# REGULAR PARTITIONS OF HYPERGRAPHS: REGULARITY LEMMAS

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Abstract. Szemerédi's regularity lemma for graphs has proved to be a powerful tool with many subsequent applications. The objective of this paper is to extend the techniques developed by Nagle, Skokan, and authors and obtain a stronger and more "user friendly" regularity lemma for hypergraphs.

## 1. Introduction

In the course of proving his celebrated *density theorem* concerning arithmetic progressions [30], Szemerédi established the *regularity lemma* for graphs [31]. This lemma turned out to be a very useful tool in extremal graph theory and theoretical computer science (see, e.g., [18, 19] for a survey). Following the work of Frankl and Rödl on 3-uniform hypergraphs [5], Gowers [10, 12] and Nagle, Skokan, and authors [20, 26] developed generalizations of the *regularity method* to $k$-uniform hypergraphs. Subsequently, Tao [32] also obtained such a generalization. Those extensions yield the following theorem (see [10, 20, 27]), which settles a conjecture of Erdős, Frankl, and Rödl [4].

**Theorem 1** (Removal lemma)**.** *Let $\ell \geq k \geq 2$ be integers and let $\varepsilon > 0$ there exist $\delta = \delta(\ell, k, \varepsilon) > 0$ and $n_0 = n_0(\ell, k, \varepsilon)$ so that the following holds.*

*Suppose $\mathcal{F}^{(k)}$ is a fixed $k$-uniform hypergraph on $\ell$ vertices and $\mathcal{H}^{(k)}$ is a $k$-uniform hypergraph on $n \geq n_0$ vertices. If $\mathcal{H}^{(k)}$ contains at most $\delta n^\ell$ copies of $\mathcal{F}^{(k)}$, then one can delete $\varepsilon n^k$ edges of $\mathcal{H}^{(k)}$ to make it $\mathcal{F}^{(k)}$-free.*

This theorem can be viewed as an extension of the theorem of Ruzsa and Szemerédi [28], which addressed the case $k = 2$ and $\mathcal{F}^{(2)}$ being a triangle, the complete graph on three vertices. In [4], Theorem 1 was verified for all graphs $\mathcal{F}^{(2)}$. It was shown by Frankl and Rödl [5], Solymosi [29], and Tengan, Tokushige, and authors [25], that Theorem 1 implies Szemerédi's density theorem [30], as well as some of its multidimensional extensions due to Furstenberg and Katznelson [7, 8]. (It is not known, however, whether Theorem 1 also yields an alternative proof of the density version of the theorem of Hales and Jewett [15], which was established by Furstenberg and Katznelson [9] using *ergodic theory*.)

In this paper we continue the line of research from [5, 20, 26] and obtain a stronger and hopefully easier to use regularity lemma for hypergraphs – Theorem 17. The proof of a corresponding counting lemma will appear in a subsequent paper [23].

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A standard application of those theorems, following the lines of [4, 5, 10, 20, 27], yields a proof of Theorem 1.

As a byproduct we obtain a result for hypergraphs, Theorem 14, which might be of independent interest. Roughly speaking, in the context of graphs Theorem 14 says that for every fixed $\nu > 0$ any graph on $n$ vertices can be approximated, by adding and deleting at most $\nu n^2$ edges, by an $\varepsilon$-regular graph on a vertex partition into $t$ parts, where $\varepsilon = \varepsilon(t)$ is an arbitrary function of $t$, and thus we may have $\varepsilon(t) \ll \frac{1}{t}$. This may perhaps be somewhat surprising, since it follows from the work of Gowers [11], that there are graphs which if not changed admit only an $\varepsilon$-regular partition with $t$ classes, where $t \gg \frac{1}{\varepsilon}$. In fact Gowers constructed graphs with number of partition classes in any $\varepsilon$-regular partition being bigger than a tower of height polynomial in $1/\varepsilon$.

## 2. Main results

In what follows, we give a precise account of Szemerédi's regularity lemma. For a graph $G = (V, E)$ and two disjoint sets $A$, $B \subset V$, let $E(A, B)$ denote the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$ and set $e(A, B) = |E(A, B)|$. We also set $d(A, B) = e(A, B)/(|A||B|)$ for the *density* of the pair $A$, $B$.

The concept central to Szemerédi's lemma is that of an *$\varepsilon$-regular pair.* Let $\varepsilon > 0$ be given. We say that the pair $A$, $B$ is $\varepsilon$-regular if $|d(A, B) - d(A', B')| < \varepsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'||B'| > \varepsilon|A||B|$.

We call a partition $\mathscr{P}^{(1)} = \{V_i \colon 0 \le i \le t\}$ of $V$ *$t$-equitable* if it satisfies $|V_0| \le t$ and $|V_i| = \lfloor |V|/t \rfloor$ for $i \in [t]$. We say the graph $G = (V, E)$ is *$\varepsilon$-regular w.r.t. $\mathscr{P}^{(1)}$* if all but $\varepsilon t^2$ pairs $V_i$, $V_j$ are $\varepsilon$-regular. Szemerédi's lemma [31] may then be stated as follows.

**Theorem 2** (Szemerédi's regularity lemma)**.** *For any positive real $\varepsilon$ and any integer $t_0$, there are positive integers $t_{\rm Sz} = t_{\rm Sz}(\varepsilon, t_0)$ and $n_{\rm Sz} = n_{\rm Sz}(\varepsilon, t_0)$ such that for every graph $G = (V, E)$ with $|V| = n \ge n_{\rm Sz}$ vertices there exists a partition $\mathscr{P}^{(1)}$ of $V$ such that*

*(i) $\mathscr{P}^{(1)} = \{V_i \colon 0 \le i \le t\}$ is $t$-equitable, where $t_0 \le t \le t_{\rm Sz}$, and*
*(ii) $G$ is $\varepsilon$-regular w.r.t. $\mathscr{P}^{(1)}$.*

*Moreover, if $(t_{\rm Sz})!$ divides $n$ then $V_0$ can be chosen to be empty.*

We note that our definition of an $\varepsilon$-regular pair differs slightly from the usual one of [31]. However, it is easy to see that both are equivalent. Also we point out that in an early version of the regularity lemma, which appeared in [30], the partition structure was a bit different and more complicated from the one stated above, which appeared in [31].

In this paper we consider two extensions of Theorem 2 to hypergraphs (see Theorem 14 and Theorem 17). To simplify the notation we will restrict to the case where $V_0$ is empty. Since our result is of asymptotic nature, dealing with hypergraphs on $n$ vertices where $n$ is very large, and since every hypergraph can be altered to satisfy $(t_{\rm Sz})!|n$ by adding or deleting a constant number (independent of $n$) of vertices this additional divisibility assumption has no essential baring.

### 2.1. Basic notation.

For real constants $\alpha$, $\beta$, and a non-negative constants $\xi$ we sometimes write

$$\alpha = \beta \pm \xi\,, \quad \text{if} \quad \beta - \xi \le \alpha \le \beta + \xi\,.$$

For a positive integer $\ell$, we denote by $[\ell]$ the set $\{1,\dots,\ell\}$. For a set $V$ and an integer $k \geq 1$, let $[V]^k$ be the set of all $k$-element subsets of $V$. We may drop one pair of brackets and write $[\ell]^k$ instead of $\big[[\ell]\big]^k$. A subset $\mathcal{H}^{(k)} \subseteq [V]^k$ is a *$k$-uniform hypergraph* on the vertex set $V$. We identify hypergraphs with their edge sets. For a given $k$-uniform hypergraph $\mathcal{H}^{(k)}$, we denote by $V(\mathcal{H}^{(k)})$ and $E(\mathcal{H}^{(k)})$ its vertex and edge set, respectively. For $U \subseteq V(\mathcal{H}^{(k)})$, we denote by $\mathcal{H}^{(k)}[U]$ the sub-hypergraph of $\mathcal{H}^{(k)}$ induced on $U$ (i.e. $\mathcal{H}^{(k)}[U] = \mathcal{H}^{(k)} \cap [U]^k$). A $k$-uniform *clique* of *order* $j$, denoted by $K_j^{(k)}$, is a $k$-uniform hypergraph on $j \geq k$ vertices consisting of all $\binom{j}{k}$ different $k$-tuples.

In this paper $\ell$-partite, $j$-uniform hypergraphs play a special rôle, where $j \leq \ell$. Given vertex sets $V_1,\dots,V_\ell$, we denote by $K_\ell^{(j)}(V_1,\dots,V_\ell)$ the *complete* $\ell$-partite, $j$-uniform hypergraph (i.e., the family of all $j$-element subsets $J \subseteq \bigcup_{i\in[\ell]} V_i$ satisfying $|V_i \cap J| \leq 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an $(m,\ell,j)$-*hypergraph* $\mathcal{H}^{(j)}$ on $V_1 \cup \dots \cup V_\ell$ is any subset of $K_\ell^{(j)}(V_1,\dots,V_\ell)$. Note that the vertex partition $V_1 \cup \dots \cup V_\ell$ is an $(m,\ell,1)$-hypergraph $\mathcal{H}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leq i \leq \ell$ and set $\Lambda_i \in [\ell]^i$, we denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_i} V_\lambda\big]$ the sub-hypergraph of the $(m,\ell,j)$-hypergraph $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda\in\Lambda_i} V_\lambda$.

For an $(m,\ell,j)$-hypergraph $\mathcal{H}^{(j)}$ and an integer $j \leq i \leq \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(j)})$ the family of all $i$-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$ of order $i$. Note that $|\mathcal{K}_i(\mathcal{H}^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{H}^{(j)}$.

Given an $(m,\ell,j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ and an $(m,\ell,j)$-hypergraph $\mathcal{H}^{(j)}$ such that $V(\mathcal{H}^{(j)}) \subseteq V(\mathcal{H}^{(j-1)})$, we say an edge $J$ of $\mathcal{H}^{(j)}$ *belongs to* $\mathcal{H}^{(j-1)}$ if $J \in \mathcal{K}_j(\mathcal{H}^{(j-1)})$, i.e., $J$ corresponds to a clique of order $j$ in $\mathcal{H}^{(j-1)}$. Moreover, $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$, i.e., every edge of $\mathcal{H}^{(j)}$ belongs to $\mathcal{H}^{(j-1)}$. This brings us to one of the main concepts of this paper, the notion of a *complex*.

*Definition* 3 (**$(m,\ell,h)$-complex**). *Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An $(m,\ell,h)$-complex $\boldsymbol{\mathcal{H}}$ is a collection of $(m,\ell,j)$-hypergraphs $\{\mathcal{H}^{(j)}\}_{j=1}^h$ such that*

- *(a) $\mathcal{H}^{(1)}$ is an $(m,\ell,1)$-hypergraph, i.e., $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$;*
- *(b) $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$ for $2 \leq j \leq h$, i.e., $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.*

*Remark* 4. We may also define hypergraphs and complexes in the same way for underlying vertex sets $V_1,\dots,V_\ell$ with different cardinalities. In such a case we will drop the $m$ and say $\mathcal{H}^{(j)}$ is an $(\ell,j)$-hypergraph or $\boldsymbol{\mathcal{H}}$ is an $(\ell,h)$-complex.

2.2. **Regular complexes.** We begin with a notion of relative density of a $j$-uniform hypergraph w.r.t. $(j-1)$-uniform hypergraph on the same vertex set.

*Definition* 5 (**relative density**). *Let $\mathcal{H}^{(j)}$ be a $j$-uniform hypergraph and let $\mathcal{H}^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the density of $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{H}^{(j-1)}$ as*

$$d\big(\mathcal{H}^{(j)}\big|\mathcal{H}^{(j-1)}\big) = \begin{cases} \frac{\left|\mathcal{H}^{(j)}\cap\mathcal{K}_j(\mathcal{H}^{(j-1)})\right|}{\left|\mathcal{K}_j(\mathcal{H}^{(j-1)})\right|} & \text{if} \quad \left|\mathcal{K}_j(\mathcal{H}^{(j-1)})\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We now define a notion of regularity of an $(m,j,j)$-hypergraph with respect to an $(m,j,j-1)$-hypergraph.

*Definition* 6. *Let reals $\varepsilon > 0$ and $d_j \geq 0$ be given along with an $(m, j, j)$-hypergraph $\mathcal{H}^{(j)}$ and an underlying $(m, j, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$. We say $\mathcal{H}^{(j)}$ is $(\varepsilon, d_j)$-regular w.r.t. $\mathcal{H}^{(j-1)}$ if whenever $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfies*

$$\big|\mathcal{K}_j(\mathcal{Q}^{(j-1)})\big| \geq \varepsilon\big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big|, \quad \text{then} \quad d\big(\mathcal{H}^{(j)}\big|\mathcal{Q}^{(j-1)}\big) = d_j \pm \varepsilon\,.$$

We extend the notion of $(\varepsilon, d_j)$-regularity from $(m, j, j)$-hypergraphs to $(m, \ell, j)$-hypergraphs $\mathcal{H}^{(j)}$.

*Definition* 7 (**$(\varepsilon, d_j)$-regular hypergraph**). *We say an $(m, \ell, j)$-hypergraph $\mathcal{H}^{(j)}$ is $(\varepsilon, d_j)$-regular w.r.t. an $(m, \ell, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ if for every $\Lambda_j \in [\ell]^j$ the restriction $\mathcal{H}^{(j)}[\Lambda_j] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$ is $(\varepsilon, d_j)$-regular w.r.t. to the restriction $\mathcal{H}^{(j-1)}[\Lambda_j] = \mathcal{H}^{(j-1)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$.*

*We sometimes write $\varepsilon$-regular to mean $\big(\varepsilon, d\big(\mathcal{H}^{(j)}\big|\mathcal{H}^{(j-1)}\big)\big)$-regular.*

Finally, we close this section with the notion of a regular complex.

*Definition* 8 (**$(\varepsilon, \boldsymbol{d})$-regular complex**). *Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \dots, d_h)$ be a vector of non-negative reals. We say an $(m, \ell, h)$-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is $(\varepsilon, \boldsymbol{d})$-regular if $\mathcal{H}^{(j)}$ is $(\varepsilon, d_j)$-regular w.r.t. $\mathcal{H}^{(j-1)}$ for every $j = 2, \dots, h$.*

2.3. **Partitions.** The regularity lemmas for $k$-uniform hypergraphs which we prove in this paper provide a well-structured family of partitions $\mathscr{P} = \big\{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\big\}$ of vertices, pairs, …, and $(k-1)$-tuples of some vertex set. We now discuss the structure of these partitions following the approach of [26]. First we define the *refinement* of a partition.

*Definition* 9 (**refinement**). *Suppose $A \supseteq B$ are sets, $\mathscr{A}$ is a partition of $A$, and $\mathscr{B}$ is a partition of $B$. We say $\mathscr{A}$ refines $\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if for every $\mathcal{A} \in \mathscr{A}$ there either exists a $\mathcal{B} \in \mathscr{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \subseteq A \setminus B$.*

Let $k$ be a fixed integer and $V$ be a set of vertices. Throughout this paper we require a family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ on $V$ to satisfy properties which we are going to describe below (see Definition 10).

Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of $V$. For every $1 \leq j \leq k$ let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing $j$-tuples $J$, i.e., the set of $j$-tuples which satisfy $|J \cap V_i| \leq 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ into $(i, i)$-hypergraphs have been defined for $1 \leq i \leq j-1$. Then for every $(j-1)$-tuple $I$ in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$ there exist a unique $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. Moreover, for every $j$-tuple $J$ in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we define the *polyad of $J$*

$$\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup\Big\{\mathcal{P}^{(j-1)}(I)\colon\ I \in [J]^{j-1}\Big\}\,.$$

In other words, $\hat{\mathcal{P}}^{(j-1)}(J)$ is the unique collection of $j$ partition classes of $\mathscr{P}^{(j-1)}$ in which $J$ spans a copy of $K_j^{(j-1)}$. Observe that $\hat{\mathcal{P}}^{(j-1)}(J)$ can be viewed as a $(j, j-1)$-hypergraph, i.e., a $j$-partite, $(j-1)$-uniform hypergraph. More generally, for $1 \leq i < j$, we set

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup\Big\{\mathcal{P}^{(i)}(I)\colon\ I \in [J]^i\Big\} \qquad \text{and} \qquad \boldsymbol{\mathcal{P}}(J) = \big\{\hat{\mathcal{P}}^{(i)}(J)\big\}_{i=1}^{j-1}\,. \tag{1}$$

Next, we define $\hat{\mathscr{P}}^{(j-1)}$ the family of all polyads

$$\hat{\mathscr{P}}^{(j-1)} = \big\{\hat{\mathcal{P}}^{(j-1)}(J)\colon\ J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\,\big\}\,.$$

Note that $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for different $j$-tuples $J$ and $J'$. We view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently, $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we have in this paper is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}\,. \tag{2}$$

In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note, that due to (2) we inductively infer that $\boldsymbol{\mathcal{P}}(J)$ defined in (1) is a $(j, j-1)$-complex.

Throughout this paper we also want to have control over the number of partition classes in $\mathscr{P}^{(j)}$, and more specifically, over the number of classes contained in $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ for a fixed polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. We render this precisely in the following definition.

*Definition* 10 (**family of partitions**). *Suppose $V$ is a set of vertices, $k \geq 2$ is an integer and $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ is a family of partitions on $V$, if it satisfies the following:*

- $(i)$ *$\mathscr{P}^{(1)}$ is a partition of $V$ into $a_1$ classes,*
- $(ii)$ *$\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ satisfying:*

$$\mathscr{P}^{(j)} \quad \text{refines} \quad \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$$

$$\text{and} \quad \big|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}\colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}\big| = a_j \quad \text{for every} \quad \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\,.$$

*Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is $t$-bounded, if $\max\{a_1, \ldots, a_{k-1}\} \leq t$.*

We now combine Definition 9 and Definition 10 and define the *refinement of a family of partitions*.

*Definition* 11 (**refinement of families**). *Suppose $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ and $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}})$ are families of partitions on the same vertex set $V$. We say $\mathscr{P}$ refines $\mathscr{R}$ and write $\mathscr{P} \prec \mathscr{R}$, if $\mathscr{P}^{(j)} \prec \mathscr{R}^{(j)}$ (cf. Definition 9) for every $j \in [k-1]$.*

2.4. **Main results.** In this paper we prove two hypergraph regularity lemmas, which may be viewed as strengthened versions of the hypergraph regularity lemma from [26]. Those new lemmas were already applied in [2, 3, 21, 22, 24]. As in Szemerédi's regularity lemma, such hypergraph regularity lemmas should ensure the existence of partitions of the edge set of a $k$-uniform hypergraph which satisfy certain properties. Besides the structural conditions discussed in the last section the partitions ensured by the main theorems in this paper will satisfy two more properties which we define below. More specifically, the family of partitions $\mathscr{P}$ have to satisfy properties analogous to $(i)$ and $(ii)$ of Theorem 2. We first extend the notion of equitability.

*Definition* 12 (**$(\eta, \varepsilon, \boldsymbol{a})$-equitable**). *Suppose $V$ is a set of $n$ vertices, $\eta$ and $\varepsilon$ are positive reals, $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ is a vector of positive integers, and $a_1$ divides $n$.*

*We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on $V$ (as defined in Definition 10) is $(\eta, \varepsilon, \boldsymbol{a})$-equitable if it satisfies the following:*

- $(a)$ $\big|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\big| \leq \eta\binom{n}{k}$;
- $(b)$ *$\mathscr{P}^{(1)} = \{V_i\colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $|V_i| = |V|/a_1$ for $i \in [a_1]$;*

(*c*) *for every $k$-tuple $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ we have that $\boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$ is an $(\varepsilon, \boldsymbol{d})$-regular $(n/a_1, k, k-1)$-complex, where $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$.*

Next, we extend (*ii*) of Theorem 2. In this paper we consider two possible extensions, which give rise to the two different regularity lemmas below.

*Definition* 13 (**perfectly $\varepsilon$-regular**). *Suppose $\varepsilon$ is some positive real. Let $\mathcal{G}^{(k)}$ be a $k$-uniform hypergraph with vertex set $V$ and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on $V$. We say $\mathcal{G}^{(k)}$ is perfectly $\varepsilon$-regular w.r.t. $\mathscr{P}$, if for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ we have that $\mathcal{G}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is $\varepsilon$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.*

The following theorem is one of the two main results in this paper.

**Theorem 14** (Regular approximation lemma)**.** *Let $k \geq 2$ be a fixed integer. For all positive constants $\eta$ and $\nu$, and every function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ there are integers $t_{\mathrm{Thm.14}}$ and $n_{\mathrm{Thm.14}}$ so that the following holds.*

*For every $k$-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \geq n_{\mathrm{Thm.14}}$ such that $(t_{\mathrm{Thm.14}})!$ divides $n$ there exist a $k$-uniform hypergraph $\mathcal{G}^{(k)}$ on the same vertex set and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that*

(*i*) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\mathrm{Thm.14}}$-bounded,*
(*ii*) *$\mathcal{G}^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$, and*
(*iii*) *$|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| \leq \nu n^k$.*

Let us briefly compare Theorem 14 for $k = 2$ with Theorem 2. Note that as discussed in [19, Section 1.8] there are graphs with irregular pairs in any partition. Therefore, due to the "perfectness" in (*ii*) of Theorem 14 one has to alter $H = \mathcal{H}^{(2)}$ to obtain $G = \mathcal{G}^{(2)}$.

The main difference between Theorem 14 for $k = 2$ and Theorem 2, however, is in the choice of $\varepsilon$ being a function of $a_1^{\mathscr{P}}$. It follows from the work of Gowers in [11] that it is not possible to regularize a graph $H$ with an $\varepsilon$ in such a way that, e.g., $\varepsilon < 1/a_1^{\mathscr{P}}$ can be ensured, where $a_1^{\mathscr{P}} = |\mathscr{P}^{(1)}|$ is the number of vertex classes. Properties (*i*) and (*iii*) of Theorem 14 assert, however, that by adding or deleting at most $\nu n^2$ edges from $H$ one can obtain a graph $G$ which admits an $\varepsilon(a_1^{\mathscr{P}})$ regular partition, with $\varepsilon(a_1^{\mathscr{P}}) < 1/a_1^{\mathscr{P}}$. Such a lemma for graphs can be also deduced from the iterated regularity lemma in [1].

The other result of this paper, Theorem 17, concerns the case in which we do not change the given hypergraph $\mathcal{H}^{(k)}$. Due to the discussion above such a lemma needs to allow exceptional pairs (or polyads for $k \geq 3$) in the partition $\mathscr{P}$. Moreover, the measure of regularity of $\mathcal{H}^{(k)}$ w.r.t. $\mathscr{P}$ (called $\delta_k$ here) cannot depend on $a_1^{\mathscr{P}}$. In fact, in our proof of Theorem 17 $\delta_k$ is a constant independent of each $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. On the other hand, as in [5, 26] we will infer that $\mathcal{H}^{(k)}$ is $(\delta_k, *, r)$-regular (defined below), where $r$ may depend on $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. We first extend Definition 7.

*Definition* 15 (**$(\delta_k, d_k, r)$-regular hypergraph**). *Let $\delta_k$ and $d_k$ be positive reals and $r$ be a positive integer. Suppose $\mathcal{H}^{(k-1)}$ is an $(m, k, k-1)$-hypergraph spanning at least one $K_k^{(k-1)}$. We say an $(m, k, k)$-hypergraph $\mathcal{H}^{(k)}$ is $(\delta_k, d_k, r)$-regular w.r.t. $\mathcal{H}^{(k-1)}$ if for every collection $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\}$ of not necessarily disjoint sub-hypergraphs of $\mathcal{H}^{(k-1)}$ which satisfy*

$$\Big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\Big| \geq \delta_k \left|\mathcal{K}_k(\mathcal{H}^{(k-1)})\right| > 0\,,$$

*we have*

$$\frac{\left|\mathcal{H}^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|}{\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|} = d_k \pm \delta_k\,.$$

*We write* $(\delta_k, *, r)$*-regular to mean* $\big(\delta_k, d\big(\mathcal{H}^{(k)}\big|\mathcal{H}^{(k-1)}\big), r\big)$*-regular. Moreover, if* $r = 1$*, then a* $(\delta_k, d_k, 1)$*-regular hypergraph is* $(\varepsilon, d_k)$*-regular with* $\varepsilon = \delta_k$ *(cf. Definition 7) and vice versa.*

Finally, we give the second extension of $(ii)$ of Theorem 2, which will be ensured by Theorem 17.

*Definition* 16 ($(\delta_k, *, r)$**-regular w.r.t.** $\mathscr{P}$). *Suppose* $\delta_k$ *is a positive real and* $r$ *is a positive integer. Let* $\mathcal{H}^{(k)}$ *be a* $k$*-uniform hypergraph with vertex set* $V$ *and* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *be a family of partitions on* $V$*. We say* $\mathcal{H}^{(k)}$ *is* $(\delta_k, *, r)$*-regular w.r.t.* $\mathscr{P}$*, if*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \mathcal{H}^{(k)} \text{ is not } (\delta_k, *, r)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\Big\}\Big| \le \delta_k |V|^k\,.$$

The following theorem is a strengthening of the main result of [26].

**Theorem 17** (Regularity lemma)**.** *Let* $k \ge 2$ *be a fixed integer. For all positive constants* $\eta$ *and* $\delta_k$*, and all functions* $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ *and* $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ *there are integers* $t_{\text{Thm.17}}$ *and* $n_{\text{Thm.17}}$ *so that the following holds.*

*For every* $k$*-uniform hypergraph* $\mathcal{H}^{(k)}$ *with* $|V(\mathcal{H}^{(k)})| = n \ge n_{\text{Thm.17}}$ *such that* $(t_{\text{Thm.17}})!$ *divides* $n$*, there exists a family of partitions* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ *so that*

(*i*) $\mathscr{P}$ *is* $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$*-equitable and* $t_{\text{Thm.17}}$*-bounded and*
(*ii*) $\mathcal{H}^{(k)}$ *is* $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$*-regular w.r.t.* $\mathscr{P}$*.*

## 3. Auxiliary results

In this section we review a few results that are essential for our proofs of Theorem 14 and Theorem 17.

The following theorem can be used to estimate the number of copies of $K_\ell^{(h)}$ in an appropriate collection of dense and regular blocks within a regular partition provided by the regular approximation lemma, Theorem 14. Moreover, it can be applied to count the number of $K_k^{(k-1)}$'s in the polyads of the partitions obtained by Theorem 14 and Theorem 17.

**Theorem 18** (Dense counting lemma)**.** *For all integers* $2 \le h \le \ell$ *and all positive constants* $\gamma$ *and* $d_0$ *there exist* $\varepsilon_{\text{DCL}} = \varepsilon_{\text{DCL}}(h, \ell, \gamma, d_0) > 0$ *and an integer* $m_{\text{DCL}} = m_{\text{DCL}}(h, \ell, \gamma, d_0)$ *so that if* $\boldsymbol{d} = (d_2, \dots, d_h) \in \mathbb{R}^{h-1}$ *satisfying* $d_j \ge d_0$ *for* $2 \le j \le h$ *and* $m \ge m_{\text{DCL}}$*, and if* $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ *is an* $(\varepsilon_{\text{DCL}}, \boldsymbol{d})$*-regular* $(m, \ell, h)$*-complex, then*

$$\left|\mathcal{K}_\ell\big(\mathcal{H}^{(h)}\big)\right| = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{\ell}{j}} \times m^\ell\,.$$

This theorem was proved by Kohayakawa, Rödl, and Skokan in [17, Theorem 6.5]. For completeness we give a short proof of a generalization of Theorem 18 in the subsequent paper [23].

The following two facts regard regularity properties of the union of regular hypergraphs. The first of those two propositions states that the union of regular $(m, j, j)$-hypergraphs which share the same underlying $(m, j, j-1)$-hypergraph is regular. The proof is straightforward and we refrain from presenting it here.

**Proposition 19.** *Let $j \geq 2$, $m$, $t$, $r \geq 1$ be fixed integers and let $\delta$ and $d(1), \dots, d(t)$ be positive reals. Suppose $\mathcal{P}_1^{(j)}, \dots, \mathcal{P}_t^{(j)}$ is a family of pairwise edge disjoint $(m, j, j)$-hypergraphs with the same underlying $(m, j, j-1)$-hypergraph $\hat{\mathcal{P}}^{(j-1)}$.*

*If $\mathcal{P}_\tau^{(j)}$ is $(\delta, d(\tau), r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ for every $\tau \in [t]$, then $\mathcal{P}^{(j)}$ is $(t\delta, d, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$, where $\mathcal{P}^{(j)} = \bigcup_{\tau\in[t]} \mathcal{P}_\tau^{(j)}$ and $d = \sum_{\tau\in[t]} d(\tau)$.* □

The next proposition gives us control when we union hypergraphs having different underlying polyads. Before we make this precise, we define the setup for our proposition.

*Setup* 20. *Let $j \geq 2$, $m$, $t \geq 1$ be fixed integers and let $\delta$ and $d$ be positive reals. Let $\{\hat{\mathcal{P}}_\tau^{(j-1)}\}_{\tau\in[t]}$ be a family of $(m, j, j-1)$-hypergraphs such that*

$$\bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)} \text{ is a } j\text{-partite } (j-1)\text{-uniform hypergraph,}$$

$$\mathcal{K}_j\Big(\bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)}\Big) = \bigcup_{\tau\in[t]} \mathcal{K}_j\big(\hat{\mathcal{P}}_\tau^{(j-1)}\big)\,, \tag{3}$$

$$\text{and}\quad \mathcal{K}_j\big(\hat{\mathcal{P}}_\tau^{(j-1)}\big) \cap \mathcal{K}_j\big(\hat{\mathcal{P}}_{\tau'}^{(j-1)}\big) = \varnothing \quad \text{for} \quad 1 \leq \tau < \tau' \leq t\,.$$

*Let $\{\mathcal{P}_\tau^{(j)}\}_{\tau\in[t]}$ be a family of $(m, j, j)$-hypergraphs such that $\hat{\mathcal{P}}_\tau^{(j-1)}$ underlies $\mathcal{P}_\tau^{(j)}$ for any $\tau \in [t]$. Set $\hat{\mathcal{P}}^{(j-1)} = \bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)}$ and $\mathcal{P}^{(j)} = \bigcup_{\tau\in[t]} \mathcal{P}_\tau^{(j)}$.*

**Proposition 21.** *Let $r \geq 1$ be a fixed integer and let $\{\mathcal{P}_\tau^{(j)}\}_{\tau\in[t]}$ and $\{\hat{\mathcal{P}}_\tau^{(j-1)}\}_{\tau\in[t]}$ satisfy Setup 20. If $\mathcal{P}_\tau^{(j)}$ is $(\delta, d, r)$-regular w.r.t. $\hat{\mathcal{P}}_\tau^{(j-1)}$ for every $\tau \in [t]$, then $\mathcal{P}^{(j)}$ is $(2\sqrt{\delta}, d, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$.* □

For $r = 1$ a proof of Proposition 21 appeared in [20] and the proof presented there works verbatim for general $r \geq 1$.

The proof of the following lemma is based on Chernoff's inequality and the fact that randomly chosen sub-hypergraphs of a regular hypergraph are regular. Similar statements were proved in [5, 26] and we will omit the technical details here.

**Proposition 22** (Slicing lemma)**.** *Let $j \geq 2$, $s_0$, $r \geq 1$ be integers and let $\delta_0$, $\varrho_0$, and $p_0$ be positive real numbers. There is an integer $m_{\mathrm{SL}} = m_{\mathrm{SL}}(j, s_0, r, \delta_0, \varrho_0, p_0)$ so that the following holds. If $m \geq m_{\mathrm{SL}}$,*

*(i) $\hat{\mathcal{P}}^{(j-1)}$ is a $(m, j, j-1)$-hypergraph satisfying $|\mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)| \geq m^j / \ln m$ and*

*(ii) $\mathcal{P}^{(j)} \subseteq \mathcal{K}_j\big(\hat{\mathcal{P}}^{(j-1)}\big)$ is an $(\delta, \varrho, r)$-regular $(m, j, j)$-hypergraph with $\varrho \geq \varrho_0 \geq 2\delta \geq 2\delta_0$.*

*Then for any positive integer $1 \leq s \leq s_0$ and all positive reals $p_1, \dots, p_s$ satisfying*

*(iii) $\sum_{\sigma\in[s]} p_\sigma \leq 1$ and $p_\sigma \geq p_0$ for $\sigma \in [s]$*

*there exists a partition $\{\mathcal{T}_0^{(j)}, \mathcal{T}_1^{(j)}, \dots, \mathcal{T}_s^{(j)}\}$ of $\mathcal{P}^{(j)}$ such that $\mathcal{T}_\sigma^{(j)}$ is $(3\delta, p_\sigma\varrho, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ for every $\sigma = 1, \dots, s$.*

*Moreover, $\mathcal{T}_0^{(j)}$ is $(3\delta, (1 - \sum_{\sigma\in[s]} p_\sigma)\varrho, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ and $\mathcal{T}_0^{(j)} = \varnothing$ if $\sum_{\sigma\in[s]} p_\sigma = 1$.* □

## 4. Outline of the proofs

Roughly speaking, our proof of both theorems, Theorem 14 and Theorem 17, is based on the following induction scheme

$$\text{Theorem 17 for } k \Longrightarrow \text{Theorem 14 for } k \Longrightarrow \text{Theorem 17 for } k+1.$$

To carry out the technical details for such an induction scheme, we need to strengthen the statements of Theorem 17 (**r**egularity **l**emma) and of Theorem 14 (**r**egularity **a**pproximation **l**emma) to more general, but, unfortunately, less esthetically pleasing statement RL($k$), Lemma 23, and RAL($k$), Lemma 25.

Before we start to discuss these more general statements we will briefly outline why they are needed. While the proof of the implication Theorem 14 for $k \Longrightarrow$ Theorem 17 for $k+1$ could follow the lines of [5, 26] (now using Theorem 14 for $k$ to regularize the witnesses, which provides the cleaner partition $\mathscr{P}$), the need for generalizing the statements comes from the implication Theorem 17 for $k \Longrightarrow$ Theorem 14 for $k$. In our proof of this implication we need to apply Theorem 17 for $k$ twice. After the first application we obtain an $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable partition $\mathscr{P}$ which is bounded. However, the hypergraph $\mathcal{H}$ will only be $\delta_k$-regular w.r.t. $\mathscr{P}$, where $\delta_k$ is a constant independent of $\boldsymbol{a}^{\mathscr{P}}$, and not $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular, as required by part (*ii*) of Theorem 14. To obtain such an $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular hypergraph $\mathcal{G}^{(k)}$, which will be "$\nu$ close to $\mathcal{H}^{(k)}$" (cf. (*iii*) of Theorem 14) we need to apply Theorem 17 again. It will be essential for us that the partition obtained in the second application of Theorem 17 will refine $\mathscr{P}$, the partition obtained in the first application. This is the reason why we will strengthen the statement of Theorem 17 (see Lemma 23). This change is due to the induction scheme requiring a corresponding strengthening of Theorem 14 (see Lemma 25).

We now state the strengthened variant of Theorem 17. It allows us to enter the regularity lemma with an initial equitable family of partitions $\mathscr{O}$ and a family of $k$-uniform hypergraphs $\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}$. It then guarantees the existence of an equitable refinement $\mathscr{P}$ of $\mathscr{O}$ for which each $\mathcal{H}_i^{(k)}$ is regular. (Since it might not be completely obvious that Theorem 17 follows from Lemma 23 stated below, we give the formal reduction after Remark 24.)

**Lemma 23** (RL($k$))**.** *For all positive integers $o$ and $s$, all positive reals $\eta$ and $\delta_k$, and all functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real $\mu_{\mathrm{RL}}$ and positive integers $t_{\mathrm{RL}}$ and $n_{\mathrm{RL}}$ such that the following holds. Suppose*

- (*a*) *$V$ is a set of cardinality $n \ge n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides $n$,*
- (*b*) *$\mathscr{O} = \mathscr{O}(k-1, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and $o$-bounded family of partitions on $V$, and*
- (*c*) *$\mathscr{H}^{(k)} = \{\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}\}$ is a partition of $[V]^k$.*

*Then there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that*

- (*P1*) *$\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\mathrm{RL}}$-bounded,*
- (*P2*) *$\mathscr{P} \prec \mathscr{O}$,*

*and for every $i \in [s]$*

- (*H*) *$\mathcal{H}_i^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.*

*Remark* 24. In the inductive proof we will apply Lemma 23 twice. In the second application in Section 5.2 it will be convenient to use a variant of Lemma 23, where assumptions (*a*) and (*b*) are replaced by

($a'$) $V = V_1 \cup \dots \cup V_k$, $|V_i| = m \geq n_{\mathrm{RL}}/k$ and $t_{\mathrm{RL}}!$ divides $m$,
($b'$) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3, \boldsymbol{d})$-regular $(m, k, k-1)$-complex, where the vertex set $\mathcal{R}^{(1)} = V_1 \cup \dots \cup V_k$ and $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$, $a_i \in \mathbb{N}$ and $a_i \leq o$ for $2 \leq i < k$.

Moreover, we weaken conclusion ($P2$) in this context, insisting only that $\mathscr{P}$ "refines" the given complex $\boldsymbol{\mathcal{R}}$, more precisely

($P2'$) $\mathscr{P}^{(1)} \prec \mathcal{R}^{(1)} = V_1 \cup \dots \cup V_k$ and for every $2 \leq j < k$ and every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ we have either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$.

Note that this version of Lemma 23 is in fact a consequence of Lemma 23.

We now verify that Lemma 23 implies Theorem 17 for the same $k$.

*Proof:* RL($k$) $\Longrightarrow$ *Theorem 17 for* $k$. Let $k$ be a fixed integer and let constants $\eta$ and $\delta_k$ and functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ be given by Theorem 17. We want to apply Lemma 23. For that we will define an auxiliary family of partitions $\mathscr{O}$. In fact any sufficiently equitable partition would do. In order to avoid trivial cases we are going to split the vertex set into $k$ parts of the same size and any part of the partition $\mathscr{O}^{(j)}$ will be isomorphic to the complete $j$-partite $j$-uniform hypergraph of the appropriate order for $2 \leq j \leq k-1$ (see (4) below). With this in mind we apply Lemma 23 with $o = k$, $s = 2$, and the given constants $\eta$ and $\delta_k$, and functions $r$ and $\delta$ to obtain $\mu_{\mathrm{RL}}$, $t_{\mathrm{RL}}$ and $n_{\mathrm{RL}}$. We then set $t_{\mathrm{Thm.17}} = t_{\mathrm{RL}}$ and $n_{\mathrm{Thm.17}} = n_{\mathrm{RL}}$.

Now let $n \geq n_{\mathrm{Thm.17}}$ be a multiple of $t_{\mathrm{Thm.17}} = (t_{\mathrm{RL}})!$ and $\mathcal{H}^{(k)}$ be a hypergraph with vertex set $V$, where $|V| = n$. Set $a_1^{\mathscr{O}} = k$, $a_j^{\mathscr{O}} = 1$ for $j = 2, \dots, k-1$, $\boldsymbol{a}^{\mathscr{O}} = (a_1^{\mathscr{O}}, \dots, a_{k-1}^{\mathscr{O}})$ and let $V = V_1 \cup \dots \cup V_{a_1^{\mathscr{O}}} = \mathscr{O}^{(1)}$ be some arbitrary equitable vertex partition. Moreover, set

$$\mathscr{O}^{(j)} = \{K_j^{(j)}(V_{i_1}, \dots, V_{i_j})\colon\ 1 \leq i_1 < \dots < i_j \leq a_1^{\mathscr{O}} = k\} \tag{4}$$

and $\mathscr{H}^{(k)} = \{\mathcal{H}^{(k)}, [V]^k \setminus \mathcal{H}^{(k)}\}$. Clearly, $\mathscr{O}$ constructed that way is $(\eta^{\mathscr{O}}, \mu, \boldsymbol{a}^{\mathscr{O}})$-equitable for some $\eta^{\mathscr{O}} > 0$ and every $\mu > 0$. Consequently, $V$, $\mathscr{O}$ and $\mathscr{H}^{(k)}$ satisfy the assumptions ($a$)–($c$) of Lemma 23 for $n_{\mathrm{RL}}$, $t_{\mathrm{RL}}$, $o = a_1^{\mathscr{O}} = k$, $s = 2$ and any $\mu_{\mathrm{RL}}$. Then, ($P1$) and ($H$) yield conclusions ($i$) and ($ii$) of Theorem 17. □

Next we state a similarly strengthened version of Theorem 14.

**Lemma 25** (RAL($k$))**.** *For all positive integers $o$ and $s$, all positive reals $\eta$ and $\nu$, and every function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real $\mu_{\mathrm{RAL}}$ and positive integers $t_{\mathrm{RAL}}$ and $n_{\mathrm{RAL}}$ such that the following holds. Suppose*

($a$) *$V$ is a set of cardinality $n \geq n_{\mathrm{RAL}}$ and $(t_{\mathrm{RAL}})!$ divides $n$,*
($b$) *$\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is a $(\eta^{\mathscr{O}}, \mu_{\mathrm{RAL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and $o$-bounded family of partitions on $V$, and*
($c$) *$\mathscr{H}^{(k)} = \{\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}\}$ is a partition of $[V]^k$ so that $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$.*

*Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that*

($P1$) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\mathrm{RAL}}$-bounded and*
($P2$) *$\mathscr{P} \prec \mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$.*

*Furthermore, there exists a partition $\mathscr{G}^{(k)} = \{\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_s^{(k)}\}$ of $[V]^k$ such that for every $i \in [s]$ the following holds*

($G1$) *$\mathcal{G}_i^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$,*

(G2) $|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \le \nu n^k$, *and*
(G3) *if* $\mathcal{H}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)})$ *then* $\mathcal{G}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)})$ *and* $\mathscr{G}^{(k)} \prec \mathscr{O}^{(k)}$.

Lemma 25 yields Theorem 14 for the same $k$ in a similar way as Lemma 23 implies Theorem 17 and we omit the details. Hence it suffices to show

$$\mathrm{RL}(2) \quad \text{and} \quad \mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1) \quad \text{for} \quad k \ge 2\,,$$

in order to establish Theorem 17 and Theorem 14 inductively.

We outline the basis of the induction, the proof of RL(2), in Section 4.1. The proofs of each of the two implications establishing the induction step are the content of Section 5 and Section 6, respectively.

4.1. **Sketch of the proof of** RL(2)**.** Observe that in the statement of RL(2), Lemma 23 for $k = 2$, the constant $\mu$ and the function $\delta$ have no bearing. Consequently, RL(2) reduces to the following statement.

**Lemma 26** (RL(2))**.** *For all positive integers $o$ and $s$, all positive reals $\eta$ and $\delta_2$, and any function $r\colon \mathbb{N} \to \mathbb{N}$ there are positive integers $t_{\mathrm{RL}}$ and $n_{\mathrm{RL}}$ such that the following holds.*

*Suppose*

(a) *$V$ is a set of cardinality $n \ge n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides $n$,*
(b) *$\mathscr{O}^{(1)}$ is a vertex partition $V_1 \cup \dots \cup V_{a_1^{\mathscr{O}}}$ of $V$, where $|V_1| = \dots = |V_{a_1^{\mathscr{O}}}|$ and $a_1^{\mathscr{O}} \le o$*
(c) *$\mathscr{H} = \{H_1, \dots, H_s\}$ is a partition of $[V]^2$ the complete graph on $n$ vertices.*

*Then there exists a partition $\mathscr{P}^{(1)} = \{W_1, \dots, W_{a_1^{\mathscr{P}}}\}$ of $V$ so that*

(P1) $|W_1| = \dots = |W_{a_1^{\mathscr{P}}}|$, $\mathrm{Cross}_2(\mathscr{P}^{(1)}) \ge (1-\eta)\binom{n}{2}$, *and* $a_1^{\mathscr{P}} \le t_{\mathrm{RL}}$,
(P2) *for every $i \in [a_1^{\mathscr{P}}]$ we have $W_i \subseteq V_j$ for some $j \in [a_1^{\mathscr{O}}]$*

*and for every $i \in [s]$*

(H) *$H_i$ is $(\delta_2, *, r(a_1^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}^{(1)}$.*

The proof of RL(2) follows closely the lines of the proof of Szemerédi's regularity lemma [31], Theorem 2. There are three differences, however. The first and the last of them are standard.

(1) Rather than one graph we have a fixed number of graphs $H_1, \dots, H_s$ to regularize. Such a regularity lemma was used in a number of applications and is discussed for example in [19, Section 1.9].
(2) This difference which regards the concept of regularity in (H) is perhaps most significant. Instead of a single pair $A' \subseteq A$, $B' \subseteq B$, $|A'||B'| \ge \varepsilon|A||B|$ that witnesses the irregularity of a bipartite graph with vertex classes $A$ and $B$, we consider here a more complicated witness; namely an $r$-tuple of pairs $(A_i, B_i)$ of sets where $A_1, \dots, A_r \subseteq A$, $B_1, \dots, B_r \subseteq B$ and $|\bigcup_{i\in[r]} A_i \times B_i| \le \varepsilon|A||B|$ (cf. Definition 15 with $k = 2$ and $\mathcal{H}^{(1)} = (A, B)$).

   We recall that the proof of Szemerédi's regularity lemma [31] is based on a procedure in which, having an initial partition $\mathscr{P}_0^{(1)}$, one constructs a sequence $\mathscr{P}_0^{(1)}, \mathscr{P}_1^{(1)}, \dots$ of partitions. To each partition a quantity (called *index*) is associated which is known to satisfy $\mathrm{ind}(\mathscr{P}^{(1)}) \le 1$ for every vertex partition $\mathscr{P}^{(1)}$. On the other hand, one proves that if $\mathscr{P}_i^{(1)}$ is irregular, then $\operatorname{ind}(\mathscr{P}_{i+1}^{(1)}) \geq \operatorname{ind}(\mathscr{P}_i^{(1)}) + \delta_2^4/10$. Consequently, one infers that after at most $10/\delta_2^4$ iterations one arrives to a partition which is $\delta_2$-regular.

While in [31], if $\mathscr{P}_i^{(1)}$ was partition into $a_1^{\mathscr{P}_i}$ parts implied that $\mathscr{P}_{i+1}^{(1)}$ is a partition into at most $4^{a_1^{\mathscr{P}_i}}$ parts, in our proof (due to the fact that the witness has $r(a_1^{\mathscr{P}_i})$ parts for each pair) we may have as many as $4^{r(a_1^{\mathscr{P}_i}) \times a_1^{\mathscr{P}_i}}$ partition classes in $\mathscr{P}_{i+1}^{(1)}$. Consequently, $t_{\mathrm{RL}}$ (which is an upper bound for the number of classes in the final partition) depends not only on $\delta_2$, but also on the function $r(\cdot)$. It is independent, however, of the cardinality of the vertex set $V$.

(*3*) In order to avoid the exceptional class $V_0$ we assume that the cardinality of $V$ is divisible by $(t_{\mathrm{RL}})!$. This allows us to redistribute all the vertices in $V_i$ which would remain from subdividing the witnesses. Such a lemma was considered, e.g., in [26].

## 5. Proof of: $\mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k)$

In order to simplify the presentation we break the proof into two parts. In the first part we deduce $\mathrm{RAL}(k)$ from $\mathrm{RL}(k)$ and the following lemma.

**Lemma 27.** *For every positive integer $s$, all positive reals $\nu$ and $\varepsilon$, and every vector $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \leq i \leq k-1$, there exist positive reals $\delta_{27}$ and $\xi_{27}$ and integers $t_{27}$ and $m_{27}$ such that the following holds. Suppose*

(*a*) *$m \geq m_{27}$ and $(t_{27})!$ divides $m$,*
(*b*) *$\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{27}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex,*
(*c*) *$\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{27}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and*
(*d*) *$\{\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}\}$ is a partition of $\mathcal{F}^{(k)}$, where each $\mathcal{H}_i^{(k)}$ is $(\nu/12, *, t_{27}^{2^k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for every $i \in [s]$.*

*Then there exists a partition $\{\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_s^{(k)}\}$ of $\mathcal{F}^{(k)}$ so that for every $i \in [s]$ the following holds*

(*i*) *$\mathcal{G}_i^{(k)}$ is $(\varepsilon, d(\mathcal{H}_i^{(k)} | \mathcal{R}^{(k-1)}))$-regular w.r.t. $\mathcal{R}^{(k-1)}$ and*
(*ii*) *$|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \leq \nu |\mathcal{K}_k(\mathcal{R}^{(k-1)})|$.*

In Section 5.1 we derive $\mathrm{RAL}(k)$ from Lemma 27 and $\mathrm{RL}(k)$, then, in Section 5.2, we give the proof of Lemma 27 which is based on another application of $\mathrm{RL}(k)$.

5.1. **Lemma 27 and $\mathrm{RL}(k)$ imply $\mathrm{RAL}(k)$.** The idea of this reduction is as follows. Let $\mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and $\mathscr{H}^{(k)}$ be given by $\mathrm{RAL}(k)$. We apply $\mathrm{RL}(k)$ to $\mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$ and $\mathscr{H}^{(k)}$. The constants will be chosen in such a way that after that application of $\mathrm{RL}(k)$ a "typical" polyad $\hat{\mathcal{P}}^{(k-1)}$ with its underlying complex $\boldsymbol{\mathcal{P}} = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$ matches the assumptions of Lemma 27 for $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{P}}$, $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{k-1})$ (where $\mathcal{O}^{(k)} \in \mathscr{O}^{(k)}$), and $\{\widetilde{\mathcal{H}}_h^{(k)} = \mathcal{H}_h^{(k)} \cap \mathcal{F}^{(k)} \colon \mathcal{H}_h^{(k)} \in \mathscr{H}^{(k)}$ and $\mathcal{H}_h^{(k)} \subseteq \mathcal{O}^{(k)}\}$. Lemma 27 then yields hypergraphs $\widetilde{\mathcal{G}}_h^{(k)}$ satisfying (*i*) and (*ii*) of Lemma 27. Repeating this for all "typical" polyads $\hat{\mathcal{P}}^{(k-1)}$ and $\mathcal{O}^{(k)} \in \mathscr{O}^{(k)}$ and taking appropriate care of the "untypical" case, then yields the promised hypergraphs $\mathcal{G}_1^{(k)} \dots \mathcal{G}_s^{(k)}$ with properties (*G1*)–(*G3*) of $\mathrm{RAL}(k)$. We give the technical details of this outline below.

*Proof:* RL($k$) $\wedge$ Lemma 27 $\Longrightarrow$ RAL($k$). Let positive constants $o_{\rm RAL}$, $s_{\rm RAL}$, $\eta_{\rm RAL}$, and $\nu_{\rm RAL}$, and a function $\varepsilon_{\rm RAL}\colon \mathbb{N}^{k-1}\to(0,1]$ be given (w.l.o.g. we may assume that $\varepsilon_{\rm RAL}$ is monotone in every coordinate). We have to determine $\mu_{\rm RAL}$, $t_{\rm RAL}$, and $n_{\rm RAL}$ (see (11)). Our proof relies on an application of RL($k$) followed by an application of Lemma 27. In order to match the assumptions of Lemma 27 the parameters for the application of RL($k$) have to match these assumptions. Consequently, "constant-wise" we first apply Lemma 27 to foresee what is needed for its application, which will be provided by RL($k$). With this in mind we set

$$s_{27}=s_{\rm RAL}\,,\qquad\qquad \nu_{27}=\nu_{\rm RAL}/2\,. \tag{5}$$

Note that for every choice of $\varepsilon$ and $d_1,\dots,d_{k-1}$ (satisfying $1/d_i\in\mathbb{N}$), Lemma 27 yields constants $\delta_{27}$, $\xi_{27}$, $t_{27}$, and $m_{27}$. Accordingly, we define functions $\delta_{\rm aux}$, $\xi_{\rm aux}\colon \mathbb{N}^{k-1}\to(0,1]$ and $t_{\rm aux}$, $m_{\rm aux}\colon \mathbb{N}^{k-1}\to\mathbb{N}$ mapping any $\boldsymbol{a}=(a_1,\dots,a_{k-1})\in\mathbb{N}^{k-1}$ to the corresponding constant from Lemma 27 with $\varepsilon=\varepsilon_{\rm RAL}(\boldsymbol{a})$ and $d_2=1/a_2,\dots,d_{k-1}=1/a_{k-1}$. More precisely, we set for $x\in\{\delta,\xi,t,m\}$

$$x_{\rm aux}(\boldsymbol{a})=x_{\rm L.27}\big(s=s_{27},\nu=\nu_{27},\varepsilon=\varepsilon_{\rm RAL}(\boldsymbol{a}),d_2=1/a_2,\dots,d_{k-1}=1/a_{k-1}\big) \tag{6}$$

where $x_{\rm L.27}(s,\nu,\varepsilon,d_2,\dots,d_{k-1})$ is given by Lemma 27 applied with constants $s$, $\nu$, $\varepsilon$, and $d_2,\dots,d_{k-1}$. Without loss of generality we assume that the functions defined in (6) are monotone in every coordinate.

We now choose the parameters for the application of RL($k$). For that we set

$$o_{\rm RL}=o_{\rm RAL}\,,\quad s_{\rm RL}=s_{\rm RAL}\,,\quad \eta_{\rm RL}=\eta_{\rm RAL}\,,\quad\text{and}\quad \delta_{k,{\rm RL}}=\min\left\{\frac{\nu_{27}}{12},\frac{\nu_{\rm RAL}}{2s_{\rm RAL}}\right\} \tag{7}$$

and consider functions $r_{\rm RL}\colon\mathbb{N}^{k-1}\to\mathbb{N}$ and $\delta_{\rm RL}\colon\mathbb{N}^{k-1}\to(0,1]$ defined for every integer vector $\boldsymbol{a}=(a_1,\dots,a_{k-1})$ by

$$r_{\rm RL}(\boldsymbol{a})=\big(t_{\rm aux}(\boldsymbol{a})\big)^{2^k}\quad\text{and} \tag{8}$$

$$\delta_{\rm RL}(\boldsymbol{a})=\min\big\{\varepsilon_{\rm RAL}(\boldsymbol{a}),\delta_{\rm aux}(\boldsymbol{a}),\varepsilon_{\rm DCL}\big(h=k-1,\ell=k,\gamma=\tfrac12,d_0=\min_{2\le i<k}a_i^{-1}\big)\big\}, \tag{9}$$

where $\varepsilon_{\rm DCL}(h,\ell,\gamma,d_0)$ is given by Theorem 18.

Having defined all input parameters of RL($k$), Lemma 23, in (7), (8) and (9), Lemma 23 now yields positive constants $\mu_{\rm RL}$, $t_{\rm RL}$, and $n_{\rm RL}$. We use $t_{\rm RL}$ to establish "worst case" estimates on the functions $\xi_{\rm aux}$, $t_{\rm aux}$, and $m_{\rm aux}$ and set

$$\xi_{\rm worst}=\xi_{\rm aux}(t_{\rm RL},\dots,t_{\rm RL})\,,\ t_{\rm worst}=t_{\rm aux}(t_{\rm RL},\dots,t_{\rm RL})\,,\qquad \text{and}\quad m_{\rm worst}=m_{\rm aux}(t_{\rm RL},\dots,t_{\rm RL}) \tag{10}$$

Finally, we define $\mu_{\rm RAL}$, $t_{\rm RAL}$, and $n_{\rm RAL}$ promised by RAL($k$). For that we set

$$\mu_{\rm RAL}=\min\left\{\mu_{\rm RL}\,,\frac{\varepsilon_{\rm RAL}(t_{\rm RL},\dots,t_{\rm RL})}{2t_{\rm RL}^{2^k}}\,,\frac{\xi_{\rm worst}}{2t_{\rm RL}^{2^k}}\right\},\quad t_{\rm RAL}=t_{\rm RL}+t_{\rm worst}\,,\quad\text{and}$$

$$n_{\rm RAL}=\max\left\{n_{\rm RL}\,,t_{\rm RL}m_{\rm worst}\,,t_{\rm RL}m_{\rm DCL}\big(h=k-1,\ell=k,\gamma=\tfrac12,d_0=t_{\rm RL}^{-1}\big)\right\}. \tag{11}$$

Note that for given input parameters $o_{\rm RAL}$, $s_{\rm RAL}$, $\eta_{\rm RAL}$, $\nu_{\rm RAL}$, and $\varepsilon_{\rm RAL}\colon\mathbb{N}^{k-1}\to(0,1]$ of RAL($k$), above in (11) we defined the corresponding output parameters. Now we need to show, that with this choice we will be able to verify RAL($k$), Lemma 25.

Let $V$, $\mathscr{O}_{\mathrm{RAL}}$, and $\mathscr{H}^{(k)}$ satisfying $(a)$–$(c)$ of RAL$(k)$, Lemma 25 be given, i.e.,

(RAL.$a$) $|V| = n \geq n_{\mathrm{RAL}}$ and $(t_{\mathrm{RAL}})!$ divides $n$,

(RAL.$b$) $\mathscr{O}_{\mathrm{RAL}} = \mathscr{O}_{\mathrm{RAL}}(k, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}}) = \{\mathscr{O}^{(j)}_{\mathrm{RAL}}\}_{j=1}^{k}$ is $(\eta^{\mathscr{O}_{\mathrm{RAL}}}, \mu_{\mathrm{RAL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RAL}}} > 0$) and $o_{\mathrm{RAL}}$-bounded, and

(RAL.$c$) $|\mathscr{H}^{(k)}| = s_{\mathrm{RAL}}$ and $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}_{\mathrm{RAL}}$.

Our objective is to find a family of partitions $\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RAL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$ on $V$ and a partition $\mathscr{G}^{(k)} = \{\mathcal{G}^{(k)}_1, \dots, \mathcal{G}^{(k)}_{s_{\mathrm{RAL}}}\}$ of $[V]^k$ so that

(RAL.$P1$) $\mathscr{P}_{\mathrm{RAL}}$ is $(\eta_{\mathrm{RAL}}, \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-equitable and $t_{\mathrm{RAL}}$-bounded,

(RAL.$P2$) $\mathscr{P}_{\mathrm{RAL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}^{(j)}_{\mathrm{RAL}}\}_{j=1}^{k-1}$,

(RAL.$G1$) $\mathcal{G}^{(k)}_i$ is perfectly $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-regular w.r.t. $\mathscr{P}_{\mathrm{RAL}}$ for every $i \in [s_{\mathrm{RAL}}]$,

(RAL.$G2$) $|\mathcal{G}^{(k)}_i \triangle \mathcal{H}^{(k)}_i| \leq \nu_{\mathrm{RAL}} n^k$ for every $i \in [s_{\mathrm{RAL}}]$, and

(RAL.$G3$) if $\mathcal{H}^{(k)}_i \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$ then $\mathcal{G}^{(k)}_i \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$ for every $i \in [s_{\mathrm{RAL}}]$ and $\mathscr{G}^{(k)} \prec \mathscr{O}^{(k)}_{\mathrm{RAL}}$.

Without loss of generality we may assume that

$$\mathcal{H}^{(k)}_i \neq \varnothing \quad \text{for every} \quad i \in [s_{\mathrm{RAL}}]\,. \tag{12}$$

Otherwise we simply set $\mathcal{G}^{(k)}_i = \varnothing$ for every $i \in [s_{\mathrm{RAL}}]$ for which $\mathcal{H}^{(k)}_i = \varnothing$ and obviously (RAL.$G1$)–(RAL.$G3$) holds for those $\mathcal{G}^{(k)}_i$ for any family of partitions $\mathscr{P}$.

As we already mentioned we want to apply RL$(k)$ to $V$, $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}^{(j)}_{\mathrm{RAL}}\}_{j=1}^{k-1}$, $\boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}} = (a_1^{\mathscr{O}_{\mathrm{RAL}}}, \dots, a_{k-1}^{\mathscr{O}_{\mathrm{RAL}}})$, and $\mathscr{H}^{(k)}$ with constants $o_{\mathrm{RL}}$, $s_{\mathrm{RL}}$, $\eta_{\mathrm{RL}}$, and $\delta_{k,\mathrm{RL}}$ defined in (7) and functions $r_{\mathrm{RL}}$ and $\delta_{\mathrm{RL}}$ defined in (8) and (9). For that we have to verify that

(RL.$a$) $|V| = n \geq n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides $n$,

(RL.$b$) $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}}) = \{\mathscr{O}^{(j)}_{\mathrm{RL}}\}_{j=1}^{k-1}$ is $(\eta^{\mathscr{O}_{\mathrm{RL}}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RL}}} > 0$) and $o_{\mathrm{RL}}$-bounded, and

(RL.$c$) $|\mathscr{H}^{(k)}| = s_{\mathrm{RL}}$.

We note that (RL.$a$) is an easy consequence of the choice of $n_{\mathrm{RAL}} \geq n_{\mathrm{RL}}$ and $t_{\mathrm{RAL}} \geq t_{\mathrm{RL}}$ in (11) and (RAL.$a$). Similarly, (RL.$b$) follows from the choice of $\mu_{\mathrm{RAL}} \leq \mu_{\mathrm{RL}}$ in (11) and (RAL.$b$), while (RL.$c$) is a consequence of (RAL.$c$) and the choice of $s_{\mathrm{RL}} = s_{\mathrm{RAL}}$ in (7). Having verified that (RL.$a$)–(RL.$c$) hold, we reason that there is a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on $V$ which satisfies properties $(P1)$, $(P2)$, and $(H)$ of Lemma 23

(RL.$P1$) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and $t_{\mathrm{RL}}$-bounded,

(RL.$P2$) $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1)$, and

(RL.$H$) $\mathcal{H}^{(k)}_i$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

We set

$$\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RL}} \quad \text{and} \quad \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}} = \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}\,. \tag{13}$$

It then follows from (RL.$P1$) and (RL.$P2$) and the choices of $\eta_{\mathrm{RL}} = \eta_{\mathrm{RAL}}$ in (7), $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leq \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ in (9), and $t_{\mathrm{RAL}} \geq t_{\mathrm{RL}}$ in (11), that

$$\mathscr{P}_{\mathrm{RAL}} \text{ satisfies (RAL.}P1\text{) and (RAL.}P2\text{)}\,. \tag{14}$$

It is left to ensure the existence of $\mathscr{G}^{(k)}$ of $[V]^k$ which satisfies (RAL.$G1$)–(RAL.$G3$).

Before we prove the existence of $\mathscr{G}^{(k)}$ we make some preparations, which simplify the presentation. We complete $\mathscr{O}^{(k)}_{\mathrm{RAL}}$ (which partitions $\mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$), to a partition of $[V]^k$. For that we set

$$\widetilde{\mathscr{O}}^{(k)} = \mathscr{O}^{(k)}_{\mathrm{RAL}} \cup \big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big)\,. \tag{15}$$

We also define for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$

$$I(\mathcal{O}^{(k)}) = \big\{i \in [s_{\mathrm{RAL}}] \colon \mathcal{H}_i^{(k)} \subseteq \mathcal{O}^{(k)} \text{ and } \mathcal{H}_i^{(k)} \neq \varnothing\big\}\,. \tag{16}$$

Note that due to (RAL.$c$), (12), and (15) the family $\{I(\mathcal{O}^{(k)}) \colon \mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}\}$ forms a partition of $[s_{\mathrm{RAL}}]$. Before we continue we make the observation.

**Claim 28.** *For every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ and $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ the following holds. Set $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, then $\mathcal{F}^{(k)}$ is $(2t_{\mathrm{RL}}^{2^k}\mu_{\mathrm{RAL}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.*

*Proof.* The claim is trivial if $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) = \varnothing$ and, hence, we assume that

$$\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing\,. \tag{17}$$

We distinguish two cases. From, $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1)$ (cf. (RL.$P2$)) we infer that either $\hat{\mathcal{P}}^{(k-1)}$ is contained in some polyad $\hat{\mathcal{O}}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$ or $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$. If $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$, then we have $\mathcal{O}^{(k)} = [V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$ (using (17)) and, consequently, $\mathcal{F}^{(k)} = \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$. Hence, $\mathcal{F}^{(k)}$ is $\xi$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $\xi > 0$ which yields the claim in that case.

On the other hand, if $\hat{\mathcal{P}}^{(k-1)} \subseteq \hat{\mathcal{O}}^{(k-1)}$ for some $\hat{\mathcal{O}}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$, then we have due to (17) and the fact that $\mathscr{O}_{\mathrm{RAL}}$ is a family of partitions (cf. Definition 10) that $\mathcal{O}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})$. Therefore, if follows from (RAL.$b$) and the definition of regularity (Definition 7) that

$$\mathcal{F}^{(k)} \text{ is } \left(\mu_{\mathrm{RAL}}\frac{|\mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})|}{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|}\right)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,. \tag{18}$$

Clearly, $|\mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})| \leq n^k$ and due to the choice of $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leq \varepsilon_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = \min_{2\leq i<k} 1/a_i^{\mathscr{P}_{\mathrm{RL}}})$ in (9), the appropriate choice of $n_{\mathrm{RAL}} \geq t_{\mathrm{RL}} \times m_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = t_{\mathrm{RL}}^{-1})$ in (11), and (RL.$P1$), by Theorem 18, we infer

$$\Big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\Big| \geq \frac{1}{2}\prod_{j=2}^{k-1}\left(\frac{1}{a_j^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{j}} \times \left(\frac{n}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^k \geq \frac{n^k}{2t_{\mathrm{RL}}^{2^k}}\,.$$

and the claim follows. □

We now continue with the proof of the existence of the partition $\mathscr{G}^{(k)}$ of $[V]^k$ which satisfies (RAL.$G1$)–(RAL.$G3$). For that we will mainly use Lemma 27 applied to the polyads of $\mathscr{P}_{\mathrm{RL}}$. However, we distinguish between two types of polyads and set

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} = \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}} \colon \mathcal{H}_i^{(k)} \text{ is } (\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ for every } i \in [s_{\mathrm{RL}}]\Big\}\,.$$

**Case 1** ($\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$)**.** In this case let $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and set $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ with $\hat{\mathcal{P}}^{(k-1)}(K) = \hat{\mathcal{P}}^{(k-1)}$ (see (1)). Let $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ be such that

$$\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing\,, \tag{19}$$

and set

$$\widetilde{\mathcal{H}}^{(k)}_i = \mathcal{H}^{(k)}_i \cap \mathcal{F}^{(k)} = \mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \qquad \text{for} \quad i \in I(\mathcal{O}^{(k)})\,. \tag{20}$$

We want to apply Lemma 27 with parameters $s = s_{27}$, $\nu = \nu_{27}$, $\varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$, and $d_i = 1/a_i^{\mathscr{P}_{\mathrm{RL}}}$ for $2 \leq i < k$. Note that due to the definition of the functions $\delta_{\mathrm{aux}}$, $\xi_{\mathrm{aux}}$, $t_{\mathrm{aux}}$, and $m_{\mathrm{aux}}$ in view of (6), Lemma 27 yields constants $\delta_{27}$, $\xi_{27}$, $t_{27}$ and $m_{27}$ which satisfy

$$\delta_{27} = \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\,, \quad \xi_{27} = \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\,,$$
$$t_{27} = t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\,, \quad \text{and} \quad m_{27} = m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\,.$$

In order to apply Lemma 27 with the chosen parameters to $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$, $\boldsymbol{\mathcal{R}}$, $\mathcal{F}^{(k)}$, and $\{\widetilde{\mathcal{H}}^{(k)}_i \colon i \in I(\mathcal{O}^{(k)})\}$ we have to verify

(L.27.$a$) $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m \geq m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ and $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides $m$,
(L.27.$b$) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{d})$-regular $(m, k, k-1)$-complex, where $\boldsymbol{d} = (1/a_2^{\mathscr{P}_{\mathrm{RL}}}, \dots, 1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$,
(L.27.$c$) $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and
(L.27.$d$) the family $\{\widetilde{\mathcal{H}}^{(k)}_i \colon i \in I(\mathcal{O}^{(k)})\}$ partitions $\mathcal{F}^{(k)}$, $|I(\mathcal{O}^{(k)})| \leq s_{27}$, and each $\widetilde{\mathcal{H}}^{(k)}_i$ is $\big(\nu_{27}/12, *, (t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))^{2^k}\big)$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for every $i \in I(\mathcal{O}^{(k)})$.

The verification of (L.27.$a$)–(L.27.$d$) is straightforward, but somewhat technical. We give the details below.

Due to (RAL.$a$), (11), (10), (RL.$b$) and the monotonicity of the function $m_{\mathrm{aux}}$ we have

$$n \geq t_{\mathrm{RL}} \times m_{\mathrm{worst}} \geq a_1^{\mathscr{P}_{\mathrm{RL}}} \times m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\,.$$

In order to verify (L.27.$a$) it is left to show that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$. For that we note that due to the definition of $t_{\mathrm{RAL}}$ in (11) we have $t_{\mathrm{RAL}} = t_{\mathrm{RL}} + t_{\mathrm{worst}}$, which due to (RAL.$a$) yields $(t_{\mathrm{RL}} + t_{\mathrm{worst}})!$ divides $n$. Consequently, $(t_{\mathrm{RL}})!(t_{\mathrm{worst}})!$ divides $n$ (to see this consider $\binom{t_{\mathrm{RL}}+t_{\mathrm{worst}}}{t_{\mathrm{worst}}}$). Hence, from $a_1^{\mathscr{P}_{\mathrm{RL}}} \leq t_{\mathrm{RL}}$ (cf. (RL.$b$)) it follows that $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m$ is divisible by $(t_{\mathrm{worst}})!$. It now follows that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides $m$ since $t_{\mathrm{worst}} \geq t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ due to the monotonicity of the function $t_{\mathrm{aux}}$.

Part (L.27.$b$) follows easily from (RL.$b$) and the choice of the function $\delta_{\mathrm{RL}}$ in (9) ensuring that $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leq \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Next we verify (L.27.$c$). It follows from the definition of $\mathcal{F}^{(k)}$ that $\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)}$ underlies $\mathcal{F}^{(k)}$. The second assertion of (L.27.$c$) follows from $2t_{\mathrm{RL}}^{2^k}\mu_{\mathrm{RAL}} \leq \xi_{\mathrm{worst}} \leq \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ (cf. (11) and (10)) and Claim 28.

Finally, it is left to verify (L.27.$d$). It follows from the definitions in (16) and (20) and the fact that $\mathscr{H}^{(k)}$ is a partition of $[V]^k$ (cf. (RAL.$d$)), that $\{\widetilde{\mathcal{H}}^{(k)}_i \colon i \in I(\mathcal{O}^{(k)})\}$ partitions $\mathcal{F}^{(k)}$. Clearly, $|I(\mathcal{O}^{(k)})| \leq s_{27}$. Moreover, from the assumption of this case ($\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$) we know that $\widetilde{\mathcal{H}}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)}$ for every $i \in I(\mathcal{O}^{(k)})$. Therefore, (L.27.$d$) follows from the choice of $\delta_{k,\mathrm{RL}} \le \nu_{27}/12$ in (7) and $r_{\mathrm{RL}}(a^{\mathscr{P}_{\mathrm{RL}}}) = (t_{\mathrm{aux}}(a^{\mathscr{P}_{\mathrm{RL}}}))^{2^k}$ in (8).

Having verified (L.27.$a$)–(L.27.$d$), we are now able to apply Lemma 27 and infer the existence of a partition $\{\widetilde{\mathcal{G}}_i^{(k)} \colon i \in I(\mathcal{O}^{(k)})\}$ of $\mathcal{F}^{(k)}$ so that for every $i \in I(\mathcal{O}^{(k)})$

(L.27.$i$) $\widetilde{\mathcal{G}}_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)} = \mathcal{R}^{(k-1)}$, and
(L.27.$ii$) $|\widetilde{\mathcal{G}}_i^{(k)} \triangle \widetilde{\mathcal{H}}_i^{(k)}| \le \nu_{27}|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|$.

For $i \in I(\mathcal{O}^{(k)})$ each $\widetilde{\mathcal{G}}_i^{(k)}$ given above defines $\mathcal{G}_i^{(k)}$ restricted to the polyad $\hat{\mathcal{P}}^{(k-1)}$. Formally we set

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \widetilde{\mathcal{G}}_i^{(k)} \quad \text{for} \quad i \in I(\mathcal{O}^{(k)})\,, \tag{21}$$

and repeat the procedure for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ satisfying (19). $\Diamond$

**Case 2** ($\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$)**.** Again, let $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and set $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ with $\hat{\mathcal{P}}^{(k-1)}(K) = \hat{\mathcal{P}}^{(k-1)}$ (see (1)). Let $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ be such that

$$\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing\,. \tag{22}$$

In this case fix some index $i_0 \in I(\mathcal{O}^{(k)})$. We then define for $i \in I(\mathcal{O}^{(k)})$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \begin{cases} \mathcal{F}^{(k)} & \text{for} \quad i = i_0\,, \\ \varnothing & \text{for} \quad i \neq i_0 \in I(\mathcal{O}^{(k)})\,. \end{cases} \tag{23}$$

For later reference we note that for every $i \in I(\mathcal{O}^{(k)})$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \subseteq \mathcal{O}^{(k)} \tag{24}$$

and

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \text{ is } \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,. \tag{25}$$

Indeed, (24) is trivial for every $i \in I(\mathcal{O}^{(k)})$ and (25) is trivial for $i \neq i_0$. In the case $i = i_0$ we have $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and (25) follows from Claim 28 and the choice of $\mu_{\mathrm{RAL}}$ in (11) ensuring $2t_{\mathrm{RL}}^{2^k} \times \mu_{\mathrm{RAL}} \le \varepsilon_{\mathrm{RAL}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}}) \le \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Again we repeat this procedure for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ satisfying (22). $\Diamond$

We note that due to the both cases above the following statement holds:

($*$) For every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ and every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ with $\mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing$ we have $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \colon i \in I(\mathcal{O}^{(k)})\}$ is a partition of $\mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$.

Now we define the partition $\mathscr{G}^{(k)}$ and verify (RAL.$G1$)–(RAL.$G3$). For that we set for $i \in [s_{\mathrm{RAL}}]$

$$\mathcal{G}_i^{(k)} = \bigcup \left\{ \mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}} \right\}. \tag{26}$$

Since $\widetilde{\mathscr{O}}^{(k)}$ is a partition of $[V]^k$ we infer from ($*$) that $\mathscr{G}^{(k)} = \{\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_{s_{\mathrm{RAL}}}^{(k)}\}$ forms a partition of $[V]^k$.

Next we verify (RAL.$G1$). From (L.27.$i$) (combined with (21)) and (25) we conclude that for all $i \in [s_{\mathrm{RAL}}]$ and all $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ the defined $\mathcal{G}_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(a^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. Consequently, the definition of $\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RL}}$ and $\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}} = \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}$ in (13) yields (RAL.$G1$).

In order to show (RAL.*G2*), let $i \in [s_{\mathrm{RAL}}]$ be fixed. It follows from (L.27.*ii*) that

$$\sum \Big\{ \big|(\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}) \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} \Big\} \overset{(5)}{\leq} \frac{1}{2}\nu_{\mathrm{RAL}} n^k \tag{27}$$

Moreover, from (RL.*H*) and Definition 15 we infer

$$\sum \Big\{ \big|(\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}) \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| \colon \hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} \Big\}$$
$$\leq \sum \Big\{ |\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| \colon \hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} \Big\} \leq s_{\mathrm{RL}} \delta_{k,\mathrm{RL}} n^k \overset{(7)}{\leq} \frac{1}{2}\nu_{\mathrm{RAL}} n^k \tag{28}$$

In view of (13) the inequalities (27) and (28) then yield (RAL.*G2*).

Finally, we consider (RAL.*G3*). For that for each $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ we set

$$J(\mathcal{O}^{(k)}) = \{ i \in [s_{\mathrm{RAL}}] \colon \mathcal{G}_i^{(k)} \cap \mathcal{O}^{(k)} \neq \varnothing \}\,.$$

Since (12) and $\mathscr{G}^{(k)}$ is a partition of $[V]^k$, the two assertions in (RAL.*G3*) are implied by the following two statements which we verify below

$$J\big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big) \subseteq I\big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big)\,, \text{and} \tag{29}$$

$$J(\mathcal{O}_1^{(k)}) \cap J(\mathcal{O}_2^{(k)}) = \varnothing \quad \text{for all} \quad \mathcal{O}_1^{(k)} \neq \mathcal{O}_2^{(k)} \in \widetilde{\mathscr{O}}^{(k)}\,. \tag{30}$$

From $(*)$ we infer for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ that if $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \cap \mathcal{O}^{(k)} \neq \varnothing$ then $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \subseteq \mathcal{O}^{(k)}$. Consequently, $(*)$ yields

$$J(\mathcal{O}^{(k)}) \subseteq I(\mathcal{O}^{(k)}) \tag{31}$$

for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$, which gives (29).

Moreover, since $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$ and, therefore, $\mathscr{H}^{(k)} \prec \widetilde{\mathscr{O}}^{(k)}$ (see (15), we have $I(\mathcal{O}_1^{(k)}) \cap I(\mathcal{O}_2^{(k)}) = \varnothing$ for all distinct $\mathcal{O}_1^{(k)}$ and $\mathcal{O}_2^{(k)}$ from $\widetilde{\mathscr{O}}^{(k)}$. Hence, (30) holds as well, and consequently (RAL.*G3*) follows.

From the discussion above and (14) we infer that $\mathscr{P}_{\mathrm{RAL}}$ defined in (13) and $\mathscr{G}^{(k)}$ defined in (26) satisfy the conclusions of RAL($k$), Lemma 25, i.e., (RAL.*P1*)–(RAL.*G3*). □

5.2. RL($k$) **implies Lemma 27.** The proof of Lemma 27 is the heart of the implication RL($k$) $\Longrightarrow$ RAL($k$) and its idea resembles the main idea from the work of B. Nagle and the authors in [20]. Before we give with the detailed proof below, we briefly discuss the main idea.

Recall that in Lemma 27 a $(\delta_{27}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ and a $\xi$-regular $k$-uniform hypergraph $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ are given. Moreover, we are given a partition $\mathscr{H}^{(k)} = \{\mathcal{H}_i^{(k)} \colon i \in [s_{27}]\}$ of $\mathcal{F}^{(k)}$, where every $\mathcal{H}_i^{(k)}$ is $(\nu, *, t_{27}^{2^k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$. We will apply RL($k$) to regularize every $\mathcal{H}_i^{(k)} \in \mathscr{H}^{(k)}$ with some appropriately chosen $\delta_k$ less than the given $\varepsilon$. For this regularization we apply the variant of RL($k$) discussed in Remark 24, which allows us to find a $t_{\mathrm{RL}}$-bounded family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) = \{\mathscr{P}^{(j)}_{\mathrm{RL}}\}_{j=1}^{k-1}$ such that for each $j = 1, \dots, k-1$ and each $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\mathrm{RL}}$ either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$. Since each $\mathcal{H}_i^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$, we will focus on the "interesting" part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$

which are subsets of $\mathcal{R}^{(k-1)}$. For that we set

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}}) = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}} \colon \hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{R}^{(k-1)}\right\}.$$

From RL($k$) we infer that for every "typical" $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$

($i$) $\mathcal{H}^{(k)}_i$ is $(\delta_k, d(\mathcal{H}^{(k)}_i|\hat{\mathcal{P}}^{(k-1)}), 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{27}]$.

Moreover, we will prove (cf. Claim 30) that for every $i \in [s_{27}]$ the typical density $d(\mathcal{H}^{(k)}_i|\hat{\mathcal{P}}^{(k-1)})$ will be "near" to the density of $\mathcal{H}^{(k)}_i$ in $\mathcal{R}^{(k-1)}$, i.e.,

($ii$) $|d(\mathcal{H}^{(k)}_i|\hat{\mathcal{P}}^{(k-1)}) - d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)})| \le \nu/6$ for "most" $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$.

Property ($ii$) is the key observation in the proof of Lemma 27. Its proof is based on our choice of $t_{27} \ge t_{\mathrm{RL}}$ and $t^{2^k}_{\mathrm{RL}} \ge |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})|$. The proof of ($ii$) then is simple. Assuming that there is a constant fraction of polyads in $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ which violate ($ii$) gives rise to a witness that is $(\nu/12, *, t^{2^k}_{27})$-irregular w.r.t. $\mathcal{R}^{(k-1)}$. (The choice of $t_{27} \ge t_{\mathrm{RL}}$ allows us to "look" into a constant fraction of polyads in $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$.)

Combining, ($i$) and ($ii$) with an appropriate use of the slicing lemma, Proposition 22, allows us to prove that for a typical $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$, $\mathcal{H}^{(k)}_i$ needs to be altered only slightly (in less than $\nu/6$ proportion of the number of cliques in $\hat{\mathcal{P}}^{(k-1)}$) to become $(\varepsilon^2/4, d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. In other words, the resulting graph, which we will denote by $\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})$, maintains large degree of regularity (we will choose $\delta_k \ll \varepsilon$), while its density will be $\sim d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)})$.

On the other hand in the rare case of an atypical polyad $\hat{\mathcal{P}}^{(k-1)}$ for which ($i$) or ($ii$) does not hold for $\mathcal{H}^{(k)}_i$ we use slicing lemma to replace $\mathcal{H}^{(k)}_i$ by a randomly chosen (and therefore extremely regular) $\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})$, with $d(\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})|\hat{\mathcal{P}}^{(k-1)}) \sim d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)})$.

For each $i \in [s_{27}]$ we then set $\mathcal{G}^{(k)}_i = \bigcup \mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})$ where the union is taken over all (typical and atypical) $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$. Since, $\mathcal{G}^{(k)}_i$ obtained that way is $(\varepsilon^2/4, d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)})$-regular for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$, Proposition 19 then yields that $\mathcal{G}^{(k)}_i$ is $(\varepsilon, d(\mathcal{H}^{(k)}_i|\mathcal{R}^{(k-1)})$-regular w.r.t. $\mathcal{R}^{(k-1)}$. Moreover, since in the typical case we changed $\mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ only "slightly" to become $\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})$ and since the atypical case, in which more drastic changes are needed, happens rarely, we will be able to prove that $|\mathcal{G}^{(k)}_i \triangle \mathcal{H}^{(k)}_i| \le \nu n^k$.

We now give the technical details of the proof of Lemma 27, sketched above.

*Proof:* RL($k$) $\Longrightarrow$ Lemma 27. Let positive reals $s_{27}$, $\nu_{27}$, and $\varepsilon_{27}$ and a vector $\boldsymbol{d}_{27} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \le i < k$ be given. Lemma 27 is trivial for $\nu_{27} > 1$. Moreover, without loss of generality we may assume that

$$\varepsilon_{27} < \nu_{27} \le 1\,. \tag{32}$$

We will apply RL($k$). For that we set[1]

$$o_{\mathrm{RL}} = \max_{2\le i<k} 1/d_i\,, \qquad s_{\mathrm{RL}} = s_{27}+1\,, \qquad \eta_{\mathrm{RL}} = 10^{-2}\,, \tag{33}$$

$$\text{and} \qquad \delta_{k,\mathrm{RL}} = \min\left\{\frac{\nu_{27}\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}}{6\times s_{27}\times k^k}\,, \frac{\varepsilon_{27}^2}{384 s_{27}}, \frac{\nu_{27}}{18}\right\} \tag{34}$$

and consider functions $r_{\mathrm{RL}}\colon \mathbb{N}^{k-1}\to\mathbb{N}$, and $\delta_{\mathrm{RL}}\colon \mathbb{N}^{k-1}\to(0,1]$ defined for every $\boldsymbol{a}=(a_1,\dots,a_{k-1})\in\mathbb{N}^{k-1}$ by

$$r_{\mathrm{RL}}(\boldsymbol{a}) = 1 \quad\text{and}\quad \delta_{\mathrm{RL}}(\boldsymbol{a}) = \varepsilon_{\mathrm{DCL}}\Big(h=k-1, \ell=k, \gamma=\tfrac{\nu_{27}}{48}, d_0=\min_{2\le i<k} a_i^{-1}\Big)\,, \tag{35}$$

where $\varepsilon_{\mathrm{DCL}}(h,\ell,\gamma,d_0)$ is given by Theorem 18.

Having defined all input parameters of RL($k$), Lemma 23, in (33), (34), and (35), Lemma 23 now yields positive constants $\mu_{\mathrm{RL}}$, $t_{\mathrm{RL}}$, and $n_{\mathrm{RL}}$. We define $\delta_{27}$, $\xi_{27}$, $t_{27}$, and $m_{27}$ promised by Lemma 27. For that we set $t_{27}=t_{\mathrm{RL}}$,

$$\delta_{27} = \min\left\{\tfrac{\mu_{\mathrm{RL}}}{3}\,, \varepsilon_{\mathrm{DCL}}\Big(h=k-1,\ell=k,\gamma=\tfrac12, d_0=\min_{2\le i<k} d_i\Big)\right\},\ \ \xi_{27} = \frac{\varepsilon_{27}^2}{192 t_{\mathrm{RL}}^{2^k}}\,, \tag{36}$$

and let $m_{27}$ be sufficiently large.

Having defined all the parameters of Lemma 27, now let $m$, $\boldsymbol{\mathcal{R}}$, $\mathcal{F}^{(k)}$, and $\mathscr{H}^{(k)}$ satisfying $(a)$–$(d)$ of Lemma 27 for these parameters be given, i.e.,

(L.27.$a$) $m\ge m_{27}$ and $(t_{27})!$ divides $m$,

(L.27.$b$) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{27},\boldsymbol{d}_{27})$-regular $(m,k,k-1)$-complex with vertex set $V=V_1\cup\dots\cup V_k$,

(L.27.$c$) $\mathcal{F}^{(k)}\subseteq\mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{27}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and

(L.27.$d$) the family $\mathscr{H}^{(k)} = \{\mathcal{H}_1^{(k)},\dots,\mathcal{H}_{s_{27}}^{(k)}\}$ is a partition of $\mathcal{F}^{(k)}$ and every $\mathcal{H}_i^{(k)}$ is $(\nu_{27}/12, *, t_{27}^{2^k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for $i\in[s_{27}]$.

We have to ensure the existence of a partition $\mathscr{G}^{(k)} = \{\mathcal{G}_1^{(k)},\dots,\mathcal{G}_{s_{27}}^{(k)}\}$ of $\mathcal{F}^{(k)}$ so that for every $i\in[s_{27}]$

(L.27.$i$) $\mathcal{G}_i^{(k)}$ is $\varepsilon_{27}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and

(L.27.$ii$) $|\mathcal{G}_i^{(k)}\triangle\mathcal{H}_i^{(k)}|\le\nu_{27}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|$.

Before we start we note for later use that due to (L.27.$b$) and the choice of $\delta_{27}\le\varepsilon_{\mathrm{DCL}}(h=k-1,\ell=k,\gamma=1/2,d_0=\min_{2\le i<k} d_i)$ in (36) we infer for sufficiently large $m$ by DCL, Theorem 18, that

$$\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big| = \left(1\pm\frac12\right)\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\times m^k\,. \tag{37}$$

Our proof is based on the variant of RL($k$), Lemma 23, discussed in Remark 24. More precisely we want to apply this variant of Lemma 23 with the constants and functions chosen in (33), (34), and (35) to $V$, $\boldsymbol{\mathcal{R}}$, and $\mathcal{H}_0^{(k)}\cup\{\mathcal{H}_1^{(k)},\dots,\mathcal{H}_{s_{27}}^{(k)}\}$, where

$$\mathcal{H}_0^{(k)} = [V]^k\setminus\mathcal{F}^{(k)} = [V]^k\setminus\bigcup_{i\in[s_{27}]}\mathcal{H}_i^{(k)}\,. \tag{38}$$

[1]Since we later are only interested in partition classes $\mathcal{P}^{(j)}$, which are sub-hypergraphs of the given $\mathcal{R}^{(j)}$ (see, e.g., (40)), the constant $\eta_{\mathrm{RL}}$ is unessential for our proof and any positive constant value would do.

We artificially add $\mathcal{H}_0^{(k)}$ only to obtain a partition of $[V]^k$, to formally match the assumption $(c)$ of RL$(k)$ (see (RL.$c$) below). We have to verify the following assumptions of Lemma 23 (see also Remark 24).

(RL.$a'$) $|V| = km \geq n_{\mathrm{RL}}$, $V = V_1 \cup \dots \cup V_k$ with $V_i = m$ and $t_{\mathrm{RL}}!$ divides $m$,

(RL.$b'$) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3, \boldsymbol{d}_{27})$-regular $(m,k,k-1)$-complex, where $\boldsymbol{d}_{27} = (d_2, \dots, d_{k-1})$, $1/d_i \in \mathbb{N}$ and $1/d_i \leq o_{\mathrm{RL}}$ for $2 \leq i < k$, and $\mathcal{R}^{(1)} = V_1 \cup \dots \cup V_k$, and

(RL.$c$) $\{\mathcal{H}_0^{(k)}, \mathcal{H}_1^{(k)}, \dots, \mathcal{H}_{s_{27}}^{(k)}\}$ is a partition of $[V]^k$ into $s_{\mathrm{RL}}$ parts.

Note that (RL.$a'$) follows from (L.27.$a$) and the choice of $t_{27}$ in (36) for sufficiently large $m$. Moreover, (RL.$b'$) is a consequence of the assumption on $\boldsymbol{d}_{27}$, and (L.27.$b$) combined with the choice of $\delta_{27}$ in (36) and $o_{\mathrm{RL}}$ in (33). Similarly, (RL.$c$) follows from (L.27.$d$) in conjunction with (38) and the choice of $s_{\mathrm{RL}}$ in (33).

Having verified that (RL.$a'$), (RL.$b'$), and (RL.$c$) hold, Lemma 23 then ensures the existence of a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on $V$ which satisfies the following properties:

(RL.$P1$) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and $t_{\mathrm{RL}}$-bounded,

(RL.$P2'$) $\mathscr{P}^{(1)} \prec \mathcal{R}^{(1)} = V_1 \cup \dots \cup V_k$ and for every $2 \leq j < k$ and every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ we have either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$, and

(RL.$H$) $\mathcal{H}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

Before we continue with the proof we make a few observations and develop some notation. To an arbitrary polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ consider its corresponding $(m/a_1^{\mathscr{P}_{\mathrm{RL}}}, k, k-1)$-complex $\boldsymbol{\mathcal{P}} = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$. (More precisely, recalling (1), $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ for any $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$). Due to (RL.$P1$) and part $(c)$ of Definition 12, $\boldsymbol{\mathcal{P}}$ is an $(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), (1/a_2^{\mathscr{P}_{\mathrm{RL}}}, \dots, 1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$-regular $(m/a_1^{\mathscr{P}_{\mathrm{RL}}}, k, k-1)$-complex. From the choice of the function $\delta_{\mathrm{RL}}$ in (35)we infer for sufficiently large $m$ by Theorem 18 that

$$\left|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\right| = \left(1 \pm \frac{\nu_{27}}{48}\right) \prod_{h=2}^{k-1} \left(\frac{1}{a_h^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{h}} \times \left(\frac{m}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^k, \tag{39}$$

holds for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$.

Since each $\mathcal{H}_i^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ for the rest of the proof we will focus to the "interesting" part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads which are sub-hypergraphs of $\mathcal{R}^{(k-1)}$. To this end we set

$$\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}}) = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)} \colon \hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{R}^{(k-1)}\right\}. \tag{40}$$

Note that due to (RL.$P2'$) and the properties of $\mathscr{P}_{\mathrm{RL}}$ we have that

$$\left\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})\right\} \text{ partitions } \mathcal{K}_k(\mathcal{R}^{(k-1)}). \tag{41}$$

To simplify the notation we set

$$d_{\mathscr{H},\mathcal{R}}(i) = d(\mathcal{H}_i^{(k)} | \mathcal{R}^{(k-1)}).$$

The following claim, ensures the existence of a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \colon i \in [s_{27}]\}$ of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for every polyad $\hat{\mathcal{P}}^{(k-1)}$ in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$ with the property that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{27}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for each $i \in [s_{27}]$. This property will enable us to use Proposition 21 to infer property (L.27.$i$) for $\mathcal{G}_i^{(k)}$ defined in the obvious way.

In order to verify (L.27.$ii$) we will need some additional information concerning the $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon i\in[s_{27}]\}$. Here our analysis splits into two cases and we define[2]

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})=\Big\{\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\colon \mathcal{H}_i^{(k)} \text{ is } (\delta_{k,\mathrm{RL}},*,r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ for every } i\in[s_{27}]\Big\}. \tag{42}$$

Below we present two claims, based on which we will ensure the existence of $\{\mathcal{G}_i^{(k)}\colon i\in[s_{27}]\}$ with the desired properties (L.27.$i$) and (L.27.$ii$). We then give the proofs of the claims.

**Claim 29.** *For any $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ there exist a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon i\in[s_{27}]\}$ of $\mathcal{F}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ such that for every $i\in[s_{27}]$*

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \textit{ is } (\varepsilon_{27}^2/4, d_{\mathscr{H},\mathcal{R}}(i))\textit{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,. \tag{43}$$

*Moreover, if $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})$, then the partition $\{\mathcal{G}_i^{(k)}\colon i\in[s_{27}]\}$ has the additional property that for every $i\in[s_{27}]$*

$$\Big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\Big| \le \big(\big|d_{\mathscr{H},\mathcal{R}}(i)-d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)})\big|+\tfrac{\nu_{27}}{6}\big)\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| \tag{44}$$

In order to verify (L.27.$ii$) we need further control over the quantity considered in (44). The following claim ensures that "typically" $|d_{\mathscr{H},\mathcal{R}}(i)-d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)})|\le\frac{\nu_{27}}{6}$. For that we define for every $i\in[s_{27}]$

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)}) =\big\{\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\colon \big|d_{\mathscr{H},\mathcal{R}}(i)-d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)})\big|>\tfrac{\nu_{27}}{6}\big\}. \tag{45}$$

**Claim 30.** *For every $i\in[s_{27}]$*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})\Big\}\Big|\le\frac{\nu_{27}}{3}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,. \tag{46}$$

We now finish the proof of Lemma 27 based on Claim 29 and Claim 30. We use Claim 29 and set $\mathcal{G}_i^{(k)}$ for every $i\in[s_{27}]$ equal to

$$\mathcal{G}_i^{(k)}=\bigcup\Big\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\Big\}\,. \tag{47}$$

From $\mathcal{F}^{(k)}\subseteq\mathcal{K}_k(\mathcal{R}^{(k-1)})$ (cf. (L.27.$c$)) combined with (41) and Claim 29 we infer that $\mathscr{G}^{(k)}=\{\mathcal{G}_i^{(k)}\colon i\in[s_{27}]\}$ defined in (47) is a partition of $\mathcal{F}^{(k)}$.

Now, we have to verify (L.27.$i$) and (L.27.$ii$) for every fixed $i\in[s_{27}]$ and this choice of $\mathscr{G}^{(k)}$. So let $i\in[s_{27}]$ be fixed.

We start with property (L.27.$i$). Due to (41) the two families $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ and $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\}$ satisfy Setup 20 for $j=k$, and $t=$

[2]Note that we exclude the artificially added hypergraph $\mathcal{H}_0^{(k)}$ in the definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})$.

$|\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})|$. Consequently, in view of (43) we can apply Proposition 21 with $r=1$, $\delta=\varepsilon_{27}^2/4$, and $d=d_{\mathscr{H},\mathcal{R}}(i)$, to infer

$$\mathcal{G}_i^{(k)} \text{ is } \big(\varepsilon_{27}, d_{\mathscr{H},\mathcal{R}}(i)\big)\text{-regular w.r.t. } \bigcup\Big\{\hat{\mathcal{P}}^{(k-1)}\colon\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\Big\}=\mathcal{R}^{(k-1)},$$

and, therefore, (L.27.$i$) holds.

We now focus on (L.27.$ii$) for a fixed $i\in[s_{27}]$. We will estimate $|\mathcal{G}_i^{(k)}\triangle\mathcal{H}_i^{(k)}|$ as the sum of the symmetric difference taken over all polyads $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$. In this sum we distinguish among polyads in which some $\mathcal{H}^{(k)}\in\mathscr{H}^{(k)}$ is "irregular", in which $\mathcal{H}_i^{(k)}$ has "bad" (atypical) density and the remaining "typical" polyads in which $\mathcal{H}_i^{(k)}$ has the correct density and every $\mathcal{H}^{(k)}\in\mathscr{H}^{(k)}$ is regular. With this in mind we set

$$\begin{aligned}\mathfrak{D}_{\mathrm{irreg}}(i)=\sum\Big\{\Big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\Big|\colon&\\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\setminus\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})&\Big\}\\ \mathfrak{D}_{\mathrm{typ}}(i)=\sum\Big\{\Big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\Big|\colon&\\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})\setminus\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})&\Big\}\\ \mathfrak{D}_{\mathrm{bad}}(i)=\sum\Big\{\Big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\Big|\colon&\\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})&\Big\}\end{aligned}$$

and note that

$$\big|\mathcal{G}_i^{(k)}\triangle\mathcal{H}_i^{(k)}\big|\le\mathfrak{D}_{\mathrm{irreg}}(i)+\mathfrak{D}_{\mathrm{typ}}(i)+\mathfrak{D}_{\mathrm{bad}}(i)\,. \tag{48}$$

In the following we bound each of the terms of (48) separately. We start with $\mathfrak{D}_{\mathrm{irreg}}(i)$. Due to (RL.$H$) and the definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})$ in (42) we have

$$\sum\Big\{\Big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\Big|\colon\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\setminus\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})\Big\}\le s_{27}\times\delta_{k,\mathrm{RL}}k^km^k\,.$$

Clearly, the left-hand side of the last inequality is an upper bound on $\mathfrak{D}_{\mathrm{irreg}}(i)$ and we infer

$$\mathfrak{D}_{\mathrm{irreg}}(i)\le s_{27}\delta_{k,\mathrm{RL}}k^km^k\overset{(34)}{\le}\frac{\nu_{27}}{6}\prod_{h=2}^{k-1}d_h^{\binom{k}{h}}\times m^k\overset{(37)}{\le}\frac{\nu_{27}}{3}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,. \tag{49}$$

We consider $\mathfrak{D}_{\mathrm{typ}}(i)$. Since in view of (45) for each $\hat{\mathcal{P}}^{(k-1)}\notin\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})$ we have $|d_{\mathscr{H},\mathcal{R}}(i)-d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)})|\le\nu_{27}/6$, we infer from (44) that for every $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}\setminus\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})$

$$\Big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\Big|\le\frac{\nu_{27}}{3}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,.$$

Consequently, directly from the definition of $\mathfrak{D}_{\mathrm{typ}}(i)$, we infer

$$\begin{aligned}\mathfrak{D}_{\mathrm{typ}}(i)&\le\frac{\nu_{27}}{3}\sum\Big\{\Big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\Big|\colon\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})\setminus\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}},\mathcal{H}_i^{(k)})\Big\}\\&\le\frac{\nu_{27}}{3}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,.\end{aligned} \tag{50}$$

Finally, we derive a bound for $\mathfrak{D}_{\rm bad}(i)$ directly from the definition of $\mathfrak{D}_{\rm bad}(i)$ and (46)

$$\mathfrak{D}_{\rm bad}(i) \leq \frac{\nu_{27}}{3}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,. \tag{51}$$

We now conclude (L.27.*ii*) from (49), (50), and (51), applied to (48). In order to complete the proof of Lemma 27 we still have to verify Claim 29 and Claim 30, which we will do below. □

*Proof of Claim 29.* Let $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\rm RL}(\boldsymbol{\mathcal{R}})$ be fixed. First we recall (39). Below we will apply the slicing lemma, Proposition 22 to sub-hypergraphs of $\hat{\mathcal{P}}^{(k-1)}$. For that, among others, we have to verify the assumption $(i)$ of Proposition 22, i.e.,

$$\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| \geq m^k/\ln m\,. \tag{52}$$

This, however, follows from (39) for sufficiently large $m$. Therefore, we don't have to verify this condition in future applications of the slicing lemma. We begin with the following consequence of the choice of $\xi_{27} \leq \varepsilon_{27}^2/(192 t_{\rm RL}^{2^k})$ in (36)

$$\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \text{ is } \big(\varepsilon_{27}^2/96, d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)})\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,. \tag{53}$$

The proof of Claim 29 splits into two main cases.

**Case 1** $(d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)}) > \varepsilon_{27}^2/16)$**.** In this case we will treat "thin" hypergraphs $\mathcal{H}_i^{(k)}$ w.r.t. $\mathcal{R}^{(k-1)}$ somewhat differently. To this end we set

$$R_{\rm THIN} = \Big\{ i \in [s_{27}]\colon\, d_{\mathscr{H},\mathcal{R}}(i) < \frac{\varepsilon_{27}^2}{192 s_{27}} \Big\}\,. \tag{54}$$

Due to the definition of $R_{\rm THIN}$ and the assumption of Case 1 we have

$$[s_{27}] \setminus R_{\rm THIN} \neq \varnothing\,. \tag{55}$$

We distinguish two sub-cases of Case 1 depending on $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{{\rm RL},\mathscr{H}\cdot{\rm reg}}(\boldsymbol{\mathcal{R}})$.

**Case 1.1** $(\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{{\rm RL},\mathscr{H}\cdot{\rm reg}}(\boldsymbol{\mathcal{R}}))$**.** In this particular case it suffices to prove the existence of a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon\, i \in [s_{27}]\}$ of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ which satisfies (43) only. For this we will simply use the slicing lemma, Proposition 22, to decompose $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ into hypergraphs with the appropriate densities (as required for (43)). More precisely, we apply Proposition 22, with

$$j = k\,,\quad s_0 = s_{27}\,,\quad r = 1\,,\quad \delta_0 = \frac{\varepsilon_{27}^2}{96}\,,\quad \varrho_0 = \frac{\varepsilon_{27}^2}{16}\,,\quad\text{and}\quad p_0 = \frac{\varepsilon_{27}^2}{192 s_{27}}\,,$$
$$\text{to}\quad \hat{\mathcal{P}}^{(k-1)}\,,\quad\text{and}\quad \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\quad\text{with}\quad s = \big|[s_{27}] \setminus R_{\rm THIN}\big|\,,$$
$$\delta = \frac{\varepsilon_{27}^2}{96}\,,\ \varrho = d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)})\,,\ \text{and}\ \Big\{ p_i = \frac{d_{\mathscr{H},\mathcal{R}}(i)}{d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)})}\colon\, i \in [s_{27}] \setminus R_{\rm THIN} \Big\}\,.$$

The conditions of Proposition 22 are immediate consequences of (52)–(54), and the assumption of Case 1.1 Proposition 22 yields a family $\mathcal{T}_0^{(k)} \cup \{\mathcal{T}_i^{(k)}\colon\, i \in [s_{27}] \setminus$

$R_{\mathrm{THIN}}\}$ satisfying the following properties

$$\mathcal{T}_0^{(k)} \cup \{\mathcal{T}_i^{(k)} \colon i \in [s_{27}] \setminus R_{\mathrm{THIN}}\} \text{ partitions } \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\,, \tag{56}$$

$$\mathcal{T}_i^{(k)} \text{ is } (\varepsilon_{27}^2/32, d_{\mathscr{H},\mathcal{R}}(i))\text{-regular w.r.t } \hat{\mathcal{P}}^{(k-1)} \text{ for } i \in [s_{27}] \setminus R_{\mathrm{THIN}} \tag{57}$$

$$\mathcal{T}_0^{(k)} \text{ is } (\varepsilon_{27}^2/32, d_{\mathcal{T}_0^{(k)}})\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,, \text{where} \tag{58}$$

$$d_{\mathcal{T}_0^{(k)}} = d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)}) - \sum\left\{d_{\mathscr{H},\mathcal{R}}(i) \colon i \in [s_{27}] \setminus R_{\mathrm{THIN}}\right\} \overset{(54)}{\leq} \frac{\varepsilon_{27}^2}{192}\,. \tag{59}$$

Fix some $i_0 \in [s_{27}] \setminus R_{\mathrm{THIN}}$ (due to (55) such an $i_0$ exists). We then define the family $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ for $i \in [s_{27}]$ as follows

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \begin{cases} \varnothing & \text{if} \quad i \in R_{\mathrm{THIN}} \\ \mathcal{T}_i^{(k)} \cup \mathcal{T}_0^{(k)} & \text{if} \quad i = i_0 \\ \mathcal{T}_i^{(k)} & \text{otherwise.} \end{cases}$$

From (56) we infer that $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \in [s_{27}]\}$ defined that way forms a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and it is left to verify (43) for every $i \in [s_{27}]$.

First, let $i \in R_{\mathrm{THIN}}$. From the definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \varnothing$ we infer that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon', 0)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for all $\varepsilon' > 0$. Since $i \in R_{\mathrm{THIN}}$, $d_{\mathscr{H},\mathcal{R}}(i) < \varepsilon_{27}^2/4$, and, consequently, the $(\varepsilon', 0)$-regularity for every $\varepsilon' > 0$ implies that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{27}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular (i.e., (43) for $i \in R_{\mathrm{THIN}}$).

If $i \in [s_{27}] \setminus R_{\mathrm{THIN}}$ and $i \neq i_0$, then (57) and the definition $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{T}_i^{(k)}$ immediately implies (43).

It is left to verify (43) for $i = i_0$. In that case Proposition 19 applied to $\mathcal{T}_{i_0}^{(k)}$ and $\mathcal{T}_0^{(k)}$ implies by (57) and (58) that $\mathcal{G}_{i_0}^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{27}^2/16)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, with density between $d_{\mathscr{H},\mathcal{R}}(i_0)$ and $d_{\mathscr{H},\mathcal{R}}(i_0) + \varepsilon_{27}^2/192$ (cf. (59)). Consequently, $\mathcal{G}_{i_0}^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $((\varepsilon_{27}^2/16 + \varepsilon_{27}^2/192), d_{\mathscr{H},\mathcal{R}}(i_0))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, which yields (43).

Having verified (43) for every $i \in [s_{27}]$, we conclude Case 1.1. $\Diamond$

**Case 1.2** **($\hat{\mathcal{P}}^{(k-1)} \in \mathscr{P}_{\mathrm{RL},\mathscr{H}\text{-reg}}^{(k-1)}(\boldsymbol{\mathcal{R}})$).** In this case we have to guarantee the existence of a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ which satisfies both (43) and (44) of Claim 29. Due to (44) we have to be more careful in defining the desired partition. On the other hand, the assumption in this case says that $\mathcal{H}_i^{(k)}$ is $\delta_{k,\mathrm{RL}}$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{27}]$. This allows us to apply the slicing lemma, to any $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$.

Below we give a short outline how we use this additional assumption. To simplify the notation we set for every $i \in [s_{27}]$

$$d_{\mathscr{H},\hat{\mathcal{P}}}(i) = d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)})\,.$$

We first consider the hypergraphs $\mathcal{H}_i^{(k)}$ which are too "fat" in $\hat{\mathcal{P}}^{(k-1)}$, i.e., we consider

$$I_{\mathrm{FAT}}(\hat{\mathcal{P}}) = \left\{i \in [s_{27}] \setminus R_{\mathrm{THIN}} \colon d_{\mathscr{H},\hat{\mathcal{P}}}(i) > d_{\mathscr{H},\mathcal{R}}(i) + \frac{\varepsilon_{27}^2}{192 s_{27}}\right\}. \tag{60}$$

We apply the slicing lemma to split each $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ into a "main" part $\mathcal{M}_i^{(k)}$ of density $d_{\mathscr{H},\mathcal{R}}(i)$ and a "leftover" $\mathcal{L}_i^{(k)}$. The $\mathcal{M}_i^{(k)}$ will be used to define $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$. Furthermore, since each $\mathcal{L}_i^{(k)}$ is regular, and since each $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for $i\in R_{\mathrm{THIN}}$ is regular, as well, (by the assumption of the case), we will infer that their union $\mathcal{U}^{(k)}=\bigcup_{i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})}\mathcal{L}_i^{(k)}\cup\bigcup_{i\in R_{\mathrm{THIN}}}(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is regular with density "very close" to

$$\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})=\sum\Big\{d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)\colon\ i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\Big\}\,, \tag{61}$$

where

$$I_{\mathrm{SLIM}}(\hat{\mathcal{P}})=\Big\{i\in[s_{27}]\setminus R_{\mathrm{THIN}}\colon\ d_{\mathscr{H},\hat{\mathcal{P}}}(i)<d_{\mathscr{H},\mathcal{R}}(i)-\frac{\varepsilon_{27}^2}{192s_{27}}\Big\}\,. \tag{62}$$

We then apply the slicing lemma again, this time to $\mathcal{U}^{(k)}$, to split it into regular pieces of densities $\{d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)\colon\ i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\}$. For $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ uniting $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ with the appropriate slice from $\mathcal{U}^{(k)}$ then gives rise to the desired partition. We now implement the technical details of this plan.

Let $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ and $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ be defined as in (60) and (62). We set

$$I_{\mathrm{OK}}(\hat{\mathcal{P}})=\Big\{i\in[s_{27}]\setminus R_{\mathrm{THIN}}\colon\ d_{\mathscr{H},\hat{\mathcal{P}}}(i)=d_{\mathscr{H},\mathcal{R}}(i)\pm\frac{\varepsilon_{27}^2}{192s_{27}}\Big\} \tag{63}$$

and note that $[s_{27}]$ is the disjoint union of $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, $I_{\mathrm{OK}}(\hat{\mathcal{P}})$, $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$, and $R_{\mathrm{THIN}}$. We will later need the following observation

$$\begin{aligned}
&\bigg|\bigg(\sum_{i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})}\big(d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)\big)+\sum_{i\in R_{\mathrm{THIN}}}d_{\mathscr{H},\hat{\mathcal{P}}}(i)\bigg)-\sum_{i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})}\big(d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)\big)\bigg|\\
&\quad=\Big|\sum\Big\{d_{\mathscr{H},\hat{\mathcal{P}}}(i)\colon\ i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})\cup R_{\mathrm{THIN}}\cup I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\Big\}\\
&\qquad\qquad-\sum\Big\{d_{\mathscr{H},\mathcal{R}}(i)\colon\ i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})\cup I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\Big\}\Big|\\
&\quad=\Big|\sum\Big\{d_{\mathscr{H},\hat{\mathcal{P}}}(i)\colon\ i\in[s_{27}]\setminus I_{\mathrm{OK}}(\hat{\mathcal{P}})\Big\}\\
&\qquad\qquad-\sum\Big\{d_{\mathscr{H},\mathcal{R}}(i)\colon\ i\in[s_{27}]\setminus\big(I_{\mathrm{OK}}(\hat{\mathcal{P}})\cup R_{\mathrm{THIN}}\big)\Big\}\Big|\\
&\quad\le\Big|d(\mathcal{F}^{(k)}|\hat{\mathcal{P}}^{(k-1)})-d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)})\Big|+\sum_{i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})}\Big|d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)\Big|+\sum_{i\in R_{\mathrm{THIN}}}d_{\mathscr{H},\mathcal{R}}(i)\,.
\end{aligned}$$

Thus in view of (61) and (53), (54), and (63) we derive the following bound on the left-hand side from above

$$\bigg|\bigg(\sum_{i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})}\Big(d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)\Big)+\sum_{i\in R_{\mathrm{THIN}}}d_{\mathscr{H},\hat{\mathcal{P}}}(i)\bigg)-\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})\bigg|\le\frac{\varepsilon_{27}^2}{48}\,. \tag{64}$$

Case 1.2 splits into two sub-cases depending on the size of $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$.

**Case 1.2.1** **($\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})>\varepsilon_{27}^2/12$).** For every $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ we have, due to the assumption of Case 1.2, that $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}},d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular w.r.t.

$\hat{\mathcal{P}}^{(k-1)}$. We apply the slicing lemma, Proposition 22, with

$$j=k\,,\quad s_0=2\,,\quad r=1\,,\quad \delta_0=\delta_{k,\mathrm{RL}}\,,\quad \varrho_0=\frac{\varepsilon_{27}^2}{192s_{27}}\,,\quad p_0=\frac{\varepsilon_{27}^2}{192s_{27}}\,,$$
$$\text{to}\quad \hat{\mathcal{P}}^{(k-1)}\,,\quad \text{and}\quad \mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\quad \text{with}$$
$$s=2\,,\ \delta=\delta_{k,\mathrm{RL}}\,,\ \varrho=d_{\mathscr{H},\hat{\mathcal{P}}}(i)\,,\ p_1=\frac{d_{\mathscr{H},\mathcal{R}}(i)}{d_{\mathscr{H},\hat{\mathcal{P}}}(i)}\,,\ \text{and}\quad p_2=\frac{d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)}{d_{\mathscr{H},\hat{\mathcal{P}}}(i)}\,.$$

For this choice of parameters the assumptions of Proposition 22 are satisfied. Indeed we have (52), $\varrho\ge\varrho_0$ since $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ and (60), $\varrho_0\ge 2\delta=\delta_{k,\mathrm{RL}}$ (cf. (34)), $p_1\ge p_0$ since $i\notin R_{\mathrm{THIN}}$ by definition of $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, and $p_2\ge p_0$ since $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$.

Since $p_1+p_2=1$, Proposition 22 yields a partition $\mathcal{M}_i^{(k)}\cup\mathcal{L}_i^{(k)}$ of $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for every $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, where

$$\mathcal{M}_i^{(k)}\ \text{is}\ (3\delta_{k,\mathrm{RL}},d_{\mathscr{H},\mathcal{R}}(i))\text{-regular w.r.t. }\hat{\mathcal{P}}^{(k-1)}\ \text{and}\tag{65}$$
$$\mathcal{L}_i^{(k)}\ \text{is}\ (3\delta_{k,\mathrm{RL}},(d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)))\text{-regular w.r.t. }\hat{\mathcal{P}}^{(k-1)}\,.\tag{66}$$

We now collect all "leftovers" and distribute them among the hypergraphs $\mathcal{H}_i^{(k)}$ which are too "slim" in $\hat{\mathcal{P}}^{(k-1)}$. For that we set

$$\mathcal{U}^{(k)}=\bigcup_{i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})}\mathcal{L}_i^{(k)}\cup\bigcup_{i\in R_{\mathrm{THIN}}}\Big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\Big)\,.$$

From (66) and the assumption of Case 1.2 we infer with Proposition 19 that $\mathcal{U}^{(k)}$ is $(3s_{27}\delta_{k,\mathrm{RL}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. Moreover, by the choice of $\delta_{k,\mathrm{RL}}$ in (34) we have $3s_{27}\delta_{k,\mathrm{RL}}\le\varepsilon_{27}^2/48$ and by (64) it follows that

$$\mathcal{U}^{(k)}\ \text{is}\ \big(\varepsilon_{27}^2/24,\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})\big)\text{-regular w.r.t. }\hat{\mathcal{P}}^{(k-1)}\,.\tag{67}$$

We then apply the slicing lemma again, this time with

$$j=k\,,\quad s_0=s_{27}\,,\quad r=1\,,\quad \delta_0=\frac{\varepsilon_{27}^2}{24}\,,\quad \varrho_0=\frac{\varepsilon_{27}^2}{12}\,,\quad p_0=\frac{\varepsilon_{27}^2}{192s_{27}}\,,$$
$$\text{to}\quad \hat{\mathcal{P}}^{(k-1)}\,,\quad \text{and}\quad \mathcal{U}^{(k)}\quad \text{with}\quad s=\big|I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\big|\,,$$
$$\delta=\frac{\varepsilon_{27}^2}{24}\,,\quad \varrho=\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})\,,\quad \text{and}\quad \left\{p_i=\frac{d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)}{\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})}\colon\ i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\right\}.$$

Here the assumptions of Proposition 22 are immediate consequences of (52) (showing $(i)$ of Proposition 22), (67) (showing that $\mathcal{U}^{(k)}$ is sufficiently regular), the assumption of Case 1.2.1 (yielding $\varrho\ge\varrho_0$), and the definition of $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ in (62) combined with $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})\le 1$ (yielding $p_i\ge p_0$).

Also, note that $\sum_{i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})}p_i=1$ and, consequently, Proposition 22 yields a partition $\{\mathcal{T}_i^{(k)}\colon\ i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\}$ of $\mathcal{U}^{(k)}$, which by (67) has density "close" to $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$, so that

$$\mathcal{T}_i^{(k)}\ \text{is}\ \big(\varepsilon_{27}^2/8,(d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i))\big)\text{-regular w.r.t. }\hat{\mathcal{P}}^{(k-1)}\,.\tag{68}$$

Finally, we are ready to define the family $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon i\in[s_{27}]\}$. Set

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\begin{cases}\varnothing & \text{if} \quad i\in R_{\mathrm{THIN}}\\ \mathcal{M}_i^{(k)} & \text{if} \quad i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})\\ \mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) & \text{if} \quad i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})\\ \big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\cup\mathcal{T}_i^{(k)} & \text{if} \quad i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\,.\end{cases}$$

It is obvious that $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\colon i\in[s_{27}]\}$ defined this way is a partition of $\mathcal{F}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$. We still have to verify (43) and (44).

We start with showing (43). First let $i\in R_{\mathrm{THIN}}$. By definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ it is $(\varepsilon',0)$-regular for every $\varepsilon'>0$ and, hence, it is $((\varepsilon'+d_{\mathscr{H},\mathcal{R}}(i)),d_{\mathscr{H},\mathcal{R}}(i))$-regular. Therefore, (43) follows from $d_{\mathscr{H},\mathcal{R}}(i)\le\varepsilon_{27}^2/(192s_{27})<\varepsilon_{27}^2/4$ (cf. (54)).

If $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, then (43) follows from (65) and $3\delta_{k,\mathrm{RL}}<\varepsilon_{27}^2/4$ (cf. (34)).

Now let $i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})$. Then $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}},d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular due to the assumption of Case 1.2. Since (63) $|d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)|\le\varepsilon_{27}^2/(192s_{27})$ and, hence, $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}+\varepsilon_{27}^2/(192s_{27}),d_{\mathscr{H},\mathcal{R}}(i))$-regular. Now (43) follows, since $\delta_{k,\mathrm{RL}}+\varepsilon_{27}^2/(192s_{27})\le\varepsilon_{27}^2/4$ (cf. (34)).

Finally, let $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$. Then Proposition 19 applied to $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and $\mathcal{T}_i$ yields that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $((\delta_{k,\mathrm{RL}}+\varepsilon_{27}^2/8),d_{\mathscr{H},\mathcal{R}}(i))$-regular (cf. assumption of Case 1.2 and (68)). Consequently, (43) follows since $\delta_{k,\mathrm{RL}}+\varepsilon_{27}^2/8\le\varepsilon_{27}^2/4$ (cf. (34)).

It is left to verify (44) for $i\in[s_{27}]$ to conclude this case, Case 1.2.1. Again our argument is different for each partition classes $R_{\mathrm{THIN}}$, $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, $I_{\mathrm{OK}}(\hat{\mathcal{P}})$, and $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ of $[s_{27}]$.

For $i\in R_{\mathrm{THIN}}$, due to notational reasons it will be easier to verify (44) in terms of the corresponding density

$$d\Big(\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\mathcal{H}_i^{(k)}\Big|\hat{\mathcal{P}}^{(k-1)}\Big)=\frac{\big|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\big|}{\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big|}\,.$$

If $i\in R_{\mathrm{THIN}}$, then $d_{\mathscr{H},\mathcal{R}}(i)\le\varepsilon_{27}^2/(192s_{27})$ and, consequently,

$$d\Big(\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\mathcal{H}_i^{(k)}\Big|\hat{\mathcal{P}}^{(k-1)}\Big)=d_{\mathscr{H},\hat{\mathcal{P}}}(i)\le|d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)|+\varepsilon_{27}^2/(192s_{27})\,.$$

Therefore, (44) follows for $i\in R_{\mathrm{THIN}}$ from $\varepsilon_{27}^2/(192s_{27})\le\nu_{27}/6$ (cf. (32)).

If $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, then by definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{M}_i^{(k)}$ we have

$$|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}))|=|\mathcal{L}_i^{(k)}|\,.$$

Moreover, due to (66) we have $|\mathcal{L}_i^{(k)}|\le(d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)+3\delta_{k,\mathrm{RL}})|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|$, which combined with the choice of $\delta_{k,\mathrm{RL}}\le\nu_{27}/18$ (cf. (34)) yields (44) for $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$.

If $i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})$, then (44) is a consequence of the definition $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, which yields that the left-hand side in (44) is 0.

Finally, we consider the case $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$. It follows from the definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ and (68) that

$$\begin{aligned}|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}))|&=|\mathcal{T}_i^{(k)}|\\&\le(d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)+\varepsilon_{27}^2/8)|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|.\end{aligned}$$

Consequently, (44) for $i \in I_{\text{SLIM}}(\hat{\mathcal{P}})$ follows from (32).

Having verified (43) and (44) for every $i \in [s_{27}]$ we conclude Claim 29 in Case 1.2.1. In order to finish Case 1.2, we have to consider the complementing and rather trivial sub-case when $\Delta_{\text{SLIM}}(\hat{\mathcal{P}})$ is small. $\Diamond$

**Case 1.2.2** ($\Delta_{\text{SLIM}}(\hat{\mathcal{P}}) \leq \varepsilon_{27}^2/12$)**.** In this case we set for every $i \in [s_{27}]$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\,.$$

Therefore, (44) of Claim 29 holds trivially, and we only have to show (43). For that we note, that due to the assumption of Case 1.2 we have $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\text{RL}}, d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ and consequently for every $i \in [s_{27}]$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \text{ is } \Big(\delta_{k,\text{RL}} + |d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)|, d_{\mathscr{H},\mathcal{R}}(i)\Big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\,. \tag{69}$$

In what follows we show that

$$\Big|d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i)\Big| \leq \frac{\varepsilon_{27}^2}{6} \quad \text{for every } i \in [s_{27}]\,, \tag{70}$$

which combined with (69) and $\delta_{k,\text{RL}} + \varepsilon_{27}^2/6 \leq \varepsilon_{27}^2/4$ (cf. (34)), yields (43) for every $i \in [s_{27}]$.

First we consider $i \in R_{\text{THIN}}$. Due to (64) and the assumption of Case 1.2.2 we have

$$\sum_{i \in I_{\text{FAT}}(\hat{\mathcal{P}})} \Big(d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)\Big) + \sum_{i \in R_{\text{THIN}}} d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leq \left(\frac{1}{12} + \frac{1}{48}\right)\varepsilon_{27}^2 < \frac{\varepsilon_{27}^2}{6}\,, \tag{71}$$

where all terms on the left-hand side are positive (cf. (60)). Therefore, $d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leq \varepsilon_{27}^2/6$ for every $i \in R_{\text{THIN}}$. Moreover, since $d_{\mathscr{H},\mathcal{R}}(i) \leq \varepsilon_{27}^2/(192 s_{27})$ for every $i \in R_{\text{THIN}}$, (70) holds for every $i \in R_{\text{THIN}}$.

If $i \in I_{\text{FAT}}(\hat{\mathcal{P}})$, then (71) yields $0 \leq d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i) \leq \varepsilon_{27}^2/6$ and consequently (70) holds for those $i$.

For $i \in I_{\text{OK}}(\hat{\mathcal{P}})$, (70) follows from the definition of $I_{\text{OK}}(\hat{\mathcal{P}})$ in (63).

Finally, we consider $i \in I_{\text{SLIM}}(\hat{\mathcal{P}})$. From the assumption of this case, Case 1.2.2, and the definition of $\Delta_{\text{SLIM}}(\hat{\mathcal{P}})$ in (61) we infer $0 \leq d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leq \varepsilon_{27}^2/12$, which clearly implies (70) for $i \in I_{\text{SLIM}}(\hat{\mathcal{P}})$.

This concludes Case 1.2.2 the last sub-case of Case 1. $\Diamond$

**Case 2** ($d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)}) \leq \varepsilon_{27}^2/16$)**.** In this case we set

$$\mathcal{G}_1^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \quad \text{and} \quad \mathcal{G}_2^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \dots = \mathcal{G}_{s_{27}}^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \varnothing\,.$$

Again we have to show (43) and (44) of Claim 29. We start with (43). Note that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{27}^2/96)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{27}]$. (This is trivial for $i \geq 2$ and follows from (53) for $i = 1$.) In order to show that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is also $(\varepsilon_{27}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular recall the assumption $d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)}) \leq \varepsilon_{27}^2/16$, which implies that $d_{\mathscr{H},\mathcal{R}}(i) \leq \varepsilon_{27}^2/16$ for every $i \in [s_{27}]$. Consequently, $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $\big((\varepsilon_{27}^2/96 + \varepsilon_{27}^2/16), d_{\mathscr{H},\mathcal{R}}(i)\big)$-regular for every $i \in [s_{27}]$ and (43) follows.

In order to infer (44) we observe that for $i \in [s_{27}]$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \triangle (\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \subseteq \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\,.$$

Moreover, due to the assumption $d(\mathcal{F}^{(k)}|\mathcal{R}^{(k-1)}) \leq \varepsilon_{27}^2/16$ and (53) we have

$$|\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| \leq (\varepsilon_{27}^2/16 + \varepsilon_{27}^2/96)|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\,.$$

Property (44) then follows from $\varepsilon_{27}^2/16 + \varepsilon_{27}^2/96 \leq \nu_{27}/6$ (see (32)). ◊

In all cases we ensured the existence of a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, which satisfies the conclusions of Claim 29. This concludes the proof of Claim 29. □

*Proof of Claim 30.* We assume the contrary, i.e.,

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}\Big| > \frac{\nu_{27}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|\,.$$

We may assume that $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ defined by

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) = \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\colon\ d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)}) > d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{6}\Big\}$$

satisfies

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}\Big| \geq \frac{\nu_{27}}{6}\big|\mathcal{K}_k(\mathcal{R}^{(k-1)})\big|\,. \tag{72}$$

(The case concerning $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,SLIM}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$instead of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ is very similar.) In what follows we will show that (72) contradicts the $(\nu_{27}/12, *, t_{27}^{2^k})$-regularity of $\mathcal{H}_i^{(k)}$ w.r.t. $\mathcal{R}^{(k-1)}$ (see (L.27.$d$)). Since

$$\big|\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\big| \leq |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}| = \prod_{h=1}^{k-1}(a_h^{\mathscr{P}_{\mathrm{RL}}})^{\binom{k}{h}} \leq t_{\mathrm{RL}}^{2^k} \overset{(36)}{=} t_{27}^{2^k}$$

this contradiction follows once we establish the following inequality

$$\frac{\Big|\mathcal{H}_i^{(k)} \cap \bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}\Big|}{\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}\Big|} \geq d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{12}\,. \tag{73}$$

By definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ we have $d(\mathcal{H}_i^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq d_{\mathscr{H},\mathcal{R}}(i) + \nu_{27}/6$ for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ and, since $\mathcal{K}_k(\hat{\mathcal{P}}_1^{(k-1)}) \cap \mathcal{K}_k(\hat{\mathcal{P}}_2^{(k-1)}) = \varnothing$ for all distinct $\hat{\mathcal{P}}_1^{(k-1)}$, $\hat{\mathcal{P}}_2^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ (cf. 41) it suffices to verify

$$\Big(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{6}\Big)\frac{\min\Big\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}}{\max\Big\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}} \geq d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{12}$$

to infer (73). In view of (39) we derive the following upper bound on the right-hand side of the last inequality

$$\Big(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{6}\Big)\frac{1-\nu_{27}/48}{1+\nu_{27}/48} \geq \Big(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{6}\Big)\Big(1 - \frac{\nu_{27}}{24}\Big) \geq d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{27}}{12}\,,$$

which concludes the proof of Claim 30. □

## 6. Proof of: RAL($k$) $\Longrightarrow$ RL($k+1$)

In what follows we give a proof of RL($k+1$), Lemma 23, based on RAL($k$), Lemma 25. The proof presented here resembles the main ideas from [5, 26, 31] combined with some techniques from [20]. In the next section we recall the concept of an *index* of a partition (cf. Definition 32) and derive some facts about it. We then give the proof of RL($k+1$) in Section 6.2.

### 6.1. The index of a partition.

The following propositions center around the notion of an *index*. Throughout this section we will work under the following setup.

*Setup* 31. *Let $\mathscr{R}_0^{(1)}$ be a fixed partition of some vertex set $V$ and $\mathscr{H}^{(k+1)}$ be a partition of $[V]^k$. Moreover, let $\mathscr{X}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, i.e., for every $\mathcal{X}^{(k)} \in \mathscr{X}^{(k)}$ we have $\mathcal{X}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ or $\mathcal{X}^{(k)} \cap \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) = \varnothing$. Let $\mathcal{U}(\mathscr{X}^{(k)}) = \bigcup\{\mathcal{X}^{(k)} \colon \mathcal{X}^{(k)} \in \mathscr{X}^{(k)}\} \supseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ be the set of $k$-tuples partitioned by $\mathscr{X}^{(k)}$.*

*For any $K \in \mathcal{U}(\mathscr{X}^{(k)})$ let $\mathcal{X}^{(k)}(K)$ be that partition class of $\mathscr{X}^{(k)}$ which contains $K$, i.e.,*

$$\mathcal{X}^{(k)}(K) = \mathcal{X}^{(k)} \in \mathscr{X}^{(k)} \quad \textit{so that} \quad K \in \mathcal{X}^{(k)} .$$

*Moreover, for every $(k+1)$-tuple $K' \in [V]^{k+1}$ satisfying $[K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})$ we set*

$$\hat{\mathcal{X}}^{(k)}(K') = \bigcup\big\{\mathcal{X}^{(k)}(K) \colon K \in [K']^k\big\}$$
$$\text{and} \quad \hat{\mathscr{X}}^{(k)} = \big\{\hat{\mathcal{X}}^{(k)}(K') \colon K' \in [V]^{k+1} \text{ s.t. } [K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})\big\} .$$

*Note that every $K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})$ satisfies $[K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})$, since $\mathscr{X}^{(k)}$ refines $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$.* $\square$

We then define the index of a partition $\mathscr{X}^{(k)}$ (satisfying the above setup) with respect to $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ as follows.

*Definition* 32 (**Index**). *Given $V$, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Setup 31. We set the index of $\mathscr{X}^{(k)}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ equal to*

$$\begin{aligned}\mathrm{ind}(\mathscr{X}^{(k)}) &= \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})} d^2\big(\mathcal{H}^{(k+1)} \big| \hat{\mathcal{X}}^{(k)}(K')\big) \\ &= \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\substack{\hat{\mathcal{X}}^{(k)} \in \hat{\mathscr{X}}^{(k)} \\ \hat{\mathcal{X}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})}} d^2\big(\mathcal{H}^{(k+1)} \big| \hat{\mathcal{X}}^{(k)}\big) \big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})\big| .\end{aligned}$$

The next observation follows straight from the definition of the index.

**Fact 33.** *For all $V$, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Setup 31, $\mathrm{ind}(\mathscr{X}^{(k)})$ is bounded between 0 and 1.* $\square$

We now derive a few more propositions related to the index , which allow a simpler presentation of the the proof of RL($k+1$).

**Proposition 34.** *Let $V$, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be given as in Setup 31. Suppose $\mathscr{X}^{(k)}=\{\mathcal{X}_1^{(k)},\dots,\mathcal{X}_s^{(k)}\}$ and $\mathscr{Y}^{(k)}=\{\mathcal{Y}_1^{(k)},\dots,\mathcal{Y}_s^{(k)}\}$ are partitions which refine $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. Moreover, let $\nu$ be a given positive real. If for every $\ell\in[s]$ we have*

*(i)* $\big|\mathcal{X}_\ell^{(k)}\triangle\mathcal{Y}_\ell^{(k)}\big|\le\nu|V|^k$ *and*

*(ii) if* $\mathcal{X}_\ell^{(k)}\subseteq\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ *then* $\mathcal{Y}_\ell^{(k)}\subseteq\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$,

*then*

$$\mathrm{ind}(\mathscr{Y}^{(k)})\ge\mathrm{ind}(\mathscr{X}^{(k)})-3(k+1)s^{k+1}|\mathscr{H}^{(k+1)}|\nu\,. \tag{74}$$

*Proof.* For every $(k+1)$-tuple $I\in[s]^{k+1}$ we set

$$\hat{\mathcal{X}}_I^{(k)}=\bigcup_{i\in I}\mathcal{X}_i^{(k)}\quad\text{and}\quad\hat{\mathcal{Y}}_I^{(k)}=\bigcup_{i\in I}\mathcal{Y}_i^{(k)}\,.$$

From $(i)$ we infer that for every $I\in[s]^{k+1}$ we have

$$\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|-\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|\Big|\le\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\triangle\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|$$
$$\le\nu(k+1)|V|^{k+1}\,. \tag{75}$$

Suppose the partition classes of $\mathscr{H}^{(k+1)}$ are labeled $\mathcal{H}_1^{(k+1)},\dots,\mathcal{H}_h^{(k+1)}$. For a more concise notation we set for every $I\in[s]^{k+1}$ and $\zeta\in[h]$

$$d(\zeta|\hat{\mathcal{X}}_I^{(k)})=d\big(\mathcal{H}_\zeta^{(k+1)}\big|\hat{\mathcal{X}}_I^{(k)}\big)\quad\text{and}\quad d(\zeta|\hat{\mathcal{Y}}_I^{(k)})=d\big(\mathcal{H}_\zeta^{(k+1)}\big|\hat{\mathcal{Y}}_I^{(k)}\big)\,.$$

The triangle inequality and (75) gives for every $I\in[s]^{k+1}$ and $\zeta\in[h]$

$$\begin{aligned}
&\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|d^2(\zeta|\hat{\mathcal{X}}_I^{(k)})-\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|d^2(\zeta|\hat{\mathcal{Y}}_I^{(k)})\Big|\\
&\qquad\le\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|d(\zeta|\hat{\mathcal{X}}_I^{(k)})-\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|d(\zeta|\hat{\mathcal{Y}}_I^{(k)})\Big|d(\zeta|\hat{\mathcal{X}}_I^{(k)})\\
&\qquad\qquad+\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|-\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|\Big|d(\zeta|\hat{\mathcal{Y}}_I^{(k)})d(\zeta|\hat{\mathcal{X}}_I^{(k)})\\
&\qquad\qquad+\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|d(\zeta|\hat{\mathcal{X}}_I^{(k)})-\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|d(\zeta|\hat{\mathcal{Y}}_I^{(k)})\Big|d(\zeta|\hat{\mathcal{Y}}_I^{(k)})\\
&\qquad\le3\Big|\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|-\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|\Big|\overset{(75)}{\le}3\nu(k+1)|V|^{k+1}\,.
\end{aligned}\tag{76}$$

Now let $\hat{\mathscr{X}}^{(k)}$ and $\hat{\mathscr{Y}}^{(k)}$ be defined as in Setup 31. Clearly, for every $\hat{\mathcal{X}}^{(k)}\in\hat{\mathscr{X}}^{(k)}$ there exist a unique $I\in[s]^{k+1}$ so that $\hat{\mathcal{X}}^{(k)}=\hat{\mathcal{X}}_I^{(k)}$, while the converse fails to be true in general. We define

$$S(\mathscr{X}^{(k)})=\Big\{I\in[s]^{k+1}\colon\ \hat{\mathcal{X}}_I^{(k)}\in\hat{\mathscr{X}}^{(k)}\text{ and }\hat{\mathcal{X}}_I^{(k)}\subseteq\mathrm{Cross}_k(\mathscr{R}_0^{(1)})\Big\}\,.$$

Then we have

$$\mathrm{ind}(\mathscr{X}^{(k)})=\frac{1}{|V|^{k+1}}\sum_{\zeta\in[h]}\sum_{I\in S(\mathscr{X}^{(k)})}\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})\big|d^2(\zeta|\hat{\mathcal{X}}_I^{(k)})$$

and applying (76) for every $\zeta\in[h]$ and $I\in S(\mathscr{X}^{(k)})$ yields

$$\mathrm{ind}(\mathscr{X}^{(k)})\le\frac{1}{|V|^{k+1}}\sum_{\zeta\in[h]}\sum_{I\in S(\mathscr{X}^{(k)})}\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big|d^2(\zeta|\hat{\mathcal{Y}}_I^{(k)})+3\nu(k+1)h\big|S(\mathscr{X}^{(k)})\big| \tag{77}$$

Due to assumption $(ii)$ we have that $\hat{\mathcal{Y}}_I^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ for every $I \in S(\mathscr{X}^{(k)})$. Consequently, $\hat{\mathcal{Y}}_I^{(k)}$ is either in $\hat{\mathscr{Y}}^{(k)}$ or $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)}) = \varnothing$ for every $I \in S(\mathscr{X}^{(k)})$ and, hence,

$$\mathrm{ind}(\mathscr{Y}^{(k)}) \geq \frac{1}{|V|^{k+1}} \sum_{\zeta\in[h]} \sum \Big\{ \big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})\big| d^2(\zeta|\hat{\mathcal{Y}}_I^{(k)}) \colon I \in S(\mathscr{X}^{(k)}) \Big\} .$$

Therefore, the last inequality combined with (77) implies

$$\mathrm{ind}(\mathscr{X}^{(k)}) \leq \mathrm{ind}(\mathscr{Y}^{(k)}) + 3\nu(k+1)hs^{k+1} ,$$

which concludes the proof of Proposition 34. □

The following proposition is a simple consequence of Jensen's inequality.

**Proposition 35.** *Suppose $\hat{\mathcal{Y}}^{(k)}$ is an $(m,k+1,k)$-hypergraph and $\{\hat{\mathcal{Z}}_1^{(k)},\dots,\hat{\mathcal{Z}}_z^{(k)}\}$ is a family of $(m,k+1,k)$-hypergraphs such that $\{\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)}) \colon i \in [z]\}$ partitions $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$, then*

$$d^2\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Y}}^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})\big| \leq \sum_{i\in[z]} d^2\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}_i^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})\big| \tag{78}$$

*for every hypergraph $\mathcal{H}^{(k+1)} \subseteq \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$.*

*Proof.* For $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$ let $\hat{\mathcal{Z}}^{(k)}(K')$ be the unique member of $\{\hat{\mathcal{Z}}_1^{(k)},\dots,\hat{\mathcal{Z}}_z^{(k)}\}$ so that $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Z}}^{(k)}(K'))$. Then we have

$$\begin{aligned} d\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Y}}^{(k)}\big) &= \frac{\sum_{i\in[z]} d\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}_i^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})\big|}{\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})\big|} \\ &= \frac{\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}^{(k)}(K')\big)}{\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})\big|} , \end{aligned}$$

and Jensen's inequality yields (78), since

$$\begin{aligned} d^2\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Y}}^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})\big| &= \frac{\Big(\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}^{(k)}(K')\big)\Big)^2}{\big|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})\big|} \\ &\leq \sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}^{(k)}(K')\big) \\ &= \sum_{i\in[z]} d^2\big(\mathcal{H}^{(k+1)}\big|\hat{\mathcal{Z}}_i^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})\big| . \end{aligned}$$

□

The following proposition is a corollary of Proposition 35 and asserts that the refinement of a family of partitions has the same or bigger index.

**Proposition 36.** *Let $V$, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Y}^{(k)}$ be given as in Setup 31 and let $\mathscr{Z}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. If $\mathscr{Z}^{(k)} \prec \mathscr{Y}^{(k)}$, then $\mathrm{ind}(\mathscr{Y}^{(k)}) \leq \mathrm{ind}(\mathscr{Z}^{(k)})$ .*

*Proof.* We observe that for every $\hat{\mathcal{Y}}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$ satisfying $\hat{\mathcal{Y}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ the family $\{\mathcal{K}_{k+1}(\hat{\mathcal{Z}}^{(k)})\colon \hat{\mathcal{Z}}^{(k)} \in \hat{\mathscr{Z}}^{(k)}$ and $\hat{\mathcal{Z}}^{(k)} \subseteq \hat{\mathcal{Y}}^{(k)}\}$ partitions $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$. Consequently, we can apply Proposition 35 to every $\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$ and $\hat{\mathcal{Y}}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$, which yields the proposition. □

In the proof of RL$(k+1)$ we will also deal with partitions which "almost" refine each other (see Definition 37 below) and we need approximations of their index (Proposition 38).

*Definition* 37. *Given $V$, $\mathscr{R}_0^{(1)}$, and $\mathscr{Z}^{(k)}$ as in Setup 31. Moreover, let $\beta \geq 0$ and let $\mathscr{T}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. We say the partition $\mathscr{T}^{(k)}$ is a $\beta$-*refinement *of $\mathscr{Z}^{(k)}$ if*

$$\sum \Big\{ \big|\mathcal{T}^{(k)}\big| \colon\, \mathcal{T}^{(k)} \in \mathscr{T}^{(k)}\,,\ \mathcal{T}^{(k)} \not\subseteq \mathcal{Z}^{(k)} \text{ for every } \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)} \Big\} \leq \beta |V|^k\,.$$

The following proposition extends Proposition 36 and its proof is very similar to [5, Lemma 3.6].

**Proposition 38.** *Let $V$, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Z}^{(k)}$ be given as in Setup 31, let $\mathscr{T}^{(k)}$ be a $\beta$-refinement of $\mathscr{Z}^{(k)}$ for some $\beta \geq 0$. Then $\mathrm{ind}(\mathscr{T}^{(k)}) \geq \mathrm{ind}(\mathscr{Z}^{(k)}) - \beta$.*

*Proof.* We first define an auxiliary partition $\mathscr{S}^{(k)}$ which is a refinement of $\mathscr{T}^{(k)}$ and $\mathscr{Z}^{(k)}$. For that set

$$\mathscr{S}^{(k)} = \Big\{ \mathcal{T}^{(k)} \cap \mathcal{Z}^{(k)} \colon\, \mathcal{T}^{(k)} \in \mathscr{T}^{(k)} \quad \text{and} \quad \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)} \Big\}\,.$$

Due to Proposition 36 we have

$$\mathrm{ind}(\mathscr{Z}^{(k)}) \leq \mathrm{ind}(\mathscr{S}^{(k)})\ . \tag{79}$$

Let $\hat{\mathscr{T}}_0^{(k)}$ be the family of polyads $\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}^{(k)}$ which are sub-hypergraphs of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ and for which there exists a $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}$ such that

$$\mathcal{T}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)} \qquad \text{and} \qquad \mathcal{T}^{(k)} \not\subseteq \mathcal{Z}^{(k)} \text{ for all } \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\,.$$

Since $\mathscr{H}^{(k+1)}$ is a partition of $[V]^{k+1}$ and $\mathscr{T}^{(k)}$ is a $\beta$-refinement of $\mathscr{Z}^{(k)}$ we have

$$\sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \ \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \ \sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d\big(\mathcal{H}^{(k+1)} | \hat{\mathcal{S}}^{(k)}\big) \big|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})\big| \leq \beta |V|^k \times |V|\,. \tag{80}$$

Note that for every $\hat{\mathcal{T}}^{(k)} \notin \hat{\mathscr{T}}_0^{(k)}$ there exist some $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)}$ such that $\hat{\mathcal{S}}^{(k)} = \hat{\mathcal{T}}^{(k)}$. Consequently,

$$\begin{aligned}
&\mathrm{ind}(\mathscr{S}^{(k)}) - \mathrm{ind}(\mathscr{T}^{(k)}) \\
&\quad = \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \ \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \Bigg( \sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d^2\big(\mathcal{H}^{(k+1)} | \hat{\mathcal{S}}^{(k)}\big) \big|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})\big| \\
&\qquad\qquad\qquad\qquad\qquad\qquad - d^2\big(\mathcal{H}^{(k+1)} | \hat{\mathcal{T}}^{(k)}\big) \big|\mathcal{K}_{k+1}(\hat{\mathcal{T}}^{(k)})\big| \Bigg) \\
&\quad \leq \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \ \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \ \sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d\big(\mathcal{H}^{(k+1)} | \hat{\mathcal{S}}^{(k)}\big) \big|\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})\big| \overset{(80)}{\leq} \beta\,,
\end{aligned}$$

and the proposition follows from (79). □

The last proposition in this section concerns the index of a family of partitions $\mathscr{R}$ failing to satisfy $(H)$ of RL$(k+1)$. It can be shown that a certain refinement of $\mathscr{R}$ has an index of at least the index of $\mathscr{R}$ plus some positive constant depending on $\delta_{k+1}$. This observation is the crucial idea in the proof of RL$(k+1)$. Since, it roughly shows (together with Fact 33) that there are only finitely many refinements which violate $(H)$. The same idea was already used in [5, 26, 31].

**Proposition 39.** *Let $V$, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be given as in Setup 31 and let $\mathscr{R}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. Moreover, let $\delta$ be a positive real and $r \geq 1$ be an integer. If*

$$\left|\,\mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})\,\right| \geq \left(1-\frac{\delta}{2}\right)\binom{|V|}{k+1} \tag{81}$$

*and if there is some $\mathcal{H}_{\mathrm{irr}}^{(k+1)} \in \mathscr{H}^{(k+1)}$ which is $(\delta,*,r)$-irregular*[3] *w.r.t. $\mathscr{R}^{(k)}$, then there exists a partition $\mathscr{X}^{(k)}$ of $[V]^{k+1}$ satisfying*

*(i)* $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}$,
*(ii)* $|\mathscr{X}^{(k)}| \leq |\mathscr{R}^{(k)}| \times 2^{r\times|\hat{\mathscr{R}}^{(k)}|}$, *and*
*(iii)* $\mathrm{ind}(\mathscr{X}^{(k)}) \geq \mathrm{ind}(\mathscr{R}^{(k)}) + \delta^4/2$.

In the proof of Proposition 39 we will use the defect form of the Cauchy–Schwarz inequality, which we state first (see, e.g., [31] for a similar statement).

**Proposition 40** (Defect Cauchy–Schwarz inequality)**.** *Suppose $\varnothing \neq J \subsetneq I$ are some index sets and $d_i \geq 0$ is some non-negative real number for every $i \in I$. If*

$$\frac{1}{|J|}\sum_{j\in J} d_j = \frac{1}{|I|}\sum_{i\in I} d_i + \alpha \tag{82}$$

*for some (not necessarily non-negative) real $\alpha$ and if $|\alpha| \geq \delta$ and $|J| \geq \delta|I|$ for some $\delta \geq 0$, then*

$$\sum_{i\in I} d_i^2 \geq \frac{1}{|I|}\left(\sum_{i\in I} d_i\right)^2 + \delta^3|I|\,.$$

□

*Proof of Proposition 39.* Let $\hat{\mathscr{R}}_{\mathrm{irr},0}^{(k)}$ be the set of those polyads $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ such that

$$\mathcal{H}_{\mathrm{irr}}^{(k+1)} \text{ is } (\delta,*,r)\text{-irregular w.r.t. } \hat{\mathcal{R}}^{(k)} \text{ and} \tag{83}$$

$$\hat{\mathcal{R}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)}). \tag{84}$$

From the definition $(\delta,*,r)$-regularity w.r.t. $\mathscr{R}^{(k)}$ (see footnote 3) and (81) we infer that

$$\sum\left\{\left|\mathcal{K}_k(\hat{\mathcal{R}}^{(k)})\right|\colon\ \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}_{\mathrm{irr},0}^{(k)}\right\} \geq \frac{\delta}{2}|V|^{k+1}\,. \tag{85}$$

[3] Strictly speaking in Definition 16 we only defined the regularity with respect to a family of partitions while here we only have a partition $\mathscr{R}^{(k)}$ of $k$-tuples. However, we can easily alter the definition based on $\hat{\mathscr{R}}^{(k)}$ meaning that $\mathcal{H}^{(k+1)}$ is $(\delta,*,r)$-regular w.r.t. $\mathscr{R}^{(k)}$ if $\left|\bigcup\{\mathcal{K}_k(\hat{\mathcal{R}}^{(k)})\colon\ \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ and $\mathcal{H}^{(k+1)}$ is not $(\delta,*,r)$-regular w.r.t. $\hat{\mathcal{R}}^{(k)}\}\right| \leq \delta|V|^{k+1}$.

For $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ there exist a witness of irregularity, i.e., there exits $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)}) = \{\hat{\mathcal{Q}}^{(k)}_1, \dots, \hat{\mathcal{Q}}^{(k)}_r\}$ such that $\hat{\mathcal{Q}}^{(k)}_i \subseteq \hat{\mathcal{R}}^{(k)}$ for every $i \in [r]$ and

$$\Big| \bigcup_{i\in[r]} \mathcal{K}_k\big(\hat{\mathcal{Q}}^{(k)}_i\big)\Big| \geq \delta \Big|\mathcal{K}_{k+1}\big(\hat{\mathcal{R}}^{(k)}\big)\Big| > 0\,, \tag{86}$$

$$d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big) = d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}\big|\hat{\mathcal{R}}^{(k)}\big)\big| + \alpha_{\hat{\mathcal{R}}^{(k)}} \text{ for } \alpha_{\hat{\mathcal{R}}^{(k)}} \text{ with } |\alpha_{\hat{\mathcal{R}}^{(k)}}| > \delta\,, \tag{87}$$

where $d\big(\mathcal{H}^{(k)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big) = \big|\mathcal{H}^{(k+1)}_{\mathrm{irr}} \cap \bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)})\big| \,/\, \big|\bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)})\big|$. Moreover we define for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ the family $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ of those sub-hypergraphs of $\mathcal{R}^{(k)}$ which are contained in some witness $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})$ with $\mathcal{R}^{(k)} \subseteq \hat{\mathcal{R}}^{(k)}$. More precisely we set

$$\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \big\{\mathcal{R}^{(k)} \cap \hat{\mathcal{Q}}^{(k)} \colon \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0} \text{ with } \mathcal{R}^{(k)} \subseteq \hat{\mathcal{R}}^{(k)} \text{ and } \hat{\mathcal{Q}}^{(k)} \in \hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big\}\,.$$

We observe that $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ might be empty (e.g., if $\mathcal{R}^{(k)} \nsubseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$), that the hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ are not necessarily disjoint, and that for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ we have the following trivial upper bound $w_{\mathcal{R}^{(k)}}$ on the number of hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$

$$w_{\mathcal{R}^{(k)}} = \big|\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})\big| \leq r \times \big|\hat{\mathscr{R}}^{(k)}\big|\,. \tag{88}$$

We now define the promised refinement $\mathscr{X}^{(k)}$ of $\mathscr{R}^{(k)}$. We construct $\mathscr{X}^{(k)}$ for each $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ separately. This partition of $\mathcal{R}^{(k)}$ will be called $\mathscr{X}^{(k)}(\mathcal{R}^{(k)})$ and is given by the atoms arising from the intersection of the hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ (i.e., the regions of the Venn diagram of the family $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$). More precisely, if $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) \neq \varnothing$ let $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \{\mathcal{W}^{(k)}_i \colon i \in [w_{\mathcal{R}^{(k)}}]\}$ be some enumeration of the elements of $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ and set

$$\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) = \bigg\{\bigcap_{i\in I} \mathcal{W}^{(k)}_i \cap \bigcap_{i\in I^c} \big(\mathcal{R}^{(k)} \setminus \mathcal{W}^{(k)}_i\big) \colon \{I, I^c\} \text{ partitions } [w_{\mathcal{R}^{(k)}}]\bigg\}\,.$$

If $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \varnothing$, then we set $\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) = \{\mathcal{R}^{(k)}\}$. Collecting "contributions" for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ in that way defines $\mathscr{X}^{(k)}$

$$\mathscr{X}^{(k)} = \bigcup \Big\{\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) \colon \mathcal{R}^{(k)} \in \mathscr{R}^{(k)}\Big\}\,.$$

Owing to the construction above, the partition $\mathscr{X}^{(k)}$ clearly refines $\mathscr{R}^{(k)}$, i.e., it satisfies $(i)$ of Proposition 39. Moreover, (88) and the construction yields $(ii)$ of the proposition.

It is left to verify $(iii)$. For that we first fix some $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ and consider the witness of irregularity $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)}) = \{\hat{\mathcal{Q}}^{(k)}_1, \dots, \hat{\mathcal{Q}}^{(k)}_r\}$. Since, $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$ it satisfies the assumptions of Setup 31 with $V$, $\mathscr{R}^{(1)}_0$, and $\mathscr{H}^{(k+1)}$. In particular, the family of polyads $\hat{\mathscr{X}}^{(k)}$ is well defined and for every $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})$ there exist a $\hat{\mathcal{X}}^{(k)}(K') \in \hat{\mathscr{X}}^{(k)}$ so that $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})$. We are heading towards an application of Proposition 40 with

$$I = \mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)}))\,, \quad J = \bigcup_{i=1}^{r} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)}_i)\,, \quad \text{and} \quad d_{K'} = d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}(K')\big) \tag{89}$$

for every $K' \in I$ and verify (82) below for $\alpha_{\hat{\mathcal{R}}^{(k)}}$ and the choice above

$$\frac{1}{|J|}\sum_{K'\in J} d_{K'} \overset{(89)}{=} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big)$$
$$\overset{(87)}{=} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{R}}^{(k)}\big) + \alpha_{\hat{\mathcal{R}}^{(k)}} \overset{(89)}{=} \frac{1}{|I|}\sum_{K'\in I} d_{K'} + \alpha_{\hat{\mathcal{R}}^{(k)}}.$$

Since, $|\alpha_{\hat{\mathcal{R}}^{(k)}}| \geq \delta$ (cf. (87)) and

$$|J| \overset{(89)}{=} \Big|\bigcup_{i=1}^{r}\mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)}_i)\Big| \overset{(86)}{\geq} \delta\Big|\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})\Big| \overset{(89)}{=} \delta|I|,$$

Proposition 40 yields

$$\sum_{K'\in I} d^2_{K'} \geq \frac{1}{|I|}\bigg(\sum_{K'\in I} d_{K'}\bigg)^2 + \delta^3|I|. \tag{90}$$

In view of (89) and since

$$\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}(K')\big) = \sum_{\substack{\hat{\mathcal{X}}^{(k)}\in\hat{\mathscr{X}}^{(k)}\\ \hat{\mathcal{X}}^{(k)}\subseteq\hat{\mathcal{R}}^{(k)}}} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}\big)\big|\mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})\big|$$
$$= d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{R}}^{(k)}\big)\big|\mathcal{K}_{k+1}(\mathcal{R}^{(k)})\big|$$

we can reformulate inequality (90) to

$$\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}(K')\big) \geq \sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})}\Big(d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{R}}^{(k)}\big) + \delta^3\Big). \tag{91}$$

Note that (91) holds for every irregular polyad $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$. Summing over all such polyads inequality (91) together with (85) yields

$$\sum_{\hat{\mathcal{R}}^{(k)}\in\hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}}\ \sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}(K')\big)$$
$$\geq \sum_{\hat{\mathcal{R}}^{(k)}\in\hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}}\ \sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{R}}^{(k)}\big) + \frac{\delta^4}{2}|V|^{k+1}.$$

Since $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$, we can apply Proposition 35 to every $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)} \setminus \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ which is contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ and we infer

$$\sum_{K'\in\mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{X}}^{(k)}(K')\big)$$
$$\geq \sum_{K'\in\mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}}|\hat{\mathcal{R}}(K')\big) + \frac{\delta^4}{2}|V|^{k+1}.$$

Finally, part (*iii*) of Proposition 39 follows from the last inequality and Proposition 35 applied to every $\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$, $\mathcal{H}^{(k+1)} \neq \mathcal{H}^{(k+1)}_{\mathrm{irr}}$ and every $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$. □

6.2. **Proof of** RL$(k+1)$**.** In what follows we give a proof of RL$(k+1)$ based on RAL$(k)$, or more precisely, based on Lemma 25. In the next section, Section 6.2.1, we define all constants involved in the proof of this implication. In Section 6.2.2 we state the so called *index pumping lemma* and deduce RL$(k+1)$ from it. We then prove the index pumping lemma in Section 6.3.

6.2.1. *Constants.* We first recall the quantification of RL$(k+1)$, Lemma 23 for $k+1$

$$\forall\, o_{\rm RL},\ s_{\rm RL},\ \eta_{\rm RL},\ \delta_{k+1,{\rm RL}},\ r_{\rm RL}\colon\ \mathbb{N}^k\to\mathbb{N},\ \delta_{\rm RL}\colon\ \mathbb{N}^k\to(0,1]\quad \exists\,\mu_{\rm RL}>0,\ t_{\rm RL},\ n_{\rm RL}\,.$$

So let positive integers $o_{\rm RL}$ and $s_{\rm RL}$, positive reals $\eta_{\rm RL}$ and $\delta_{k+1,{\rm RL}}$, and positive functions $r_{\rm RL}$ and $\delta_{\rm RL}$ be given. Without loss of generality we assume that

$$\eta_{\rm RL}\le\delta_{k+1,{\rm RL}}/2 \text{ and } r_{\rm RL} \text{ and } \delta_{\rm RL} \text{ are monotone in every variable.} \tag{92}$$

For the definition of the promised constants $\mu_{\rm RL}$, $t_{\rm RL}$, and $n_{\rm RL}$ we need auxiliary sequences of constants $t_i$, $o_i$, $s_i$, $\eta_i$, and $\nu_i$ and a sequence of functions $\varepsilon_i\colon\ \mathbb{N}^{k-1}\to(0,1]$ for $i\ge 0$. First we define $t_0$

$$t_0=\min\Big\{t\ge\Big\lceil\tfrac{(k+1)^{k+1}}{2\eta_{\rm RL}}\Big\rceil\colon\ (o_{\rm RL})!\text{ divides } t\Big\}>o_{\rm RL}\,. \tag{93}$$

Without loss of generality we may assume that the given function $\delta_{\rm RL}$ is bounded for every $\boldsymbol{a}=(a_1,\dots,a_k)\in\mathbb{N}^k$ by

$$\delta_{\rm RL}(\boldsymbol{a})\le\frac{\delta^4_{k+1,{\rm RL}}}{24t_0}<\frac{2}{t_0}\,,\quad\text{and}\quad \delta_{\rm RL}(\boldsymbol{a})\le\frac{3}{2a_k}\,. \tag{94}$$

For convenience we define the following integer-valued function $f\colon\mathbb{N}\to\mathbb{N}$

$$f(s)=\min\Big\{x\in\mathbb{N}\colon\ x\ge\frac{24t_0s}{\delta^4_{k+1,{\rm RL}}}\text{ and } (t_0)!\text{ divides } x\Big\}\,. \tag{95}$$

We then define $o_i$, $s_i$, $\eta_i$, and $\nu_i$ in terms of $t_i$, $\delta_{k+1,{\rm RL}}$, $\eta_{\rm RL}$, and $r_{\rm RL}(t_i,\dots,t_i)$

$$o_i=t_0\,,\ \ s_i=t_i^{2^k}2^{r_{\rm RL}(t_i,\dots,t_i)t_i^{2^{k+1}}}\,,\ \ \eta_i=\eta_{\rm RL}\,,\quad\text{and}\quad \nu_i=\tfrac{\delta^4_{k+1,{\rm RL}}}{12(k+1)s_{\rm RL}s_i^{k+1}}\,. \tag{96}$$

Moreover, for $i\ge 0$ we define the function $\varepsilon_i\colon\ \mathbb{N}^{k-1}\to(0,1]$ defined for every $\boldsymbol{a}=(a_1,\dots,a_{k-1})\in\mathbb{N}^{k-1}$ as

$$\varepsilon_i(\boldsymbol{a})=\min\Big\{\tfrac{\delta_{\rm RL}(a_1,\dots,a_{k-1},f(s_i))}{18s_i}\,,\varepsilon_{\rm DCL}\big(k-1,k,\tfrac12,\min_{2\le j\le k-1}\tfrac{1}{a_j}\big)\,,\tfrac{1}{2f(s_i)}\,,\tfrac{\delta^4_{k+1,{\rm RL}}}{72s_it_0}\Big\}, \tag{97}$$

where $\varepsilon_{\rm DCL}$ is given by Theorem 18. Moreover, with out loss of generality we assume that $\varepsilon_i$ is monotone in every variable.

We then define $t_{i+1}$ using $t_{\rm RAL}(o,s,\eta,\nu,\varepsilon(\cdot,\dots,\cdot))$ given by Lemma 25 and set

$$t_{i+1}=\max\Big\{t_i\,,t_{\rm RAL}\big(o_i,s_i,\eta_i,\nu_i,\varepsilon_i(\cdot,\dots,\cdot)\big)\,,f(s_i)\Big\}\overset{(95)}{\ge}s_i\,. \tag{98}$$

This concludes the definition of the sequences $t_i$, $o_i$, $s_i$, $\eta_i$, $\nu_i$ and $\varepsilon_i\colon\ \mathbb{N}^{k-1}\to(0,1]$ for $i\ge 0$. We note that the sequence $t_i$ is monotone by definition. In a similar way we define the monotone sequences $\mu_i$ for $i\ge 1$ by setting $\mu_1=\delta_{\rm RL}(t_0,\dots,t_0)$ and

$$\mu_{i+1}=\min\Bigg\{\mu_i\,,\mu_{\rm RAL}\big(o_i,s_i,\eta_i,\nu_i,\varepsilon_i(\cdot,\dots,\cdot)\big)\,,\frac{\delta_{\rm RL}(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}},f(s_i))}{12t_{i+1}^{2^k}}\Bigg\} \tag{99}$$

and we define $n_i$ by setting $n_1 = 1$ and

$$n_{i+1} = \max\Big\{n_i\,, n_{\mathrm{RAL}}\big(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)\big)\,, t_{i+1}m_{\mathrm{DCL}}\big(k-1, k, \tfrac{1}{2}, \tfrac{1}{t_{i+1}}\big)\,,$$
$$t_{i+1}m_{\mathrm{SL}}\Big(k, f(s_i), 1, \varepsilon_i\big(t_{i+1},\dots,t_{i+1}\big), \tfrac{1}{f(s_i)}, \tfrac{1}{f(s_i)}\Big)\,, \tag{100}$$
$$t_{i+1}m_{\mathrm{SL}}\Big(k, f(s_i), 1, \tfrac{1}{3}\delta_{\mathrm{RL}}\big(t_{i+1},\dots,t_{i+1}, f(s_i)\big),$$
$$\tfrac{1}{2}\delta_{\mathrm{RL}}\big(t_{i+1},\dots,t_{i+1}, f(s_i)\big), \tfrac{1}{f(s_i)}\Big)\Big\}\,.$$

We also define auxiliary constants

$$\mu^* = \min\Big\{\min_{2\le j\le k}\big\{\varepsilon_{\mathrm{DCL}}\big(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0}\big)\big\}\,,\ \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}\Big\}\,,$$
$$n^* = \max_{2\le j\le k}\max\Big\{t_0 m_{\mathrm{DCL}}\big(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0}\big)\,,\ t_0 m_{\mathrm{SL}}\big(j+1, o_{\mathrm{RL}}, 1, \tfrac{\mu^*}{3}, 1, \tfrac{1}{o_{\mathrm{RL}}}\big)\Big\}\,. \tag{101}$$

Finally, we fix the constants $\mu_{\mathrm{RL}}$, $t_{\mathrm{RL}}$, and $n_{\mathrm{RL}}$ promised by Lemma 23 in the following way

$$\mu_{\mathrm{RL}} = \mu^*/(2t_0^{2^k})\,, \quad t_{\mathrm{RL}} = t_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}\,, \quad \text{and} \quad n_{\mathrm{RL}} = \max\big\{n^*, n_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}\big\}\,. \tag{102}$$

For the rest of this section let all constants and functions be fixed as stated in (93)–(102).

6.2.2. *The index pumping lemma.* Now let a set $V$, a family of partitions $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and a family of $(k+1)$-uniform hypergraphs $\mathscr{H}^{(k+1)}$ satisfying the assumptions $(a)$–$(c)$ of RL$(k+1)$ be given, i.e.,

(RL.$a$) $|V| = n \ge n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divided $n$,
(RL.$b$) $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and $o_{\mathrm{RL}}$-bounded family of partitions on $V$, and
(RL.$c$) $\mathscr{H}^{(k+1)} = \{\mathcal{H}_1^{(k+1)}, \dots, \mathcal{H}_{s_{\mathrm{RL}}}^{(k+1)}\}$ is a partition of $[V]^{k+1}$.

The main idea of the proof is to inductively define a sequence of families of partitions $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ on $V$ for $i \ge 0$, which will satisfy

($\mathrm{R}_0$.*1*) $\mathscr{R}_0 = \{\mathscr{R}_0^{(j)}\}_{j=1}^k$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable and $t_0$-bounded,
($\mathrm{R}_0$.*2*) $\mathscr{R}_0 \prec \mathscr{O}$,
($\mathrm{R}_i$.*1*) $\mathscr{R}_i = \{\mathscr{R}_i^{(j)}\}_{j=1}^k$ is an $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i}), \boldsymbol{a}^{\mathscr{R}_i})$-equitable and $t_i$-bounded, and
($\mathrm{R}_i$.*2*) $\mathscr{R}_i \prec \mathscr{R}_0$.

Note that due to the fact that $\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \le \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0})$ (cf. (99)), a family of partitions $\mathscr{R}_0$ which satisfies ($\mathrm{R}_0$.*1*) and ($\mathrm{R}_0$.*2*) also satisfies ($\mathrm{R}_i$.*1*) and ($\mathrm{R}_i$.*2*) for $i = 0$.

Moreover, we will show that if there is a hypergraph $\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$ which is not $(\delta_{k+1,\mathrm{RL}}, *, r(\boldsymbol{a}^{\mathscr{R}_i}))$-regular w.r.t. $\mathscr{R}_i$, then $\mathscr{R}_{i+1}$ can be chosen in such a way that the index increases by $\delta^4_{k+1,\mathrm{RL}}/8$, More precisely we will show the following so-called *index pumping lemma*, which proof merges some ideas from [26] and [20, Cleaning Phase I].

**Lemma 41** (Index pumping lemma)**.** *Let $0 \leq i < \lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil$ be an integer and let $\mathscr{R}_0$ be a family of partitions satisfying* ($\mathrm{R}_0$.*1*) *and* ($\mathrm{R}_0$.*2*)*.*

*If $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ satisfies* ($\mathrm{R}_i$.*1–2*)*, but fails to satisfy* (*H*) *of* $\mathrm{RL}(k+1)$ *for* $r(\boldsymbol{a}^{\mathscr{R}_i})$*, then there exists a family of partitions $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ satisfying* ($\mathrm{R}_{i+1}$.*1*) *and* ($\mathrm{R}_{i+1}$.*2*) *and*

$$\mathrm{ind}(\mathscr{R}_{i+1}^{(k)}) \geq \mathrm{ind}(\mathscr{R}_i^{(k)}) + \delta_{k+1,\mathrm{RL}}^4/8\,, \tag{103}$$

*where the index is defined with respect to $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ (cf. Definition 32).*

Next we deduce $\mathrm{RL}(k+1)$ (i.e., Lemma 23) from Lemma 41. We then give the proof of Lemma 41 in Section 6.3.

*Proof of Lemma 23.* Suppose all constants are fixed as in Section 6.2.1 and let $V$, $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$, and $\mathscr{H}^{(k+1)}$ satisfying (RL.*a*)–(RL.*c*) be given. We have to ensure the existence of a family of partitions $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}^{\mathscr{P}})$ on $V$ satisfying

- (RL.*P1*) $\mathscr{P}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\mathrm{RL}}$-bounded,
- (RL.*P2*) $\mathscr{P} \prec \mathscr{O}$, and
- (RL.*H*) $\mathcal{H}_i^{(k+1)}$ is $(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$ for every $i \in [s_{\mathrm{RL}}]$.

**Construction of a family $\mathscr{R}_0$.** In view of Lemma 41 we first need an appropriate family of partitions $\mathscr{R}_0$. We distinguish two cases depending on the size of $\eta^{\mathscr{O}}$.

**Case 1** ($\eta^{\mathscr{O}} \leq \eta_{\mathrm{RL}}$)**.** In this case we simply set $\mathscr{R}_0 = \mathscr{O}$. It then follows from (RL.*b*) that $\mathscr{R}_0$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable, since $\mu_{\mathrm{RL}} \leq \mu^* \leq \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}$ by (101) and (102). Also $\mathscr{R}_0 = \mathscr{O}$ is $o_{\mathrm{RL}}$-bounded by (RL.*b*) and, hence, it is $t_0$-bounded by (93). Therefore, $\mathscr{R}_0$ chosen this way satisfies ($\mathrm{R}_0$.*1*). Moreover, ($\mathrm{R}_0$.*2*) holds trivially. ♢

**Case 2** ($\eta^{\mathscr{O}} > \eta_{\mathrm{RL}}$)**.** We construct a refinement $\mathscr{R}_0$ of $\mathscr{O}$ so that $|\,\mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})\,| \geq (1-\eta_{\mathrm{RL}})\binom{n}{k+1}$. We construct $\mathscr{R}_0 = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(k)}\}$ inductively. More precisely we show for every $j = 1, \dots, k$ that the following statement ($\mathfrak{S}_j$) holds.

($\mathfrak{S}_j$) there is a $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}, \dots, a_j^{\mathscr{R}_0})\big)$-equitable, $t_0$-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(j)}\}$ on $V$, which refines $\mathscr{O}(j) = \{\mathscr{O}^{(1)}, \dots, \mathscr{O}^{(j)}\}$.

Since, $\mu^* \leq \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}$ it then follows that there is a family of partitions $\mathscr{R}_0$ so that ($\mathrm{R}_0$.*1*) and ($\mathrm{R}_0$.*2*) are satisfied.

**Induction start $j = 1$.** We split each vertex class $W \in \mathscr{O}^{(1)}$ into $t_0/a_1^{\mathscr{O}}$ classes of size $n/(a_1^{\mathscr{O}} t_0)$, where $t_0$ is given in (93). Note that $t_0/a_1^{\mathscr{O}}$ is an integer by definition of $t_0$ and $o_{\mathrm{RL}} \geq a_1^{\mathscr{O}}$. Moreover, $n/(a_1^{\mathscr{O}} t_0)$ is an integer due to the choice of $t_{\mathrm{RL}} \geq t_0 > o_{\mathrm{RL}} \geq a_1^{\mathscr{O}}$ (cf. (102) and (93)) and (RL.*a*). This defines the partition $\mathscr{R}_0^{(1)}$ with $a_1^{\mathscr{R}_0} = t_0$. Note that

$$\Big|[V]^{k+1} \setminus \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})\Big| \leq t_0 \binom{n/t_0}{2} n^{k-1} \leq \frac{n^{k+1}}{2t_0} \overset{(93)}{\leq} \eta_{\mathrm{RL}} \binom{n}{k+1}.$$

Consequently, $\mathscr{R}_0^{(1)}$ is an $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0})\big)$-equitable, $t_0$-bounded refinement of $\mathscr{O}^{(1)}$, which establishes the induction start.

**Induction step.** Assume there exist a $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}, \dots, a_j^{\mathscr{R}_0})\big)$-equitable, $t_0$-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(j)}\}$ refining $\mathscr{O}(j)$. We define

$\mathscr{R}_0^{(j+1)}$ for each polyad $\hat{\mathcal{R}}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$. We set $a_{j+1}^{\mathscr{R}_0} = a_{j+1}^{\mathscr{O}}$ and in view of statement ($\mathfrak{S}_{j+1}$) we have to show that for every $\hat{\mathcal{R}}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$ there exists a partition $\{\mathcal{R}_a^{(j+1)} \colon a \in [a_{j+1}^{\mathscr{R}_0}]\}$ of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ so that for every $a \in [a_{j+1}^{\mathscr{R}_0}]$ the following two assertions hold

- (I) $\mathcal{R}_a^{(j+1)}$ is $(\mu^*, 1/a_{j+1}^{\mathscr{R}_0})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$ and
- (II) either $\mathcal{R}_a^{(j+1)} \subseteq \mathrm{Cross}_{j+1}(\mathscr{R}_0^{(1)}) \setminus \mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$ or $\mathcal{R}_a^{(j+1)} \subseteq \mathcal{O}^{(j+1)}$ for some $\mathcal{O}^{(j+1)} \in \mathscr{O}^{(j+1)}$.

So let $\hat{\mathcal{R}}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$ and let $\boldsymbol{\mathcal{R}}$ be the corresponding $(n/a_1^{\mathscr{R}_0}, j+1, j)$-complex, i.e., $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}(J') = \{\hat{\mathcal{R}}^{(h)}(J')\}_{h=1}^{j}$ for any $J' \in \mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ (see (1)). From the induction assumption we infer that $\boldsymbol{\mathcal{R}}$ is an $(\mu^*, (1/a_1^{\mathscr{R}_0}, \dots, 1/a_j^{\mathscr{R}_0}))$-regular complex. Therefore, by the choice of $\mu^*$ and $n_{\mathrm{RL}} \geq n^*$ in (101) and (102) we can apply Theorem 18 and infer that

$$\left|\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})\right| \geq \frac{1}{2} \prod_{h=2}^{j} \left(\frac{1}{a_h^{\mathscr{R}_0}}\right)^{\binom{j+1}{h}} \times \left(\frac{n}{a_1^{\mathscr{R}_0}}\right)^{j+1} \geq \frac{n^{j+1}}{2t_0^{2^{j+1}}}\,. \tag{104}$$

**Case 2.1** ($\hat{\mathcal{R}}^{(j)} \not\subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$)**.** In this case we simply apply the slicing lemma, Proposition 22, with

$$j_{\mathrm{SL}} = j+1\,,\ s_{0,\mathrm{SL}} = o_{\mathrm{RL}}\,,\ r_{\mathrm{SL}} = 1\,,\ \delta_{0,\mathrm{SL}} = \mu^*/3\,,\ \varrho_{0,\mathrm{SL}} = 1\,,\ \text{and } p_{0,\mathrm{SL}} = 1/o_{\mathrm{RL}}\,,$$
$$\text{to}\quad \hat{\mathcal{P}}_{\mathrm{SL}}^{(j)} = \hat{\mathcal{R}}^{(j)}\,,\quad \text{and}\quad \mathcal{P}_{\mathrm{SL}}^{(j+1)} = \mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})\quad \text{with}$$
$$s_{\mathrm{SL}} = a_{j+1}^{\mathscr{O}}\,,\quad \delta_{\mathrm{SL}} = \mu^*/3\,,\quad \varrho_{\mathrm{SL}} = 1\,,\quad \text{and}\quad \left\{p_{\xi,\mathrm{SL}} = 1/a_{j+1}^{\mathscr{O}} \colon \xi \in [a_{j+1}^{\mathscr{O}}]\right\}.$$

It follows from (104) and the choice of $n^*$ in (101) that all assumptions of Proposition 22 are satisfied for this choice of parameters. Consequently, there exist a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ into $a_{j+1}^{\mathscr{O}}$ distinct $(n/a_1^{\mathscr{R}_0}, j+1, j+1)$-hypergraphs which are $(\mu^*, 1/a_{j+1}^{\mathscr{R}_0})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$, i.e., (I) holds. Moreover, since we assume $\hat{\mathcal{R}}^{(j)} \not\subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$ each of these $(n/a_1^{\mathscr{R}_0}, j+1, j+1)$-hypergraphs is contained in $\mathrm{Cross}_{j+1}(\mathscr{R}_0^{(1)}) \setminus \mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$ and (II) holds. ♢

**Case 2.2** ($\hat{\mathcal{R}}^{(j)} \subseteq \mathrm{Cross}_j(\mathscr{O}^{(1)})$)**.** Then there exists some $\hat{\mathcal{O}}^{(j)} \in \hat{\mathscr{O}}^{(j)}$ such that $\hat{\mathcal{R}}^{(j)} \subseteq \hat{\mathcal{O}}^{(j)}$, since $\mathscr{R}_0(j) \prec \mathscr{O}(j)$ by induction assumption. Moreover, there exists a family $\{\mathcal{O}_1^{(j+1)}, \dots, \mathcal{O}_{a_{j+1}^{\mathscr{O}}}^{(j+1)}\} \subseteq \mathscr{O}^{(j+1)}$ of $(\mu_{\mathrm{RL}}, 1/a_{j+1}^{\mathscr{O}})$-regular (w.r.t. $\hat{\mathcal{O}}^{(j)}$) $(n/a_1^{\mathscr{O}}, j+1, j+1)$-hypergraphs which partition $\mathcal{K}_{j+1}(\hat{\mathcal{O}}^{(j)})$. Hence (104) yields that $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)}) \cap \mathcal{O}_a^{(j+1)}$ is $(2t_0^{2^{j+1}}\mu_{\mathrm{RL}}, 1/a_{j+1}^{\mathscr{O}})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$ for every $a \in [a_{j+1}^{\mathscr{O}}]$. Therefore, from the choice of $\mu_{\mathrm{RL}}$ in (102) we infer that

$$\{\mathcal{R}_a^{(j+1)} = \mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)}) \cap \mathcal{O}_a^{(j+1)} \colon a \in [a_{j+1}^{\mathscr{O}}]\}$$

is a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ which satisfies (I). Moreover, (II) holds trivially. ♢

In both cases, Case 2.1 and Case 2.2, we defined a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ which satisfies (I) and (II). Repeating the argument for every $\hat{\mathcal{R}}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$ gives rise to $\mathscr{R}_0^{(j+1)}$ and establishes the induction step. Consequently, there exist a partition $\mathscr{R}_0$ which satisfies ($\mathrm{R}_0$.*1*) and ($\mathrm{R}_0$.*2*) in this case, Case 2. ♢

Having constructed and appropriate family of partitions $\mathscr{R}_0$, the rest of the proof of Lemma 23 is based on successive applications of Lemma 41. This idea was introduced by Szemerédi in [31] and also used in [5, 6, 10, 13, 14, 16, 26].

Since $\mathscr{R}_0$ was constructed in such a way that ($\mathrm{R}_0.1$) and ($\mathrm{R}_0.2$) hold, we note that due to

$$\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \overset{(99)}{\leq} \delta_{\mathrm{RL}}(t_0,\dots,t_0) \overset{(92)}{\leq} \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0}),$$

and $t_0 \leq t_{\mathrm{RL}}$ (cf. (98) and (102)) the partition $\mathscr{P} = \mathscr{R}_0$ satisfies (RL.*P1*) and (RL.*P2*). If (RL.*H*) holds as well, then we are done.

Otherwise we iterate Lemma 41 and infer the existence of a sequence of partitions $\mathscr{R}_i$ for $i \geq 0$, which satisfy ($\mathrm{R}_i.1$) and ($\mathrm{R}_i.2$). It then follows from Fact 33 and (103) that there must be some $0 \leq i_0 \leq \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$ such that $\mathscr{R}_{i_0}$ also admits (RL.*H*) for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i_0}})$. Since $t_i \leq t_{\mathrm{RL}}$ (cf. (98) and (102)) and and $\mathscr{R}_i \prec \mathscr{R}_0 \prec \mathscr{O}$ (cf. ($\mathrm{R}_i.2$) and ($\mathrm{R}_0.2$)) for every $0 \leq i \leq \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$, $\mathscr{P} = \mathscr{R}_{i_0}$ satisfies (RL.*P1*), (RL.*P2*), and (RL.*H*). This concludes the proof of Lemma 23 based on Lemma 41. □

6.3. **Proof of the index pumping lemma.** We prove Lemma 41 in this section. The proof is mainly based on Lemma 25 and the propositions developed in Section 6.1.

*Proof of Lemma 41.* Recall the definition of the constants and functions in (93)–(102) in Section 6.2.1. Let $0 \leq i < \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$ be some integer and suppose $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ satisfies ($\mathrm{R}_i.1$) and ($\mathrm{R}_i.2$) and fails to satisfy ($H$) of RL($k+1$) for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$. In other words

($\neg H_i$) there exist some $s_0 \in [s_{\mathrm{RL}}]$ such that $\mathcal{H}^{(k+1)}_{s_0}$ is $\big(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})\big)$-irregular w.r.t. $\mathscr{R}_i$.

Then $V$, $\mathscr{R}^{(1)}_0$, $\mathscr{H}^{(k+1)}$, and $\mathscr{R}^{(k)}_i$ satisfy the assumptions of Proposition 39 with $\delta = \delta_{k+1,\mathrm{RL}}$ and $r = r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$, due to ($\mathrm{R}_0.1$) combined with (92) and ($\neg H_i$). Consequently, there exists a partition $\mathscr{X}^{(k)}$ of $[V]^k$ satisfying the conclusions (*i*)–(*iii*) of Proposition 39, i.e.,

(P.39.*i*) $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}_i \prec \mathscr{R}^{(k)}_0$ (cf. ($\mathrm{R}_i.2$) for the second '$\prec$'),

(P.39.*ii*) $|\mathscr{X}^{(k)}| \leq |\mathscr{R}^{(k)}_i| \times 2^{r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})\times|\hat{\mathscr{R}}^{(k)}_i|} \leq s_i$ (cf. $t_i$-boundedness of $\mathscr{R}_i$ in ($\mathrm{R}_i.1$), the monotonicity of $r_{\mathrm{RL}}(\cdot,\dots,\cdot)$ in (92), and the definition of $s_i$ in (96)), and

(P.39.*iii*) $\mathrm{ind}(\mathscr{X}^{(k)}) \geq \mathrm{ind}(\mathscr{R}^{(k)}_i) + \delta^4_{k+1,\mathrm{RL}}/2$.

The next step is to apply RAL($k$), Lemma 25 to $V$, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants $o_i$, $s_i$, $\eta_i$, $\nu_i$, and the function $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$ defined in (93)–(97). For this we have to check the assumptions of Lemma 25;

(RAL.*a*) $|V| \geq n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot))$ and $(t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)))!$ divides $n$,

(RAL.*b*) $\mathscr{R}_0 = \mathscr{R}_0(k, \boldsymbol{a}^{\mathscr{R}_0})$ is a $(\eta', \mu', \boldsymbol{a}^{\mathscr{R}_0})$-equitable family of partitions (for some $\eta' > 0$ and $\mu' \leq \mu_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)))$ and $o_i$-bounded, and

(RAL.*c*) $s' = |\mathscr{X}^{(k)}| \leq s_i$ and $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}_0$.

Property (RAL.*a*) is implied by (RL.*a*) and the fact that for $i < \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$

$$n_{\mathrm{RL}} \overset{(102)}{\geq} n_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \overset{(100)}{\geq} n_{i+1} \overset{(100)}{\geq} n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot))$$

and that the same line of inequalities holds with $n$ replaced by $t$.

It follows from the definition of $o_i$ in (96) and (R$_0$.*1*) that $\mathscr{R}_0$ is $o_i$-bounded. Moreover, (R$_0$.*1*) and (99) imply the required equitability of $\mathscr{R}_0$, which yields (RAL.*b*).

Finally, (RAL.*c*) follows immediately from (P.39.*i*) and (P.39.*ii*).

Consequently, we can apply Lemma 25 to $V$, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants $o_i$, $s_i$, $\eta_i$, $\nu_i$, and $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$. Lemma 25 then asserts that there exist a family of partitions $\mathscr{S} = \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$, and a partition $\mathscr{Y}^{(k)} = \{\mathcal{Y}^{(k)}_1, \dots, \mathcal{Y}^{(k)}_{s'}\}$ of $[V]^k$ so that

- (RAL.*S1*) $\mathscr{S}$ is $(\eta_i, \varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), \boldsymbol{a}^{\mathscr{S}})$-equitable and $t_{i+1}$-bounded family of partitions (since $t_{i+1} \geq t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))$ by (98)),
- (RAL.*S2*) $\mathscr{S} \prec \mathscr{R}_0(k-1) = \{\mathscr{R}^{(j)}_0\}_{j=1}^{k-1}$
- (RAL.*Y1*) $\mathcal{Y}^{(k)}_\ell$ is perfectly $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\mathscr{S}$ for every $\ell \in [s']$,
- (RAL.*Y2*) $\big|\mathcal{Y}^{(k)}_\ell \triangle \mathcal{X}^{(k)}_\ell\big| \leq \nu_i n^k$ for every $\ell \in [s']$, and
- (RAL.*Y3*) if $\mathcal{X}^{(k)}_\ell \subseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ then $\mathcal{Y}^{(k)}_\ell \subseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ for every $\ell \in [s']$ and $\mathscr{Y}^{(k)} \prec \mathscr{R}^{(k)}_0 \prec \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$.

In particular, (P.39.*i*) and (RAL.*Y3*) show that $\mathscr{X}^{(k)}$ and $\mathscr{Y}^{(k)}$ refine $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$, respectively. Hence, due to (RAL.*Y2*) and the first part of (RAL.*Y3*) the assumptions of Proposition 34 are satisfied for $V$, $\mathscr{R}^{(1)}_0$, $\mathscr{H}^{(k+1)}$, $\mathscr{X}^{(k)}$, $\mathscr{Y}^{(k)}$, $s_{\mathrm{P}.34} = s' \leq s_i$, and $\nu_{\mathrm{P}.34} = \nu_i$ and, consequently, Proposition 34 yields

$$\begin{aligned}\mathrm{ind}(\mathscr{Y}^{(k)}) &\geq \mathrm{ind}(\mathscr{X}^{(k)}) - 3(k+1)s_{\mathrm{RL}}s_i^{k+1}\nu_i \\ &\overset{(96)}{\geq} \mathrm{ind}(\mathscr{X}^{(k)}) - \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4} \overset{(\mathrm{P}.39.iii)}{\geq} \mathrm{ind}(\mathscr{R}^{(k)}_i) + \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4}\,. \end{aligned} \tag{105}$$

Our next temporary goal is to construct a partition $\mathscr{Z}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ which forms a family of partitions together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$. This means, that such an $\mathscr{Z}^{(k)}$ has to satisfy two conditions – it must partition $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and it must refine $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$. The partition $\mathscr{Y}^{(k)}$ fails to satisfy any of these two requirements. It partitions all of $[V]^k$ (rather than only $\mathrm{Cross}_k(\mathscr{S}^{(1)})$) and, more importantly, we cannot ensure that it refines $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$. However, we easily "fix" these shortcomings of $\mathscr{Y}^{(k)}$ and define $\mathscr{Z}^{(k)}$ as follows

$$\mathscr{Z}^{(k)} = \big\{\mathcal{Y}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon\ \mathcal{Y}^{(k)} \in \mathscr{Y}^{(k)} \text{ and } \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\big\}\,. \tag{106}$$

For convenience we set for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$

$$\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}) = \{\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\colon\ \mathcal{Z}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \neq \varnothing\}\,. \tag{107}$$

The partition $\mathscr{Z}^{(k)}$ has the following properties which we verify below.

- (*Z1*) $\mathscr{Z}^{(k)}$ partitions $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and $\mathscr{Z}^{(k)} \prec \{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon\ \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$,
- (*Z2*) $|\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})| \leq s_i$ for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$, and
- (*Z3*) for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ and $\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})$ we have that $\mathcal{Z}^{(k)}$ is $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$,
- (*Z4*) $\mathscr{Z}^{(k)} \prec \mathscr{R}^{(k)}_0 \prec \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$, and
- (*Z5*) $\mathrm{ind}(\mathscr{Z}^{(k)}) \geq \mathrm{ind}(\mathscr{R}^{(k)}_i) + \delta^4_{k+1,\mathrm{RL}}/4\,.$

Property (*Z1*) follows from the fact that $\mathscr{Y}^{(k)}$ partitions all of $[V]^k$ and the definition of $\mathscr{Z}^{(k)}$ in (106). Assertion (*Z2*) is an immediate consequence of (107) and $|\mathscr{Y}^{(k)}| = s' \leq s_i$ (cf. (RAL.*c*)). We also note that (*Z3*) is simply a reformulation of (RAL.*Y1*).

Hence, it is only left to verify (*Z4*) and (*Z5*). First we consider (*Z4*). For that we first note that $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$ partitions a superset of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ due to (RAL.*S2*). Consequently, (*Z4*) follows from the definition of $\mathscr{Z}^{(k)}$ in (106) and (RAL.*Y3*).

Finally we focus on (*Z5*). For that we consider the restriction of $\mathscr{Y}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, i.e.,

$$\mathscr{Y}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \big\{\mathcal{Y}^{(k)} \in \mathscr{Y}^{(k)}\colon\ \mathcal{Y}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\ \big\}\,.$$

It follows from Definition 32 that the index of $\mathscr{Y}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ satisfies

$$\mathrm{ind}\big(\mathscr{Y}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) = \mathrm{ind}(\mathscr{Y}^{(k)}) \overset{(105)}{\geq} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \tfrac{\delta_{k+1,\mathrm{RL}}^4}{4}\,. \tag{108}$$

On the other hand, in view of (*Z4*) the restriction of $\mathscr{Z}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$

$$\mathscr{Z}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \big\{\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\colon\ \mathcal{Z}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\ \big\}$$

is a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ and, therefore, $\mathrm{ind}(\mathscr{Z}^{(k)}) = \mathrm{ind}\big(\mathscr{Z}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big)$ . Moreover, we observe that $\mathscr{Z}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} \prec \mathscr{Y}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$ due to (106).

Finally, Proposition 36 then yields (*Z5*)

$$\begin{aligned}\mathrm{ind}(\mathscr{Z}^{(k)}) &= \mathrm{ind}\big(\mathscr{Z}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big)\\ &\geq \mathrm{ind}\big(\mathscr{Y}^{(k)}\big|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) \overset{(108)}{\geq} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \tfrac{\delta_{k+1,\mathrm{RL}}^4}{4}\,.\end{aligned}$$

Having verified (*Z1*)–(*Z5*) we come to the last part of the proof and define the family of partitions $\mathscr{R}_{i+1}$. The careful reader (who managed not to get lost in details so far) will note that due to (*Z1*) the partition $\mathscr{Z}^{(k)}$ together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ forms a family of partitions on $V$. Moreover, due to (RAL.*S2*) and (*Z4*) it satisfies ($\mathrm{R}_{i+1}$.*2*) and due to (RAL.*S1*), (*Z2*), and (*Z3*) it "almost" satisfies ($\mathrm{R}_{i+1}$.*1*). But unfortunately, the densities of the $\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}$ vary and thus this family of partitions $\mathscr{Z}^{(k)} \cup \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is not equitable. In the final step of this proof we derive $\mathscr{R}_{i+1}$ from $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}}) \cup \mathscr{Z}^{(k)}$ by "cleaning the imperfections" of $\mathscr{Z}^{(k)}$ mentioned above. For that we will use the following claim, which somewhat dry proof is based on repeated applications of Proposition 22.

**Claim 42.** *There exist a partition $\mathscr{T}^{(k)}$ of* $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ *such that*

- (*T1*) $\mathscr{T}^{(k)} \prec \{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon\ \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$,
- (*T2*) $\big|\big\{\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}\colon\ \mathcal{T}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\big\}\big| = f(s_i)$ *for every fixed* $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$,
- (*T3*) *every* $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}$ *is* $\big(\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}}, f(s_i)\big), 1/f(s_i)\big)$*-regular w.r.t. the unique* $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ *which satisfies* $\mathcal{T}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$,
- (*T4*) $\mathscr{T}^{(k)} \prec \mathscr{R}_0^{(k)}$, *and*
- (*T5*) $\mathscr{T}^{(k)}$ *is a* $(\delta_{k+1,\mathrm{RL}}^4/8)$*-refinement of* $\mathscr{Z}^{(k)}$.

We first finish the proof of Lemma 41 and give the proof of Claim 42, which makes use of (*Z1*)–(*Z4*), afterwards. In order to conclude the proof of Lemma 41 we have to define a family of partitions $\mathscr{R}_{i+1}$ on $V$, which satisfies ($\mathrm{R}_{i+1}.1$), ($\mathrm{R}_{i+1}.2$), and (103). With this in mind we set

$$\boldsymbol{a}^{\mathscr{R}_{i+1}} = \left(a_1^{\mathscr{S}}, \dots, a_{k-1}^{\mathscr{S}}, f(s_i)\right),$$
$$\mathscr{R}_{i+1}^{(j)} = \begin{cases} \mathscr{S}^{(j)} & \text{for } j \in [k-1] \\ \mathscr{T}^{(k)} & \text{for } j = k \end{cases} \quad \text{and} \quad \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}}) = \left\{\mathscr{R}_{i+1}^{(j)}\right\}_{j=1}^{k}.$$

We now first show that $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ is a family of partitions on $V$. Due to the fact that $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is a family of partitions on $V$, we only have to verify that $\mathscr{R}_{i+1}^{(k)} = \mathscr{T}^{(k)}$ fulfills both requirements of part (*ii*) of Definition 10. However, this is immediate from (*T1*) and (*T2*).

Next we consider ($\mathrm{R}_{i+1}.1$). Note that (RAL.*S1*) (combined with (97)) and (*T3*) show that $\mathscr{R}_{i+1}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i+1}}), \boldsymbol{a}^{\mathscr{R}_{i+1}})$-equitable. Moreover, (RAL.*S1*) and the choice of $t_{i+1} \geq f(s_i)$ in (98) imply that $\max_{j\in[k]} a_j^{\mathscr{R}_{i+1}} \leq t_{i+1}$. In other words, $\mathscr{R}_{i+1}$ is $t_{i+1}$-bounded and ($\mathrm{R}_{i+1}.1$) holds.

The property ($\mathrm{R}_{i+1}.2$) follows from (RAL.*S2*) and (*T4*) and (103) is a consequence of (*Z5*) and (*T5*), combined with Proposition 38.

Hence $\mathscr{R}_{i+1}$ has the desired properties and we conclude the proof of Lemma 41 based on Claim 42. □

*Proof of Claim 42.* We have to show that there is a partition $\mathscr{T}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ satisfying (*T1*)–(*T5*). For technical reasons we first extend the partition $\mathscr{R}_0^{(k)}$ from a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ to a partition of $[V]^k$ and we set

$$\widetilde{\mathcal{R}}^{(k)} = [V]^k \setminus \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) \quad \text{and} \quad \widetilde{\mathscr{R}}_0^{(k)} = \mathscr{R}_0^{(k)} \cup \widetilde{\mathcal{R}}^{(k)}. \tag{109}$$

In view of (*T1*) and (*T4*) it seems natural to define $\mathscr{T}^{(k)}$ separately for every pair

$$\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}, \quad \mathcal{R}^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)} \quad \text{satisfying} \quad \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)} \neq \varnothing. \tag{110}$$

In fact, we will prove the following claim.

**Claim 42′.** *For every pair $\hat{\mathcal{S}}^{(k-1)}$, $\mathcal{R}^{(k)}$ satisfying (110) there exists a partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ of $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$ satisfying the following properties*

(*T2′*)

$$\left|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})\right| = \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if} \quad \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}, \text{[4]} \\ f(s_i) & \text{if} \quad \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}, \end{cases}$$

(*T3′*) *every $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ is $\left(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\right)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$,*

(*T5′*) *and*

$$\left|\bigcup\left\{\mathcal{T}^{(k)} \colon \mathcal{T}^{(k)} \in \mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) \text{ and } \mathcal{T}^{(k)} \nsubseteq \mathcal{Z}^{(k)}\ \forall \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\right\}\right| \leq \frac{\delta_{k+1,\mathrm{RL}}^4}{8}\left|\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}\right|.$$

[4] Note that $f(s_i)/a_k^{\mathscr{R}_0}$ is an integer since $a_k^{\mathscr{R}_0} \leq t_0$ (cf. ($\mathrm{R}_0.1$)) and due to the fact that the definition of the function $f(\cdot)$ in (95) ensures that $f(s_i)$ is a multiple of $(t_0)!$.

Before we verify Claim 42′, we deduce Claim 42 from it. So let $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ be given for every $\hat{\mathcal{S}}^{(k-1)}$, $\mathcal{R}^{(k)}$ satisfying (110). We then set

$$\mathscr{T}^{(k)}=\bigcup\Big\{\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})\colon\ \hat{\mathcal{S}}^{(k-1)} \text{ and } \mathcal{R}^{(k)} \text{ satisfy (110)}\Big\}\,.$$

Clearly, $\mathscr{T}^{(k)}$ is a partition of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$, since $\widetilde{\mathscr{R}}_0^{(k)}$ is a partition of $[V]^k$ (cf. (109)). Furthermore, (*T1*) and (*T4*) are immediate since we constructed $\mathscr{T}^{(k)}$ separately on $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$. Moreover, it is easy to see that (*T2*), (*T3*), and (*T5*) are implied by its "prime" counterpart.

This finishes the reduction of Claim 42 to Claim 42′, which is the last missing piece in the proof of the implication $\mathrm{RAL}(k)\Longrightarrow\mathrm{RL}(k+1)$. □

Below we prove Claim 42′. The proof resembles some ideas from [20, Section 5]. The main tool in that proof is the somewhat technical slicing lemma, Proposition 22 and we first give an informal outline to convey the idea.

Suppose $\hat{\mathcal{S}}^{(k-1)}$ and $\mathcal{R}^{(k)}$ satisfy (110). Let $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ be the collection of those partition classes $\mathcal{Z}^{(k)}$ of $\mathscr{Z}^{(k)}$ which are contained in $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$, i.e.,

$$\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})=\big\{\mathcal{Z}^{(k)}\in\mathscr{Z}^{(k)}\colon\ \mathcal{Z}^{(k)}\subseteq\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}\big\}\,. \tag{111}$$

Note that due to (*Z1*) and (*Z4*)

$$\big\{\mathcal{Z}^{(k)}\colon\ \mathcal{Z}^{(k)}\in\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})\big\} \text{ partitions } \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}\,. \tag{112}$$

Indeed by (*Z1*), $\mathscr{Z}^{(k)}$ has each of its partition classes completely within or outside $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$ and by (*Z4*) the same is true for $\mathcal{R}^{(k)}$.

We will use the slicing lemma twice. In the first round we apply the slicing lemma separately to each $\mathcal{Z}^{(k)}\in\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ to slice it in such a way that all but at most one slice ("leftover" part) has density $1/f(s_i)$ w.r.t. $\hat{\mathcal{S}}^{(k-1)}$. On the other hand, we infer from the choice of $\mu_{i+1}\geq\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}$ and ($\mathrm{R}_0$.*1*) that $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$ is still $(\delta',1/a_k^{\mathscr{R}_0})$-regular with $\delta'\ll\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i))$ (cf. (99)). (In the special case $\mathcal{R}^{(k)}=\widetilde{\mathcal{R}}^{(k)}$ we have $(\delta',1)$-regularity for any $\delta'>0$.) Consequently, the union of the earlier produced "leftovers" must have a density very close to a multiple of $1/f(s_i)$, since it is $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$ minus regular pieces of density $1/f(s_i)$. Therefore, we can use the slicing lemma again (second round) to "recycle" the "leftovers", splitting it into regular pieces of density $1/f(s_i)$ and the "recycled" partition will satisfy (*T2′*) and (*T3′*). Finally, we will show that it also exhibits (*T5′*) since we chose $f(\cdot)$ in (95) in such a way that $|\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})|\times 1/f(s_i)\leq s_i/f(s_i)\ll\delta_{k+1,\mathrm{RL}}/8$, which is an upper bound on the density of the union of the "leftovers".

Below we give the technical details of the plan outlined above.

*Proof of Claim 42′.* Let $\hat{\mathcal{S}}^{(k-1)}\in\hat{\mathscr{S}}^{(k-1)}$ and $\mathcal{R}^{(k)}\in\widetilde{\mathscr{R}}_0^{(k)}$ satisfying (110) be given. We start with a few observations. From the choice of the function $\varepsilon_i$ in (97) and $n_i$ in 100 combined with (*S1*) we infer by Theorem 18 that

$$\big|\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\big|\geq\frac{1}{2}\prod_{j=2}^{k-1}\left(\frac{1}{a_j^{\mathscr{S}}}\right)^{\binom{k}{j}}\times\left(\frac{n}{a_1^{\mathscr{S}}}\right)^k\geq\frac{n^k}{2t_{i+1}^{2^k}}\,. \tag{113}$$

Suppose $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$. Let $\hat{\mathcal{R}}^{(k-1)}$ be the polyad in $\hat{\mathscr{R}}_0^{(k-1)}$ such that $\mathcal{R}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)})$. Since, $\mathscr{S}^{(k-1)} \prec \mathscr{R}_0^{(k-1)}$ (cf. (RAL.*S2*)) and $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \neq \varnothing$ (cf. (110)) we have that $\hat{\mathcal{S}}^{(k-1)} \subseteq \hat{\mathcal{R}}^{(k-1)}$. Consequently, we infer from (113) and (R$_0$.*1*) that if $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$ then

$$\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \text{ is } \big(2t_{i+1}^{2^k}\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, 1/a_k^{\mathscr{R}_0}\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}\,.$$

Moreover, if $\mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}$, then assumption (110) yields that $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \subseteq \mathcal{R}^{(k)}$ and $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$ is $(\delta', 1)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$ for every $\delta' > 0$. Therefore, in view of

$$\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \le \mu_{i+1} \overset{(99)}{\le} \frac{\delta_{\mathrm{RL}}(t_{i+1},\dots,t_{i+1},f(s_i))}{12t_{i+1}^{2^k}} \overset{(\mathrm{RAL}.S1)}{\le} \frac{\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i))}{12t_{i+1}^{2^k}}$$

we have for every $\mathcal{R}^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)}$ that

$$\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \text{ is } \big(\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i)), d_{\mathcal{R}^{(k)}}\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}\,, \tag{114}$$

where

$$d_{\mathcal{R}^{(k)}} = \begin{cases} 1/a_k^{\mathscr{R}_0} & \text{if } \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)} \\ 1 & \text{if } \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)} \end{cases}. \tag{115}$$

Furthermore, we infer from (114) combined with $d_{\mathcal{R}^{(k)}} \ge 1/a_k^{\mathscr{R}_0} \ge 1/t_0$ (cf. (R$_0$.*1*)) that

$$d\big(\mathcal{R}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \ge \frac{1}{t_0} - \frac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i)) \overset{(94)}{>} \max\left\{\frac{1}{3}\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}},f(s_i)\big)\,,\, \frac{2}{3t_0}\right\}. \tag{116}$$

Recall the definition $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ from (111) and let $\{\mathcal{Z}_1^{(k)},\dots,\mathcal{Z}_z^{(k)}\}$ be an enumeration of its members. Clearly, $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)}) \subseteq \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})$ (cf. (107)) and due to (*Z2*) we have

$$z \le s_i\,. \tag{117}$$

Our plan is to apply the slicing lemma to every member $\mathcal{Z}_j^{(k)}$ of $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$. For that we have to satisfy the assumptions of the slicing lemma among which we have to ensure that $d(\mathcal{Z}_j^{(k)}|\hat{\mathcal{S}}^{(k-1)})$ is not "too small". However, since the $\mathcal{Z}_j^{(k)}$ arose from an application of RAL$(k)$, we only have limited control over their densities, which leads to the following definition

$$Z_{\mathrm{THIN}} = \Big\{j \in [z]\colon\ d\big(\mathcal{Z}_j^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) < \tfrac{1}{f(s_i)}\Big\}\,. \tag{118}$$

Moreover, for every $j \in [z]$ we set

$$\zeta_j = \big\lfloor f(s_i) \times d\big(\mathcal{Z}_j^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big)\big\rfloor\,. \tag{119}$$

Clearly, $\zeta_j > 0$ if and only if $j \notin Z_{\mathrm{THIN}}$. We now apply the slicing lemma, Proposition 22, for every $j \in [z] \setminus Z_{\mathrm{THIN}}$ separately with

$$j_{\mathrm{SL}} = k\,,\quad s_{0,\mathrm{SL}} = s_{27}\,,\quad r_{\mathrm{SL}} = 1\,,\quad \delta_{0,\mathrm{SL}} = \varepsilon_i\big(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}}\big)\,,\quad \varrho_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}\,,$$

$$\text{and}\quad p_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}\,,\text{to}\quad \hat{\mathcal{P}}_{\mathrm{SL}}^{(k-1)} = \hat{\mathcal{S}}^{(k-1)}\,,\quad\text{and}\quad \mathcal{P}_{\mathrm{SL}}^{(k)} = \mathcal{Z}_j^{(k)}\quad \text{with} s_{\mathrm{SL}} = \zeta_j\,,$$

$$\delta_{\mathrm{SL}} = \varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\,,\quad \varrho_{\mathrm{SL}} = d(\mathcal{Z}_j^{(k)}|\hat{\mathcal{S}}^{(k-1)})\,,\ \text{and}\ \left\{p_{\xi,\mathrm{SL}} = \frac{1/f(s_i)}{d(\mathcal{Z}_j^{(k)}|\hat{\mathcal{S}}^{(k-1)})}\colon\ \xi \in [\zeta_j]\right\}.$$

It follows from (113) that the assumption $(i)$ of Proposition 22 is satisfied for $\hat{\mathcal{P}}^{(k-1)}_{\mathrm{SL}} = \hat{\mathcal{S}}^{(k-1)}$.

Moreover, $(ii)$ is a consequence of $(Z3)$ (yielding the $(\delta_{\mathrm{SL}}, \varrho_{\mathrm{SL}}, r_{\mathrm{SL}})$-regularity of $\mathcal{P}^{(k)}_{\mathrm{SL}} = \mathcal{Z}^{(k)}_j$), the definition of $Z_{\mathrm{THIN}}$ (yielding $\varrho_{\mathrm{SL}} \geq \varrho_{0,\mathrm{SL}}$), (RAL.$S1$) and the monotonicity of the function $\varepsilon_i$ (yielding $\delta_{\mathrm{SL}} \geq \delta_{0,\mathrm{SL}}$), and the choice of $\varepsilon_i$ in (97) (yielding $\varrho_{0,\mathrm{SL}} \geq 2\delta_{\mathrm{SL}}$). Furthermore, assumption $(iii)$ of Proposition 22 is a consequence of the fact that $d(\mathcal{Z}^{(k)}_j|\hat{\mathcal{S}}^{(k-1)}) \leq 1$ (yielding $p_{\xi,\mathrm{SL}} \geq p_{0,\mathrm{SL}}$ for $\xi \in [\zeta_j] = \{1, \dots, \zeta_j\}$ and the choice of the integer parameter $\zeta_j$ in (119) (yielding $\sum_{\xi\in[\zeta_j]} p_{\xi,\mathrm{SL}} \leq 1$).

Having verified the assumptions of Proposition 22 for every $j \in [z] \setminus Z_{\mathrm{THIN}}$, we infer that for every such $j$ there exists a family $\{\mathcal{T}^{(k)}_{j,0}, \mathcal{T}^{(k)}_{j,1}, \dots, \mathcal{T}^{(k)}_{j,\zeta_j}\}$ such that

$$\left\{\mathcal{T}^{(k)}_{j,0}, \mathcal{T}^{(k)}_{j,1}, \dots, \mathcal{T}^{(k)}_{j,\zeta_j}\right\} \text{ partitions } \mathcal{Z}^{(k)}_j\,, \tag{120}$$

$$\mathcal{T}^{(k)}_{j,\xi} \text{ is } \big(3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), 1/f(s_i)\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)} \text{ for every } \xi = 1, \dots, \zeta_j\,, \text{and} \tag{121}$$

$$\mathcal{T}^{(k)}_{j,0} \text{ is } (3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), d_{j,0})\text{-regular}$$
$$\text{for some } 0 \leq d_{j,0} \leq d(\mathcal{Z}^{(k)}_j|\hat{\mathcal{S}}^{(k-1)}) - \tfrac{\zeta_j}{f(s_i)} \overset{(119)}{\leq} \tfrac{1}{f(s_i)}\,. \tag{122}$$

Unfortunately, the "leftover" hypergraph $\mathcal{T}^{(k)}_{j,0}$ might not be empty and has a density differing from $1/f(s_i)$. Moreover, in general $Z_{\mathrm{THIN}}$ is not empty and we have to recycle the "leftovers" $\mathcal{T}^{(k)}_{j,0}$ with $j \notin Z_{\mathrm{THIN}}$ and the hypergraphs $\mathcal{Z}^{(k)}_j$ with $j \in Z_{\mathrm{THIN}}$. For that we consider their union

$$\mathcal{U}^{(k)} = \bigcup_{j\in[z]\setminus Z_{\mathrm{THIN}}} \mathcal{T}^{(k)}_{j,0} \cup \bigcup_{j\in Z_{\mathrm{THIN}}} \mathcal{Z}^{(k)}_j\,. \tag{123}$$

Clearly, $\mathcal{U}^{(k)}$ is the complement of $\bigcup_{j\in[z]\setminus Z_{\mathrm{THIN}}} \bigcup_{\xi\in[\zeta_j]} \mathcal{T}^{(k)}_{j,\xi}$ in $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$. Consequently, in view of (121), $|Z_{\mathrm{THIN}}| \leq z \leq s_i$ (cf. (117)), and (114) an application of Proposition 19 yields that

$$\mathcal{U}^{(k)} \text{ is } \Big(\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i, d_{\mathcal{R}^{(k)}} - \tfrac{\sum_{j\notin Z_{\mathrm{THIN}}} \zeta_j}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}\,. \tag{124}$$

Recall, that $d_{\mathcal{R}^{(k)}}$ is an integer multiple of $1/f(s_i)$. (This is obvious if $\mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}$ since $f(s_i)$ is an integer-valued function. Moreover, if $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$, then $d_{\mathcal{R}^{(k)}} = 1/a^{\mathscr{R}_0}_k$ which is a multiple of $1/f(s_i)$ since $(t_0)!$ divides $f(s_i)$ (cf. (95)) and $t_0 \geq a^{\mathscr{R}_0}_k$ (cf. (R$_0$.$1$)).) Consequently,

$$d_{\mathcal{R}^{(k)}} - \frac{\sum_{j\notin Z_{\mathrm{THIN}}} \zeta_j}{f(s_i)} = \frac{u}{f(s_i)} \text{ for some integer } 0 \leq u \leq f(s_i)\,. \tag{125}$$

This observation and the choice of the function $\varepsilon_i$ in (97) allows us to rewrite (124)

$$\mathcal{U}^{(k)} \text{ is } \Big(\tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \tfrac{u}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}\,. \tag{126}$$

The further treatment of $\mathcal{U}^{(k)}$ depends on the value of and we consider two cases.

**Case 1** $(u = 0)$**.** Note that the assumption $u = 0$ and (126) not necessarily implies that $\mathcal{U}^{(k)} = \varnothing$. It rather yields, that

$$d\big(\mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \leq \tfrac{1}{3}\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}}, f(s_i)\big)\,.$$

On the other hand, by (116)

$$d\big(\mathcal{R}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) > \tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))\,.$$

Therefore, from (112), we infer that $Z_{\mathrm{THIN}} \neq [z]$ and there exist some $j_0 \in [z] \setminus Z_{\mathrm{THIN}}$ with $\zeta_{j_0} \geq 1$ and hence $\mathcal{T}^{(k)}_{j_0,1}$ exists. We then define, the promised partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ as follows

$$\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) = \Big\{\mathcal{T}^{(k)}_{j,\xi}\colon\ j \in [z] \setminus Z_{\mathrm{THIN}},\ j \neq j_0, \text{ and } \xi \in [\zeta_j]\Big\} \\ \cup \Big\{\mathcal{T}^{(k)}_{j_0,\xi}\colon\ \xi = 2, \dots, \zeta_j\Big\} \cup \Big\{\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)}\Big\}\,.$$

It follows from the definition of $\mathcal{U}^{(k)}$ in (123) in conjunction with (120) and (112) that $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ defined above indeed partitions $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$. We conclude this case with the verification of properties (*T2′*), (*T3′*), and (*T5′*).

First we consider (*T2′*). Clearly, $\big|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})\big| = \sum_{j\in[z]\setminus Z_{\mathrm{THIN}}} \zeta_j$. So in view of (125), we infer from the assumption $u = 0$ in this case, that

$$\big|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})\big| = \sum_{j\in[z]\setminus Z_{\mathrm{THIN}}} \zeta_j = d_{\mathcal{R}^{(k)}} \times f(s_i) \overset{(115)}{=} \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } \ \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)} \\ f(s_i) & \text{if } \ \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)} \end{cases},$$

which is (*T2′*).

Since $\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leq \frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$ (cf. (97)), (121) guarantees (*T3′*) for all members of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ with exception $\mathcal{T}^{(k)}_{j_0} \cup \mathcal{U}^{(k)}$. Consequently, verifying (*T3′*) reduces to showing that

$$\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)} \text{ is } \big(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}\,. \tag{127}$$

However, this follows from (121) and (126) by Proposition 19, since $u = 0$ and since by the choice in (97) we have $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leq \frac{2}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$.

Finally, we consider (*T5′*). Here we note that due to the definition of the partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ it suffices to show that

$$d\big(\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \leq \frac{\delta^4_{k+1,\mathrm{RL}}}{8} d\big(\mathcal{R}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big)\,. \tag{128}$$

For that we first derive from (127) combined with the definition of the function $f(\cdot)$ in (95) and the bound from (94) that

$$d\big(\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \overset{(127)}{\leq} \frac{1}{f(s_i)} + \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) \leq \frac{\delta^4_{k+1,\mathrm{RL}}}{12 t_0}\,. \tag{129}$$

Therefore, (128) follows from (129) and (116). This concludes the proof of Claim 42′ in this case. ♢

**Case 2** ($u > 0$)**.** Recall, due to (126), is $\mathcal{U}^{(k)}$ $\big(\frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \frac{u}{f(s_i)}\big)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$. In this case we are going to apply the slicing lemma to "recycle" the edges of $\mathcal{U}^{(k)}$, i.e., to partition it into regular pieces of density $1/f(s_i)$. More precisely, we apply Proposition 22 with

$$j_{\rm SL}=k\,,\ \ s_{0,{\rm SL}}=f(s_i)\,,\ \ r_{\rm SL}=1\,,\ \ \delta_{0,{\rm SL}}=\tfrac{1}{3}\delta_{\rm RL}(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}},f(s_i))\,,$$

$$\varrho_{0,{\rm SL}}=\tfrac{1}{f(s_i)}\,,\ \text{and}\ p_{0,{\rm SL}}=\tfrac{1}{f(s_i)}\,,\text{to}\quad \hat{\mathcal{P}}^{(k-1)}_{\rm SL}=\hat{\mathcal{S}}^{(k-1)}\,,\ \text{and}\ \mathcal{P}^{(k)}_{\rm SL}=\mathcal{U}^{(k)}\ \text{with}$$

$$s_{\rm SL}=u\,,\quad \delta_{\rm SL}=\frac{1}{3}\delta_{\rm RL}(\boldsymbol{a}^{\mathscr{S}},f(s_i))\,,\quad \varrho_{\rm SL}=\frac{u}{f(s_i)}\,,\quad\text{and}\quad \left\{p_{\xi,{\rm SL}}=\frac{1}{u}\colon\ \xi\in[u]\right\}.$$

It follows from (113) that the assumption $(i)$ of Proposition 22 is satisfied for $\hat{\mathcal{P}}^{(k-1)}_{\rm SL}=\hat{\mathcal{S}}^{(k-1)}$. Moreover, $(ii)$ follows from (126) (yielding the $(\delta_{\rm SL},\varrho_{\rm SL},r_{\rm SL})$-regularity of $\mathcal{P}^{(k)}_{\rm SL}=\mathcal{U}^{(k)}_j$), the assumption of the case $u\geq 1$ (yielding $\varrho_{\rm SL}\geq\varrho_{0,{\rm SL}}$), (RAL.*S1*) and the monotonicity (cf. (92)) of the function $\delta_{\rm RL}$ (yielding $\delta_{\rm SL}\geq\delta_{0,{\rm SL}}$), and of (94) (yielding $\varrho_{0,{\rm SL}}\geq 2\delta_{\rm SL}$). Furthermore, note that $p_{\xi,{\rm SL}}\geq p_{0,{\rm SL}}$, since $u\leq f(s_i)$ (cf. (125)) and that that

$$\sum_{\xi\in[u]}p_{\xi,{\rm SL}}=1\,. \tag{130}$$

Consequently, assumption $(iii)$ of Proposition 22 holds for the choice of parameters above.

Having verified the assumptions of Proposition 22, we infer that there exists a family $\{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\}$ (note that due to (130) there is "leftover" class $\mathcal{U}^{(k)}_0$) such that

$$\{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\}\ \text{partitions}\ \mathcal{U}^{(k)} \tag{131}$$

$$\mathcal{U}^{(k)}_\xi\ \text{is}\ \big(\delta_{k,{\rm RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i)),1/f(s_i)\big)\text{-regular w.r.t. }\hat{\mathcal{S}}^{(k-1)}\ \text{for every}\ \xi\in[u]\,. \tag{132}$$

We finally define the required family $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ in a straightforward manner

$$\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})=\left\{\mathcal{T}^{(k)}_{j,\xi}\colon j\in[z]\setminus Z_{\rm THIN}\ \text{and}\ \xi\in[\zeta_j]\right\}\cup\left\{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\right\}. \tag{133}$$

Again it directly follows from the definition of $\mathcal{U}^{(k)}$ in (123) in conjunction with (131) and (112) that $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ defined above partitions $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$ and it is left to verify $(T2')$, $(T3')$, and $(T5')$ for this partition.

First we consider $(T2')$. By the definition of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ we have

$$\big|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})\big|=\sum_{j\in[z]\setminus Z_{\rm THIN}}\zeta_j+u\overset{(125)}{=}d_{\mathcal{R}^{(k)}}f(s_i)\overset{(115)}{=}\begin{cases}\frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } \mathcal{R}^{(k)}\neq\widetilde{\mathcal{R}}^{(k)}\\ f(s_i) & \text{if } \mathcal{R}^{(k)}=\widetilde{\mathcal{R}}^{(k)}\end{cases},$$

which is $(T2')$.

Property $(T3')$ is immediate from (132) and (121) combined with the choice of the function $\varepsilon_i$ in (97), which easily ensures $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\leq\delta_{\rm RL}(\boldsymbol{a}^{\mathscr{S}},f(s_i))$.

Finally, we discuss property $(T5')$. Due to (120) and due to the definition of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ in (133), the left-hand side of $(T5')$ is bounded by $|\mathcal{U}^{(k)}|$ and thus it suffices to show that

$$d\big(\mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big)\leq\frac{\delta^4_{k+1,{\rm RL}}}{8}d\big(\mathcal{R}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big)\,. \tag{134}$$

From the definition of $\mathcal{U}^{(k)}$ in (123), combined with (122) and the definition of $Z_{\text{THIN}}$ in (118) we infer that

$$d\big(\mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \leq \big|[z]\setminus Z_{\text{THIN}}\big|\Big(\frac{1}{f(s_i)}+3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\Big)+\big|Z_{\text{THIN}}\big|\frac{1}{f(s_i)} \overset{(117)}{\leq} \frac{s_i}{f(s_i)}+3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i\,,$$

which by definition of $f(\cdot)$ in (95) and (97) gives $d\big(\mathcal{U}^{(k)}|\hat{\mathcal{S}}^{(k-1)}\big) \leq \delta^4_{k+1,\text{RL}}/(12t_0)$.

Therefore, (134) follows from the last inequality and (116). This verifies (*Z5′*), which finishes the proof of Claim 42′ in this case. ♢

In both cases we constructed a partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ of $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\cap\mathcal{R}^{(k)}$, which exhibits properties (*T2′*), (*T3′*), and (*T5′*), as required in Claim 42′. □

## References


[1] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy, *Efficient testing of large graphs*, Combinatorica **20** (2000), no. 4, 451–476. 2.4

[2] C. Avart, V. Rödl, and M. Schacht, *Every monotone 3-graph property is testable*, SIAM J. Discrete Math., to appear. 2.4

[3] O. Cooley, N. Fountoulakis, D. Kühn, and D. Osthus, *Embeddings and Ramsey numbers of sparse k-uniform hypergraphs*, submitted. 2.4

[4] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs Combin. **2** (1986), no. 2, 113–121. 1, 1

[5] P. Frankl and V. Rödl, *Extremal problems on set systems*, Random Structures Algorithms **20** (2002), no. 2, 131–164. 1, 1, 2.4, 3, 4, 6, 6.1, 6.1, 6.2.2

[6] E. Friedgut, V. Rödl, A. Ruciński, and P. Tetali, *A sharp threshold for random graphs with a monochromatic triangle in every edge coloring*, Mem. Amer. Math. Soc. **179** (2006), no. 845, vi+66. 6.2.2

[7] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. **34** (1978), 275–291. 1

[8] ———, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Analyse Math. **45** (1985), 117–168. 1

[9] ———, *A density version of the Hales-Jewett theorem*, J. Anal. Math. **57** (1991), 64–119. 1

[10] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem*, submitted. 1, 1, 6.2.2

[11] ———, *Lower bounds of tower type for Szemerédi's uniformity lemma*, Geom. Funct. Anal. **7** (1997), no. 2, 322–337. 1, 2.4

[12] ———, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1-2, 143–184. 1

[13] B. Green, *A Szemerédi-type regularity lemma in abelian groups, with applications*, Geom. Funct. Anal. **15** (2005), no. 2, 340–376. 6.2.2

[14] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. (2), to appear. 6.2.2

[15] A. W. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229. 1

[16] Y. Kohayakawa, *Szemerédi's regularity lemma for sparse graphs*, Foundations of computational mathematics (Rio de Janeiro, 1997), Springer, Berlin, 1997, pp. 216–230. 6.2.2

[17] Y. Kohayakawa, V. Rödl, and J. Skokan, *Hypergraphs, quasi-randomness, and conditions for regularity*, J. Combin. Theory Ser. A **97** (2002), no. 2, 307–352. 3

[18] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, *The regularity lemma and its applications in graph theory*, Theoretical aspects of computer science (Tehran, 2000), Lecture Notes in Comput. Sci., vol. 2292, Springer, Berlin, 2002, pp. 84–112. 1

[19] J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. 1, 2.4, 4.1

[20] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular $k$-uniform hypergraphs*, Random Structures Algorithms **28** (2006), no. 2, 113–179. 1, 1, 3, 5.2, 6, 6.2.2, 6.3

[21] ———, *Extremal hypergraph problems and the regularity method*, Topics in discrete mathematics, Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 247–278. 2.4

[22] B. Nagle, O. Sayaka, V. Rödl, and M. Schacht, *On the Ramsey number of sparse 3-graphs*, submitted. 2.4

[23] V. Rödl and M. Schacht, *Regular partitions of hypergraphs: Counting Lemmas*, Combin. Probab. Comput., to appear. 1, 3

[24] V. Rödl, M. Schacht, M. Siggers, and N. Tokushige, *Integer and fractional packings of hypergraphs*, J. Combin. Theory Ser. B **97** (2007), no. 2, 245–268. 2.4

[25] V. Rödl, M. Schacht, E. Tengan, and N. Tokushige, *Density theorems and extremal hypergraph problems*, Israel J. Math. **152** (2006), 371–380. 1

[26] V. Rödl and J. Skokan, *Regularity lemma for $k$-uniform hypergraphs*, Random Structures Algorithms **25** (2004), no. 1, 1–42. 1, 1, 2.3, 2.4, 2.4, 2.4, 3, 4, 4.1, 6, 6.1, 6.2.2, 6.2.2

[27] ———, *Applications of the regularity lemma for uniform hypergraphs*, Random Structures Algorithms **28** (2006), no. 2, 180–194. 1, 1

[28] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam, 1978, pp. 939–945. 1

[29] J. Solymosi, *A note on a question of Erdős and Graham*, Combin. Probab. Comput. **13** (2004), no. 2, 263–267. 1

[30] E. Szemerédi, *On sets of integers containing no $k$ elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik. 1, 1, 2

[31] ———, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401. 1, 2, 2, 4.1, 6, 6.1, 6.1, 6.2.2

[32] T. Tao, *A variant of the hypergraph removal lemma*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1257–1280. 1



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