1}(z))\, v_{\mathrm{lin}}(f^n(z)) + Df^{2n-1}(f(z))\, v_{\mathrm{lin}}(z), \quad z \in V.$$

(Note that the right-hand side of this equation is formally equal to $DP(f)v_{\mathrm{lin}}$ assuming that v_{lin} is extended outside $f^n(V)$ by zero.) To solve (4.3), let us linearize f^n in V. Let $\lambda = Df^n(q)$, $L(s) = \lambda s$, S be a neighborhood of 0 such that $L(S) \supset S$, and let $\phi : (L(S), 0) \to (f^n(V), q)$ be a conformal map satisfying $\phi \circ L = f^n \circ \phi$ for $s \in S$.

In this linear coordinate s the above equation can be written in the form (4.2), with $a = w \circ \phi$, $b = Df^n \circ f^n \circ \phi$ (so that $b(0) = \lambda$), and

$$u = (v_{\mathrm{lin}} \circ \phi \circ L) \cdot (Df^{n-1} \circ f \circ \phi \circ L).$$

Using Lemma 4.7, we obtain a solution $u \in \mathcal{B}_S$ of (4.2). Passing back to V we obtain a solution v_{lin} of (4.3).

Let v_n be an even polynomial vector field which is close to v_{lin} on $f^n(V)$, and let v_0 be an even polynomial vector field close to

$$\frac{w(z) - Df^{n-1}(f^{n+1}(z))\, v_n(f^n(z))}{Df^{2n-1}(f(z))}$$

in a neighborhood of $\overline{U_0 \cup V}$. Notice that v_0 is close to v_n on V.

Let us now construct a vector field interpolating between v_0 and v_n. Since we want even vector fields, we first symmetrize the domains. Let $-U_n$ and $-V$ be the central reflections of respectively U_n and V about the origin, and let

$$U_n^{\mathrm{sym}} = U \cup (-U), \quad V^{\mathrm{sym}} = V \cup (-V)$$

(see Fig. 2). Let $K \subset V^{\mathrm{sym}}$ be a compact 0-symmetric neighborhood of $\overline{U}_0 \cap \overline{U_n^{\mathrm{sym}}}$.

Using a partition of unity, construct an even smooth (C^∞) vector field v_{sm} on $\mathbb{C}$ with the following properties: $v_{\mathrm{sm}} = v_0$ on $\overline{U}_0 \setminus K$, $v_{\mathrm{sm}} = v_n$ on $U_n^{\mathrm{sym}} \setminus K$, and v_{sm} is C^1 close to v_n on K (this is possible because v_0 is C^1 close to v_n on $K \subset V$).

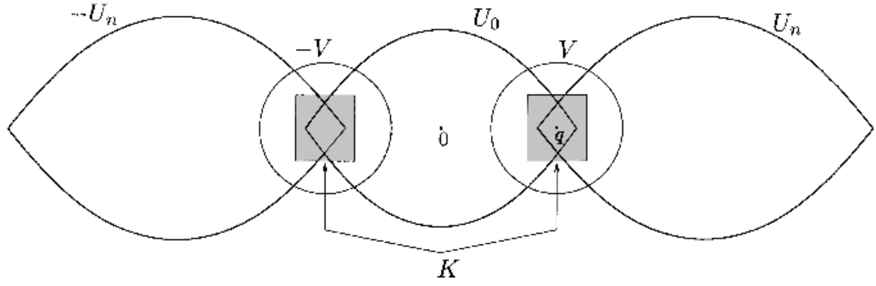

Fig. 2.

Consider now a normalized qc vector field α such that $\overline{\partial}\alpha|U_0 \cup U_n^{\mathrm{sym}} = \overline{\partial}v_{\mathrm{sm}}$ and $\overline{\partial}\alpha|\mathbb{C} \setminus (U_0 \cup U_n^{\mathrm{sym}}) = 0$ (see §2.6.1). Notice that $\|\overline{\partial}\alpha\|_\infty$ is close to 0 and so by the Second Compactness Lemma, α is close to 0. Then $v_{\mathrm{sm}} - \alpha \in \mathcal{B}^0_{U_0 \cup U_n^{\mathrm{sym}}}$ interpolates between v_0 and v_n.

By the Mergelyan Theorem, there is an even polynomial vector field v which is close to v_0 on U_0, to v_n on U_n and to 0 on the U_i, $i \neq 0, n$. Then $DP(f)v$ is close to w in $\mathcal{B}_U$. □

Lemmas 4.5 and 4.6 imply:

Lemma 4.8. *DR is transversally non-singular.*

4.3. Local laminations near infinitely renormalizable maps.

Theorem 4.9. *Let $f \in \mathcal{U}_a$ be an infinitely renormalizable map. Then some neighborhood $\mathcal{V} \subset \mathcal{U}_a$ of f is foliated by the real hybrid classes, which are codimension-one real analytic submanifolds in $\mathcal{V}$.*

Proof. Consider the above renormalization operator $R : \mathcal{V} \to \mathcal{Q}^{\mathbb{R}}_W$. Let $g = R(f)$. By Theorem 4.1, $\mathcal{Q}^{\mathbb{R}}_W = T_g \oplus V_g^{\mathrm{tr}}$, where T_g is the tangent space to the real hybrid class at g and V_g^{tr} is the transverse vertical line. Moreover, g has a neighborhood $\mathcal{W} \subset \mathcal{Q}^{\mathbb{R}}_W$ foliated by the real hybrid classes of quadratic-like maps, which are graphs of real analytic functions $T_g \to V_g^{\mathrm{tr}}$.

If the neighborhood $\mathcal{W}$ is sufficiently small, then the hybrid class $\mathcal{H}^{\mathbb{R}}_g \cap \mathcal{W}$ is the zero-set of some real analytic submersion $\phi : \mathcal{W} \to \mathbb{R}$. By Lemma 4.8, there exists a vector $v \in \mathcal{A}^{\mathbb{R}}_a$ such that $w \equiv DR(f)v \notin T_g$. Hence the composition $\phi \circ R : \mathcal{V} \to \mathbb{R}$ is a submersion at f. By the Implicit Function Theorem, its zero-set $X_f = R^{-1}\mathcal{H}_g$ is a codimension-one submanifold near f.

Since R is real analytic and the horizontal space T_G varies continuously for $G \in \mathcal{W}$, $DR(F)v \notin T_{R(F)}$ for all $F \in \mathcal{V}$ sufficiently close to f. By shrinking $\mathcal{V}$, we can assume that this is valid for all $F \in \mathcal{V}$. Applying the Implicit Function Theorem as above we see that the preimages $X_F = R^{-1}\mathcal{H}_{R(F)}$, $F \in \mathcal{V}$, are real analytic codimension-one submanifolds in $\mathcal{V}$.

Moreover, these submanifolds are closed in $\mathcal{V}$ and are transverse to v at any point $F \in \mathcal{V}$. Hence they form a real analytic lamination of $\mathcal{V}$.

Observe finally that the leaves X_F of this lamination coincide with the real hybrid classes. Indeed, if two non-hyperbolic maps F and $\tilde{F}$ in $\mathcal{V}$ are topologically equivalent on the real line then so are their renormalizations $R(F)$ and $R(\tilde{F})$. By Theorem 2.21, these two maps are hybrid equivalent. If F and $\tilde{F}$ are hyperbolic with the same multiplier then so are their renormalizations.

Vice versa, if $R(F)$ and $R(\tilde{F})$ are topologically equivalent on the real line then the maps F and $\tilde{F}$ have the same kneading sequence. Hence, by Remark 2.10, if $R(F)$ and $R(\tilde{F})$ are hybrid equivalent, then so do F and $\tilde{F}$. $\square$

Remark 4.1. Since the hybrid class of an infinitely renormalizable map f has just been shown to be codimension-one submanifolds, the transverse non-singularity of the renormalization operator at f (and the fact that hybrid classes are preserved by the renormalization) implies that the image by DR of any vector transverse to $\mathcal{H}_f^{\mathbb{R}}$ is transverse to $\mathcal{H}_{R(f)}^{\mathbb{R}}$.

4.4. Regular or stochastic property near infinitely renormalizable parameter values.

Theorem 4.10. *Let $\{f_t\} \subset \mathcal{U}_a$ be a one-parameter real analytic family of quasi-quadratic maps such that $f \equiv f_{t_0}$ is infinitely renormalizable. If $\{f_t\}$ is transverse to the real hybrid class $\mathcal{H}_f^{\mathbb{R}}$ then for almost all t near t_0, the map f_t is either regular or stochastic.*

Proof. Let us consider the above renormalization operator $R : \mathcal{V} \to \mathcal{Q}_W^{\mathbb{R}}$. By Remark 4.1, $DR(f)v \notin T_{R(f)}$ if v is not tangent to $\mathcal{H}_f^{\mathbb{R}}$. Since R is analytic, $\{g_t = R(f_t)\}$ is a real analytic family of quadratic-like maps transverse to the hybrid class $\mathcal{H}_g^{\mathbb{R}} \cap \mathcal{Q}_W^{\mathbb{R}}$, where $g = g_0$.

By Theorem 4.1, the foliation by the real hybrid classes in $\mathcal{Q}^{\mathbb{R}}$ is transversally quasisymmetric with connected leaves, so the straightening $g_t \mapsto q_{\chi(t)}$ is quasisymmetric near 0. By Theorem 2.28 and the argument of §3.2.10 (detailed in §9.4), this implies the Regular or Stochastic property for the family $\{g_t\}$ near 0.

Since the map g_t is regular or stochastic if and only if the corresponding map f_t is, the conclusion follows. $\square$

4.5. Straightening near infinitely renormalizable parameters.

Theorem 4.11. *Let $\{f_t\}$ be a one-parameter real analytic family of unimodal maps such that $f = f_{t_0}$ is infinitely renormalizable. If $\{f_t\}$ is transverse to $\mathcal{H}_f^{\mathbb{R}}$ at f then the straightening χ of this family is quasisymmetric near f.*

Proof. Consider the renormalization operator $R : \mathcal{V} \to \mathcal{Q}$ in a neighborhood $\mathcal{V}$ of f described in §4.2. Since R is transversally non-singular, it diffeomorphically maps the family $\{f_t\}$ near f onto its image $\{Rf_t\}$.

Let $q = \chi(f)$ be the straightening of f. By Proposition 4.3, the quadratic family q_τ is transverse to the hybrid lamination. For the same reason as above, the renormalization R diffeomorphically maps the quadratic family near q onto its image $\{Rq_\tau\} \subset \mathcal{Q}$.

By Proposition 4.2, there is a well-defined quasisymmetric holonomy h from the family $\{Rf_t\}$ to the family $\{Rq_\tau\}$ in the space $\mathcal{Q}$ of quadratic-like maps. Since $\chi = R^{-1} \circ h \circ R$ (the renormalization operator acts on hybrid classes, see Remark 2.10), we conclude that χ is quasisymmetric as well.

□

We will consider next the case of at most finitely renormalizable maps.

5. Tangent space and puzzle maps

5.1. Tangent space. To prove that a hybrid class is a Banach submanifold we have to find a candidate for its tangent space. We will now associate to each map f a complex vector space whose intersection with the real slice will be the tangent space to the hybrid class whenever the map is real and unimodal.

As in the unimodal case, a map $g \in \mathcal{A}_a$ is called preperiodic if there exists k such that c_k is a repelling periodic orbit. It is called hyperbolic if $\omega(0)$ is an attracting cycle.

Remark 5.1. For the class of maps that we are considering, there could be (complex) attracting cycles that do not attract the critical point. However, whenever we refer to an attracting cycle, we mean the one which does attract the critical point.

The following three propositions will motivate the definition of the tangent space to a hybrid class.

Proposition 5.1. *Let $f \in \mathcal{A}_a$ and $v \in \mathcal{B}_a$. Let $\{f_t\}$ be a smooth curve in $\mathcal{A}_a$ such that $f_0 = f$ and*

$$\frac{d}{dt} f_t \bigg|_{t=0} = v.$$

Then, for each n, the curve $c_n(t) = f_t^n(0) \in \mathbb{C}$ is smooth in a neighborhood of $t = 0$. If there is an equivariant vector field α on $\mathrm{orb}_f(0)$ *then*

$$\alpha(c_n) = \frac{d}{dt} c_n(t) \bigg|_{t=0} = v^n(0).$$

Proof. The second equation follows from (2.9). Since α is also equivariant by (f^n, v^n) (see §2.6.2), we have:

$$\alpha(c_n) = \alpha(c_n) - Df^n(0)\alpha(0) = v^n(0).$$

□

Remark 5.2. If the orbit of zero is infinite, there is always a (perhaps not even continuous) vector field α on $\operatorname{orb}_f(0)$ satisfying (2.5). If the critical orbit of f is periodic or preperiodic, the set T_f of vector fields v for which there exists an α satisfying (2.5) is a codimension-one subspace. Moreover, α on the orbit of the critical value is uniquely determined by v on the orbit of the critical point.

Recall that O_f stands for the postcritical set $\overline{\operatorname{orb}(c_1)}$.

Definition 5.1. *Let us consider a map $f \in \mathcal{U}_a$. We denote by $\mathcal{G}_f$ the set of qc vector fields on the closed set O_f endowed with the norm $\|\cdot\|_{\mathrm{qc}}$.*

Proposition 5.2. *Let f_λ be a holomorphic family in $\mathcal{A}_a$ through $f \equiv f_0$, with non escaping critical orbit. Assume there is a holomorphic motion h_λ satisfying the equation $h_\lambda \circ f \circ h_\lambda^{-1} = f_\lambda$ on the critical orbit. Let*

$$v = \frac{d}{d\lambda} f_\lambda \bigg|_{\lambda=0}$$

and

$$\alpha = \frac{d}{d\lambda} h_\lambda \bigg|_{\lambda=0}.$$

Then α is a qc vector field that satisfies equation (2.5) on the critical orbit.

Proof. Differentiating the equation

$$h_\lambda \circ f(c_n) = f_\lambda \circ h_\lambda(c_n)$$

at $\lambda = 0$ we conclude that

$$\alpha(c_{n+1}) = f'(c_n)\alpha(c_n) + v(c_n)$$

for every $n \geq 0$, which is clearly equivalent to equation (2.5). Quasiconformality of α follows from Lemma 2.10. □

If $f \in \mathcal{A}_a$ is a preperiodic map we denote by $\mathcal{C}_f \subset \mathcal{A}_a$ the connected component of f in the set of preperiodic maps with the same relation on the critical orbit. If f is hyperbolic we denote by $\mathcal{C}_f$ the connected component of f in the set of hyperbolic maps with the same multiplier of the attractor as f.

Definition 5.2. *Let f be either a non-hyperbolic map in $\mathcal{U}_a$ or a complex map in $\mathcal{A}_a$ with preperiodic or periodic critical point. We denote by T_f the set of vectors $v \in T\mathcal{A}_a$ such that there exists a qc vector field α equivariant on the critical orbit.*

By Remark 5.2, for each $v \in T_f$, there exists a unique qc vector field α on O_f which extends to a qc vector field equivariant on the critical orbit. It is clear that the correspondence $L_f : T_f \to \mathcal{G}_f$ that to each v associates such an α is a linear map.

Remark 5.3. We will later see that L_f is a *continuous* linear map. This is part of the content of the Key estimate (Lemma 6.12) and is one of the main results of our analysis.

Proposition 5.3. *Let $f \in \mathcal{A}_a$ be a preperiodic map. Then $\mathcal{C}_f$ is a complex codimension-one submanifold in $\mathcal{A}_a$ whose tangent space at f is T_f.*

Proof. If f is a preperiodic then there exists minimal $p, q > 0$ such that $c_p = c_{p+q}$. By the Implicit Function Theorem, the solutions of the equation $c_p = c_{p+q}$ form a codimension-one submanifold whose tangent space at f is T_f. □

With this motivation we define T_f for a hyperbolic but not superattracting map $f \in \mathcal{A}_a$ as the tangent space to $\mathcal{C}_f$, which is a codimension-one complex submanifold by the Implicit Function Theorem. The following simple proposition shows that in this case, T_f cannot be described by Definition 5.2.

Proposition 5.4. *If $f \in \mathcal{A}_a$ is hyperbolic but not superattracting and $v \in \mathcal{B}_a$, then there is a qc vector field on $\mathbb{C}$ which satisfies the equation (2.5) on the critical orbit.*

Proof. Since f is hyperbolic but not superattracting, for λ small enough f is conjugate to $f_\lambda \equiv f + \lambda v$ in a neighborhood V of the attractor. Since there are only a finite number of n such that $c_n \notin V$ we conclude that f is conjugate to f_λ on the critical orbit. We can now consider a holomorphic motion h_λ such that $h_\lambda(f^n(0)) = f_\lambda^n(0)$. The conclusion follows from Proposition 5.2. □

If f is hyperbolic but not superattracting and $v \in T_f$ we define $L_f(v) = \alpha \in \mathcal{G}_f$ satisfying (2.5) which exists by the above proposition (and is unique by Remark 5.2).

It is easy to see that if f is a hyperbolic map with a periodic attractor p of period n then vector fields v tangent to $\mathcal{C}_f$ at f satisfy the following formula,

$$Dv^n(p)(Df^n(p) - 1) = v^n(p)D^2 f^n(p), \tag{5.1}$$

since this formula implies that the multiplier of the continuation of p along $f+tv$ (given infinitesimally by $p+tv^n(p)(1-Df^n(p))^{-1}$) does not change infinitesimally.

5.2. Puzzle maps. Let $D \subset \mathbb{C}$ be a topological disk. We say that D has L-bounded shape around $x \in D$ if there exists $r > 0$ such that

$$\mathbb{D}_r(x) \subset D \subset \mathbb{D}_{Lr}(x).$$

Let $U \subset \mathbb{C}$ be an open set and let $X \subset U$ be measurable. We say that X is *thin* in U if there exist $L, \varepsilon > 0$ with the following property: any $x \in X$ has a neighborhood $D \subset U$ with L-bounded shape around x, such that $\operatorname{mod}(U \setminus D) > \varepsilon$ and $\operatorname{meas}(D \setminus X)/\operatorname{meas} D > \varepsilon$. Notice that this notion is qc invariant: if X is thin in U and $f : U \to \mathbb{C}$ is a qc map then $f(X)$ is thin in $f(U)$. It is also invariant with respect to the lifting by branched coverings: if $f : V \to U$ is a holomorphic finite branched covering and X is thin in U then $f^{-1}(X)$ is thin in V.

Definition 5.3. *A holomorphic map $f\colon U \to \mathbb{C}$ of class $A^1(U)$ is called a puzzle map if:*

- *U is a countable union of quasidisks U_i, $i \geq 0$, called puzzle pieces, with pairwise disjoint closures, and $U_0 \ni 0$;*
- *For $i > 0$, f is a diffeomorphism of the closure of U_i onto the closure of some U_j;*
- *There exists a sequence $n_i \geq 0$ such that $f^{n_i}|U_i$ is a diffeomorphism onto U_0;*
- *0 is a critical point of f and f' does not vanish on ∂U_0;*
- *$f|U_0$ is a double covering onto the image;*
- *For any i, either $\overline{U_i}$ is contained in $f(U_0)$ or it does not intersect $\overline{f(U_0)}$.*

And furthermore the collection U_i satisfy the following geometric conditions:

(1) $\inf_{U_i \subset f(U_0)} \operatorname{mod}(f(U_0) \setminus U_i) > 0$*;*
(2) $\overline{\cup U_i} \cap f(U_0)$ *is thin in* $f(U_0)$*;*
(3) $\lim_{i\to\infty} \operatorname{diam}(U_i) = 0$.

We will use notation $U_1 \equiv U(c_1)$ for the puzzle piece containing the critical value c_1 (whenever $c_1 \in U$). We will also use the notation $\mathcal{P} = \mathcal{P}_f$ for the collection of puzzle pieces $\{U_i\}$.

Definition 5.4. *The filled Julia set $K(f)$ of a puzzle map $f\colon U \to \mathbb{C}$ is the set of points which do not escape U.*

We say that a puzzle map $f : U \to \mathbb{C}$ is hyperbolic if there is an attracting periodic cycle $\overline{p}$ contained in U. For hyperbolic puzzle maps, the immediate basin of attraction of $\overline{p}$ is compactly contained in U and contains the critical orbit.

Definition 5.5. *Let $f \in \mathcal{U}_a$. A geometric puzzle for f is a countable collection of topological disks U_i (puzzle pieces) with piecewise smooth boundary such that:*

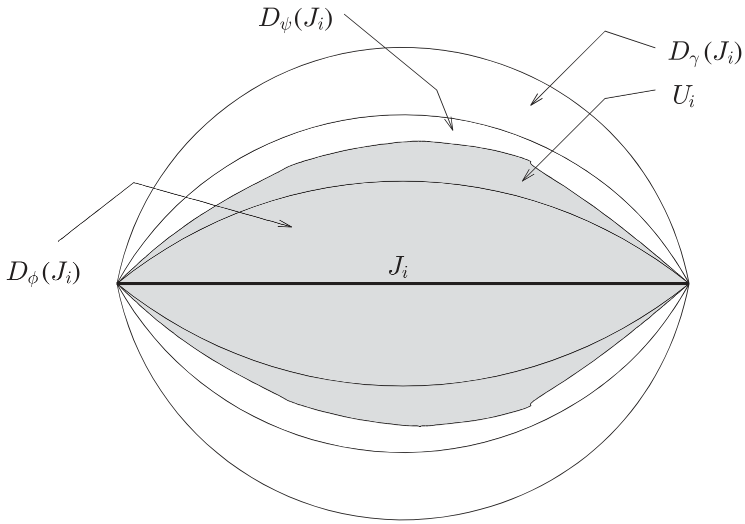

Fig. 3. Geometric domains

- *f restricted to the union of the puzzle pieces is a puzzle map;*
- *There is a real puzzle P such that the real slices $J_i = U_i \cap \mathbb{R}$ are puzzle pieces of P;*
- *The U_i are $\mathbb{R}$-symmetric;*
- *The closure of the union of the J_i contains a neighborhood of the interval $[-1, c_1]$ in $[-1, 1]$.*

And furthermore, there exist $0 < \phi < \psi < \gamma < \pi/2$ and $k > 0$ such that

(1) $D_\phi(J_i) \subset U_i \subset D_\psi(J_i)$;
(2) For each i, either the closure of $D_\gamma(J_i)$ is contained in $f(U_0)$ or it does not intersect the closure of $f(U_0)$;
(3) If U_i is contained in $f(U_0)$ then $\mathrm{mod}(f(U_0) \setminus U_i) > k$;
(4) There exists $\tilde{U}_0 \supset U_0$ with $\mathrm{mod}(\tilde{U}_0 \setminus U_0) > k$, such that if f^{n_i} maps U_i diffeomorphically onto U_0 then there is an $\tilde{U}_i \supset U_i$ which is mapped diffeomorphically onto $\tilde{U}_0$. Furthermore $\tilde{U}_i \subset f(U_0)$ or $\tilde{U}_i \cap f(U_0) = \emptyset$.

We will call (ϕ, ψ, γ, k) the geometric parameters of the puzzle.

Remark 5.4. The geometric conditions 1–4 of Definition 5.5 of a geometric puzzle map automatically imply the geometric conditions 1–3 of Definition 5.3 of a puzzle map.

If U is a geometric puzzle for f, then $f : U \to \mathbb{C}$ is naturally called a geometric puzzle map.

Definition 5.6. *We say that the puzzle $\mathcal{P}_f$ persists in a neighborhood $\mathcal{V}$ of f in $\mathcal{A}_a$ if there exists a normalized holomorphic motion $H_g : \mathbb{C} \to \mathbb{C}$, $g \in \mathcal{V}$,*

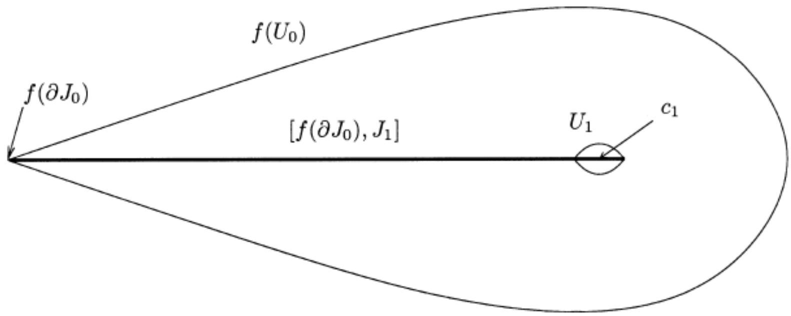

Fig. 4. Domain $f(U_0)$

such that $g(H_g(z)) = H_g(f(z)), \forall z \in \partial U^f$. *We denote* $H_g(U^f) = U^g$. *Notice that* $g|U^g$ *is automatically a puzzle map.*

Lemma 5.5. *Let* $0 < \phi < \psi < \gamma < \pi/2$ *and let* $k > 0$ *be arbitrarily big. If* $f \in \mathcal{U}_a$ *is a Yoccoz map, then there exists a puzzle for* f *with geometric parameters* (ϕ, ψ, γ, k).

Proof. Let $g = f^n : J \to J$ be the last pre-renormalization of f and let p be the fixed point of g in the interior of J. Let T^i be the principal nest for g.

Assume first that the critical point is recurrent. By Theorem 2.17, $|T^{i-1}|/|T^i|$ is unbounded. Let i_0 be such that $|T^{i_0-1}|/|T^{i_0}|$ is big. Letting $J_0 = T^{i_0}$, consider the associated real puzzle $\{J_i\}$. Let $\tilde{J}_0 = T^{i_0-1}$. In what follows, we discard the puzzle pieces to the right of the puzzle piece J_1. We still have that $\overline{\cup J_i}$ is a neighborhood of $[-1, c_1]$ in $[-1, 1]$ (since the set of points that never enter J_0 is a Cantor set, see Remark 2.8).

By Theorem 2.16, there are intervals $\tilde{J}_i \supset J_i$ such that f^{n_i} is a diffeomorphism from $\tilde{J}_i$ to $\tilde{J}_0$ and $\tilde{J}_i \cap J_0 = \emptyset$ for $i \neq 0$.

Let $\theta = (\phi + \psi)/2$ and let $U_0 = D_\theta(J_0)$. By Lemma 2.18, there exist topological disks $U_i \supset J_i$ such that for each i, f^{n_i} maps U_i onto U_0 and extends to a univalent map onto some $\tilde{U}_0$ with big $\mathrm{mod}(\tilde{U}_0 \setminus U_0)$. By the Koebe Distortion Theorem, each U_i satisfies $D_\phi(J_i) \subset U_i \subset D_\psi(J_i)$. In particular the domains U_i are pairwise disjoint.

Let us show that if U_i intersects $V \equiv f(U_0)$, then U_i is well inside V (see Fig. 4). Note first that in this case, $J_i \subset V \cap \mathbb{R}$. Otherwise there would be an interval J_k such that $f(J_k) = J_i$, $J_k \cap J_0 = \emptyset$, but $U_k \cap U_0 \neq \emptyset$ contradicting disjointness of the U_j. It follows that J_i is contained in the convex hull of $f(\partial J_0)$ and J_1 (since we assume there are no J_i to the right of J_1).

Notice also that there exists a constant ρ that only depends on θ and the initial bounds on f such that V contains $D_\rho(V \cap \mathbb{R})$. It follows that there is a κ (depending only on the same data) such that for any $x \in [f(\partial J_0), c_1]$, $\mathrm{dist}(x, f(\partial J_0)) < \kappa\, \mathrm{dist}(x, \partial V)$.

Furthermore, the interior of $\tilde{J}_i$ does not intersect $f(\partial J_0)$ since the orbit of $f(\partial J_0)$ never returns to the interior of $\tilde{J}_0$. Since $\mathrm{dist}(J_i, \partial \tilde{J}_i)/|J_i|$ is big when $|\tilde{J}_0|/|J_0|$ is big, we conclude that $|J_i|$ is much smaller than $\mathrm{dist}(J_i, f(\partial J_0))$. It follows that $|J_i|$ is much smaller than $\mathrm{dist}(J_i, \partial V)$. Thus, U_i is well inside V as was asserted. All other properties of a geometric puzzle are easily supplied.

Suppose now that the critical point is non-recurrent. Let T^i be the last interval of the principal nest, so that the iterates of 0 never return to the interior of T^i. It follows that the set of points in T^i which do not return to the interior of T^i accumulate on 0. Let us consider a small nice interval J_0 whose endpoints do not return to T^i and the associated real puzzle $\{J_i\}$.

Let $\tilde{J}_0 = T^i$. By Theorem 2.14, there are intervals $\tilde{J}_i \supset J_i$ such that f^{n_i} is a diffeomorphism from $\tilde{J}_i$ onto $\tilde{J}_0$. Notice that c_1 is accumulated from the right by intervals J_i, so we can discard all J_i outside a small neighborhood of $[-1, c_1]$ and still get a neighborhood of $[-1, c_1]$ in $[-1, 1]$. We can now argue exactly as before. □

Remark 5.5. Notice that the above construction can be adapted to construct a (non-geometric) puzzle for f with any central puzzle piece U_0 with a reasonable shape. More precisely, given $0 < \phi < \psi < \pi$, if $|T_n|/|T_{n-1}|$ is small enough, then any U_0 trapped in between $D_\phi(T_n)$ and $D_\psi(T_n)$ (i.e., $D_\phi(T_n) \subset U_0 \subset D_\psi(T_n)$) generates a puzzle $\{U_j\}$ whose real trace $\{J_j\}$ form a real puzzle for f.

Lemma 5.6. *Let $0 < \phi < \psi < \gamma < \pi/2$ and let $k > 0$ be arbitrarily big. If $f \in \mathcal{U}_a$ is a Yoccoz map, then there exists a puzzle for f with geometric parameters (ϕ, ψ, γ, k) which persists on a neighborhood $\mathcal{V} \subset \mathcal{A}_a$ of f.*

Proof. We will show that the puzzle given by Lemma 5.5 is a persistent puzzle. We will keep the notation of that lemma. We show how to define a holomorphic motion H_g in a neighborhood of f.

We first observe that the Cantor set Q of points which never enter J_0 is contained in a persistent Markov family $\{M_j\}$, that is, f restricted to $\cup M_j$ is strictly Markov (see Remark 2.8). So by Proposition 2.11 there is a holomorphic motion $h_g : \mathbb{C} \to \mathbb{C}$ over a neighborhood of f such that $h_g \circ f = g \circ h_g$ on $\cup M_j$.

Let us show that there is a neighborhood $\mathcal{V}$ of f such that for any i there is a holomorphic motion h_g^i of $Q \cup U_i$ such that $h_g^i|Q = h_g$ and $h_g \circ f^{n_i+1} = g^{n_i+1} \circ h_g^i$ in U_i. It is clear that for any fixed i we can get a neighborhood $\mathcal{V}_i$ where such a holomorphic motion is defined.

To deal with all i at the same time we notice that all but finitely many U_i are compactly contained in $\cup M_j$ since the U_i accumulate on Q and Q is compactly contained in $\cup M_j$. Let $\mathcal{I}$ be the set of U_i which are not compactly contained in $\cup M_j$. Let $\mathcal{J}$ be the set of U_i which are compactly contained in $\cup M_j$ but $f(U_i) \in \mathcal{I}$ (in particular $\mathcal{J}$ is also finite). Shrinking the neighborhood $\mathcal{V}$ if needed, we may suppose that for $g \in \mathcal{V}$, h_g^i is defined for $U_i \in \mathcal{I} \cup \mathcal{J}$, and $h_g^i(U_i) \subset \cup h_g(M_j)$ for $U_i \in \mathcal{J}$.

Now, if $U_i \notin \mathcal{I} \cup \mathcal{J}$, there is a unique k such that $f^k(U_i) = U_j$ belongs to $\mathcal{J}$. We then define $h_g^i|U_i$ so that $g^k \circ h_g^i = h_g^j \circ f^k$ in U_i. These are the desired holomorphic motions h_g^i.

We extend each of the holomorphic motions h_g^i to normalized holomorphic motions of $\mathbb{C}$.

The property that $\operatorname{orb}(f(\partial J_0))$ never enters $\operatorname{int} \tilde{J}_0$ implies that $\tilde{J}_j \cap J_i = \emptyset$, for all $i \neq j$ with $n_i \leq n_j$ (compare the proof of the previous Lemma). Hence $\operatorname{dist}(J_j, \partial J_i)$ is much bigger than $|J_j|$. By the condition 1 in the definition of geometric parameters, we have for all k, $U_k \subset D_{\pi/2}(J_k)$, hence

$$\inf_{i \neq j} \operatorname{dist}_S(U_i, U_j) > 0, \tag{5.2}$$

where $S = \mathbb{C} \setminus Q$ (use the Schwarz Lemma and $\partial J_j \subset Q$). Moreover, by condition 4 in the definition of geometric parameters, for all k there exists $\tilde{U}_i$ with $\tilde{U}_i \cap \partial V = \emptyset$ and $\operatorname{mod}(\tilde{U}_i \setminus U_i) > k$, so by similar considerations,

$$\inf_i \operatorname{dist}_S(U_i, \partial V) > 0, \quad \text{where } V \equiv f(U_0). \tag{5.3}$$

By the Quasiconformality Lemma (see §2.5), $\mathcal{V}$ can be shrunk so that the dilatation of the motions h_g and h_g^i will be close to 1. By Lemma 2.3,

$$\begin{aligned} &\inf_{i \neq j} \operatorname{dist}_{h_g(S)} \left(h_g^i(U_i), h_g^j(U_j)\right) > 0, \\ &\inf_i \operatorname{dist}_{h_g(S)} \left(h_g^i(U_i), h_g(\partial V)\right) > 0. \end{aligned} \tag{5.4}$$

So we can define a holomorphic motion H_g which agrees with h_g on $Q \cup \partial V$ and with h_g^i on U_i. This concludes the proof of the lemma. $\square$

Remark 5.6. The above proof also shows that the holomorphic motion H_g, $g \in \mathcal{V}$ associated to the persistent puzzle $\mathcal{P}_f$ may be taken $\mathbb{R}$-symmetric. More precisely, we can choose $\mathcal{V}$ to be a small ball around f and construct H_g with the property that $H_g(\overline{z}) = \overline{H_{\operatorname{conj}(g)}(z)}$, where $\operatorname{conj}(g)(\overline{z}) = \overline{g(z)}$. To see this, notice that all the constructive (dynamical) steps in the above proof respect the $\mathbb{R}$-symmetry. If we also choose all neighborhoods of f to be small balls centered at f, then the successive applications of the Extension Lemma in the above argument yield $\mathbb{R}$-symmetric holomorphic motions by Remark 2.2.

From now on, when considering a holomorphic motion associated to a persistent geometric puzzle we will always assume it is $\mathbb{R}$-symmetric. In particular, if $g \in \mathcal{U}_a \cap \mathcal{V}$ then H_g is an $\mathbb{R}$-symmetric qc map.

Remark 5.7. The above construction is refined in Lemma 7.9. For a related but different construction see also Lemma 12.5 of [LS2].

6. Pullback arguments and the key estimate

6.1. Macroscopic pullback argument. Let $f : U^f \to \mathbb{C}$ and $\tilde{f} : U^{\tilde{f}} \to \mathbb{C}$ be two puzzle maps, and let h be a homeomorphism of $\mathbb{C}$ such that $h(U^f) = U^{\tilde{f}}$, equivariant on ∂U^f. If $h(f(0)) = \tilde{f}(0)$, then there is a unique homeomorphism h_1 coinciding with h on $\mathbb{C} \setminus U^f$ and such that $h \circ f = \tilde{f} \circ h_1$ on U^f. It is called the *lift* of h.

Definition 6.1. *We say that a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ is a combinatorial equivalence between f and $\tilde{f}$ if it is equivariant on ∂U^f and the lift h_1 of h is homotopic to h rel $\partial U^f \cup \mathrm{orb}_f(0)$.*

Lemma 6.1. *Let h be a qc combinatorial equivalence between f and $\tilde{f}$. Then the lift h_1 of h is a qc combinatorial equivalence between f and $\tilde{f}$ and* $\mathrm{Dil}(h_1) \le \mathrm{Dil}(h)$.

Proof. Let H^t be a homotopy between h and h_1 rel $\partial U^f \cup \mathrm{orb}_f(0)$, and let H_1^t be the lift of H^t. Then H_1^t is a homotopy rel $\partial U^f \cup \mathrm{orb}_f(0)$ between h_1 and its lift h_2.

Define a sequence of homeomorphisms ψ_k as follows

$$\psi_k = \begin{cases} h_1 & \text{on } \cup_{j=0}^k U_j^f \\ h & \text{otherwise.} \end{cases}$$

We notice that ψ_k is quasiconformal on $\mathbb{C} \setminus (\cup_{j=0}^k \partial U_j \cup \{0\})$, since it coincides with h on the complement of $\cup_{j=0}^k U_j$ and is the conformal lift of h on $\cup_{j=0}^k U_j \setminus \{0\}$. Since quasiarcs and points are qc removable, ψ_k is quasiconformal on the whole complex plane. Since quasiarcs have zero Lebesgue measure, the estimate $\mathrm{Dil}(\psi_k) \le \mathrm{Dil}(h)$ follows from the fact that conformal lifts preserve the norm of Beltrami differentials.

Since $\psi_k \to h_1$ pointwise, the result follows from the First Compactness Lemma. □

The Pullback Argument in the context of quadratic-like maps was formulated by Sullivan, see [MS], Chap. 6, Sect. 4. The following result adapts it to the context of puzzle maps.

Theorem 6.2. *Let us consider two puzzle maps f and $\tilde{f}$ with all periodic orbits hyperbolic. Let h be a qc combinatorial equivalence between f and $\tilde{f}$. Then there is a qc homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $H \circ f = \tilde{f} \circ H$ on U^f, $H = h$ on $\mathbb{C} \setminus U^f$, and* $\mathrm{Dil}(H) \le \mathrm{Dil}(h)$. *If there are no invariant line fields on $K(f)$ or if f and $\tilde{f}$ are hyperbolic maps with the same multiplier, then* $\mathrm{Dil}(H) \le \mathrm{Dil}(h|\mathbb{C} \setminus U^f)$.

Proof. Assume first that all periodic orbits are repelling. In this case $K(f)$ has empty interior by Lemma A.21.

Let $h_0 = h$ and define by induction h_{k+1} as the lift of h_k. By the previous lemma all h_k are qc maps with $\mathrm{Dil}(h_k) \le \mathrm{Dil}(h)$. By the First Compactness Lemma, there exists an accumulation point H of h_k which is quasiconformal and whose dilatation is bounded by the dilatation of h. Since the $h_k(z)$ are eventually the same for any $z \in \mathbb{C} \setminus K(f)$, H is a conjugacy between f and $\tilde{f}$ on $U^f \setminus K(f)$. Since $K(f)$ has empty interior, H is a conjugacy between f and $\tilde{f}$ on U^f.

Moreover, $\mathrm{Dil}(H|\mathbb{C} \setminus K(f) \le \mathrm{Dil}(h|\mathbb{C} \setminus U^f)$. Hence, if there are no invariant line fields on $K(f)$, then the dilatation of H on the entire complex plane has the same bound.

If f and $\tilde{f}$ are hyperbolic maps with the same multiplier we can use Lemma A.26 to modify h inside $K(f)$ in order to have $\overline{\partial} h|K(f) = 0$. □

Remark 6.1. We will show in the Appendix A that the assumption that a puzzle map f has only hyperbolic periodic orbits is satisfied for all complex preperiodic or hyperbolic maps (Lemma A.23), as well as for the puzzle extensions of unimodal maps without parabolic points on I (Lemma A.20).

Lemma 6.3. *There exists a constant $L > 0$ with the following property. Let $f \in \mathcal{U}_a$ be a Yoccoz map, and let $\mathcal{P}_f$ be a geometric puzzle which persists in an ε-neighborhood of f in $\mathcal{A}_a$ (which exists by Lemma 5.6). Let $\mathcal{V}$ be an $\varepsilon/2$-neighborhood of f. If $g \in \mathcal{U}_a \cap \mathcal{V}$ is preperiodic or hyperbolic and $\tilde{g}$ belongs to the same connected component of $\mathcal{C}_g \cap \mathcal{V}$, then there is a normalized L-qc homeomorphism $h : \mathbb{C} \to \mathbb{C}$ equivariant with respect to g and $\tilde{g}$ on U^g.*

Proof. Let H_g be the holomorphic motion of $\mathcal{P}_f$. By the λ-lemma, the dilatation of the motion restricted to $\mathcal{V}$ is bounded by some constant L.

Consider some preperiodic or hyperbolic $g \in \mathcal{U}_a \cap \mathcal{V}$ and a holomorphic path $g_\lambda, \lambda \in \mathbb{D}$, in $\mathcal{C}_g \cap \mathcal{V}$ connecting $g_0 = g$ with $\tilde{g}$ (such path exists since $\mathcal{C}_g$ is a codimension-one analytic submanifold). Let $H_\lambda = H_{g_\lambda} \circ H_g^{-1}$ denote the corresponding holomorphic motion of the puzzle $\mathcal{P}_{g_\lambda}$ with base point $\lambda = 0$.

Let $h_\lambda = H_\lambda$ on the complement of U^g and $h_\lambda(g^n(0)) = g_\lambda^n(0)$ whenever $g^n(0) \in U^g$.

Assume first that the critical orbit of g never escapes U^g. Then

$$g^n(0) \notin \partial U^g, \quad n = 0, 1, \dots. \tag{6.1}$$

Let us consider the set W of points $\lambda_0 \in \mathbb{D}$ such that h_λ is injective in a neighborhood of λ_0. Then $0 \in W$. Indeed, by (6.1), any curve $\lambda \mapsto g_\lambda^n(0)$ does not collide with ∂U^{g_λ} for λ sufficiently close to 0 (depending on n). This makes the statement obvious in the preperiodic case. In the hyperbolic case, there exist $N \in \mathbb{N}$ and neighborhoods $V \ni 0$ and $\Lambda \ni 0$ such that $g_\lambda|V$

is injective and $g_\lambda(V) \Subset V$ for $\lambda \in \Lambda$, and $g_\lambda^N(0) \in V$. Then the points $g_\lambda^n(0)$, $n \geq N$, do not collide neither with ∂U^{g_λ}, nor with each other.

Let W_0 be the connected component of W containing 0. We claim that for each $\lambda_0 \in W_0$, h_{λ_0} extends to a qc combinatorial equivalence $\hat{h}_{\lambda_0}$ between g and g_{λ_0}. To see it, let us consider a simply connected domain $D \subset W_0$ containing 0 and λ_0. Restricting the motion h_λ to $\lambda \in D$ and using Slodkowski's Theorem (see Remark 2.3)[5], we obtain an extension $\hat{h}_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in D$. By the Quasiconformality Lemma, $\hat{h}_{\lambda_0}$ is qc. Moreover, since $\hat{h}_{\lambda_0}$ comes from a holomorphic motion, it is a combinatorial equivalence between g and g_{λ_0}[6]. This concludes the proof of the claim.

By Theorem 6.2 (together with Remark 6.1), there exists an L-qc map of $\mathbb{C}$ which conjugates g_λ to g on U^g. By the First Compactness Lemma, for any $\lambda \in \overline{W_0} \cap \mathbb{D}$, g_λ is L-qc conjugate to g on U^g.

Vice versa, if g_λ is qc conjugate to g on U^g, then there exists a neighborhood of λ where h_λ is injective, so $\lambda \in W$. Hence $\overline{W_0} \cap \mathbb{D} \subset W_0$, so that W_0 is open and closed. Thus, $W_0 = \mathbb{D}$, so that h_λ is always injective and g_λ is always L-qc conjugate to g on U^g.

Assume now that the critical orbit of g escapes U. Since g is real, there is a smallest n such that $g^n(0)$ belongs to the invariant Cantor set $Q \equiv \partial U^g \cap \mathbb{R}$ (see Proposition 2.15). The holomorphic motion H_λ restricts to the dynamical motion of Q given by Proposition 2.11. Hence $H_\lambda(g^m(0)) = g_\lambda^m(0)$, for $m \geq n$, so that the maps h_λ are equivariant on $\mathrm{orb}_g(0) \cup \partial U_g$.

By the same argument of the non-escaping case, the points $g_\lambda^m(0) = h_\lambda(g^m(0))$, $m < n$, do not collide with $H_\lambda(\mathbb{C} \setminus U^g)$, which yields the desired statement. □

Remark 6.2. In the above Lemma we constructed, for each $\tilde{g}$ in the connected component of g in $\mathcal{C}_g \cap \mathcal{V}$, a qc map $h_{\tilde{g}}$ which has the three following properties:

- $h_{\tilde{g}}|\mathbb{C} \setminus U^g = H_{\tilde{g}} \circ H_g^{-1}$ is a holomorphic motion of the puzzle with base point g;
- $h_{\tilde{g}}$ is equivariant on U_g;
- $\overline{\partial} h_{\tilde{g}} = 0$ on $K(g)$.

It is easy to see that such a qc map $h_{\tilde{g}}$ is uniquely defined by the above properties (once the motion $H_{\tilde{g}}$ is fixed). It follows that the map $\tilde{g} \mapsto h_{\tilde{g}}|\mathbb{C} \setminus \mathrm{int}\, K(g)$ depends holomorphically on $\tilde{g}$ along the connected component of g in $\mathcal{C}_g \cap \mathcal{V}$. Indeed, this is automatic in the case where $K(g)$ has empty interior. By Lemma A.22, since g is $\mathbb{R}$-symmetric, if $K(g)$ has non-empty interior then $\mathrm{int}\, K(g)$ is the basin of attraction of an attracting

[5] We can use the Extension Lemma instead to construct a piecewise holomorphic motion joining 0 with λ_0, see Remark 2.4.

[6] If we denote the lift of $\hat{h}_\lambda$ by $\hat{h}_{\lambda,1}$ and let $s : [0, 1] \to D$ be any path connecting 0 to λ_0, then a homotopy between $\hat{h}_{\lambda_0}$ and its lift can be written explicitly as $\hat{h}_{\lambda_0} \circ \hat{h}_{s(t)}^{-1} \circ \hat{h}_{s(t),1}$, $t \in [0, 1]$.

periodic cycle. The holomorphic dependence of the linearizing coordinate of an attracting periodic cycle implies that $h_{\tilde{g}}$ depends holomorphically in int $K(g)$ as well.

6.2. Infinitesimal Pullback Argument. The "Infinitesimal Pullback Argument" introduced in this section will allow us to reconstruct, by means of consecutive liftings (see §2.6.2), a qc vector field which is equivariant on some essential part of the dynamical domain into a qc vector field which is equivariant on the whole domain. One step of the pullback argument is given by the following lemma:

Lemma 6.4. *Let $\Omega \ni 0$ be a quasidisk. Consider a map $f \in A^1(\Omega)$ whose derivative does not vanish on $\overline{\Omega} \setminus \{0\}$. Assume that $f : \overline{\Omega} \to f(\overline{\Omega})$ is either a diffeomorphism or a double branched covering ramified at 0. Let $v \in \mathcal{B}_\Omega$. Let α and β be qc vector fields on $\mathbb{C}$ such that $\beta|\partial\Omega$ is the lift of α by (f, v). Moreover, if f is a double branched covering, we assume that $v(0) = \alpha(f(0))$. Then there exists a qc vector field γ such that $\gamma|\Omega$ is the lift of α by (f, v), $\gamma|\mathbb{C} \setminus \Omega = \beta$, and*

$$\|\bar{\partial}\gamma\|_\infty \le \max\{\|\bar{\partial}\alpha\|_\infty, \|\bar{\partial}\beta\|_\infty\}. \tag{6.2}$$

Proof. Define a continuous vector field γ on $\mathbb{C} \setminus \{0\}$ by letting $\gamma = \beta$ on $\mathbb{C} \setminus \Omega$ and letting $\gamma = (\alpha \circ f - v)/f'$ on $\Omega \setminus \{0\}$.

If f is a diffeomorphism then γ clearly extends to 0. Assume f is a branched double covering. Since the modulus of continuity of qc vector fields is $\phi(x) = -x \ln(x)$ (see §2.6.1), we have for z near 0:

$$|\alpha(f(z)) - \alpha(f(0))| = O(\phi(|f(z) - f(0)|)) = O(\phi(|z|^2)).$$

Since $v(0) = \alpha(f(0))$ and f' has a simple root at 0, we have

$$\gamma(z) = \frac{v(0) - v(z)}{f'(z)} + O(\phi(|z|)),$$

where the first term is a regular holomorphic function. Hence γ admits a continuous extension to 0.

It is clear that γ is quasiconformal on $\mathbb{C} \setminus (\partial\Omega \cup \{0\})$. Since quasiarcs and isolated points are qc removable, γ is quasiconformal on the whole complex plane.

Since the lifts preserve the qc norm of vector fields, we have $\|\bar{\partial}\gamma\|_\infty = \|\bar{\partial}\alpha\|_\infty$ on Ω, while we have $\|\bar{\partial}\gamma\|_\infty = \|\bar{\partial}\beta\|_\infty$ on $\mathbb{C} \setminus \Omega$. Since quasiarcs are removable, (6.2) follows. □

Remark 6.3. This lemma extends to branched coverings of any degree n with some extra conditions on the derivatives of v (which assure that the critical point does not bifurcate infinitesimally along $f + tv$).

Theorem 6.5. *Let $f : U \to \mathbb{C}$ be a puzzle map whose critical point does not escape $\overline{U}$, and let v be a tangent vector field at f. Assume there exists a normalized qc vector field β on $\mathbb{C}$ which is equivariant on $\partial U \cup \operatorname{orb}(0)$. Then there exists an equivariant (on U) qc vector field α with $\|\bar\partial\alpha\|_\infty \le \|\bar\partial\beta\|_\infty$ which coincides with β on $\mathbb{C} \setminus U$. Furthermore, if there are no invariant line fields on $K(f)$, then $\|\bar\partial\alpha\|_\infty \le \|\bar\partial\beta \mid \mathbb{C} \setminus U\|_\infty$.*

Proof. This proof is the essence of the "infinitesimal pullback argument".

Using Lemma 6.4, let us lift the vector field β to the puzzle piece U_0 by means of $f : U_0 \to \mathbb{C}$. We obtain a qc vector field γ_0 on $\mathbb{C}$ with $\|\bar\partial\gamma_0\|_\infty \le \|\bar\partial\beta\|_\infty$, coinciding with β on $\mathbb{C} \setminus U_0$. Let $U^m = (f|U \setminus U_0)^{-m}(U_0)$. Define inductively a sequence of qc vector fields γ_m on $\mathbb{C}$ by letting γ_m on U^m be the lift of γ_{m-1} and letting $\gamma_m = \gamma_{m-1}$ on $\mathbb{C} \setminus U^m$. Since holomorphic lifts preserve the qc norm, $\|\bar\partial\gamma_n\|_\infty \le \|\bar\partial\gamma_{n-1}\|_\infty$.

By The Second Compactness Lemma, this sequence of vector fields is precompact. Since it stabilizes pointwise, it converges to a qc vector field α_1. This vector field satisfies the following properties:

- $\alpha_1|U_0$ is the lift of β;
- $\alpha_1|U \setminus U_0$ is equivariant;
- $\alpha_1|\mathbb{C} \setminus U = \beta$;
- $\|\bar\partial\alpha_1\|_\infty \le \|\bar\partial\beta\|_\infty$.

Replace now $\beta \equiv \alpha_0$ with α_1 and repeat the procedure. In this way we will construct a sequence of qc vector fields α_n with the following properties

(i) $\alpha_n|U_0$ is the lift of α_{n-1};
(ii) $\alpha_n|U \setminus U_0$ is equivariant;
(iii) $\alpha_n|\mathbb{C} \setminus U = \alpha_{n-1}$ (so by induction $\alpha_n|\mathbb{C} \setminus U = \beta$ for all n);
(iv) $\|\bar\partial\alpha_n\|_\infty \le \|\bar\partial\alpha_{n-1}\|_\infty$.

Taking any Cesaro limit of this sequence (which exists by the Second Compactness Lemma),

$$\alpha = \lim \frac{1}{n_i} \sum_{k=0}^{n_i-1} \alpha_k,$$

we obtain a qc vector field α with $\|\bar\partial\alpha\|_\infty \le \|\bar\partial\beta\|_\infty$. Properties (i)–(iii) imply that α is equivariant everywhere on U.

Note that α on $\mathbb{C} \setminus K(f)$ is obtained by consecutive liftings of β on $\mathbb{C} \setminus U$. Hence

$$\|\bar\partial\alpha \mid \mathbb{C} \setminus K(f)\|_\infty \le \|\bar\partial\beta \mid \mathbb{C} \setminus U\|_\infty.$$

If there are no invariant line fields on $K(f)$ then

$$\|\bar\partial\alpha\|_\infty = \|\bar\partial\alpha \mid \mathbb{C} \setminus K(f)\|_\infty,$$

and the last assertion follows (see Lemma 2.30). □

Remark 6.4. The same proof also applies to the case when the critical point escapes. More precisely, let f be a puzzle map and $\Omega \subset U$ be a union of puzzle pieces. Let $n = \min\{k \geq 0,\ c_k \notin \Omega\}$ (the escaping time of the critical point). Assuming the critical point escapes Ω, we have $n < \infty$. If β is a vector field equivariant on $\partial\Omega$ and on $\{c_k,\ 0 \leq k \leq n-1\}$, then there exists a vector field α equivariant on Ω coinciding with β outside of Ω and such that $\|\overline{\partial}\alpha\|_\infty \leq \|\overline{\partial}\beta\|_\infty$. By Lemma A.23, $K(f|\Omega)$ has zero Lebesgue measure, so $\|\overline{\partial}\alpha\|_\infty \leq \|\overline{\partial}\beta|\mathbb{C} \setminus U\|_\infty$.

6.2.1. Gluing qc vector fields. Assume now that we have two qc vector fields β_i on $\mathbb{C}$ which respect some dynamical data on sets $X_i \subset \mathbb{C}$, $i = 0, 1$. Let us say that these vector fields can be *glued* if there is a qc vector field on $\mathbb{C}$ coinciding with β_i on X_i. The case which will interest us is when $X_1 = O_f$, $X_0 = \mathbb{C} \setminus U$.

Note first that if X_0 and X_1 have disjoint closures, the vector fields β_i can be obviously glued using partition of unity. Together with Theorem 6.5 this implies:

Lemma 6.6. *Let f be a puzzle map with non-escaping critical point such that the postcritical set O_f is disjoint from ∂U, and let $v \in \mathcal{B}_U$. Assume that there are qc vector fields β_0 and β_1 on $\mathbb{C}$ such that β_1 is equivariant on* $\mathrm{orb}_f(0)$ *and β_0 is equivariant on ∂U. Then there exists an equivariant qc vector field α on U coinciding with β_0 on $\mathbb{C}\setminus U$ and coinciding with β_1 on O_f. If there are no invariant line fields on $K(f)$, then $\|\overline{\partial}\alpha\|_\infty \leq \|\overline{\partial}\beta_0 \,|\, \mathbb{C}\setminus U\|_\infty$.*

If the postcritical set intersects the boundary of the puzzle, the gluing method is much more delicate. It will be based on a pullback construction.

Recall that $U(x)$ is the connected component of U containing x.

Lemma 6.7. *Let f be a puzzle map such that $c_1 \in U$. Let $v \in \mathcal{B}_U$, and let β_0 be a qc vector field on $\mathbb{C}$ equivariant on ∂U. Then there are constants C_1 and C_2 with the following property. Consider a point $x \in U$ such that* $0 \in \omega(x)$. *If there exists a bounded vector field β equivariant on* $\mathrm{orb}(x)$ *then there exists a qc vector field α on $\mathbb{C}$ such that*

- $\|\overline{\partial}\alpha\|_\infty \leq C_1 + C_2\|\beta\|_\infty$;
- $\alpha(x) = \beta(x)$;
- *α coincides with β_0 outside of* $U(x)$.

Proof. Let $W = (f|U_0)^{-1}(U)$ (the domain of the first return map to U_0) and $W_0 = W(0)$ (the central domain of the first return map).

Let β_1 be a qc vector field on $\mathbb{C}$ coinciding with β_0 on $\mathbb{C} \setminus U_1$ and such that $v(0) = \beta_1(c_1)$. By Lemma 6.4 there exists a qc vector field ζ coinciding with β_0 on $\mathbb{C} \setminus U_0$ and such that ζ is the lift of β_1 on U_0. So ζ is equivariant on $\partial U \cup \partial W$.

Since $\overline{W_0} \subset U_0$, there exist qc vector fields θ_0, θ_1 such that $\theta_0|W_0 = 1$, $\theta_0|\mathbb{C} \setminus U_0 = 0$, and $\theta_1|W_0 = 0$, $\theta_1|\mathbb{C} \setminus U_0 = \beta_0$. We let

$$C_1 = \max\{\|\overline{\partial}\beta_0\|_\infty, \|\overline{\partial}\zeta\|_\infty, \|\overline{\partial}\theta_1\|_\infty\} \quad \text{and} \quad C_2 = \|\overline{\partial}\theta_0\|_\infty.$$

Let s_i denote the sequence of landing moments of x, that is,

$$s_0 = \min\{k \geq 0 | f^k(x) \in U_0\} \quad \text{and} \quad s_{i+1} = \min\{k > s_i | f^k(x) \in U_0\}.$$

Let $m = \min\{k \geq 0 | f^{s_k}(x) \in W_0\}$, so that $f^{s_m}(x)$ is the first landing of x on W_0 ($m < \infty$ since $0 \in \omega(x)$). For each $k \leq m$ there exists a unique quasidisk V_k around x such that $f^{s_k}|\overline{V}_k$ is a diffeomorphism onto $\overline{U}_0$.

Let $\eta = \theta_1 + \beta(f^{s_m}(x))\theta_0$ (we identify freely vector fields and functions when defined on subsets of $\mathbb{C}$), so that

$$\eta|\mathbb{C} \setminus U_0 = \beta_0, \quad \eta(f^{s_m}(x)) = \beta(f^{s_m}(x)), \quad \|\overline{\partial}\eta\|_\infty \leq C_1 + C_2\|\beta\|_\infty.$$

Let γ_m be the lift to V_m of η by (f^{s_m}, v^{s_m}). For each $k < m$, let γ_k be the lift to V_k of ζ by (f^{s_k}, v^{s_k}). So we have constructed, for $k \leq m$, qc vector fields γ_k on V_k such that $\|\overline{\partial}\gamma_k\|_\infty \leq C_1 + C_2\|\beta\|_\infty$.

Notice that the equivariance of ζ on $\partial U \cup \partial W$ implies that, for $k < m$, γ_k and γ_{k+1} coincide on ∂V_{k+1}. It also follows that β_0 coincides with γ_0 on ∂V_0.

Let $\alpha = \beta_0$ outside V_0, $\alpha = \gamma_k$ on $V_k \setminus V_{k+1}$, $k < m$, and $\alpha = \gamma_m$ on V_m. Then α is a continuous vector field on $\mathbb{C}$ and since quasiarcs are removable, α is a qc vector field with $\|\overline{\partial}\alpha\|_\infty \leq C_1 + C_2\|\beta\|_\infty$. By equivariance of β on $\mathrm{orb}(x)$, $\alpha(x) = \beta(x)$. □

Remark 6.5. The proof above uses that $0 \in \omega(x)$ only to assure that there exists an $m < \infty$ such that $f^{s_m}(x) \in W_0$. If this is not the case (but x is still non-escaping from U), then one can show that the conclusion still holds under the (weak) assumption $|Df^{s_k}(x)| \to \infty$.

Lemma 6.8. *Let f be a puzzle map with a recurrent critical point such that* $\mathrm{orb}(0)$ *intersects infinitely many puzzle pieces, and let $v \in \mathcal{B}_U$. Assume there are qc vector fields β_0 and β_1 on $\mathbb{C}$ such that β_1 is equivariant on* $\mathrm{orb}(0)$ *and β_0 equivariant on ∂U. Then there is a qc vector field α equivariant on U coinciding with β_0 on $\mathbb{C} \setminus U$ (and coinciding with β_1 on* $\mathrm{orb}(f(0))$*). Furthermore, if there are no invariant line fields on $K(f)$, then $\|\overline{\partial}\alpha\|_\infty \leq \|\overline{\partial}\beta_0|\mathbb{C} \setminus U\|_\infty$.*

Proof. By the assumption, there exists a sequence $k(j) \to \infty$ and a sequence V_j of components of U such that

$$c_{k(j)} \in V_j \quad \text{but} \quad c_i \notin V_j,\ i < k(j).$$

Let $\Omega_j = U \setminus V_j$.

We first construct a uniformly bounded sequence of vector fields α_j equivariant only on Ω_j. This is done in three steps.

In the first, we modify the vector field β_0 inside V_j to obtain the correct value (given by β_1) on $c_{k(j)}$, uniform bounds come from Lemma 6.7. In other words, there exists a constant C and a sequence γ_j of vector fields such that $\|\overline{\partial}\gamma_j\|_\infty \leq C$, $\gamma_j(c_{k(j)}) = \beta_1(c_{k(j)})$, and γ_j coincides with β_0 on the complement of V_j.

In the second step, we prepare ourselves for a pullback argument by modifying γ_j inside Ω_j to obtain the correct values in the finite set $\{c_i, 0 \leq i < k(j)\}$. More precisely, we define a sequence ζ_j of qc vector fields on $\mathbb{C}$ such that ζ_j coincides with γ_j on $\mathbb{C} \setminus \Omega_j$ and ζ_j coincides with β on c_i, $0 \leq i \leq k(j)$. In this step we lose uniform estimates, which are recovered in the next one.

The third step is the pullback argument truncated to Ω_j. Since ζ_j is equivariant on $\partial\Omega_j$ and on c_i, $0 \leq i < k(j)$, we can apply the Infinitesimal Pullback Argument (or rather, its escaping version outlined in Remark 6.4) to obtain a qc vector field α_j equivariant on Ω_j and coinciding with ζ_j on $\mathbb{C} \setminus \Omega_j$ (hence $\alpha_j|\mathbb{C} \setminus \Omega_j = \gamma_j$). Moreover, $\|\overline{\partial}\alpha_j\|_\infty \leq C$.

To obtain the desired vector field equivariant on all of U we just need to take a limit α of the vector fields α_j (using the Second Compactness Lemma). If there are no invariant line fields, the estimate on α follows as before.

Notice that the equivariance of α and β_1 on the critical orbit determines uniquely their values on the orbit of the critical value, so $\alpha|\operatorname{orb}(f(0)) = \beta_1$. □

Remark 6.6. In the above proof, it was only used that the vector field β_1 is bounded on $\operatorname{orb}(0)$. No assumption of quasiconformality or even continuity is necessary.

Lemmas 6.6 and 6.8 immediately imply:

Theorem 6.9. *Let f be a puzzle map with a recurrent critical point, and let $v \in \mathcal{B}_U$. Assume there are qc vector fields β_0 and β_1 such that β_1 is equivariant on* $\operatorname{orb}(0)$ *and β_0 is equivariant on ∂U. Then there is a qc vector field α equivariant on U and coinciding with β_0 on $\mathbb{C} \setminus U$. Furthermore, if there are no invariant line fields on $K(f)$, then $\|\overline{\partial}\alpha\|_\infty \leq \|\overline{\partial}\beta_0|\mathbb{C} \setminus U\|_\infty$.*

Let us finish this section with a discussion of infinitesimal deformations of hyperbolic puzzle maps.

Lemma 6.10. *Let f be a hyperbolic puzzle map. If v satisfies equation (5.1) then there exists a qc vector field α on $\mathbb{C}$ which is conformal and equivariant on a neighborhood of* $\overline{\operatorname{orb}(0)}$.

Proof. Let p be the attracting periodic point whose immediate basin of attraction contains 0, and let n stand for its period. Take some f^n-invariant topological disks $V \Subset V' \ni p$ containing $\overline{\operatorname{orb}_{f^n}(0)}$. It is enough to find a solution of the equation $v^n(z) = \alpha(f^n(z)) - \alpha(z)Df^n(z)$ which is conformal and equivariant in V, since such solution can be spread (by means of $n-1$ lifts) around the rest of the orbit of 0 and then extended arbitrarily to a qc vector field on $\mathbb{C}$.

Equation (5.1) tells us that v^n is an infinitesimal deformation of $f^n|V'$ preserving the multiplier of the attracting periodic point. Hence there is a holomorphic family $g_\lambda : V \to \mathbb{C}$, $g_0 = f^n$, tangent to v^n, such that each g_λ has an attracting fixed point p_λ (depending analytically on λ) with multiplier $Df^n(p)$. By the standard local theory, there is a holomorphic family of conformal maps

$$\psi_\lambda : (V, 0) \to (V_\lambda, 0)$$

conjugating f^n to g_λ. Then $\alpha = d\psi/\partial\lambda|_{\lambda=0}$ is the desired vector field. □

Remark 6.7. The above equation can also be solved directly using the Böttcher or linearizing coordinate near p. In these coordinates, it is reduced to either $u(s) = a(s^2) - 2sa(s)$ (superattracting case) or $u(s) = a(\lambda s) - \lambda a(s)$ (simply attracting case). Both equations can be easily analyzed by means of power expansions.

Lemma 6.11. *Let f be a hyperbolic puzzle map and let $v \in \mathcal{B}_U$ be a vector field satisfying equation (5.1). Let β_0 be a qc vector field on $\mathbb{C}$ equivariant on ∂U. Then there exists a vector field α on $\mathbb{C}$ equivariant on U such that $\|\overline{\partial}\alpha\|_\infty \le \|\overline{\partial}\beta_0|\mathbb{C} \setminus U\|_\infty$. Moreover, α is holomorphic on the basin of attraction of the attracting cycle.*

Proof. Let β be the vector field given by Lemma 6.10. We can create a vector field γ by gluing β_0 on the complement of U with β on a neighborhood of $\operatorname{orb}(0)$. Applying the Infinitesimal Pullback Argument (Theorem 6.5), we obtain a vector field α which is equivariant on U, conformal on the basin of attraction $\operatorname{int} K(f)$ and satisfies

$$\|\overline{\partial}\alpha|\mathbb{C} \setminus K(f)\|_\infty \le \|\overline{\partial}\beta_0|\mathbb{C} \setminus U\|_\infty.$$

By Lemma A.21, $\partial K(f)$ has zero Lebesgue measure for a hyperbolic puzzle map, so the estimate follows. □

6.3. Key estimate. We say that a preperiodic or hyperbolic complex map g has *special combinatorics* with respect to $\mathcal{V}$, a complex neighborhood of g, if the connected component of g in $\mathcal{C}_g \cap \mathcal{V}$ contains a real map.

Recall that L_g is the linear map which associates to any tangent vector field $v \in T_g$ the unique qc vector field α on the postcritical set such that $v = \alpha \circ f - \alpha f'$ and $v(0) = \alpha(c_1)$.

Lemma 6.12 (Key estimate). *Let $f \in \mathcal{U}_a$ be a Yoccoz map. There exists a neighborhood $\mathcal{V}$ of f in $\mathcal{A}_a$ and a constant $C > 0$ such that, for any g with special combinatorics with respect to $\mathcal{V}$, the operator norm of L_g is bounded by C.*

Proof. Consider a persistent puzzle for f given by Lemma 5.6. Take an $\varepsilon > 0$ such that this puzzle persists in an ε-neighborhood of f, and let H_g be the associated holomorphic motion. Let $\mathcal{V}$ be an $\varepsilon/2$-neighborhood of f

and let $C = 2/\varepsilon$. Given $g \in \mathcal{V}$ with special combinatorics and $v \in T_g$ with $\|v\|_a = 1$, let $h_\lambda = H_{g+\lambda v} \circ H_g^{-1}$, $\lambda \in \mathbb{D}_{\varepsilon/2}$. Let

$$\beta_0 = \frac{d}{d\lambda} h_\lambda \bigg|_{\lambda=0}.$$

Notice that β_0 is equivariant on ∂U^g with respect to (g, v). By Theorem 2.7, μ_{h_λ} is a holomorphic function on $\mathbb{D}_{\varepsilon/2}$ with values in the unit ball of $L^\infty(\mathbb{C})$. By Lemma 2.10,

$$\overline{\partial}\beta_0 = \frac{d}{d\lambda} \mu_{h_\lambda} \bigg|_{\lambda=0}.$$

Since $\mu_{h_0} = 0$, we can apply the Schwarz Lemma to get $\|\overline{\partial}\beta_0\|_\infty \leq 2/\varepsilon$.

Assume g is preperiodic. By Lemma A.23, g has no invariant line fields on $K(g)$.

By Lemma 6.3, the special combinatorics assumption implies that $g|U^g$ is qc conjugate to $\tilde{g}|U^{\tilde{g}}$ for some real map $\tilde{g} \in \mathcal{V}$; in particular, the critical orbit does not escape $\overline{U^g}$. By Theorem 6.5, there exists a qc vector field α equivariant on U^g, coinciding with β_0 on $\mathbb{C} \setminus U$, and such that $\|\overline{\partial}\alpha\|_\infty \leq C$.

In the hyperbolic case we proceed as above using Lemma 6.11. □

Remark 6.8. It is possible to show that for each $f \in \mathcal{U}_a$, L_f has a bounded operator norm. Near infinitely renormalizable maps, we can reduce to a persistent quadratic-like renormalization, and our argument works unchanged. For the hyperbolic and parabolic case, one can obtain explicit estimates, using the formulas for the tangent space. However, in the parabolic case those explicit estimates are less stable than the ones in our arguments, and do not seem to allow to obtain estimates for nearby maps.

7. Transverse direction

In this section we will construct a transverse direction to the tangent space T_f for at most finitely renormalizable maps f. Later on, in §8.3, we will identify T_f with the genuine tangent space to $\mathcal{H}_f$. Let us start with simple cases.

7.1. Non-recurrent cases. In the hyperbolic case T_f was actually defined as the tangent space to the hybrid class $\mathcal{H}_f$, which is a codimension-one submanifold (see §5.1). This tangent space is explicitly given by equation (5.1), so that the transverse vector fields are those which satisfy the inequality

$$Dv^n(p)(Df^n(p) - 1) - v^n(p)D^2 f^n(p) \neq 0, \tag{7.1}$$

where $\overline{p}$ is the attracting periodic orbit of period n.

In the parabolic case with $Df^n(p) = 1$, Definition 5.2 implies that $v^n(p) = 0$ for $v \in T_f$. Obviously this condition specifies a codimension-one subspace.

The parabolic case with $Df^n(p) = -1$ is more delicate:

Lemma 7.1. *Let $f \in \mathcal{U}_a$ be a parabolic map such that the multiplier of its parabolic orbit is -1 and let $v \in T_f$. Then v satisfies (5.1), that is*

$$2Dv^n(p) + v^n(p)D^2 f^n(p) = 0.$$

Proof. Let v be a vector field in T_f and let α be a qc vector field of $\mathbb{C}$, equivariant on O_f. Since 0 is attracted by $\overline{p}$, we may assume $\lim_{k\to\infty} c_{kn} = p$.

Equivariance allows us to write

$$v^n(c_{kn}) = \alpha(c_{(k+1)n}) - Df^n(c_{kn})\alpha(c_{kn}).$$

By continuity of α on O_f, $\alpha(c_{kn}) - \alpha(p) = o(1)$. Thus,

$$\begin{aligned} &v^n(p) + Dv^n(p)(c_{kn} - p) = \\ &\qquad \alpha(c_{(k+1)n}) + \alpha(c_{kn}) - D^2 f^n(p)\alpha(p)(c_{kn} - p) + o(1)(c_{kn} - p). \end{aligned}$$

Taking the difference between two consecutive equations, we obtain:

$$(Dv^n(p)+D^2 f^n(p)\alpha(p))(c_{(k+1)n} - c_{kn}) = \alpha(c_{(k+2)n}) - \alpha(c_{kn}) + o(c_{kn} - p).$$

Since p is parabolic with multiplier -1, $c_{(k+1)n} - c_{kn} = -2(c_{kn} - p) + o(c_{kn} - p)$. Thus,

$$\lim_{k\to\infty} \frac{\alpha(c_{(k+2)n}) - \alpha(c_{kn})}{c_{kn} - p} = 2Dv^n(p) + 2D^2 f^n(p)\alpha(p).$$

Equivariance gives $v^n(p) = 2\alpha(p)$, so to obtain (5.1) we just have to show that

$$\lim_{k\to\infty} \frac{\alpha(c_{(2k+2)n}) - \alpha(c_{2kn})}{c_{2kn} - p} = 0.$$

To see this, notice that $c_{2kn} - p$ approaches p from one of the sides, that is, for big k, all $c_{2kn} - p$ have the same sign. Since p is parabolic with multiplier -1, $c_{2kn} - p \sim \eta n^{-1/2}$ for some $\eta \neq 0$ (see Lemma B.3), so $\sum_k (c_{2kn} - p)$ diverges. Since α is uniformly bounded on orb(0),

$$\sum_k \alpha(c_{(2k+2)n}) - \alpha(c_{2kn})$$

is bounded. Since the limit

$$\lim_{k\to\infty} \frac{\alpha(c_{(2k+2)n}) - \alpha(c_{2kn})}{c_{2kn} - p}$$

exists, it must be 0, proving the lemma. □

By Lemma 7.1, T_f is contained in a codimension-one subspace, and transverse directions may be obtained again by (7.1).

We will now construct the transverse direction in the Misiurewicz case. Let δ_x denote the Dirac measure concentrated on x. Let

$$\nu_n = \sum_{k=0}^{n} \frac{\delta_{c_k}}{(f^k)'(c_1)},$$

which are signed measures on the interval I or equivalently bounded functionals acting on $C^0(I)$. By Proposition 2.15, in the Misiurewicz case f is expanding on O_f, so that $|(f^n)'(c_1)|$ grows exponentially fast. Therefore, the ν_n converge in the weak-$*$ topology to a signed measure ν.

Lemma 7.2. *There exists a polynomial $p \in T_f \mathcal{A}_a^{\mathbb{R}}$ such that $\nu(p) \neq 0$.*

Proof. Since the critical point is non-recurrent, $\nu(J) = 1$ for a small interval J around 0 such that $c_j \notin J$ for $j \geq 1$. So the signed measure ν does not vanish and hence there exists a continuous function $\phi \in C^0(I)$ such that $\nu(\phi) \neq 0$. Since any continuous function can be approximated by a polynomial, the assertion follows. □

Lemma 7.3. *If $\nu(v) \neq 0$, then $v \notin T_f$.*

Proof. Let $v \in T_f$, and let α be a qc vector field equivariant on $\mathrm{orb}_f(0)$. Then $\alpha(c_{n+1}) = v^{n+1}(0) = (f^n)'(c_1)\nu_n(v)$, so that

$$\lim \frac{|\alpha(c_{n+1})|}{|(f^n)'(c_1)|} = |\nu(v)|.$$

Since α is a qc vector field, $\alpha(c_{n+1})$ is uniformly bounded, so $\nu(v) = 0$. □

The existence of the transverse direction follows from Lemmas 7.2 and 7.3. Once we show that T_f has codimension-one, it will follow that $T_f = \mathrm{Ker}(\nu)$.

Remark 7.1. More generally, this analysis allows to obtain transverse directions for maps f which satisfy the summability condition

$$\sum_{k=0}^{\infty} \frac{1}{|Df^k(c_1)|} < \infty.$$

7.2. Smooth transverse vector field. We will now proceed with analysis of puzzle maps satisfying certain geometric assumptions, which will allow us to handle the case of at most finitely renormalizable maps with a recurrent non-periodic critical point. In this section we will construct a smooth transverse vector field, which is holomorphic on the puzzle piece containing the critical value U_1 and vanishes on the other puzzle pieces.

Definition 7.1. *An equipped holomorphic family of puzzle maps over some neighborhood $\mathcal{V} \ni 0$ in a complex Banach space is a pair (f_λ, h_λ), $\lambda \in \mathcal{V}$, where $f_\lambda : \cup U_i^\lambda \to \mathbb{C}$ is a puzzle map and $h_\lambda : \mathbb{C} \to \mathbb{C}$ is a holomorphic motion of $\mathbb{C}$ that maps U_i^0 onto U_i^λ and such that*

$$h_\lambda \circ f_0(z) = f_\lambda \circ h_\lambda(z) \tag{7.2}$$

for $z \in \cup(\partial U_i^0)$.

Let $U \subset \mathbb{C}$ be an $\mathbb{R}$-symmetric open set which intersects the real line in some interval J. We denote by $\Upsilon(U)$ the Banach space of holomorphic vector fields of class $A^1(U)$ that vanish together with their derivatives at ∂J, endowed with the A^1 norm $\|\cdot\|_1$, that is $\|v\|_1 = \sup_{z \in U} |v'|$, see (2.2) ($\|\cdot\|_1$ is a norm in $\Upsilon(U)$ because of the boundary conditions). We will extend those vector fields to the whole real line as 0 outside J. (The case that will interest us most is $U = D_\gamma(J)$.)

Recall that for a puzzle map $f : \cup U_i \to \mathbb{C}$, we denote by J_i the real slices of the domains U_i.

Proposition 7.4. *Let $f : \cup U_i \to \mathbb{C}$ be a geometric puzzle map with parameters (ϕ, ψ, γ, k). Then there is an $\varepsilon > 0$ depending only on ψ and γ, and there exists an equipped holomorphic family of puzzle maps (f_v, h_v) over the Banach ball $B_\varepsilon(\Upsilon(D_\gamma(J_1)))$ such that:*

- *h_v is the identity on U_j, $j \neq 1$, on $\partial(f(U_0))$, and is normalized;*
- *$(\mathrm{id} + v) \circ h_v$ is the identity in U_1;*
- *$f_v = f$ in U_j, $j \neq i$;*
- *$f_v = f \circ (\mathrm{id} + v)$ in $U_1^v = h_v(U_1)$.*

Proof. Let $\varepsilon > 0$ and $v \in \Upsilon(D_\gamma(J_1))$ such that $\|v\|_1 < \varepsilon$. If $\varepsilon < 1$ then $I_v \equiv (\mathrm{id} + v)|\overline{D_\gamma(J_1)}$ is a diffeomorphism fixing ∂J_1 with $I_v'|\partial J_1 = 1$ (using the convexity of $D_\gamma(J)$). If ε is small (depending only on ψ and γ, then $I_v(D_\gamma(J_1))$ contains $D_\psi(J_1) \supset U_1$. Hence $I_v^{-1}(U_1) \subset D_\gamma(J_1)$. By definition of geometric parameters, the domain $I_v^{-1}(U_1)$ is disjoint from U_j, $j \neq i$, and from $\partial(f(U_0))$.

Thus, we can define f_v in $I_v^{-1}(U_1)$ as $f \circ I_v$. We then define h_v as the identity on U_j, $j \neq 1$, and on $\partial(f(U_0))$, and we let $h_v|U_1 = I_v^{-1}|U_1$. Since $U_1 \subset I_v(D_\gamma(J_1))$, $I_v^{-1}|U_1$ depends holomorphically on v. We obtain a holomorphic motion h_v on $U \cup \partial(f(U_0))$. By the Extension Lemma, it can be extended to a normalized holomorphic motion of the whole sphere over a Banach ball of radius $\varepsilon/3$. □

To motivate the next statement, note that the tangent vector to the curve $t \mapsto f_{tv}$ at $t = 0$ is given by the vector field $v(z)f'(z)$ on U_1 (and 0 elsewhere). This vector field is tangent to the hybrid class of f if there exists a qc vector field α on the orbit of the critical value c_1 such that $\alpha(c_1) = 0$ and

$$v(z)f'(z) = \alpha(f(z)) - \alpha(z)f'(z).$$

This equation can be written in the following concise form:

$$v = f^*\alpha - \alpha, \tag{7.3}$$

where $f^*\alpha$ is the pullback of α by f, compare equation (2.6).

If U is a hyperbolic domain and v is a tangent vector at a point $x \in U$, we denote $\|v\|^U_{\text{hyp}}$ the hyperbolic length of v.

Lemma 7.5. *Given (ψ, γ), there exists an $L > 0$ with the following property. Let f be a geometric puzzle map without invariant line fields on $K(f)$ with parameters (ϕ, ψ, γ, k). Let $v \in \Upsilon(D_\gamma(J_1))$ be a holomorphic vector field such that there is a qc vector field α on the closed set O_f with $\alpha(c_1) = 0$, satisfying (7.3).*

Then α has a normalized $L\|v\|_1$-qc extension, which satisfies (7.3) on U and vanishes on $\partial(U \cap \mathbb{R})$.

Proof. Without loss of generality we can assume that v is normalized: $\|v\|_1 = 1$. Take an $\varepsilon > 0$ as in the above proposition and consider the corresponding equipped holomorphic family of puzzle maps $(f_{\lambda v}, h_{\lambda v})$, $|\lambda| < \varepsilon$. Let

$$\alpha_0 = \frac{d}{d\lambda} h_{\lambda v}\bigg|_{\lambda=0}.$$

Since $h_{\lambda v}$ satisfies (7.2) for $z \in \partial U$, α_0 satisfies (7.3) for $z \in \partial U$.

As in Lemma 6.12 we conclude that $\|\alpha_0\| \le L\|v\|_1$ with $L = 1/\varepsilon$.

Now we can use Theorem 6.9 to conclude that there exists a qc vector field β that coincides with α on the orbit of the critical value, coincides with α_0 on the complement of U and satisfies the equation (7.3) on U. Moreover, by Lemma A.24, there are no invariant line fields on $K(f)$, so we conclude that $\|\bar\partial\beta\|_\infty \le L\|v\|_1$.

This vector fields vanishes on the $\partial(U \cap \mathbb{R})$, since $h_{\lambda v}|\partial(U \cap \mathbb{R}) = \text{id}$, $|\lambda| < \varepsilon$. □

Lemma 7.6. *Let f be a geometric puzzle map with geometric parameters (ϕ, ψ, γ, k) such that $c_1 \in U$. Then there exists a vector field $v \in \Upsilon(D_\gamma(J_1))$ such that $\|v\|_1 = 1$ and $\|v(c_1)\|^{U_1}_{\text{hyp}} > 1/7$.*

Proof. Let

$$w_n(z) = (1 - z^2)(1 - e^{-2n}) + \frac{2}{n}(e^{-n(1+z)} + e^{-n(1-z)} - e^{-2n} - 1).$$

Then $w_n \in \Upsilon(\mathbb{D})$. Also, $\|w_n\|_1 < 6$ and $w_n(z) \to 1 - z^2$ pointwise in $\mathbb{D}$ as $n \to \infty$. Hence the hyperbolic norm of $w_n(z)$ in $\mathbb{D}$ goes to the value $|1 - z^2|/(1 - |z^2|) \ge 1$. Take a big n and let w be the restriction of w_n to $D_\gamma(I)$ normalized so that $\|w\|_1 = 1$. Rescaling $D_\gamma(I)$ to $D_\gamma(J_1)$, we obtain a vector field v in the latter domain with desired properties, because both the norm $\|\cdot\|_1$ and the hyperbolic norm are scaling invariant. □

Lemma 7.7. *Given ϕ and k_0, there exists a $\kappa > 0$ with the following property. Let $f : U \to \mathbb{C}$ be a geometric puzzle map with parameters (ϕ, ψ, γ, k), with $k > k_0$, such that the critical value c_1 returns to U_1, and let $v \in \Upsilon(D_\gamma(J_1))$ be a vector field from Lemma 7.6. If there exists a qc vector field α on $\mathbb{C}$, satisfying (7.3) on U, then there exists $q \in f^{-1}(U_1) \cap \mathbb{R}$ such that $|\alpha(q)| > \kappa\, d(q, \partial J_0)$.*

Proof. Let $n > 1$ be minimal such that $c_n \in U_1$. Let us show that

$$\|\alpha(c_{n-1})\|_{\text{hyp}}^{U_0} \geq \|v(c_1)\|_{\text{hyp}}^{U_1}. \tag{7.4}$$

Since $n-2$ is the first landing time of c_1 in $f^{-1}(U_1)$, there exists a domain $W \subset U_1$ containing c_1, which is univalently mapped onto U_0 by f^{n-2} (see e.g., [L4, Lemma 3.5]). Moreover the orbit of this domain, $f^k(W)$, $k = 1, 2, \ldots, n-2$, does not intersect U_1.

Equation (7.3) implies

$$(f^{n-2})^*\alpha - \alpha = \sum_{k=0}^{n-3} (f^k)^* v.$$

Evaluating it at c_1 we obtain $(f^{n-2})^*\alpha(c_1) = v(c_1)$ (since v vanishes outside U_1 and $\alpha(c_1) = 0$). Since $f^{n-2} : W \to U_0$ is a hyperbolic isometry,

$$\|\alpha(c_{n-1})\|_{\text{hyp}}^{U_0} = \|v(c_1)\|_{\text{hyp}}^{W}.$$

This equation implies (7.4) by the Schwarz Lemma.

Let $q = c_{n-1}$. Then $[-q, q] \subset V \equiv f^{-1}(U_1)$. By condition 4 of the definition of geometric puzzle map, $\operatorname{mod}(U_0 \setminus V) > k/2$, hence there exists a constant depending only on k which bounds $\operatorname{dist}(q, \partial J_0)/\operatorname{diam} J_0$ from below. Since $\|\alpha(q)\|_{\text{hyp}}^{U_0} > 1/7$ and $U_0 \supset D_\phi(J_0)$, there is a constant κ depending on ϕ and k only such that $|\alpha(q)| > \kappa \operatorname{dist}(q, \partial J_0)$. □

Lemma 7.8. *Given (ϕ, ψ, γ), there exists a $k > 0$ with the following property. Let $f : U \to \mathbb{C}$ be a geometric puzzle map with parameters (ϕ, ψ, γ, k) such that the critical value c_1 returns back to the domain U_1. Assume that f does not have invariant line fields on $K(f)$. Then there exists a vector field $v \in \Upsilon(D_\gamma(J_1))$ such that there are no qc vector fields α satisfying (7.3) on the critical orbit. Moreover, $v|J_1$ is real.*

Proof. Let L be the constant from Lemma 7.5 associated to ψ and γ. Fix some $k_0 > 0$ and let κ be the constant from Lemma 7.7 associated to ϕ and k_0. Let $T = f^{-1}J_1 \cap \mathbb{R}$ and let C be the constant from Corollary 2.9 corresponding to L and T.

If $k > k_0$ is big enough then $|T| \leq \kappa C^{-1} \operatorname{dist}(T, \partial J)$. Let v be the vector field given by Lemma 7.7. Suppose by contradiction that equation (7.3) is satisfied on the critical orbit for some qc vector field. Then by Lemma 7.5, this equation is satisfied by some normalized L-qc vector

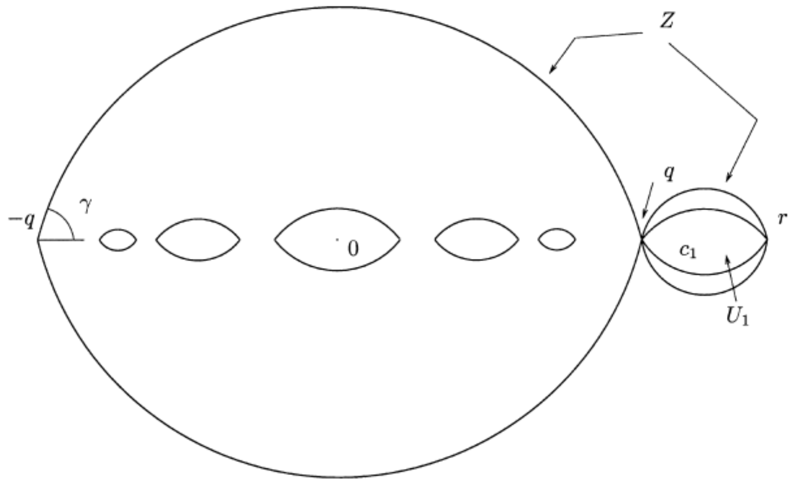

Fig. 5. Set Z and the puzzle

field α on $\cup U_j$ which vanishes at $\partial(\cup \operatorname{int} J_j)$. By (7.3), α vanishes at ∂T as well. By Corollary 2.9, $\alpha(z) \le C|T| \le \kappa \operatorname{dist}(T, \partial J_0)$ for $z \in T$. On the other hand, Lemma 7.7 yields existence of a point $q \in J$ such that $|\alpha(q)| > \kappa \operatorname{dist}(q, J_0)$. This is a contradiction. □

7.3. Puzzle motion. In this section we will show that the puzzle moves holomorphically over an appropriate neighborhood of a geometric puzzle map. First, let us introduce some notations.

Let us consider a unimodal map $f \in \mathcal{U}_a$ supplied with a geometric puzzle $\{U_i\}$ with parameters (ϕ, ψ, γ, k). Every puzzle piece U_i is univalently mapped onto U_0 by some iterate of f, $f^{n_i} : U_i \to U_0$. Recall that the (closed) real slices of the U_i are denoted by J_i and that J_1 stands for the interval containing the critical value c_1. Let $J_1 = [q, r]$, where $q < r$, and let $N = [-q, q]$. Let us consider the union of two (open) hyperbolic disks based upon the intervals N and J_1 together with their real boundary points:

$$Z \equiv Z_f = D_\gamma(N) \cup D_\gamma(J_1) \cup \{-q, q, r\}.$$

In what follows we will consider only puzzle pieces U_i which intersect Z. Notice that this family of puzzle pieces is forward invariant, except that $f(U_0)$ does not belong to it.

Let $E \equiv E_f = \operatorname{orb}(\partial J_0) \cup \{0\}$ (notice that this is a finite set). Let $\Lambda \equiv \Lambda_f$ be the space of odd vector fields $v \in A^1(Z)$ which vanish to the first order on E (where "odd" means that $v(z) = -v(-z)$ whenever both z and $-z$ belong to Z).

Notice that $\Upsilon(D_\gamma(J_1)) \subset \Lambda$ and the inclusion is an isometry.

We assume that the angle γ is so small that f is defined on some neighborhood of $\overline{Z}$. For $v \in \Lambda$, we let $f_v = f \circ (\mathrm{id} + v)$. For small enough v, this map is well defined on $\overline{Z}$, $f_v \in A^1(Z)$, and depends continuously on v.

Let $Q \equiv Q_f = \partial(\cup \operatorname{int} J_i)$ be the maximal invariant set of $f|(I \setminus \operatorname{int} J_0)$.

Lemma 7.9 (Puzzle Motion). *Let f as above be a geometric puzzle map with parameters (ϕ, ψ, γ, k) which is a restriction of a map in $\mathcal{U}_a$. Assume that $f|Q$ is hyperbolic. Then for some $\varepsilon > 0$ there is an equipped holomorphic family of puzzle maps (f_v, h_v), $v \in B_\varepsilon(\Lambda)$, such that h_v is the identity on $\partial(f(U_0)) \cup \partial Z$ and is normalized. In particular, $f_v^{n_i+1} \circ h_v = h_v \circ f^{n_i+1} = f^{n_i+1}$ on each ∂U_i.*

The proof of this lemma will be based on several lemmas. By Lemma 2.12, there exists a smooth metric $d\nu = \rho|dz|$ which is expanding on Q. By approximating ρ near Q with a piecewise constant function, we obtain an expanding piecewise Euclidean metric near Q. So, we can assume that ν is piecewise Euclidean near Q in the first place.

Lemma 7.10. *The map $f_v = f \circ (\mathrm{id} + v)$ has an invariant expanding Cantor set $Q_v \subset Z$, $Q_0 \equiv Q$, which moves holomorphically over some neighborhood $\mathcal{V} \subset \Lambda_f$ of 0.*

Proof. For simplicity, let us assume that f is non-renormalizable (the general case is dealt in the same way, but the notation is more complicated). Notice that the orientation reversing fixed point p of f belongs to $\mathrm{orb}_f(\partial J_0)$. In particular, p is also a fixed point for f_v with the same multiplier.

We will construct a holomorphically moving Markov partition $\mathcal{M}^v$ for f_v whose elements are contained in Z. To this end let us use the linearizing coordinate $L_v : (D, p) \to (\mathbb{C}, 0)$ near the orientation reversing fixed point $p_v \equiv p$ of f_v normalized by $L_v'(p_v) = 1$. The L_v-preimages of the straight rays landing at 0 will be called "rays" of f_v landing at p_v:

$$R_v(\zeta) = L_v^{-1}\{t\zeta : \ 0 \le t \le 1\}.$$

Since the multiplier of p_v is real, the second iterate of any ray "overflows" itself. Since the linearizing coordinate depends holomorphically on v, the ray $R_v(\zeta)$ with a given ζ moves holomorphically over some neighborhood $\mathcal{V} \subset \Lambda_f$. Moreover, this ray (viewed as a smooth arc in $\mathbb{C}$) smoothly depends on v.

Let x_v be a preimage of p_v under some iterate of f_v such that the whole $\mathrm{orb}_{f_v}(x_v)$ is close to Q (in particular, the smooth metric ν selected before is expanded along the orbit of x_v). If x_v is different from the special points q, $-q$ and r, then all sufficiently short rays landing at p_v can be pulled back to x_v providing us with "rays" landing at x_v.

Assume $x_v = q$. Let $f^l(q) = p$. Let us consider the union $S(q)$ of two $\mathbb{R}$-symmetric Euclidean sectors of size $2\gamma/3$ with vertex at q. Then the map $f_v^{n_1}$ is holomorphic on $S(q)$ near q and $f^l(S(q))$ contains the union $S(p)$ of two sectors centered at p of size $\gamma/2$. Hence we can take the preimages of the rays landing at p within the sectors $S(p)$ and obtain "rays" landing at q.

Similarly we construct rays landing at points $-q$ and r within appropriate sectors with vertices at those points (one sector for each of the points).

Clearly, all the rays we have constructed move holomorphically with v and smoothly depend on v. We will fix now $\theta < \gamma/2$ and only consider rays which make this angle with the real line.

Let us now defined "h-rays" for f as the rays truncated on the height $\pm h$ with respect to the smooth expanding metric ν fixed before. Here "height" denotes the vertical distance to the real line (recall that we chose ν such that in small scales near Q, ν is just a multiple of the Euclidean metric). For any point x_v, they are well-defined provided h is small enough. We define h-rays for f_v as the holomorphic motions of these rays. Since the metric ν is expanding, this family of rays satisfies the following *overflowing property*: If Γ is an h-ray landing at x_v at angle θ with the real line, then the image $f_v\Gamma$ strictly contains the h-ray landing at $f_v(x_v)$ at angle θ or $\pi - \theta$.

Given some n, we will now consider a real Markov covering $\{M_j\}$ of Q by partitioning $[-1, 1]$ by the set $Q \cap \cup_{k=0}^n f^{-k}(p)$ and taking only those (closed) intervals of the partition which intersect Q. For n big enough, this covering is contained in a small neighborhood of Q.

Let us now complexify this covering. The complex domains of the coverings will be $\mathbb{R}$-symmetric (smooth) hexagons H_j based upon the intervals M_j. Take some $M_j = [a, b]$. Consider four h-rays landing at points a and b at angle θ with the interval M_j. We obtain four sides of H_j. Join two endpoints of the rays lying in the upper half-plane with a horizontal interval on height h, and similarly in the lower half-plane. Since the h-rays are smoothly close to the straight rays for h sufficiently small, we obtain a smooth hexagon H_j (see Fig. 6). Since the h-rays move holomorphically with v, the hexagons also do (on the top and bottom sides of the hexagon the motion can be defined by linear interpolation). This defines hexagons H_j^v.

The overflowing property of the family of rays implies that the family of hexagons H_j^v is Markov family. By Proposition 2.11 (and the remark following it), the set of non-escaping points,

$$Q_v = \{z : \; f_v^n z \in \cup H_j^v, \; n = 0, 1, \dots\},$$

is a holomorphically moving invariant Cantor set.

Smooth dependence of the rays on v implies that all the hexagons are contained in Z provided v is sufficiently small. Hence $Q_v \subset Z$ as well. $\square$

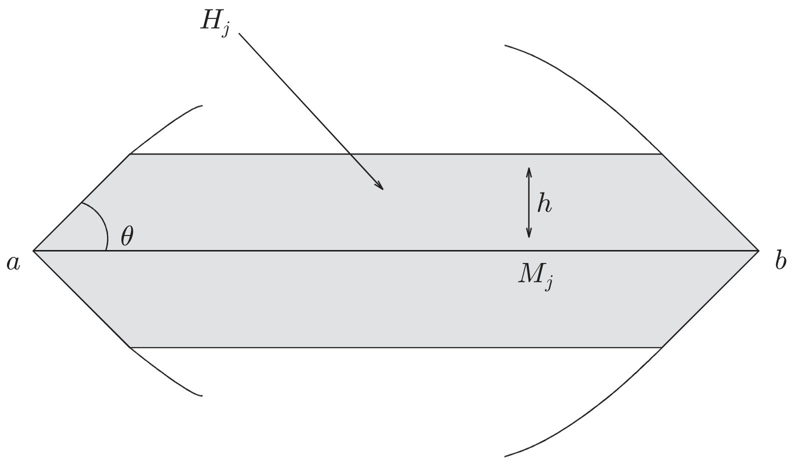

Fig. 6. A hexagonal Markov piece

Let h_v be the holomorphic motion of the Markov partition $\{H_j\}$ in the above lemma.

Lemma 7.11. *There exists a neighborhood $\mathcal{V}$ of 0 in Λ such that for each U_i there is a holomorphic motion h_v^i of $U_i \cup \partial Z \cup Q$ such that $h_v \circ f^{n_i+1} = f_v^{n_i+1} \circ h_v^i$ in U_i and $h_v^i = h_v$ in $\partial Z \cup Q$.*

Proof. Suppose first that $i = 1$. Notice that f^{n_1+1} is a ramified double covering map over a neighborhood of $f(U_0)$. Using that both f and f_v are symmetric, we conclude that for any $v \in \Lambda$ sufficiently small we can associate $w \in \Upsilon(D_\gamma(J_1))$ such that $f_v^{n_1+1} = f^{n_1+1} \circ (\mathrm{id} + w)$ and this association is continuous. We can then argue exactly like in Lemma 7.4 that for $w \in \Upsilon(D_\gamma(J_1))$ small enough, $\mathrm{id} + w$ is a diffeomorphism from $D_\gamma(J_1)$ onto a set which contains U_1, which allows us to define the motion as $(\mathrm{id} + w)^{-1}$ in U_1.

If $i \neq 1$ is fixed, it is clear that there exists a neighborhood $\mathcal{V}_i$ where such holomorphic motion is defined.

We now argue exactly as in Lemma 5.6. We notice that all but finitely many U_i are compactly contained in the domain of the persistent Markov partition $\{H_j\}$. Let $\mathcal{I}$ be the set of U_i which are not contained in any H_j. Let $\mathcal{J}$ be the set of U_i which are contained in some H_j but $f(U_i)$ is in $\mathcal{I}$. Shrinking the neighborhood if needed, we may suppose that in $\mathcal{V}$, h_v is defined, h_v^i is defined for $U_i \in \mathcal{I} \cup \mathcal{J}$ (a finite set of domains) and $h_v^i(U_i) \subset \cup h_v(H_j)$ for $U_i \in \mathcal{J}$. This allows us to construct the motion of all U_i in $\mathcal{V}$ by pulling back, as in Lemma 5.6. □

Let $S = \mathbb{C} \setminus Q$. As in Lemma 5.6 we conclude the estimates (5.2) and (5.3), which imply by Lemma 2.3 the estimates (5.4).

So we can define a holomorphic motion H_v which agrees with h_v on $Z_f \cup \partial f(U_0) \cup Q$ and with h_v^i on U_i. This conclude the proof of the Lemma 7.9.

7.4. Existence of transverse direction. Let P_f be the set of all $p \in \Lambda_f$ such that pf' is a polynomial.

Lemma 7.12. *$P_f^{\mathbb{R}}$ is dense in $\Lambda_f^{\mathbb{R}}$.*

Proof. This follows from the Mergelyan Polynomial Approximation Theorem (see [R], Theorem 20.5, p. 423). □

Lemma 7.13. *Given f as above, there exists an L with the following property. Assume $v \in \Lambda_f$ and there is a qc vector field α satisfying equation (7.3) on* $\operatorname{orb}_f(0)$. *Then α has a normalized $L\|v\|_1$-qc extension.*

Proof. We argue as in Lemma 6.12.

Let $\mathcal{V}$ be a neighborhood of 0 in Λ given by Lemma 7.9. Arguing as in Lemma 6.12 we conclude that $\beta = \frac{d}{dt} h_{tv}\big|_{t=0}$ is a normalized qc vector field such that $\|\beta\|_{qc} \le L\|v\|_1$ for some constant L. Applying Theorem 6.9 and Lemma A.24 we get a normalized qc vector field α with the same estimates. □

Corollary 7.14. *The set of $v \in \Lambda_f$ satisfying equation (7.3) is closed.*

Proof. Follows from Lemma 7.13 and the Second Compactness Lemma. □

Theorem 7.15. *If $f \in \mathcal{U}_a$ is at most finitely renormalizable then there exists a vector field $v \in \mathcal{A}_a^{\mathbb{R}}$ which does not belong to T_f.*

Proof. The non-recurrent case was treated in §7.1, so let us assume that f has a recurrent critical point. By Lemma 5.6 there exists a geometric puzzle for f satisfying the assumptions of Lemma 7.8. By Lemmas 7.8, 7.12, and Corollary 7.14, there exists a real polynomial pf' such that pf' does not belong to T_f. □

Remark 7.2. Existence of the transverse direction can be proved for maps with minimal postcritical set (which includes infinitely renormalizable and some finitely renormalizable combinatorics) by different means.

This construction is based on a renormalization approach: the assumption of minimality is used to obtain a renormalization which is polynomial-like with finitely many branches. The first step is the construction of the transverse direction for the renormalization through a variation of the construction in the case of quadratic-like maps in §4.5 of [L6]. This is possible since polynomial-like maps still enjoy a tame "external structure".

The second step is to show that the derivative of the renormalization operator has dense image and is based on the construction of Lemma 4.5.

This allows to generalize this work for classes of unimodal maps without any decay of geometry and will be elaborated elsewhere (for maps such that the closure of the critical orbit is not a minimal set, there is some decay of geometry and the construction of the transverse direction we develop here can be applied).

8. Local laminations

8.1. Transverse cone field.

Lemma 8.1. *Let f be a Yoccoz map and consider two sequences f_n, g_n in $\mathcal{U}_a$ such that f_n is hybrid equivalent to g_n and $\lim f_n = \lim g_n = f$. Assume that*

$$\lim \frac{g_n - f_n}{\|g_n - f_n\|_a} = w.$$

Then $w \in T_f$.

Proof. Denote by $\mathcal{V}_\varepsilon$ an ε-neighborhood of f in $\mathcal{A}_a$.

Consider a geometric puzzle $\mathcal{P}_f$ for f which persists in $\mathcal{V}_\varepsilon$ (see Lemma 5.6), and let $\psi_g : \mathbb{C} \to \mathbb{C}$ be a normalized holomorphic motion of the puzzle. For $F, G \in \mathcal{V}_\varepsilon$, denote by $\mu(F, G)$ the Beltrami coefficient of $\psi_G \circ \psi_F^{-1}$. It follows from the Quasiconformality Lemma that there exists a constant C such that $\|\mu(F, G)\|_\infty \leq C\|G - F\|_a$, provided F and G belongs to $\mathcal{V}_{\varepsilon/2}$.

By Remark 5.6, $\psi_{g_n} \circ \psi_{f_n}^{-1}$ is an $\mathbb{R}$-symmetric map. By Lemma A.25, there exists a normalized qc map H_n, equivariant with respect to f_n and g_n on U^{f_n}, such that

$$\|\mu_{H_n}\|_\infty \leq \|\mu(f_n, g_n)\|_\infty \leq C\|g_n - f_n\|_a.$$

Let $\mu_{H_n} = \lambda_n \mu_n$ with $\|\mu_n\|_\infty = 1$. It follows that $f_n^*(\mu_n) = \mu_n$ on U^{f_n}. We may assume also that $\tau \equiv \lim \lambda_n / \|g_n - f_n\|_a$ exists, so that $|\tau| \leq C$.

Let $H_{n,\lambda}$ be a normalized holomorphic motion over $\mathbb{D}$ with Beltrami coefficient $\lambda\mu_n$. Let $f_{n,\lambda} : H_{n,\lambda}(U^{f_n}) \to \mathbb{C}$ be defined as

$$f_{n,\lambda} = H_{n,\lambda} \circ f_n \circ H_{n,\lambda}^{-1}.$$

Since μ_n is invariant by f_n, we conclude that $f_{n,\lambda}$ is holomorphic on $H_{n,\lambda}(U^{f_n})$. Moreover, we have $f_{n,\lambda_n}|U^{g_n} = g_n$.

By Lemma 2.8, passing to a subsequence, we may assume that the $H_{n,\lambda}$ converge to some normalized holomorphic motion H_λ uniformly on compacts of $\mathbb{D}$.

Let $f_\lambda : H_\lambda(U^f) \to \mathbb{C}$ be defined as

$$f_\lambda = H_\lambda \circ f \circ H_\lambda^{-1}.$$

It follows that for each $\lambda \in \mathbb{D}$, $f_{n,\lambda}$ converges to f_λ uniformly on compacts of $H_\lambda(U^f)$, so that f_λ is holomorphic.

For any fixed compact set $K \subset U^f$, there exists $\delta > 0$ such that $H_\lambda^{-1}(K) \subset U^f$, $\lambda \in \mathbb{D}_\delta$. It follows that $(\lambda, z) \mapsto f_{n,\lambda}(z)$ converges uniformly to $(\lambda, z) \mapsto f_\lambda(z)$ on $\mathbb{D}_{\delta/2} \times K$. By uniform convergence of derivatives of holomorphic maps, we conclude that for any $z \in U^f$,

$$\begin{aligned}\tau \frac{d}{d\lambda} f_\lambda(z)\bigg|_{\lambda=0} &= \lim \frac{\lambda_n}{\|g_n - f_n\|_a} \lim \frac{f_{n,\lambda_n}(z) - f_{n,0}(z)}{\lambda_n} \\ &= \lim \frac{g_n - f_n}{\|g_n - f_n\|_a} \\ &= w(z).\end{aligned}$$

Since $\|w\|_a = 1$, it does not vanish identically, so that $\tau \neq 0$. Let

$$\alpha = \frac{d}{d\lambda} H_\lambda\bigg|_{\lambda=0}.$$

By the argument of Proposition 5.2, α is a qc vector field equivariant with respect to $(f, w/\tau)$ on orb(0), so $w \in T_f$. □

From now on, let us fix a Yoccoz map f, and let $v \in \mathcal{A}_a^{\mathbb{R}}$ be a transverse vector field given by Lemma 7.15 such that $\|v\|_a = 1$.

Corollary 8.2. *Let $\Sigma_\varepsilon = \{f + tv | t \in (-\varepsilon, \varepsilon)\}$. Then there exists an ε such that Σ_ε intersects each hybrid class in at most one point.*

Proof. If this is not the case, there exist sequences $t_{1,n}, t_{2,n} \to 0$ such that $t_{1,n} \neq t_{2,n}$ with $f + t_{1,n}v$ hybrid conjugate to $f + t_{2,n}v$. Applying Lemma 8.1 we conclude that $v \in T_f$, which is a contradiction. □

Remark 8.1. It follows that the straightening $\chi : \Sigma_\varepsilon \to [1/2, 2]$ is a homeomorphism onto the image. Since hyperbolic maps are dense in the quadratic family, they are dense in Σ_ε as well.

Below we use the notion of special combinatorics defined in §6.3.

Lemma 8.3. *There exists a neighborhood $\mathcal{V}$ of f such that if $f_n \to f$ is a sequence of maps with special combinatorics with respect to $\mathcal{V}$, then* $\limsup T_{f_n} \subset T_f$.

Proof. Consider a geometric puzzle for f which persists in a neighborhood of f and take a neighborhood $\mathcal{V}$ of f as in Lemma 6.12. Let $f_n \to f$ be a sequence of maps with special combinatorics with respect to $\mathcal{V}$, and let $v_n \in T_{f_n}$ be a sequence of vector fields converging to some vector field v. Since $\|v_n\|_a$ is uniformly bounded, and the operator norm of L_{f_n} is uniformly bounded, there exists a qc vector field α_n, equivariant on $\mathrm{orb}_{f_n}(0)$, and such that $\|\alpha_n\|_{\mathrm{qc}} < C$. Denote by β_n some normalized extension of α_n to $\mathbb{C}$ with $\|\overline{\partial}\beta_n\|_\infty < C$. By the Second Compactness Lemma, we may assume that β_n converges to some vector field β. It is easy to see that β is equivariant with respect to (f, v) on orb(0). So $v \in T_f$, and since T_f is a vector space, $\limsup T_{f_n} \subset T_f$. □

Corollary 8.4. *There exists a neighborhood $\mathcal{V}$ of f in $\mathcal{A}_a$ and an open cone $\mathcal{K}$ such that for any $g \in \mathcal{V}$ which has special combinatorics with respect to $\mathcal{V}$, T_g is transverse to $\mathcal{K}$.*

Proof. Suppose that the statement is false. Let $\mathcal{V}_n \subset \mathcal{A}_a$ be a $1/n$-neighborhood of f and let $\mathcal{W}_n \subset T\mathcal{A}_a$ be a $1/n$ neighborhood of the transverse vector field v. Since the cone generated by $\mathcal{W}_n$ is an open cone, there exists a sequence $f_n \in \mathcal{V}_n$, with special combinatorics with respect to $\mathcal{V}_n$, such that $T_{f_n} \cap \mathcal{W}_n \neq \emptyset$. So $v \in \limsup T_{f_n}$ and thus $\limsup T_{f_n} \not\subset T_f$.

To obtain a contradiction with Lemma 8.3, we observe that, for a fixed neighborhood $\mathcal{V}$ of f, $\mathcal{V}_n \subset \mathcal{V}$ for all n big enough, so that f_n has special combinatorics also with respect to $\mathcal{V}$. □

Corollary 8.5. *T_f is a codimension-one subspace of $T\mathcal{A}_a$.*

Proof. By Remark 8.1, there exists a sequence $f_n \to f$ with special combinatorics. For each f_n, T_{f_n} is codimension-one, so by Lemma 2.32, either $\limsup T_{f_n}$ is codimension-one or is equal to $T\mathcal{A}_a$. By Lemma 8.3, $T_f \supset \limsup T_{f_n}$, so T_f must be a codimension-one subspace, since the transverse vector field v does not belong to T_f. □

8.2. Proof of Theorem A. The construction of the lamination in the infinitely renormalizable case was carried out in Theorem 4.9. The lamination near hyperbolic maps is trivial to construct and parabolic combinatorics are codimension-one submanifolds by the Implicit Function Theorem (notice that parabolic points of quasiquadratic maps must be non-degenerate). We will construct the lamination near a Yoccoz map f.

Fix a puzzle for f and let $\mathcal{V}$ and $\mathcal{K}$ be as in Lemmas 6.3, 6.12 and Corollary 8.4 and v be as in last section. Let $\Pi_1 : T\mathcal{A}_a \to T_f$, $\Pi_2 : T\mathcal{A}_a \to \mathbb{C}$ be the linear projections along T_f and v, that is, $w = \Pi_1(w) + \Pi_2(w)v$. From Lemma 2.31, there exists a constant C such that for any $w \in T\mathcal{A}_a \setminus \mathcal{K}$, $\Pi_2(w) \leq C\|\Pi_1(w)\|_a$. In particular, tangent spaces as in Corollary 8.4 are C-Lipschitz graphics from T_f to the transverse one-dimensional subspace spanned by v.

Fix ε very small and let Λ be an ε-neighborhood of 0 in T_f and let $\Sigma = \Sigma_\varepsilon$ be as in Corollary 8.2.

Let $g \in \mathcal{A}_a$. We say that $\gamma : \mathcal{W} \to \mathcal{A}_a$ is a *g-graphic* (over $\mathcal{W}$) if $\mathcal{W}$ is a neighborhood of 0 in T_f and if $\gamma(0) = g$ and $\Pi_1 \circ \gamma - \mathrm{id} = \Pi_1(g)$. We say that a g-graphic is C-Lipschitz if $\|\Pi_2 \circ D\gamma\| \leq C$. We say that a g-graphic is contained in some set X if $\gamma(\mathcal{W}) \subset X$.

We say that a set $X \subset \mathcal{A}_a$ is a *definite* g-graphic if there is a g-graphic over Λ onto X.

Lemma 8.6. *If $g \in \Sigma$ has special combinatorics and $\gamma : \mathcal{W} \to \mathcal{C}_g$, $\mathcal{W} \subset \Lambda$ is a g-graphic, then γ is C-Lipschitz.*

Proof. This is an immediate consequence of Corollary 8.4. □

Lemma 8.7. *If $g \in \Sigma$ has special combinatorics then $\mathcal{C}_g$ contains a definite g-graphic.*

Proof. Since $\mathcal{C}_g$ is a codimension-one submanifold transverse to v at g, there exists a g-graphic contained in $\mathcal{C}_g$.

Let $\mathcal{W} \subset \Lambda$ be the set of all w such that there exists a neighborhood $\mathcal{W}_w$ of $[0, w]$ in T_f and a g-graphic over $\mathcal{W}_w$ contained in $\mathcal{C}_g$.

Since $\mathcal{C}_g$ is a codimension one submanifold, it is also clear that the continuation property is valid, that is, if γ_1 is a g_1-graphic over $\mathcal{W}_1$ and γ_2 is a g_2-graphic over $\mathcal{W}_2$ which are contained in $\mathcal{C}_g$, and $\mathcal{W}_1 \cap \mathcal{W}_2$ is simply connected, then $\gamma_1|\mathcal{W}_1 \cap \mathcal{W}_2 = \gamma_2|\mathcal{W}_1 \cap \mathcal{W}_2$. It follows that there exists a g-graphic $\gamma : \mathcal{W} \to \mathcal{C}_g$. So we just have to show that $\Lambda = \mathcal{W}$.

If this is not the case, there exists $w_0 \in \Lambda \setminus \{0\}$ such that $[0, w_0) \subset \mathcal{W}$, but $w_0 \notin \mathcal{W}$. By Lemma 8.6, γ is C-Lipschitz, so that $\gamma(\mathcal{W})$ is contained in a small neighborhood of g, and $\lim_{w \to w_0} \gamma_w$ exists and will be denoted by $\gamma(w_0)$. By Lemma 6.3, for any $w \in \mathcal{W}$, $\gamma(w)$ is L-qc conjugate to g on U^g. Using the First Compactness Lemma, we conclude that $\gamma(w_0)$ is L-qc conjugate to g on U^g, so $w_0 \in \mathcal{C}_g$ and w_0 has special combinatorics with respect to $\mathcal{V}$. By Corollary 8.4, v is transverse to $T_{\gamma(w_0)}$, so there exists a $\gamma(w_0)$-graphic contained in $\mathcal{C}_g$. By the continuation property, $w_0 \in \mathcal{W}$, which is a contradiction. □

Let γ_g be the definite g-graphic contained in $\mathcal{C}_g$. Let $\Delta_g = \gamma_g(\Lambda) \subset \mathcal{C}_g$.

By the continuation property, if $g_1, g_2 \in \Sigma$ have special combinatorics and are different, then $\Delta_{g_1} \cap \Delta_{g_2} = \emptyset$.

By Remark 8.1 and the Extension Lemma for holomorphic motions, there is a unique extension of the lamination Δ_g whose leaves pass through every point of Σ. Given now $g \in \Sigma$ not necessarily with special combinatorics, we let Δ_g be the leaf of the extended lamination. It is clear that each Δ_g is still a definite C-Lipschitz g-graphic.

It follows that the slices of the Δ_g by $\mathcal{U}_a$ form a lamination with codimension-one real analytic leaves. Moreover, by the First Compactness Lemma, the maps in the same leaf are L-qc conjugate, and hence the leaves coincide with the local hybrid classes. This completes the proof of Theorem A.

Remark 8.2. The present proof for the existence of the laminations given the existence of the transverse direction (see Remark 7.2) also works in the infinitely renormalizable case.

This proof also apply to a class of maps considered in [LS2], namely covering maps of the circle with a unique critical point of inflection type (for which the existence of the transverse direction is automatic). This application will be elaborated elsewhere.

8.3. Characterization of the tangent space. Let us recall that each Δ_g constructed above consists of maps $\tilde{g}$ which are L-qc conjugate to g on U^g. Moreover, the conjugacy between g and $\tilde{g}$ can be chosen to vary holomorphically inside Δ_g:

Proposition 8.8. *For each $g \in \Sigma$ there is a normalized holomorphic motion $h_{\tilde{g}}$, $\tilde{g} \in \Delta_g$ such that $h_{\tilde{g}}(U_g) = U_{\tilde{g}}$ and $h_{\tilde{g}}$ is equivariant with respect to g and $\tilde{g}$ on U_g.*

Proof. Let us fix a holomorphic motion H_g of the puzzle on $\mathcal{V}$. For g with special combinatorics, it follows from Remark 6.2 that there exists a holomorphic motion $h_{\tilde{g}}$ over Δ_g, equivariant on U_g and coinciding with $H_{\tilde{g}}$ on $\mathbb{C} \setminus U_g$.

In general, Δ_g is the limit of a sequence Δ_{g_n}, g_n with special combinatorics. By Lemma 2.8, there exists a holomorphic motion over Δ_g which is a limit of the holomorphic motions over Δ_{g_n}. By continuity, this holomorphic motion is also equivariant. □

Though the above proof only applies for our construction of the lamination in a neighborhood of a Yoccoz map, a similar statement still holds near infinitely renormalizable maps. In this case instead of using persistent puzzles, the situation can be reduced to the quadratic-like case by means of the renormalization. The hyperbolic case can be dealt in an easier way with a construction based on the persistence of the basin of attraction (in a similar argument to Proposition 5.4). Similarly, the case of parabolic maps can be dealt using the persistence of attracting petals along the submanifold where the parabolic point persists. All those cases are summarized below:

Proposition 8.9. *Let $g \in \mathcal{U}_a$. There exists a codimension-one complex submanifold $\Delta_g \subset \mathcal{A}_a$, such that $\Delta_g \cap \mathcal{U}_a \subset \mathcal{H}_g$ and a normalized holomorphic motion $h_{\tilde{g}}$, $\tilde{g} \in \Delta_g$, which is equivariant on I. Moreover, the Δ_g form a lamination near any non-parabolic map $f \in \mathcal{U}_a$.*

Recall that except in the hyperbolic but not superattracting case, the "tangent space" T_f was defined as the set of vector fields admitting a representation $v = \alpha \circ f - f'\alpha$ on $\mathrm{orb}_f(0)$ for a qc vector field α. The following proposition shows that this choice was completely justified:

Theorem 8.10. $T_f = T_f\mathcal{H}_f$.

Proof. First we notice that T_f contains the tangent space to the real hybrid class of f. Indeed, if $v \in T_f\mathcal{H}_f$ then there exists a path $f_\lambda \in \mathcal{H}_f$ through f tangent to v at f. By Proposition 8.9, f_λ can be equipped with an equivariant holomorphic motion of the interval. By Proposition 5.2, $v \in T_f$.

We obtained in §7 a transverse vector field to T_f for at most finitely renormalizable maps f. Since $T_f\mathcal{H}_f$ is codimension-one, $T_f = T_f\mathcal{H}_f$.

Let now f be infinitely renormalizable, and let R be the renormalization operator of §4, so that $R(f) : U \to U'$ is a quadratic-like map. Let v be a vector field transverse to $T_f\mathcal{H}_f$. By transverse non-singularity of R, $DR(f)v \notin T_{R(f)}\mathcal{H}_{R(f)}$ (see Remark 4.1).

By the characterization of the tangent space to the hybrid class of a quadratic-like map (see §4.1.2), there are no qc vector fields α equivariant with respect to $(R(f), DR(f)v)$ on U satisfying $\overline{\partial}\alpha | K(R(F)) = 0$.

However, this last condition is vacuous since there are no invariant line fields on $K(R(f))$ (Theorem 2.22). By the Infinitesimal Pullback Argument (in the quadratic-like setting), we see that this is equivalent to non-existence of a qc vector field α equivariant just on $\mathrm{orb}_{R(f)}(0)$. In particular, there is no qc vector field equivariant with respect to (f, v) on $\mathrm{orb}_f(0)$, so $v \notin T_f$. $\square$

9. Regular or stochastic theorem

9.1. Tangencies between holomorphic curves and holomorphic laminations. Let us consider a codimension-one holomorphic lamination $\mathcal{F}$ in an open set $\mathcal{V}$ of a complex Banach space $\mathcal{B}$. For a point $a \in \mathcal{V}$ in the support of the lamination, denote by L_a the leaf of $\mathcal{F}$ through a and by T_a the tangent space to this leaf at a. Let $\gamma : \mathbb{D} \to \mathcal{V}$ be a holomorphic curve. We say that γ has a tangency with $\mathcal{F}$ at some parameter value λ_0 if $\gamma(\lambda_0)$ belongs to the support of the lamination and

$$\frac{d}{d\lambda}\gamma(\lambda)\bigg|_{\lambda=\lambda_0} \in T_{\gamma(\lambda_0)}.$$

The set of tangencies is clearly closed in $\mathbb{D}$.

Lemma 9.1 (A. Douady). *If the curve γ is not contained in any leaf of $\mathcal{F}$ then the set of tangencies is discrete.*

Proof. We may assume that we have flow box coordinates $\mathcal{W} \oplus \mathbb{C}$, in other words, the leaves of the lamination are graphs over $\mathcal{W}$. Let's consider a parameter λ_0 where γ has a tangency with the lamination.

By a change of coordinates we may assume $\lambda_0 = 0$ and that the leaf containing $\gamma(0)$ is the graph of the zero function from $\mathcal{W}$ to $\mathbb{C}$. If γ is not contained in a leaf, we may write for z near 0, $\gamma(z) = (\phi(z), z^n\psi(z))$ where $\psi(0) \neq 0$ and $n \geq 2$.

Let $S : \mathcal{W} \oplus \mathbb{C} \to \mathcal{W} \oplus \mathbb{C}$ be defined by $S(z, w) = (z, w^n)$ and $\tilde{\mathcal{F}}$ be the lamination in $\mathcal{W} \oplus \mathbb{C}$ whose leaves are connected components of preimages by S of leaves of $\mathcal{F}$.

It is easy to see that if $\tilde{\gamma}$ is a path in $\mathcal{W} \oplus \mathbb{C}$ and if $S \circ \tilde{\gamma}$ has a tangency at λ with $\mathcal{F}$ then either $\tilde{\gamma}$ has a tangency at λ with $\tilde{\mathcal{F}}$ or $\tilde{\gamma}(\lambda) \in \mathcal{W} \times \{0\}$.

Let then $\tilde{\gamma} = (\phi(z), z\psi(z)^{1/n})$ for z near 0. Then $\tilde{\gamma}$ is transverse to $\tilde{\mathcal{F}}$ at 0, and so there is no tangency in a neighborhood of 0. We conclude that 0 is the unique tangency of $\gamma = S \circ \tilde{\gamma}$ in a neighborhood of 0. $\square$

9.2. Connectivity of some hybrid classes.

Theorem 9.2. *If f is a Yoccoz map, then there exists a one parameter real analytic family $\{f_t\}_{t\in[0,1]} \subset \mathcal{U}_a$ in the hybrid class of f connecting $f_0 = f$ with the quadratic map $f_1 = q_{\chi(f)}$.*

Proof. It follows from Theorem B.1 that f is qc conjugate to the quadratic polynomial $q_{\chi(f)}$ in a neighborhood of the interval I. Moreover, this conjugacy fixes 0 and 1, and commutes with the reflections with respect to 0 and $\mathbb{R}$ (we will call such maps "symmetric"). Extend it to a global symmetric qc homeomorphism $h : \mathbb{C} \to \mathbb{C}$. Let μ be its Beltrami differential. Solving the Beltrami equation with differential $\lambda\mu$, we obtain a holomorphic family

$$h_\lambda : \mathbb{C} \to \mathbb{C},\ |\lambda| < 1/\|\mu\|_\infty,$$

of symmetric qc maps fixing 0 and 1. Let $f_\lambda : h_\lambda(\Omega_a) \to \mathbb{C}$ be defined by $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$. Then there is some $a' < a$ such that $\{f_t\}_{t\in[0,1]}$ is a real analytic curve in $\mathcal{U}_{a'}$ contained in the hybrid class of f.

Let us show that this curve can be approximated, in the topology of $\mathcal{U}_{a'}$ by a similar curve taking values in $\mathcal{U}_a$. To this end, let us consider a one-parameter real analytic family of vector fields $\{v_t\}_{t\in[0,1]}$ in $T\mathcal{U}_{a'}$ along the family $\{f_t\}_{t\in[0,1]}$ such that for any t, v_t is transverse to the hybrid class of f in $\mathcal{U}_{a'}$ (that is $v_t \notin T_{f_t}\mathcal{H}_{f,a'}$). Consider a 2-parameter family

$$G(t,s) = f_t + sv(t) \text{ in } \mathcal{U}_{a'},\ (t,s) \in [0,1] \times [-1,1].$$

Letting $\zeta_0 : [0,1] \to [0,1] \times [-1,1]$ be the natural inclusion $\zeta_0(t) = (t,0)$, we have $f_t = G \circ \zeta_0(t)$.

Let us now consider a real analytic family $F : [0,1] \times [-1,1] \to \mathcal{U}_a$ such that $F(0,0) = G(0,0)$ and $F(1,0) = G(1,0)$. We may also require that $\Pi_{a,a'} \circ F$ is C^1 close to G, where $\Pi_{a,a'} : \mathcal{U}_a \to \mathcal{U}_{a'}$ is the inclusion. By the Implicit Function Theorem, there exists a real analytic curve $\zeta : [0,1] \to [0,1] \times [-1,1]$, C^1 close to ζ_0, such that $\Pi_{a,a'} \circ F \circ \zeta$ is contained in $\mathcal{H}_{f,a'}$. In particular, $F \circ \zeta : [0,1] \to \mathcal{U}_a$ is a real analytic path connecting f_0 to f_1 in $\mathcal{H}_{f,a}$. □

9.3. Proof of Theorem C. Let $\{f_t\}_{t\in J}$ be a real analytic family in $\mathcal{U}_a$ defined over some (open or closed) interval $J \subset \mathbb{R}$. Assume $\{f_t\}$ is non-trivial. By Theorem A each hybrid class is a real analytic codimension-one submanifold. Let $T \subset \mathbb{R}$ be the union of the set of parabolic parameters and the set of tangencies of f_t with the hybrid classes.

Lemma 9.3. *T is countable.*

Proof. We notice that the set of parabolic parameters is countable. Indeed, those are associated with countably many analytic equations of the type $f_t^k(x) = x$, $|Df_t^k(x)| = 1$, where $k > 0$ is an integer. Since $\{f_t\}$ is not contained in a leaf, each of these equations corresponds to a discrete set of parameters.

So if T is not countable there is a parameter t such that f_t is not parabolic and any neighborhood of t intersects T in infinite many points. By Theorem A, since f_t is not parabolic there exists a neighborhood of f_λ in $\mathcal{A}_a$ which is holomorphically laminated by the hybrid classes. By Lemma 9.1, the set of tangencies in a smaller neighborhood is finite. This is a contradiction. □

To conclude the proof of Theorem C, it is enough to show that each parameter in the complement of T has a neighborhood where the straightening is quasisymmetric.

Consider then a parameter $\lambda_0 \notin T$. If f_{λ_0} is hyperbolic, χ is clearly analytic in a neighborhood of λ_0. If f_{λ_0} is infinitely renormalizable, we can apply Theorem 4.11. Assume then that f_{λ_0} is a Yoccoz map. In this case, there is a path connecting it to the quadratic family and we can just use the transversality at λ_0 to conclude the quasisymmetry of χ, since codimension-one real analytic laminations are transversally quasisymmetric.

9.4. Proof of Theorem B. Let X be the complement of the set of parameters of the quadratic family which are either hyperbolic or are Yoccoz maps and have at most a finite number of central levels in the principal nest of their last renormalization. By Theorem 2.20, a map which is neither regular or stochastic is hybrid equivalent to a parameter in X. In view of Theorem C, it is enough to prove the following:

Lemma 9.4. *The image of X by any qs map has zero Lebesgue measure.*

Proof. We first decompose $X = \mathcal{I} \cup \mathcal{F} \cup \mathcal{P}$ where $\mathcal{I}$ are infinitely renormalizable parameters, $\mathcal{F}$ are Yoccoz parameters contained in X, and $\mathcal{P}$ are parabolic. Let h be a qs map.

Then $|h(\mathcal{P})| = 0$ since $\mathcal{P}$ is countable, and $|h(\mathcal{I})| = 0$ follows from Theorem 2.27, since the property of having definite gaps everywhere is preserved by quasisymmetric maps.

Parameters in $\mathcal{F}$ can be further decomposed as a countable union $\cup \mathcal{F}_j$, where parameters in each $\mathcal{F}_j$ have the same combinatorics for their smallest renormalization interval. Let us show that each $h(\mathcal{F}_j)$ has zero Lebesgue measure. For simplicity, let us consider the case $\mathcal{F}_0 \subset \mathcal{N}$ of non-renormalizable maps, the general case reduces to this one by renormalization.

According to §2.15, for each n there exists a covering of $\mathcal{N}$ by disjoint intervals $\Delta_i^n \subset [1/2, 2]$ ("real parapuzzle pieces of level n"), and each of the Δ_i^n contains a central interval Π_i^n satisfying (2.16). Furthermore, a parameter belongs to $\mathcal{F}_0$ if and only if it belongs to infinitely many Π_i^n. By Lemma 2.5, there exists constants $\tilde{C} > 0$ and $0 < \tilde{\rho} < 1$ such that

$$\frac{|h(\Pi_i^n)|}{|h(\Delta_i^n)|} \leq \tilde{C}\tilde{\rho}^n,$$

so that $\sum_n \sum_i |h(\Pi_i^n)| < \infty$. By the Borel-Cantelli Lemma, $|h(\mathcal{F}_0)| = 0$. $\square$

Appendix A. Complex return maps

The dynamics of certain classes of complex return maps was described in the works of Branner-Hubbard [BH], Yoccoz [H], and Lyubich [L2], [L4].

The precise hypothesis on the dynamics change from work to work. In this appendix we adapt those ideas for our setting, collecting the results needed for the analysis of puzzle maps.

A.1. Definitions. Let W be a quasidisk and let $\{W_j\}$ be a family of at least 2 quasidisks inside W with pairwise disjoint closures such that $0 \in W_0$. Assume further that

$$\inf \operatorname{mod}(W \setminus \overline{W_j}) > 0, \tag{A.1}$$

that $\overline{\cup W_j}$ is thin in W (see definition in the beginning of §5.2), and $\operatorname{diam}(W_j) \to 0$.

An *R-map* is a holomorphic map $F : \cup W_j \to W$ such that for any $j \neq 0$, $F|W_j$ is a univalent map onto W, and $F|W_0$ is a double covering onto W branched at 0 ("R" stands for "Return"). We let $W^n = F^{-n}(W)$ and we define the *filled Julia set* $K(F)$ as $\cap W^n$.

The components of $F^{-n}(W)$ are called *puzzle pieces of depth* n. For $x \in F^{-n}(W)$, we let $P^n(x)$ be the puzzle piece of depth n containing x. Puzzle pieces containing 0 are called *critical*.

An R-map F is called *renormalizable* if there exists a puzzle piece $V = P^n(0)$, $n \geq 1$, and an integer $p > 0$ such that $V \subset F^p(V)$, the puzzle pieces $F^j(V)$ $1 \leq j \leq p$ are pairwise disjoint, and $F^{mp}(0) \in V$, $m > 0$. The map $R(F) = F^p|P^n(0)$ with minimal n as above will be called the *renormalization* of F. It is a quadratic-like map with connected Julia set.

Let $F : \cup W_j \to W$ and $\tilde{F} : \cup \tilde{W}_j \to W$ be two R-maps, and let h be a homeomorphism of $\mathbb{C}$ equivariant on $\cup \partial W_j$. If $h(F(0)) = \tilde{F}(0)$, then for each j there is a unique homeomorphism $\psi_j : \operatorname{cl} W_j \to \operatorname{cl} \tilde{W}_j$ coinciding with h on ∂W_j and such that $h \circ F = \tilde{F} \circ \psi_j$ on W_j. Let

$$h_1 = \begin{cases} \psi_j & \text{on } W_j \\ h & \text{on } \mathbb{C} \setminus \cup W_j. \end{cases}$$

Since $\operatorname{diam} W_j \to 0$, h_1 is a homeomorphism of $\mathbb{C}$. It is called the *lift* of h (compare §6.1).

We say that a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ is a *combinatorial equivalence* between F and $\tilde{F}$ if it is equivariant on $\cup\, \partial W_j$ and the lift h_1 of h is homotopic to h rel $\cup\, \partial W_j \cup \operatorname{orb}_F(0)$.

The notions of topological/qc/hybrid equivalence between two R-maps are self-evident.

A.2. Divergence property. For $x \in K(F)$, we let

$$A^n(x) = P^n(x) \setminus \overline{W^{n+1}}.$$

We define $\operatorname{mod} A^n(x)$ as the extremal length of the family of curves in $A^n(x)$ joining $\partial P^n(x)$ to $\partial W^{n+1} \cap P^n(x)$.

Theorem A.1. *Assume that $F : \cup_{j=1}^m W_j \to W$ is a non-renormalizable R-map defined on the union of finitely many domains W_j. Then for any $x \in K(F)$,*

$$\sum \operatorname{mod} A^n(x) = \infty. \tag{A.2}$$

Proof. Fix $x \in K(F)$ and let $\mu_k^n = \operatorname{mod} A^n(F^k(x))$. These numbers satisfy the following rules: $\mu_k^n = \mu_{k+1}^{n-1}$ if $P^n(x)$ is not critical and $2\mu_k^n = \mu_{k+1}^{n-1}$ otherwise.

Since there are only finitely many W_j, $\overline{\cup W_j} \subset W$. This implies that μ_k^0 (which in fact does not depend on k) is positive. Since F is non-renormalizable, the "tableau" $\{P_k^n(0)\}_{n,k}$ is aperiodic in k. By the work of Branner and Hubbard, Theorem 4.3, p. 264 of [BH], these properties imply (A.2). □

A compact set $X \subset \mathbb{C}$ is called *removable* if any qc map $H : \mathbb{C} \setminus X \to \mathbb{C}$ extends to a qc homeomorphism of $\mathbb{C}$. The following remark is due to Jeremy Kahn.

Corollary A.2. *Under the assumptions of the previous theorem, the filled Julia set $K(F)$ is removable and hence has zero Lebesgue measure.*

Proof. Since there are only finitely many domains W_j, the filled Julia set $K(F)$ is compact. Now the result follows from the divergence property (A.2) by [SN], §1 (see also McMullen in [BH], §5.4). □

A.3. Geometry of puzzle pieces. Let us say that $x \in K(F)$ *shadows the critical orbit* if for any k and all $m \geq m(k)$, there exists $j \leq m$ such that the puzzle piece $P^{k+j}(F^{m-j}(x))$ is critical (in other words, the map $F^m : P^{k+m}(x) \to P^k(F^m(x))$ is not univalent). In particular, for $m \geq m(k)$, $P^k(F^m(x))$ intersects the critical orbit. Moreover, for all k, $\operatorname{orb}_F(x) \cap P^k(0) \neq \emptyset$.

Lemma A.3. *Let F be an R-map and let $x \in K(F)$. If*

$$\operatorname{mod}(P^n(x) \setminus \overline{P^{n+1}(x)}) \to 0$$

then x shadows the critical orbit.

Proof. Assume that x does not shadow the critical orbit. Then there exist k and arbitrarily big m such that the map $F^m : P^{k+m}(x) \to P^k(F^m(x))$ is univalent. Hence

$$\operatorname{mod}(P^{k+m}(x) \setminus \overline{P^{k+m+1}(x)}) = \operatorname{mod}(P^k(F^m(x)) \setminus \overline{P^{k+1}(F^m(x))}).$$

Taking k more iterates of $F^m(x)$, we conclude that

$$\operatorname{mod}(P^k(F^m(x)) \setminus \overline{P^{k+1}(F^m(x))}) \geq \frac{1}{2^k} \operatorname{mod}(W \setminus \overline{P^1(F^{m+k}(x))}).$$

By the definition of R-map, the later modulus is bounded away from 0. This is a contradiction. □

Lemma A.4. *If F is renormalizable and x shadows the critical orbit then there exists k such that $F^k(x) \in K(R(F))$.*

Proof. Let $R(F) = F^p|P^n(0)$ and let $U_i = P^n(F^i(0))$. Then $\{U_i\}_{i=0}^{p-1}$ is a disjoint cover of the orbit of 0. Let $K_j = F^j(K(R(F)))$ so that

$$K_j = \left\{x \in U_j : F^k(x) \in \cup_{i=0}^{p-1} U_i, k = 0, 1, ...\right\}.$$

Let $x \in K(F)$ shadow the critical orbit and let m_0 be such that

$$P^n(F^m(x)) \cap \mathrm{orb}_F(0) \neq \emptyset \quad \text{for } m \geq m_0.$$

Hence, $P^n(F^m(x)) \subset \cup U_i$ for all $m \geq m_0$, and the conclusion follows. □

Theorem A.5. *Let F be an R-map and let $x \in K(F)$ satisfy*

$$\sum \mathrm{mod}(P^n(x) \setminus \overline{P^{n+1}(x)}) < \infty.$$

Then F is renormalizable and $F^k(x) \in K(R(F))$ for some k.

Proof. For $x \in K(F)$, let $B_n(x) = \mathrm{mod}(P_n(x) \setminus \overline{P^{n+1}(x)})$. Let x be a point such that $\sum B_n(x) < \infty$. By Lemma A.3, $P^k(0)$ intersects $\mathrm{orb}_F(x)$ for every k. In particular, if m_k is minimum such that $F^{m_k}(x) \in P^{k+1}(0)$, $B_{k+m_k}(x) = B_k(0)$. Since the sequence $k + m_k$ is strictly increasing, we conclude that $\sum B_k(0) < \infty$.

Assume first that $\mathrm{orb}_F(0)$ intersects infinitely many domains W_j. Then $F(0)$ does not shadow the critical orbit: if m is minimal with $F^m(F(0)) \in W_j$, the map $F^m : P^{m+1}(F(0)) \to P^1(F^m(F(0)))$ is a diffeomorphism. In particular, by Lemma A.3, $B_k(F(0))$ does not converge to 0, so neither does $B_k(0)$, and this contradicts $\sum B_n(0) < \infty$.

Thus, $\mathrm{orb}_F(0)$ intersects only finitely many W_j. Let $\tilde{F}$ be the R-map obtained by restricting F to those. The sequence $B_k(0)$ does not change by taking this restriction. The Divergence Property of Theorem A.1 implies that $\tilde{F}$ is renormalizable. So F is also renormalizable, and since x shadows the critical orbit we have $F^k(x) \in K(R(F))$ for some k by Lemma A.4. □

Corollary A.6. *If F is a non-renormalizable puzzle map, for every $x \in K(F)$, $\cap P^n(x) = \{x\}$.*

Lemma A.7. *Let F be an R-map such that* $\mathrm{int}\, K(F) \neq \emptyset$. *Then F is renormalizable and* $\mathrm{int}\, K(F) = \cup F^{-n}(\mathrm{int}\, K(R(F)))$. *Furthermore, $R(F)$ has a non-repelling periodic orbit.*

Proof. Let $x \in \mathrm{int}\, K(F)$. Notice that $\partial P^n(x)$ is not contained in $K(F)$, so $\cap P^n(x)$ contains a neighborhood V of x. By Theorem A.5, F is renormalizable, and $F^k(x) \in K(R(F))$ for some k. Since $K(R(F)) = \cap P^n(0)$, we have:

$$F^k(V) \subset F^k(\cap P^n(x)) \subset K(R(F)).$$

Hence $F^k(x) \in \operatorname{int} K(R(F))$. By the classification of Fatou components, see [Mc1] p. 37, a quadratic polynomial whose filled-in Julia set has non-empty interior must have either a hyperbolic or parabolic periodic orbit or an indifferent periodic orbit associated to a Siegel disk, so in all cases it must have a non-repelling periodic orbit. The same conclusion holds for $R(F)$ via the Straightening Theorem of Douady-Hubbard (see §2.12). □

A.4. Measure of the Julia set. In this section we generalize the results of [L2] and [Sh] on the Lebesgue measure of the Julia sets of quadratic polynomials to the setting of R-maps.

Lemma A.8. *Let F be an R-map. Then almost every $x \in K(F)$ shadows the critical orbit.*

Proof. Consider a point $x \in K(F)$ which does not shadow the critical orbit. Then there exist k and a sequence $m_j \to \infty$ such that each map $F^{m_j} : P^{m_j+k}(x) \to P^k(F^{m_j}(x))$ is univalent. Applying a few more iterates of F, we will find a critical puzzle piece $P^{k-n_j}(0)$ such that the map $F^{m_j+n_j} : P^{m_j+k} \to P^{k-n_j}(0)$ is also univalent.

Since the property of being thin is invariant under lifts by branched coverings, the filled Julia set $K(F)$ is thin in each $P^l(0)$, $0 \le l \le k$. By the Koebe Distortion Theorem, $K(F)$ is also uniformly thin in all puzzle pieces $P^{m_j+k}(x)$. Since the puzzle pieces $P^{m_j+k}(x)$ shrink to x (by Lemma A.3), x is not a density point of $K(F)$. The conclusion follows from the Lebesgue Density Points Theorem. □

Corollary A.9. *If F has an escaping critical point then* meas $K(F) = 0$.

Proof. In this case there are no points which shadow the critical orbit. □

Theorem A.10. *Let F be an R-map. If* meas $K(F) > 0$*, then F is renormalizable and $K(F) \setminus \cup F^{-n} K(R(F))$ has zero Lebesgue measure.*

Proof. Assume F is non-renormalizable. Consider a point $x \in K(F)$ which shadows the critical orbit. Recall that $\operatorname{orb}_F(x)$ intersects all critical puzzle pieces.

Assume that the critical point intersect infinitely many W_j, and let $j_k \to \infty$ be a sequence such that $\operatorname{orb}_F(0) \cap W_{j_k} \neq \emptyset$. Let n_k be the first landing time of the orbit of 0 at W_{j_k}, and let m_k be the first landing time of x at $P^{n_k+1}(0)$. Notice that the maps $F^{n_k} : P^{n_k}(F(0)) \to W$ and $F^{m_k} : P^{n_k+m_k+1}(x) \to P^{n_k+1}(0)$ are univalent.

Since $\cup W_j$ is thin in W, $K(F)$ is uniformly thin in $P^{m_k}(F(0))$ by the Koebe Distortion Theorem. Pulling back by the double covering $F|P^1(0)$, we conclude that $K(F)$ is uniformly thin in $P^{m_k+1}(0)$. Pulling back again, we get that $K(F)$ is uniformly thin in $P^{n_k+m_k+1}(x)$. Since the $P^n(x)$ shrink to x (Corollary A.6), x is not a density point of $K(F)$. By the Lebesgue Density Point Theorem and Lemma A.8, $K(F)$ has zero Lebesgue measure.

Assume now that $\mathrm{orb}_F(0)$ intersects only finitely many W_j. Let us consider the R-map $F_\#$ which is the restriction of F to the W_j visited by the critical point. It follows that $\cup F^{-n}(K(F_\#))$ contain all points $x \in K(F)$ which shadow the critical orbit. By Corollary A.2, meas $K(F_\#) = 0$, and by Lemma A.8, $K(F)$ has zero Lebesgue measure.

Thus, F is renormalizable, and the last assertion follows from Lemma A.3 and Lemma A.4. □

A.5. Periodic orbits.

Lemma A.11. *Let F be an R-map and let $\overline{p}$ be a non-repelling periodic orbit. Then for all n, $\overline{p}$ intersects $P^n(0)$.*

Proof. Let k be the period of p and assume that $\overline{p}$ does not intersect $P^n(0)$. Then $F^k|P^{n+k}(p)$ is univalent onto $P^n(p)$. Since $\overline{P^{n+k}} \subset P^n(p)$, p is repelling by the Schwarz Lemma. □

Lemma A.12. *Let F be an R-map and assume F has a non-repelling periodic orbit $\overline{p}$. Then F is renormalizable, $\overline{p}$ intersects $K(R(F))$, and $\overline{p}$ is the unique non-repelling periodic orbit of F. Furthermore, if F is $\mathbb{R}$-symmetric, $\omega(0) = \overline{p}$, so $\overline{p} \subset \mathbb{R}$.*

Proof. By the previous lemma, $\overline{p}$ intersects $\cap P^n(0)$. By Corollary A.6, if F is non-renormalizable then $\cap P^n(0) = \{0\}$, so $0 \in \mathrm{orb}_F(p)$. Since 0 is periodic, F is renormalizable, contradiction. So F is renormalizable, and in this case $\cap P^n(0) = K(R(F))$, so $K(R(F))$ intersects $\overline{p}$. By [D2], a quadratic-like map has at most one non-repelling periodic orbit, and this yields uniqueness of $\overline{p}$.

If F is $\mathbb{R}$-symmetric, then $R(F)$ is $\mathbb{R}$-symmetric and the straightening map can also be chosen $\mathbb{R}$-symmetric. By the classical theory of Fatou and Julia, a non-repelling periodic orbit of a $\mathbb{R}$-symmetric quadratic polynomial must be contained in $\mathbb{R}$, and this yields the conclusion for F. □

Lemma A.13. *Let F be an R-map which has an attracting hyperbolic periodic orbit $\overline{p}$. Then $K(F) \setminus \mathrm{int}\, K(F)$ has zero Lebesgue measure and $\mathrm{int}\, K(F)$ is the basin of attraction of $\overline{p}$.*

Proof. By the previous lemma, F is renormalizable and $R(F)$ is a quadratic-like map with a unique attracting periodic orbit (the intersection of $\mathrm{orb}_F(p)$ and $K(R(F))$). It is well known that this implies that $\mathrm{int}\, K(R(F))$ is the basin of attraction of $\mathrm{orb}_F(p) \cap K(R(F))$ and $\partial K(R(F))$ has zero Lebesgue measure. By Lemma A.7, it follows that $\mathrm{int}\, K(F)$ is the basin of attraction of $\overline{p}$. By Theorem A.10, $K(F) \setminus \mathrm{int}\, K(F)$ is equal to the union of preimages of $\partial K(R(F))$ up to some set of zero Lebesgue measure, so it has zero Lebesgue measure. □

A.6. Rigidity. The following lemma is analogous to Lemma 6.1.

Lemma A.14. *Let h be a combinatorial equivalence between two R-maps F and $\tilde{F}$. Then the lift h_1 of h is a combinatorial equivalence between F and $\tilde{F}$. If h is quasiconformal then h_1 is quasiconformal and* $\mathrm{Dil}(h_1) \le \mathrm{Dil}(h)$.

Lemma A.15. *Let $h : \mathbb{C} \to \mathbb{C}$ be a qc map which is a combinatorial equivariance between two R-maps F and $\tilde{F}$. Assume* $\mathrm{int}\, K(F) = \emptyset$. *Then there exists a qc conjugacy $H : \mathbb{C} \to \mathbb{C}$ such that $H|\mathbb{C} \setminus \cup W_j = h$ and* $\mathrm{Dil}(H) \le \mathrm{Dil}(h)$.

Proof. As in the puzzle case (Lemma 6.2) we can obtain a qc map $H : \mathbb{C} \to \mathbb{C}$ as a limit of a series of lifts of h. This qc map automatically satisfies $\mathrm{Dil}(H) \le \mathrm{Dil}(h)$ and is equivariant on $\mathbb{C} \setminus K(F)$. Since $\mathrm{int}\, K(F) = \emptyset$, the conclusion follows. □

Theorem A.16. *Let F and $\tilde{F}$ be non-renormalizable R-maps which are combinatorially equivalent and let h be a combinatorial equivalence between them. Assume that $h|\mathbb{C} \setminus \cup W_j$ extends to a qc map of $\mathbb{C}$. Then there exists a hybrid conjugacy $H : \mathbb{C} \to \mathbb{C}$ such that $H|\mathbb{C} \setminus \cup W_j = h$.*

Proof. Let us assume first that the domain of F has finitely many components W_j. We let $h_0 = h$ and define h_{i+1} be the lift of h_i. Then $h_i(z)$ is eventually constant for every $z \notin K(F)$. Let H be the limit of h_i on $W \setminus \overline{K(F)}$. Then H is a qc homeomorphism and by Corollary A.2 H extends to a qc homeomorphism of $\mathbb{C}$. Since $K(F)$ has empty interior and H is equivariant on $\mathbb{C} \setminus \overline{K(F)}$, the result follows.

Assume now that there are infinitely many W_j, but $\mathrm{orb}_F(0)$ intersects only finitely many of them (in particular this covers the case of escaping critical orbit). Consider the restrictions $F_\#$ and $\tilde{F}_\#$ to the union of those puzzle pieces. Let $\hat{h} : \mathbb{C} \to \mathbb{C}$ be a qc extension of $h|\mathbb{C} \setminus \cup W_j$. For every $j \neq 0$ let ψ_j be the lift of $\hat{h}$ to W_j. Let us define Ψ as follows

$$\Psi = \begin{cases} \psi_j & \text{on } W_j, \text{ for } W_j \cap \mathrm{orb}_F(0) = \emptyset \\ h & \text{otherwise.} \end{cases}$$

Then Ψ is a combinatorial equivalence between $F_\#$ and $\tilde{F}_\#$ which is quasiconformal on the complement of the domain of $F_\#$. By the previous case, we obtain a qc conjugacy $H_\#$ between $F_\#$ and $\tilde{F}_\#$. Moreover, $H_\# : \mathbb{C} \to \mathbb{C}$ is also a combinatorial equivalence between F and $\tilde{F}$, so by Lemma A.15, it can be turned into the desired qc conjugacy.

Now let us assume that the critical orbit intersects infinitely many puzzle pieces. Since W is a quasidisk, the qc homeomorphism $h|\mathbb{C} \setminus W$ can be extended to a qc homeomorphism $\hat{h} : \mathbb{C} \to \mathbb{C}$ taking $F(0)$ to $\tilde{F}(0)$. As above, let ψ_j be the lift of $\hat{h}$ to W_j and let

$$\Psi = \begin{cases} \psi_j & \text{on } W_j \\ h & \text{otherwise.} \end{cases}$$

For $x \in W_0$, $y \in \tilde{W}_0$, let $h_{x,y}$ be a qc map coinciding with h on $\mathbb{C} \setminus W$ and taking x to y. It is clear that we may choose $h_{x,y}$ in such way that

$$\sup_{x \in W_0, y \in \tilde{W}_0} \mathrm{Dil}(h_{x,y}) < \infty.$$

There exist $j_k, n_k \to \infty$ such that n_k is the first landing moment of the critical orbit at W_{j_k}. Let m_k be the first landing moment of $F^{n_k}(0)$ at W_0. Assume first that $m_k < \infty$. Let $V_k^j = P^j(F^{n_k}(0))$. Define

$$\psi_{j,k} : V_k^j \setminus V_k^{j+1} \to \tilde{V}_k^j \setminus \tilde{V}_k^{j+1}, \quad 0 \le j \le m_k - 1,$$

as the lift of Ψ by the pair $(F^j, \tilde{F}^j)$. Let $\phi_k : V_k^{m_k} \to \tilde{V}_k^{m_k}$ be the lift of $h_{F^{m_k}(0), \tilde{F}^{m_k}(0)}$ by the pair $(F^{m_k}, \tilde{F}^{m_k})$. Define Ψ_k as follows:

$$\Psi_k = \begin{cases} \psi_{j,k} & \text{on } V_k^j \setminus V_k^{j+1}, \quad 0 \le j \le m_k - 1 \\ \phi_k & \text{on } V_k^{m_k} \\ h & \text{on } \mathbb{C} \setminus W. \end{cases}$$

Then the Ψ_k are qc maps such that $\Psi_k(F^{n_k}(0)) = \tilde{F}^{n_k}(0)$ and $\sup \mathrm{Dil}(\Psi_k) < \infty$.

If $m_k = \infty$, define

$$\Psi_k = \begin{cases} \psi_{j,k} & \text{on } V_k^j \setminus V_k^{j+1}, \quad j \ge 0 \\ \tilde{F}^{n_k}(0) & \text{at } F^{n_k}(0) \\ h & \text{on } \mathbb{C} \setminus W. \end{cases}$$

Notice that since $\cap_j V_k^j = \{F^{n_k}(0)\}$, this map has the same properties as before.

Let now $F_k = F | \cup_{j \neq j_k} W_j$ and $\tilde{F}_k = \tilde{F} | \cup_{j \neq j_k} W_j$. Modify Ψ_k on finitely many domains W_j which contain points $f^l(0)$, $l = 0, ..., n_k - 1$, so that it becomes a qc combinatorial equivalence Φ_k between F_k and $\tilde{F}_k$. We are now in the escaping case considered above, so that we can turn Φ_k into a qc conjugacy H_k between F_k and $\tilde{F}_k$. Moreover,

$$\mathrm{Dil}(H_k) \le \mathrm{Dil}(\Phi_k | \mathbb{C} \setminus \cup W_j) \le \mathrm{Dil}(\Psi_k),$$

since $K(F_k)$ has zero Lebesgue measure. Take some limit H of the H_k. This is the desired qc conjugacy.

Since meas $K(F) = 0$ (see Theorem A.10), H is automatically a hybrid conjugacy. □

Remark A.1. Note that the above theorem leads to a simple proof of the Yoccoz Rigidity Theorem for non-renormalizable quadratic polynomials (see [H]). Indeed such a quadratic map can be renormalized in a generalized sense to an R-map (see [L4], §3). The conjugacy between the corresponding R-maps lifts to a conjugacy between the quadratic polynomials by means of the pullback argument.

The following lemma is a slight modification of well known results, see [DH1].

Lemma A.17. *Let $F : U \to U'$ and $\tilde{F} : \tilde{U} \to \tilde{U}'$ be quadratic-like maps with connected Julia set which are qc conjugate and let ψ be a qc conjugacy on a neighborhood of $K(F)$. Let $h : \mathbb{C} \to \mathbb{C}$ be a qc map equivariant on ∂U. Then there exists a qc homeomorphism $H : \mathbb{C} \to \mathbb{C}$ such that $H|\mathbb{C} \setminus U = h$ and $H|K(F) = \psi$.*

Proof. Let $\Phi : \mathbb{C}\setminus K(F) \to \mathbb{C}\setminus K(\tilde{F})$ be obtained as a limit of a series of lifts of h by the pair of unbranched double coverings $F : U \setminus K(F) \to U' \setminus K(F)$ and $\tilde{F} : \tilde{U} \setminus K(\tilde{F}) \to \tilde{U}' \setminus K(\tilde{F})$. Then Φ is quasiconformal and equivariant on $U \setminus K(F)$.

Let $V' = F^{-n}(U)$ be a small neighborhood of $K(F)$ which is contained in the domain of ψ and let $V = F^{-1}(V')$. Let $A = \overline{V}' \setminus V$. Consider a homotopy $\Phi_t : A \to \tilde{U} \setminus K(\tilde{F})$ between $\Phi|A$ and ψ which is equivariant on ∂V (the existence of such a homotopy follows from the fact that F has degree 2).

By means of successive lifts, we obtain a homotopy $\Phi_t : \overline{V}' \setminus K(F) \to \tilde{U} \setminus K(\tilde{F})$, between $\Phi|\overline{V}' \setminus K(F)$ and ψ, equivariant on $\overline{V} \setminus K(F)$.

Let us supply $\tilde{U} \setminus K(\tilde{F})$ with the hyperbolic metric. Given a point $x \in V' \setminus K(F)$, let $c(x)$ be the length of the hyperbolic geodesic joining $\Phi_0(x) = \Phi(x)$ and $\Phi_1(x) = \psi(x)$, homotopic to the path $t \to \Phi_t(x)$. By compactness, $c(x)$ is bounded on $V' \setminus V$.

Let $x \in V$ and let $F^m(x)$ be the first landing time of x on $V' \setminus V$. By the Schwarz Lemma, $c(x) \leq c(F^n(x))$, so $c(x)$ is uniformly bounded in $V' \setminus K(F)$ as well. In particular, if x is close to $K(F)$, $|\psi(x) - \Phi(x)|$ is close to 0.

Let

$$H = \begin{cases} \Phi & \text{on } \mathbb{C} \setminus K(F) \\ \psi & \text{on } K(F). \end{cases}$$

By the previous estimate, H is continuous, so by the Gluing Lemma (see [B], Lemma 2, p. 93), H is quasiconformal. It is also clearly a conjugacy. □

Lemma A.18. *Consider two renormalizable R-maps F and $\tilde{F}$. Assume that they are combinatorially equivalent and let h be a combinatorial equivalence between them such that $h|\mathbb{C} \setminus W^1$ admits a qc extension to $\mathbb{C}$. Assume $R(F)$ and $\tilde{R}(F)$ are qc (resp. hybrid) conjugate. Then there exists a qc (resp. hybrid) conjugacy H between F and $\tilde{F}$ such that $H|\mathbb{C} \setminus W^1 = h$.*

Proof. Let $R(F) = F^p|P^n(0)$. Let ψ be the $(n+p)$-fold lift of h. Then ψ is equivariant on $W^1 \setminus W^{n+p}$. Let $V_p = P^n(0)$, $V_0 = P^{n+p}(0)$. Let $\psi_p : V_p \to \tilde{V}_p$ be a qc conjugacy between $R(F)$ and $R(\tilde{F})$, homotopic to ψ rel $\mathbb{C} \setminus V_0$ (which can be obtained using Lemma A.17).

Let $V_j = P^{n+p-j}(F^j(0))$, $1 \le j < p$. Let $\psi_j : V_j \to \tilde{V}_j$ be the lift of ψ_p by $(F^{p-j}, \tilde{F}^{p-j})$. Then ψ_0 coincides with $\psi_p|V_0$. Let

$$\Psi = \begin{cases} \psi_j & \text{on } V_j,\ 1 \le j \le p \\ \psi & \text{on } \mathbb{C} \setminus \cup V_j. \end{cases}$$

Let us show that Ψ is a combinatorial equivalence between F and $\tilde{F}$. Let $\Psi_1 : \mathbb{C} \to \mathbb{C}$ be the lift of Ψ by $(F, \tilde{F})$. Then Ψ and Ψ_1 coincide on $\mathbb{C} \setminus W^{n+p}$ by equivariance, as well as on V_j, $0 \le j < p$ by construction. The set $W^{n+p} \setminus \cup_{j=0}^{p-1} V_j$ is a countable union of Jordan disks with shrinking diameters (by the thin condition), so Ψ is homotopic to Ψ_1 rel $\mathbb{C} \setminus (W^{n+p} \setminus \cup_{j=0}^{p-1} V_j)$. Since $\mathrm{orb}_F(0) \subset \cup_{j=0}^{p-1} V_j$, Ψ is a combinatorial equivalence between F and $\tilde{F}$.

Let ϕ be a qc map homotopic to Ψ rel $(\mathbb{C} \setminus W^1) \cup \cup_{j=1}^{p} V_j$. Then ϕ is a qc combinatorial equivalence between F and $\tilde{F}$. We can obtain a qc conjugacy H as a limit of a series of lifts as before. Notice that $H|K(R(F)) = \psi_0$, so by Theorem A.10, $\mathrm{Dil}(H|K(F)) = \mathrm{Dil}(\psi_0|K(R(F))$, and if ψ_0 is hybrid then so is H. □

A.7. Non-critical *R*-maps. Let W_j and W be as in the definition of R-maps except that $0 \notin \cup W_j$. A holomorphic map $F : \cup W_j \to W$ will be called a *non-critical R-map* if $F|W_j$ is univalent onto W for all j. The definition of puzzle pieces and filled Julia set goes as before. By a trivialization of the arguments for R-maps we obtain:

Lemma A.19. *Let F be a non-critical R-map. Then* $\mathrm{meas}\, K(F) = 0$ *and for every $x \in K(F)$, $\cap P^n(x) = \{x\}$. Given another non-critical R-map $\tilde{F}$ and a qc map $h : \mathbb{C} \setminus \cup W_j \to \mathbb{C}$ equivariant on $\cup \partial W_j$ there exists a qc conjugacy H such that $H|\mathbb{C} \setminus \cup W_j = h$.*

A.8. Application to puzzle maps. Let $f : U \to \mathbb{C}$ be a puzzle map and let $Q(f)$ be the first return map to U_0. Then its domain consist of connected components of $(f|U_0)^{-1}(U)$ and $Q(f)$ is either an R-map (if $f(0) \in U$) or a non-critical R-map (otherwise). We will now use the results on complex return maps (applied to $Q(f)$) to understand f.

It is easy to see that $K(f) = \cup (f^{n_i}|U_i)^{-1}(K(Q(f))$. As a consequence, the following three lemmas can be deduced for puzzle maps f from the corresponding statements for $Q(f)$ (respectively Lemmas A.12, A.7 and A.13).

Lemma A.20. *Let f be a puzzle map and assume it has a non-repelling periodic orbit. Then this non-repelling periodic orbit is unique and $Q(f)$ is renormalizable. If f is $\mathbb{R}$-symmetric then such an orbit is necessarily real.*

Lemma A.21. *Let f be a puzzle map. If* $\mathrm{int}\, K(f) \ne \emptyset$ *then f has a non-repelling periodic orbit.*

Lemma A.22. *Let f be a hyperbolic puzzle map. Then $K(f) \setminus \operatorname{int} K(f)$ has zero Lebesgue measure and* $\operatorname{int} K(f)$ *coincides with the basin of attraction of the attracting cycle.*

The next lemmas extend well-known properties of quadratic-like maps.

Lemma A.23. *Let f be a puzzle map whose critical point either escapes or is preperiodic but not periodic. Then* $\operatorname{meas} K(f) = 0$ *and all periodic orbits are repelling.*

Proof. Assume that $\operatorname{meas} K(f) > 0$ or that f has a non-repelling periodic orbit. By Lemmas A.12 and A.10, $Q(f)$ is renormalizable and $\operatorname{meas} K(R(Q(f))) > 0$ or $R(Q(f))$ has a non-repelling periodic orbit. Since $Q(f)$ is renormalizable, the critical point of f is non-escaping.

It is well known that if F is a quadratic-like map which has a preperiodic but not periodic critical point then $\operatorname{meas} K(F) = 0$ and all periodic orbits are repelling. In particular, $R(Q(f))$ (and hence f) cannot have a preperiodic but not periodic critical point. □

Lemma A.24. *Let f be an $\mathbb{R}$-symmetric puzzle map. Then either f has an attracting or parabolic periodic orbit or $K(f)$ has no invariant line fields.*

Proof. Since $K(f)$ is the union of preimages of $K(Q(f))$, if $K(f)$ has an invariant line field, then so does $K(Q(f))$.

In particular $Q(f)$ is renormalizable and $R(Q(f))$ is a quadratic-like map with invariant line-field. Since f is real, the result follows from Theorem 2.22. □

Lemma A.25. *Let f, g be $\mathbb{R}$-symmetric puzzle maps in the same hybrid class and h be an $\mathbb{R}$-symmetric qc homeomorphism equivariant on ∂U^f. Then there is an $\mathbb{R}$-symmetric conjugacy H between f and g on U^f coinciding with h on $\mathbb{C} \setminus U^f$ and such that $\overline{\partial} H = 0$ in $K(f)$. In particular,* $\operatorname{Dil}(H) \le \operatorname{Dil}(h)$.

Proof. Given h we first redefine it inside $\cup U_i^f$ so that it coincides with the topological conjugacy between f and $\tilde{f}$ on $\cup J_i^f$. The resulting map can still be required to be a homeomorphism (we use that h is real and $\operatorname{diam}(J_i^f) \to 0$).

We can now lift this map to U_0^f to obtain a homeomorphism Ψ. It turns out that Ψ is a combinatorial equivalence between $Q(f)$ and $Q(\tilde{f})$. Indeed, Ψ coincides with its lift Ψ_1 outside $(f|U_0)^{-1}(U) \setminus \mathbb{R}$. This set is a union of Jordan disks with diameter going to 0, so Ψ is homotopic to Ψ_1 rel its boundary.

If $Q(f)$ is renormalizable, then $R(Q(f))$ is hybrid conjugate to $R(Q(\tilde{f}))$ by Theorem 2.21. By Lemma A.18 and Theorem A.16, there exists a hybrid conjugacy between $Q(f)$ and $Q(\tilde{f})$ coinciding with h on $\partial U_0^f \setminus (f|U_0^f)^{-1}(U^f)$. Let ψ_j be the lift of this conjugacy by $(f^{n_j}|U_j^f, \tilde{f}^{n_j}|U_j^{\tilde{f}})$.

We let H be defined as

$$H = \begin{cases} \psi_j & \text{on } U_j^f \\ h & \text{on } \mathbb{C} \setminus U^f. \end{cases}$$

□

Lemma A.26. *Let $f : U \to \mathbb{C}$ and $\tilde{f} : \tilde{U} \to \mathbb{C}$ be qc conjugate hyperbolic puzzle maps with the same multiplier and let $h : \mathbb{C} \to \mathbb{C}$ be a qc conjugacy between them. Then there exists a qc conjugacy $H : \mathbb{C} \to \mathbb{C}$, such that $H|\mathbb{C} \setminus K(f) = h$ and $\overline{\partial} h|K(f) = 0$.*

Proof. In this case, $Q(f)$ is renormalizable, and h is a qc conjugacy between $Q(f)$ and $Q(\tilde{f})$. Since $R(Q(f))$ is qc conjugate to $R(Q(\tilde{f}))$ and both are hyperbolic with the same multiplier, it follows that they are hybrid equivalent. By Lemma A.18, there is a hybrid equivalence H between $Q(f)$ and $Q(\tilde{f})$ coinciding with h outside $K(Q(f))$, and this equivalence can be lifted to a hybrid equivalence between f and $\tilde{f}$ with the desired properties. □

Appendix B. Quasiconformal conjugacies for Yoccoz maps

The main aim of this appendix is to prove the following theorem:

Theorem B.1. *Let $f, \tilde{f} \in \mathcal{U}_a$ topologically conjugate Yoccoz maps. Then there is a qc map $h : \mathbb{C} \to \mathbb{C}$, symmetric with respect to the real line and* 0 *which is equivariant with respect to f and $\tilde{f}$ in a neighborhood of I.*

This is stronger than the statement that f and $\tilde{f}$ are qs conjugate on I. Difficulties arise essentially because of the lack of a nice external structure (if f and $\tilde{f}$ have quadratic-like extension, both statements are equivalent using Sullivan's pullback argument). To compensate for this, we will provide some geometrical external structure, constructed by hand. This particular construction is based on existence of small scaling factors in the principal nest, and does not immediately adapt to the case of, say, infinitely renormalizable maps of bounded type.

To simplify the exposition we will assume that the Yoccoz maps we are dealing with are in fact non-renormalizable and have a recurrent critical point. The finitely renormalizable case is analogous and the proof applies with obvious modifications to the Misiurewicz case.

B.1. Compatible external structures. We have so far concentrated mostly on the description of puzzle structures, which is suitable to analyze the set of points which eventually land in a given nice interval. We have also in Lemma 7.10 used a Markov structure to treat the points which never land on the nice interval. We will here describe how both constructions can be made compatible in the presence of good geometric parameters.

Let p be the fixed point of f. Fix a sufficiently deep stage in the principal nest in a way that $|T_n|/|T_{n-1}|$ is very small, and let $J_0 = T_n = [-q, q]$.

Let Q be the set of points in I that never land on J_0 and Q' be a truncation of Q, so that $Q' = Q \cap [-1, r]$, where r is the right endpoint of J_1. So Q' is a hyperbolic Cantor set and we may use Lemma 2.12 to obtain a real analytic conformal metric ν which is expanded by f. We assume ν to be symmetric with respect to the real line and to 0. In what follows we will only measure distances with respect to the metric ν.

Let $Q_m = Q' \cap \cup_{j=0}^{m} f^{-j}(p) \cup \{-1\}$, so Q_m is a finite forward invariant set contained in Q.

Lemma B.2. *Fix a small $\varepsilon > 0$. For all sufficiently small h, there exists a family of curves (called υ-segments) $\Gamma(x)$, $x \in Q$, with the following properties:*

(i) $\Gamma(x)$ *is an $\mathbb{R}$-symmetric continuous path;*
(ii) $\Gamma(x)\backslash x$ *is a C^∞ curve whose tangent is ε-close to the vertical direction;*
(iii) $\operatorname{dist}_\nu(\partial\Gamma(x), x) = h$*;*
(iv) *overflowing:* $\Gamma(f(x)) \subset f(\Gamma(x))$.

Proof. Let us consider a complex strictly Markov covering $\{V_j\}$ of Q, which is $\mathbb{R}$-symmetric and 0-symmetric. Let $V^k = \{x \mid f^j(x) \in V^0,\ 0 \le j \le k\}$, $V^\infty = \cap V^k$ (note that V^∞ contains Q but does not necessarily coincide with it).

Let us define $g = g_k : V^k \to \mathbb{C}$ which is $\mathbb{R}$-symmetric, 0-symmetric, and for each connected component V_j^k, there exists $z \in V_j^k$ such that $g(w) = f(z) + f'(z)(w - z)$, $w \in V_j^k$. It is easy to see that $g_k : V^k \to \mathbb{C}$ is a strictly Markov map provided k is big enough. The map g is a linear model for f. This lemma is obvious for g, since we can take vertical segments for υ-segments.

Let $H : \mathbb{C} \to \mathbb{C}$ be a smooth diffeomorphism, $\mathbb{R}$-symmetric, 0-symmetric, which coincides with the identity near ∂V^k and is equivariant near ∂V^{k+1}. We may select H arbitrarily close to id choosing k sufficiently big. By Proposition 2.11, we can turn H into a qc map h equivariant on V^{k+1} and coinciding with H on $\mathbb{C} \setminus V^{k+1}$.

Notice that $\Im(Df(x))/|Df(x)| = O(\Im(x))$. Since $f|V^0$ is expanding,

$$\text{(B.1)} \qquad \frac{|\Im(Dh(x))|}{|Dh(x)|} < \varepsilon/2, \quad x \in \mathbb{C} \setminus V^\infty,$$

provided k is sufficiently big. For each $x \in Q$, consider the arc $\hat{\Gamma}(x)$, the preimage by h of the vertical line through $h(x)$. Let $\Gamma(x)$ be the truncation of $\hat{\Gamma}(x)$ at height h. Properties (i) and (iii) of this family of curves are obvious. Property (ii) follows from B.1. The overflowing property (iv) follows from the corresponding property of the linear model and the expanding property of the metric ν. $\square$

Let us now describe the central puzzle piece U_0 of the external puzzle structure. It is an $\mathbb{R}$-symmetric hexagon whose real trace is J_0. Its boundary consists of the arcs $\Gamma(-q)$, $\Gamma(q)$ (the *wall* of U_0) and four straight segments (the *roof* of U_0) constructed as follows. Let $\Gamma(-q)$ have as upper endpoint $(-x, y)$. Then one of the straight segments joins $(-x, y)$ to $(0, x + y)$ and the others are obtained by symmetries with respect to the real line and 0. Let Λ be the roof of U_0.

Since $f|Q'$ expands υ, there is a neighborhood V of q and a constant $C' > 1$ such that $f|V$ expands distances by C'. By construction, $\mathrm{dist}(f(\Lambda) \setminus f(V), [-1, r])$ is uniformly bounded from below, independent of how small h is chosen. On the other hand, $\mathrm{dist}(\Lambda, I) = h(1 + o(1))$ (since υ is conformal and continuous). So we conclude that there exists $1 < C < C'$ such that

$$\mathrm{dist}(f(\Lambda), [-1, r]) > Ch \tag{B.2}$$

for h small enough.

By Remark 5.5, U_0 generates a complex puzzle $\mathcal{P}\{U_j\}$. The boundary of the of the puzzle pieces U_j can be described in terms of walls and roofs which are taken by f^{n_j} onto the wall and the roof of U_0. By the overflowing property of Lemma B.2, the walls of U_j consist of truncated υ-segments.

Let us now construct an external Markov structure. Let $\{M_j^{\mathbb{R}}\}$ be the family of those components of $[-1, r] \setminus Q_m$ which are different from the puzzle pieces J_j. We want m to be so big that the diameter of any $M_j^{\mathbb{R}}$ is smaller than $h/10$.

The complexification of each $M_j^{\mathbb{R}} = [a, b]$ is obtained by taking four arcs, $\Gamma(a)$, $\Gamma(b)$ and straight almost horizontal segments linking the upper endpoints as well as the lower endpoints. The closed cell bounded by those arcs will be denoted M_j.

Notice that if m is big enough, then $\{-q, q\} \subset Q_m$. Furthermore, whenever the boundary of some J_m^j intersects Q_m, its boundary is contained in Q_m. By the combinatorics of the construction, given $M_k^{\mathbb{R}}$ and J_j, there are two possibilities for intersection: either J_j is compactly contained in $\mathrm{int}\, M_k^{\mathbb{R}}$, or their interiors do not intersect. Similarly, given M_k and U_j, either U_j is compactly contained in $\mathrm{int}\, M_k$, or their interiors are disjoint. Furthermore, if their boundaries do intersect, the intersection is one of the walls of U_j. This follows from the property that no puzzle piece U_j has the roof crossing the boundary of some Markov piece M_k (they can only touch at an endpoint of the roof).

To see this, notice that the roof of U_j cannot cross laterals of Markov pieces by geometric considerations (using that the angle of a U-segment is nearly $\pi/4$ and a Γ-segment is nearly vertical). As a consequence, the roof of U_j cannot intersect ∂M_k if J_j is not contained in $M_k^{\mathbb{R}}$. Furthermore, the base of M_k is at least ten times smaller then its height, while the height of U_j is almost half the size of its base since it is mapped in U_0 with small

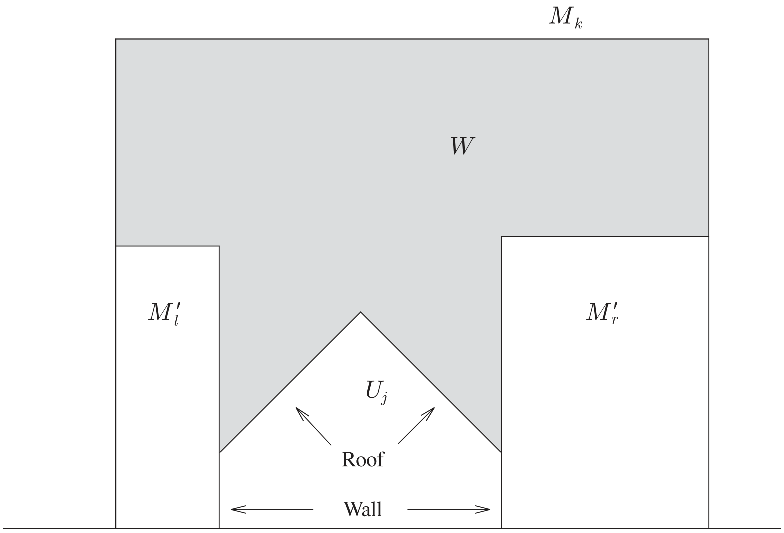

Fig. 7. Inside a Markov piece

distortion (depending on $|T_n|/|T_{n-1}|$), assuring that its roof do not intersect the top of M_k if $J_j \subset M_k$.

Observe also that all Markov pieces are in a neighborhood of $[-1, r]$ of size near $h(1 + o(1))$ (by choosing m big), so they do not intersect $f(\Lambda)$ by (B.2).

We construct the family $\{M'_j\}$ as the set of all $(f|M_k)^{-1}(M_l)$ for pieces M_k and M_l such that $M_l \cap \operatorname{int} f(M_k) \neq \emptyset$. We let $M = \cup \operatorname{int} M_j$, $M' = \cup \operatorname{int} M'_j$, and $U = \cup \operatorname{int} U_j$.

Let $W = M \backslash \overline{M' \cup U}$. This is the union of Jordan domains with piecewise smooth boundary. The maximal smooth segments contained in the boundary of those Jordan domains will be called W-segments. The boundaries of the Jordan domains can intersect, but only along a W-segment. A W-segment is either contained in ∂M, $\partial M'$ or is the roof of some U_j.

Figure 7 shows the interior of a piece M_k which contains two pieces M'_l and M'_r, with a unique U_j whose interior is contained in $M_k \setminus (M'_l \cup M'_r)$. The component of W inside M_k (shown shaded) is bounded by segments of ∂M_k, segments of $\partial M'_l$ and $\partial M'_r$, and U-segments of ∂U_j. The picture is skewed to show better the details, actual Markov pieces are much narrower (their bases being at least ten times smaller then the heights).

Notice in conclusion that if $x \in \overline{M} \setminus U$, then either $f^n(x) \in \overline{W}$ for some n, or $x \in Q'$. For this reason, $\overline{W}$ will work as a fundamental domain for the map $f : M' \setminus U \to M \setminus U$.

B.2. Construction of the conjugacy. Let us now consider a map $\tilde{f}$ topologically conjugate to f. Then it is possible to make the above construction simultaneously, the only care we have to take is choose n and m the same for both maps. Each object for f (say, a given segment in $\partial M \setminus \partial M'$) has a corresponding object for $\tilde{f}$, said to have "the same combinatorics". We will mark the corresponding objects for $\tilde{f}$ with a $\sim$ e.g., $\tilde{U}$, $\tilde{M}$, $\tilde{\Lambda}$.

The construction has essentially two parts: first we obtain a qc map relating the fundamental domains W and $\tilde{W}$ (and some additional structures), and then use the Macroscopic pullback argument to obtain a conjugacy on a neighborhood of the interval.

Let us first define an $\mathbb{R}$-symmetric homeomorphism h_0 on some relevant parts of the boundary of the puzzle and Markov pieces. The construction below will be $\mathbb{R}$-symmetric and also 0-symmetric "where defined" (that is, if x and $-x$ belong to the domain of h_0 then $-h_0(-x) = h_0(x)$).

Let us start the construction with the ray $\Gamma(p)$ through the fixed point p. It is well-known that conformal maps are locally qc conjugate near repelling fixed points in such a way that the conjugacy is smooth outside the fixed points themselves. In particular, f near p is qc conjugate to $\tilde{f}$ near $\tilde{p}$ so that the conjugacy is smooth in the punctured neighborhoods. Moreover, it is easy to select a conjugacy so that it maps $\Gamma(p)$ onto $\tilde{\Gamma}(\tilde{p})$. Similarly we can construct a local qc conjugacy near -1 carrying $\Gamma(-1)$ to $\Gamma(-1)$. Let h_0 stand for the restriction of the above local conjugacies to $\Gamma(p) \cup \Gamma(-1)$.

Recall that points of $Q_{m+1} \setminus \{-1\}$ are preimages of p. By the overflowing property of the family of rays $\Gamma(x)$, it is easy to extend h_0 to a homeomorphism

$$h_0 : \Delta \equiv \bigcup_{x \in Q_{m+1}} \Gamma(x) \to \bigcup_{\tilde{x} \in \tilde{Q}_{m+1}} \tilde{\Gamma}(\tilde{x})$$

which admits a qc extension to a neighborhood of Δ, is piecewise smooth on $\Delta \setminus \mathbb{R}$, and is equivariant on $\Delta \cap \overline{M}'$.

Thus, the map h_0 is defined on the laterals of M and M' (which are the arcs of Δ) and is equivariant on the laterals of M'. Extend it to the tops of M and M' so that it is equivariant on $\partial M'$. Then extend it to the roof of U_0 and to $f(\partial U_0) \setminus M$ (this set is the image of the roof of U_0 and the part of its wall that is not contained in M'), so that h_0 is now equivariant also on ∂U_0.

Notice that there are only a finite number of roofs which are not contained in M'. By a finite number of lifts, extend h_0 to those roofs. This concludes the definition of h_0. It is defined on the set Z_1 consisting of $f(\partial U_0)$, ∂M, $\partial M'$ and a finite number of roofs not contained in M'. Moreover, this map is equivariant on the set Z_2 consisting of ∂U_0, $\partial M'$ and the same union of roofs.

Notice that each embedded arc on Z_1 is a quasiarc since it consists of smooth arcs meeting at a positive angle. Hence the notion of a quasisymmetric map on Z_1 makes an obvious sense. The map h_0 being piecewise smooth is quasisymmetric on Z_1. The set $\overline{\mathbb{C}} \setminus Z_1$ is a finite union of quasidisks, so by Ahlfors-Beurling criteria (see [LV]), h_0 has a qc extension to $\mathbb{C}$, denoted by h_1, which can be required to be also $\mathbb{R}$-symmetric.

Since h_1 is equivariant on $\partial M'$, we can apply the simple version of the Macroscopic pullback argument (see Remark 2.5) to $f : M' \to M$. It provides us with a qc map (still denoted by h_1) which is equivariant on M'. In the course of this pullback procedure h_1 is not modified on Z_2. Hence it is equivariant on $\partial U \subset M' \cup Z_2$.

By Lemma A.25, there is a qc map h_2 which is equivariant on U and coincides with h_1 on $\mathbb{C} \setminus U$. Since this does not modify h_1 on $M \setminus U$, h_2 is also equivariant on $M' \setminus U$. Thus, h_2 is equivariant on $M' \cup U$. On the left, this set is bounded by the ray $\Gamma(-1)$, so it fails to be a neighborhood of the repelling fixed point -1. It is easy to modify h_2 in the complement of $M \cup U$ in order to make it equivariant on a small neighborhood V of -1 as well.

The last problem is that the set $V \cup M \cup U$ is a neighborhood of $[-1, f(0)]$ rather than of $[-1, 1]$. By lifting h_2 to the neighborhood $f^{-1}(V \cup M \cup U)$ of $[-1, 1]$, we obtain the desired map h. This concludes the proof of Theorem B.1.

B.3. Quasisymmetric conjugacies for real analytic quasiquadratic maps. We will now give a proof of Theorem 2.23. Obviously Theorem B.1 covers the case of Yoccoz maps, so we will concentrate on the remaining cases.

B.3.1. Infinitely renormalizable case. Let h be a topological conjugacy between f and $\tilde{f}$. If T is a deep enough renormalization interval (of period m), Theorem 2.24 implies that $f^m : T \to T$ and $\tilde{f}^m : h(T) \to h(T)$ extend to quadratic-like maps, and Theorem 2.21 can be applied so that $h|T$ is quasisymmetric.

Let us say that a triple (l, m, r) is κ-commensurable if

$$\kappa^{-1} \leq \frac{r - m}{m - l} \leq \kappa.$$

In this notation, h is κ-qs if and only if it takes a 1-commensurable triple to a κ-commensurable triple.

Fix a 1-commensurable triple (l, m, r), that is, m is the midpoint of the interval $J = [l, r] \subset I$. Let T' be a symmetric interval slightly smaller than T, and let $\delta = (|T| - |T'|)/2$. Let $k \geq 0$ be maximal with $f^j(J) \cap T' = \emptyset$ for $j < k$. It follows from the result of Guckenheimer [G] and Mañé [M] that $f^k|J$ and $\tilde{f}^k|h(J)$ have distortion bounded by some constant $\ln C$ independent of J. In particular $(f^k(l), f^k(m), f^k(r))$ is a C-commensurable triple. Two possibilities arise:

(1) $|f^k(J)| \geq \delta$. In this case both intervals $[f^k(l), f^k(m)]$ and $[f^k(m), f^k(r)]$ have length at least $\delta/2C$. Since h is a homeomorphism, there exists $\varepsilon > 0$ such that the image by h of any interval of length $\delta/2C$ has length at least ε. It follows that $(h(f^k(l)), h(f^k(m)), h(f^k(r)))$ is a $2\varepsilon^{-1}$-commensurable triple, and hence $(h(l), h(m), h(r))$ is a $2C\varepsilon^{-1}$-commensurable triple.

(2) $|f^k(J)| < \delta$. In this case, $f^k(J) \subset T$. Since $h|T$ is quasisymmetric, there exists C' (depending only on C, and thus independent of J) such that C-commensurable triples in T are taken by $h|T$ to C'-commensurable triples. Thus $(h(f^k(l)), h(f^k(m)), h(f^k(r)))$ is a C'-commensurable triple, and hence $(h(l), h(m), h(r))$ is CC'-commensurable.

This shows that h is κ-qs with $\kappa = \max\{2C\varepsilon^{-1}, CC'\}$.

B.3.2. Attracting case. If f and $\tilde{f}$ are in the same hyperbolic hybrid class then (using the linearizing coordinate) one obtains a topological conjugacy $h : I \to I$ which is *real analytic* on a symmetric interval T contained in the basin of attraction of the attracting periodic orbit p (here and in the parabolic case, p will denote the non-repelling periodic point which is *closest to* 0) and such that $[-p, p] \subset \operatorname{int} T$. The same argument for the infinitely renormalizable case can now be applied (where one should select a symmetric interval T' slightly smaller than T but bigger than $[-p, p]$).

B.3.3. Parabolic case. Let us first recall the standard fact concerning the asymptotics of the orbits converging to parabolic points.

Lemma B.3. *Let $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a real analytic germ near a non-degenerate*[7] *parabolic fixed point* 0. *Fix $x_0 \neq 0$ such that $x_n = g^n(x_0) \to 0$.*

(1) *If $Dg(0) = 1$ then $|x_n - x_{n+1}| \sim \eta n^{-2}$ for some constant $\eta > 0$.*
(2) *If $Dg(0) = -1$ then $|x_n - x_{n+2}| \sim \eta n^{-3/2}$ for some constant $\eta > 0$.*

If f has a parabolic point p (of period q) with multiplier -1, it is still true (as in the attracting case) that there exists a symmetric interval T with $[-p, p] \subset \operatorname{int} T$, and $f^q(T) \subset \operatorname{int} T$. This implies that we can choose a topological conjugacy which is smooth in $T \setminus \{-p, p\}$. It follows from Lemma B.3 that h is bi-Lipschitz near p. The dynamics outside T is uniformly expanding (by the result of Guckenheimer and Mañé mentioned above), so the same argument as before shows that h is quasisymmetric.

The case where f has a parabolic point p (of period q) with multiplier 1 is more delicate. For $s > q$ big, let $\hat{P}_s$ be the set of all $x \in I$ such that $f^j(x) = p$ for some $j \leq s$. Let $\{\hat{M}^s_j\}_j$ be the collection of connected components of $I \setminus \hat{P}_s$ which are not contained in the basin of attraction of p. There are exactly q elements of $\{\hat{M}^s_j\}_j$ whose closure intersects $\operatorname{orb}_f(p)$, which we will label $\hat{M}^s_1,\ldots,\hat{M}^s_q$. Since $f(\hat{P}_s) \subset \hat{P}_s$, the collection $\{\hat{M}^s_j\}_j$ is a Markov partition. However, the action of f on this Markov partition is not uniformly expanding.

Let P_s be the union of $\hat{P}_s$ with the set of all $x \in I$ such that $f^k(x) \in \hat{P}_s$ for some $k \geq 1$, and $f^j(x) \in \cup_{j=1}^q \hat{M}^s_j$ for $j < k$. Notice that $P_s \setminus \hat{P}_s$ is

[7] In the case of multiplier 1 it means that $D^2 g(0) \neq 0$. In the case of multiplier -1, it means that $D^3 g^2(0) \neq 0$. Parabolic points of quasiquadratic maps are always non-degenerate.

an infinite set which accumulates on $\mathrm{orb}_f(p)$. Let $\{M_j^s\}_j$ be the collection of connected components of $\cup_j \hat{M}_j^s \setminus P_s$. Since $f(P_s) \subset P_s$, the collection $\{M_j^s\}_j$ is an "infinite Markov partition". Let us show that this infinite Markov partition is "uniformly expanding" in some sense.

Lemma B.4. *If s is sufficiently big, then for any $N > 0$, and for any $\delta > 0$ sufficiently small (depending on s and N) the following holds. Let $J \subset I$ be any interval and let $k \geq 0$ be the smallest number such that either*

(1) *$f^k(J) \cap [-p, p] \neq \emptyset$, or*
(2) *$f^k(J)$ contains N different intervals of the collection $\{M_j^s\}_j$, or*
(3) *$|f^k(J)| \geq \delta$.*

Then the distortion of $f^k|J$ is at most $\ln 2$.

Proof. Let $\varepsilon(s) = \sum_j |M_j^s|$. By [M], Theorem 4, the complement of the basin of attraction of p has Lebesgue measure zero. This implies that $\lim_{s\to\infty} \varepsilon(s) = 0$.

Let $\phi \equiv f^n : L \to M_j^s$ be any branch of the first return map to some M_j^s. It follows from the Markov property that ϕ is surjective, so it has a fixed point r in the closure of its domain. Notice that $\mathrm{orb}_f(p) \cap \overline{M_j^s} = \emptyset$, so r is not parabolic and by [MMS], Theorem B, $|D\phi(p)| \geq e^{\rho}$, where $\rho > 0$ only depends on f.

Let us choose s so big that $4C\varepsilon(s) \leq \rho$ where

$$C = \sup_{x\in I\setminus[-p,p]} \ln \frac{|D^2 f(x)|}{|Df(x)|}.$$

It is easy to see that the intervals $f^i(L)$, $0 \leq i \leq n-1$ are disjoint and contained in $\cup_j \hat{M}_j^s$. In particular, the distortion of ϕ can be bounded by

$$C \sum_{i=0}^{n-1} |f^i(L)| \leq C\varepsilon(s) \leq \frac{\rho}{4},$$

so $|D\phi| \geq e^{\rho/2}$ on its entire domain.

Let now J and k be as in the statement of the lemma. Let $k_0 \leq k$ be maximal such that for $i \leq k_0$ we have $f^i(J) \subset \overline{\cup_j \hat{M}_j^s}$ (it follows that $k - k_0 \leq s$). Using that $f^{k_0-1}(J)$ is contained in the union of $N+1$ intervals $\overline{M_j^s}$, we see that

$$\sum_{i=0}^{k_0-1} |f^i(J)| \leq \frac{(N+1)}{1-e^{-\rho/2}} \sum_j \min\left\{\left|\hat{M}_j^s\right|, \delta\right\} \equiv \zeta(\delta, N, s).$$

It is clear that with N and s fixed we have $\lim_{\delta\to 0} \zeta(\delta, N, s) = 0$. Choose $\delta > 0$ so small that $C(\zeta(\delta, N, s) + \delta s) < \ln 2$. Then the distortion of $f^k|J$

is bounded by $C\sum_{i=1}^{k-1}|f^i(J)| = C(\sum_{i=0}^{k_0-1}|f^i(J)| + \sum_{i=k_0}^{k-1}|f^i(J)|) \leq C(\zeta(\delta, N, s) + \delta s) < \ln 2$ as required. $\square$

Lemma B.5. *Let h be a topological conjugacy between f and $\tilde{f}$. For every $s > q$, there exists $N > 0$, $K > 1$, such that if (l, m, r) is a 2-commensurable triple in I and $J = [l, r]$ intersects more than N intervals of the collection $\{M_j^s\}_j$ then $(h(l), h(m), h(r))$ is a K-commensurable triple.*

Proof. It is enough to consider the case when $|J|$ is small. Since the only accumulation points of P_s are in $\mathrm{orb}_f(p)$, we may assume (by taking N big) that J is close to $\mathrm{orb}_f(p)$.

The set $P_s \cup \mathrm{orb}_f(0)$ is forward invariant and its only accumulation points are in $\mathrm{orb}_f(p)$. It follows from Lemma B.3 that $h|(P_s \cup \mathrm{orb}_f(0))$ extends to a bi-Lipschitz map $h_1 : I \to I$.

Let L, M, and R be the connected components of $I \setminus (P_s \cup \mathrm{orb}_f(0))$ containing points l, m, and r respectively (in case some of these three points lies in $P_s \cup \mathrm{orb}_f(0)$, we let the corresponding component be empty). Then $|L| + |M| + |R|$ is much smaller than $|J|$ (assuming $|J|$ is small and N is large). This implies that $h|\{l, m, r\}$ extends to a bi-Lipschitz map $I \to I$ with constants almost as good as for h_1, which implies the commensurability statement. $\square$

Let us now fix a topological conjugacy h which is smooth in $(-p, p)$. Then it follows from Lemma B.3 that $h|[-p, p]$ is bi-Lipschitz. Let s be as in Lemma B.4, $N = N(s)$ be as in Lemma B.5, and let $\delta = \delta(s, N)$ be given by Lemma B.4. Let (l, m, r) be a 1-commensurable triple and let $J = [l, r]$. Let k be as in Lemma B.4, so that $f^k|J$ has distortion bounded by $\ln 2$ and at least one of the conditions (1), (2), or (3) hold. We may assume that the distortion of $\tilde{f}|h(J)$ is also bounded by $\ln 2$ (after some increasing of s and decreasing of δ). Let us show that in each of the three cases we can conclude that $(h(l), h(m), h(r))$ is κ-commensurable for some universal constant κ.

If (3) holds, we can argue as in the infinitely renormalizable case. If (2) holds, then we can apply Lemma B.5. If (1) holds then either $f^k(J) \subset [-p, p]$ or $\mathrm{int}\, f^k(J) \cap \{-p, p\} \neq \emptyset$. In the first case we just use that $h|[-p, p]$ is bi-Lipschitz. In the second case, $p \in \mathrm{int}\, f^{k+q}(J)$, so $f^{k+q}(J)$ intersects infinitely many intervals of the collection $\{M_j^s\}_j$, and Lemma B.5 implies the assertion.

Appendix C. Non-symmetric maps

So far we restricted our attention to a class of symmetric (i.e. even) unimodal maps. Below we will show how to extend our results to an appropriate space of asymmetric unimodal maps. The idea is to reduce one case to the other by means of a transversally non-singular projection from the space of asymmetric maps to the space of symmetric maps.

Let $\tilde{\mathcal{U}}^k$, $k \geq 3$ be the space of (not necessarily symmetric) C^k unimodal maps $f : I \to I$, that is, $f(-1) = f(1) = -1$ and f has a unique non-degenerate critical point at 0. Let $\tilde{\mathcal{U}} \subset \tilde{\mathcal{U}}^3$ be the set of quasiquadratic unimodal maps, that is, maps f which have a neighborhood $\mathcal{V}$ such that for all $g \in \mathcal{V}$, g is topologically conjugate to a quadratic map. Let $\tilde{\mathcal{E}}_a \subset \mathcal{B}_{\Omega_a}$ be the subspace of holomorphic maps v with $v(-1) = v(1) = 0$. Let $\tilde{\mathcal{A}}_a = q_2 + \tilde{\mathcal{E}}_a$ and $\tilde{\mathcal{U}}_a = \tilde{\mathcal{U}} \cap \tilde{\mathcal{A}}_a$.

Denote by c_f the critical point of f and let q_f be the quadratic unimodal map with the same critical value as f. Define $\Theta : \tilde{\mathcal{U}}^\infty \to C^\infty(I)$ which associates to each f the unique diffeomorphism of I such that $f = q_f \circ \Theta(f)$. Define the projection $\Pi : \tilde{\mathcal{U}}^\infty \to \mathcal{U}^\infty$ by $\Pi(f) = \Theta(f) \circ q_f$.

Obviously, $\Pi(f)$ is a symmetric unimodal map. It is conjugate to f, since $\Pi(f) = \Theta(f) \circ f \circ \Theta(f)^{-1}$. Moreover, any unimodal map $g \in \tilde{\mathcal{U}}^3$ near $\Pi(f)$ is conjugate to some unimodal map near f, namely, to $\Theta(f)^{-1} \circ g \circ \Theta(f) \in \tilde{\mathcal{U}}^3$. Hence $\Pi(f)$ is quasiquadratic. Clearly, if f is analytic then $\Pi(f)$ analytic as well. We conclude that Π acts from $\tilde{\mathcal{U}}$ to $\mathcal{U}$.

Moreover, if Γ is an analytic family of unimodal maps in some $\tilde{\mathcal{U}}_a$ then $\Pi \circ \Gamma$ is an analytic family of unimodal maps in some $\mathcal{U}_{a'}$. It follows that Theorems B and C can be "lifted" to the asymmetric setting.

To "lift" Theorem A we need some extra information: the derivative of Π (restricted to appropriate Banach spaces of analytic unimodal maps) has a dense image, and therefore it is transversally non-singular with respect to the hybrid lamination we constructed in the symmetric case. This is analogous to what we did in the infinitely renormalizable case in preparation for the proof of Theorem 4.9, where we used the renormalization operator instead of Π.

Lemma C.1. *Let $f \in \tilde{\mathcal{U}}_a$. Then there exist $b > 0$ and a neighborhood $\mathcal{V} \subset \tilde{\mathcal{U}}_a$ of f such that $\Pi(\mathcal{V}) \subset \mathcal{U}_b$, $\Pi : V \to \mathcal{U}_b$ is real analytic, and $D\Pi(f) : T\tilde{\mathcal{U}}_a \to T\mathcal{U}_b$ has a dense image.*

Proof. Let $a > a' > 0$ be such that $\Theta(f) \in \mathcal{B}_{\Omega_{a'}}$. Let $\mathcal{V} \subset \mathcal{U}_{a'}$ be a small neighborhood of f. If $b > 0$ is small enough, then $\Pi(\mathcal{V}) \subset \mathcal{U}_b$ and $\Pi : V \to \mathcal{U}_b$ is real analytic. Since the derivative of the inclusion from $\mathcal{U}_a$ into $\mathcal{U}_{a'}$ has a dense image, it is enough to prove that $D\Pi(f) : T\tilde{\mathcal{U}}_{a'} \to T\mathcal{U}_b$ has a dense image.

To this end, let us consider a polynomial vector field $w_0 \in T\mathcal{U}_b$, and let us try to find $v \in T\tilde{\mathcal{U}}_{a'}$ such that

$$D\Pi(f)v = w_0. \tag{C.1}$$

It is easier to compute $D\Pi(f)v$ for the case of a vector field v such that $v(c_f) = 0$. In this case the quadratic polynomial q_f does not vary infinitesimally, and we obtain:

$$w \equiv D\Theta(f)v = \frac{v}{q'_f \circ \Theta(f)}, \tag{C.2}$$
$$D\Pi(f)v = w \circ q_f.$$

Since w_0 is even, we can represent it in the form $w_0 = w \circ q_f$ where w is a polynomial vector field. Let $v = w \cdot (q'_f \circ \Theta(f))$. Then $v \in \mathcal{B}_{\Omega_{a'}}$ and $v(c_f) = 0$, and (C.1) follows from (C.2). □

Using this Lemma the argument of Theorem 4.9 can be applied to obtain Theorem A for asymmetric maps.

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