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Regular poles for the p-adic group GSp_4

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Abstract: We compute the regular poles of the L-factors of the admissible and irreducible representations of the group GSp_4, which admit a nonsplit Bessel functional and have a Jacquet module length of at most 2 with respect to the unipotent radical of the Siegel parabolic, over a non-Archimedean local field of odd characteristic. We also compute the L-factors of the generic representations of GSp_4.

Key words: L-function, L-factor, GSp(4), regular pole

1. Introduction

Let k be a non-Archimedean local field. There are 2 main types of L-functions for $GSp_4(k)$, standard (degree 5) and spinor (degree 4). Spinor L-functions of the representations of $GSp_4(k)$ can be defined by using 3 different constructions given by Novodvorsky [5], Shahidi [11], and Piatetski-Shapiro [6]. The techniques to determine the local L-factors for these constructions are different and in this paper we wish to investigate the L-factors for the Piatetski-Shapiro construction.

Let us first briefly explain what these constructions are. Novodvorsky integrals in [5], defined only for generic representations, are one of the integral representations for the spinor L-functions of $GSp_4(k)$. Local L-factors defined using the Novodvorsky integrals were computed in [12] by determining the germ expansions of the Whittaker functions and using the local coefficients.

In [11], Shahidi defined the L-functions by using the intertwining operators for generic representations of $GSp_4(k)$. In [3], this definition was extended to all nongeneric and nonsupercuspidal representations by the Langlands classification. Computation of the local L-factors of these L-functions was done in [3] by using the multiplicativity of the local L-factors.

Integral representations in [6] were defined by Piatetski-Shapiro for all infinite dimensional representations of $GSp_4(k)$, where the characteristic of k is odd. Piatetski-Shapiro's definition for the spinor L-functions is more general than Novodvorsky's in that it treats nongeneric representations. The local L-factors of these L-functions were computed in [6] and [7] for only special cases.

The main goal of this paper is to determine the regular poles of the local L-factors of the representations of $GSp_4(k)$, which admit a nonsplit Bessel functional and have a Jacquet module length of at most 2 with respect to the unipotent radical of the Siegel parabolic. Our results agree with the results of [12], [3], and the local Langlands conjecture. We also find the L-factors for generic representations as well, since, in this case, all poles of the L-factors are regular.

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We now describe Piatetski-Shapiro's construction briefly. Details will be given in Section 2. Let S be the unipotent radical of the Siegel parabolic subgroup of $GSp_4(k)$ and let ψ be any nondegenerate character of S. We can realize the group $GL_2(k)$ in the Levi subgroup of the Siegel parabolic subgroup. Let T be the connected component of the stabilizer of ψ in $GL_2(k)$. Then T is isomorphic to the units of a semisimple algebra K over k of index $(K:k)=2$ and $R=TS$ is a subgroup of $GSp_4(k)$. Let Λ be any character of T. Then define $\alpha_{\Lambda,\psi}(r) =: \Lambda(t)\psi(s)$ for $r \in R, s \in S, t \in T$ and $r = st$.

Let (Π, V_Π) be an infinite-dimensional, irreducible, and admissible representation of $GSp_4(k)$. The dimension of the space $\mathrm{Hom}_R(V_\Pi, \alpha_{\Lambda,\psi})$ is at most one, and if it is nonzero, a nonzero element of this space is called a Bessel functional for Π. Furthermore, by Frobenius reciprocity, Π can be embedded into the space $\mathrm{Ind}_R^{GSP_4(k)}\alpha_{\Lambda,\psi}$ and its image is called a Bessel model.

Piatetski-Shapiro defines the local factors by using the integrals

$$L(s; W_v, \Phi, \mu) = \int_{N\backslash G} W_v(g)\Phi[(0,1)g]\mu(\mathrm{detg})|detg|_k^{s+\frac{1}{2}}dg,$$

where $\Phi \in C_c^\infty(K^2)$, μ is a character of k^*, $v \in V_\Pi$, and W_v is an element of the Bessel model. The L-factor is defined to be the greatest common divisor of the family of these integrals. The poles of the L-factor, coming from an integral with a Schwartz function vanishing at zero, are called regular poles.

Let us now explain our method. In Proposition 2.5, we show that the regular poles of the L-factors are the poles of the meromorphic continuation of the integrals given by

$$\int_{k*} \varphi_v(x)\mu(x)|x|^{s-3/2}d^*x,$$

where $\varphi_v(x) := W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix}$. Hence, to compute the regular poles, one needs to find the asymptotic behavior of $\varphi_v(x)$.

For generic representations, the integral above is very similar to Novodvorsky integrals, so one would expect to use methods given in [12] to find the regular poles. However, since local coefficients for Bessel model have not been completely understood, this is not possible.

The asymptotic behavior of $\varphi_v(x)$ depends on the structure of the Jacquet module. In Proposition 3.1 we show that if the length of the Jacquet module is zero then φ_v has a compact support in k^* and there is no regular pole.

If the length of Jacquet module is one, then by Proposition 3.2 for $|x|$ sufficiently small we have

$$\varphi_v(x) = C\chi(x)$$

for a constant C and character χ. Hence, by Lemma 3.4, a regular pole is the pole of $CL(s,\chi)$.

As a corollary of Proposition 3.2 and by Proposition 3.5, if the length of the Jacquet module is 2, then we have

$$\varphi_v(x) = C_1\chi_1(x) + C_2\chi_2(x)$$

or

$$\varphi_v(x) = C_1\chi(x) + C_2\chi(x)v_k(x),$$

where C_1 and C_2 are constants; χ_1, χ_2, and χ are characters of k^*; and v_k is the valuation of k. Hence, by Lemmas 3.4 and 3.7, regular poles are poles of $C_1C_2L(s,\chi_1)L(s,\chi_2)$ or the least common multiple of $C_1L(s,\chi)$ and $C_2L(s,\chi)^2$.

Next we determine whether the constants C, C_1, and C_2 above are zero or not. In Proposition 5.8 and Proposition 5.11 we show that this depends on whether the space of homomorphisms from the constituents of the Jacquet module to the character Λ is zero or not and, in most cases, this depends on the Bessel existence conditions. By using these results in Theorem 5.9 and Theorem 5.16 we find the regular poles of each representation, which we consider separately.

This paper is organized as follows. In Section 2, we give the subgroups of $GSp_4(k)$, definitions of Bessel model, local L-factors, and regular poles. In Section 3, we find the asymptotic behavior of φ_v by using Jacquet module structure. In Section 4, we determine the subspace $\{v \in V_\Pi : \varphi_v \in C_c^\infty(k^*)\}$ by following the methods of Godement in [4] for $GL_2(k)$. In Section 5, we show that there is a relation between the asymptotic behavior of the φ_v and the homomorphisms from the constituents of the Jacquet module to the character Λ. Then we compute the regular poles of each representation separately. The results of Section 5 with exceptional (nonregular) poles as expected by the local Langlands conjecture and semisimplifications of the Jacquet modules are given in the Appendix.

2. Definitions and preliminaries

We fix some notations.
k is a non-Archimedean local field of odd characteristic.
v_k is the valuation of k.
ν is the absolute value on k.
$\mathcal{O}$ is the ring of integers of k.
$\mathcal{P}$ is the unique maximal prime ideal of $\mathcal{O}$.
ϖ is a fixed generator of $\mathcal{P}$.
q is the cardinality of the residue field of k.
ψ is a nontrivial additive character of k with conductor $\mathcal{O}$.
$dx = d_\psi x$ is the self-dual Haar measure on k.
If ξ is a representation of a group, then its space and central character are denoted by V_ξ and ω_ξ, respectively.

2.1. $\mathbf{GSp_4(k)}$ and its subgroups

In this section, we give the definitions of the subgroups of $GSp_4(k)$ as in [6], which will be needed in this paper.

Let $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $J = \begin{pmatrix} & w \\ -w & \end{pmatrix}$. Define

$$GSp_4(k) = \left\{ g \in GL_4(k) : g^t J g = \lambda(g) J \text{ for some } \lambda(g) \in k^* \right\},$$

where g^t is the transpose of the g. Let

$$P = \left\{ g \in GSp_4(k) : g = \begin{pmatrix} A & B \\ & D \end{pmatrix}, A, B, D \in M_2(k) \right\}$$

be the Siegel parabolic subgroup of $GSp_4(k)$. $P = MS$ is the Levi decomposition of P with the reductive part

$$M = \left\{ \begin{pmatrix} A & \\ & \lambda(A')^{-1} \end{pmatrix} : A \in GL_2(k), \lambda \in k^* \right\}$$

and the unipotent radical

$$S = \left\{ \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} : Y = Y' \right\},$$

where $X' = w(X^t)w$ for $X \in M_2(k)$. Any character of S is of the form

$$\psi_\beta \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} = \psi[\mathrm{tr}(\beta Y)]$$

for some $\beta = \beta'$. ψ_β is called nondegenerate if $\beta \in GL_2(k)$.

Let ψ_β be a nondegenerate character of S and let T be the connected component of the stabilizer of ψ_β in M; then there is a unique semisimple algebra K over k of index $(K : k) = 2$ and $T \cong K^*$. K is either a quadratic extension of k and $K = k(\sqrt{\rho})$ for some $\rho \notin (k^*)^2$ or $K = k \oplus k$. If K is a field then T is called nonsplit. Otherwise it is called split.

In this paper, we consider the nonsplit case and choose ρ such that $|\rho| \in \{1, \frac{1}{q}\}$. Each orbit of M in the set of nondegenerate characters contains a character ψ_β with $\beta = \begin{pmatrix} & 1 \\ -\rho & \end{pmatrix}$.

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in K^2 define an antisymmetric form on K^2 by

$$B(x, y) = \mathrm{Tr}_{K/k}(x_1 y_2 - x_2 y_1).$$

By considering K^2 as a 4-dimensional vector space over k define

$$GSp_B(k) = \{g \in GL_4 : B(xg, yg) = \lambda(g)B(x, y) \text{ for some } \lambda(g) \in k^* \}.$$

Let

$$G = \{g \in GL_2(K) : \det g \in k^*\},$$

where G acts on K^2 from the right. Since $g \in G$ preserves B up to $\det(g)$, we can realize G in GSp_B. There is also an isomorphism φ between $GSp_B(k)$ and $GSp_4(k)$ such that $\varphi(G) \cap R = TN$ [6] and

$$N = \left\{ \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} \in S : \mathrm{tr}(\beta Y) = 0 \right\}.$$

If $K = k(\sqrt{\rho})$, then the image of G in $GSp_4(k)$ consists of all elements of the form

$$\left(\begin{array}{cc|cc} a & b & c & d \\ b\rho & a & d\rho & c \\ \hline e & f & m & n \\ f\rho & e & n\rho & m \end{array}\right),$$

where $a+b\sqrt{\rho}, c+d\sqrt{\rho}, e+f\sqrt{\rho}, m+n\sqrt{\rho} \in K^*, an+bm=cf+de.$ The center of $GSp_4(k)$ is

$$Z=\left\{\left(\begin{array}{cc|cc} a & & & \\ & a & & \\ \hline & & a & \\ & & & a \end{array}\right) : a \in k^*\right\},$$

$$T=\left\{\left(\begin{array}{cc|cc} a & b & & \\ b\rho & a & & \\ \hline & & a & -b \\ & & -b\rho & a \end{array}\right) : a+b\sqrt{\rho} \in K^*\right\},$$

$$N=\left\{\left(\begin{array}{cc|cc} 1 & & u & t \\ & 1 & t\rho & u \\ \hline & & 1 & \\ & & & 1 \end{array}\right) : u,t \in k\right\}.$$

Let

$$S' := \left\{\left(\begin{array}{cc|cc} 1 & & & 0 \\ & 1 & c & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) : c \in k\right\}.$$

and

$$H := \left\{\left(\begin{array}{cc} xI_2 & \\ & I_2 \end{array}\right) : x \in k^*\right\}.$$

Note that $S=NS'$.

2.2. Bessel Model, L-factor and Regular Poles

By abuse of the notation, we shall omit β in ψ_β. For a character Λ of T and nondegenerate character ψ of S, define a character $\alpha_{\Lambda,\psi}$ of $R=TS$ by

$$\alpha_{\Lambda,\psi}(r)=\alpha_{\Lambda,\psi}(ts)=\Lambda(t)\psi(s),$$

where $r=ts \in R$ for $t \in T$ and $s \in S$. We now give the existence and uniqueness results in order to define the Bessel model.

Theorem 2.1 *[6] Let (Π, V_Π) be an irreducible smooth, admissible, and preunitary representation of $Gsp_4(k)$; then*

$$dim[Hom_R(\Pi, \alpha_{\Lambda,\psi})] \leq 1.$$

If Π is infinite dimensional, then for appropriately chosen Λ, ψ, and R

$$dim[Hom_R(\Pi, \alpha_{\Lambda,\psi})] = 1.$$

Let Π be as in the theorem above. Choose Λ, ψ and R such that $\mathrm{Hom}_R(\Pi, \alpha_{\Lambda,\psi})$ is nonzero and let l be a nonzero element of this space. For $v \in V_\Pi$ define Bessel function W_v on $GSp_4(k)$ by

$$W_v(g) := l(\Pi(g)v).$$

The space of such functions is denoted by $\mathbf{W}^{\Lambda,\psi}$ and is called the Bessel model. If a representation of $GSp_4(k)$ is defined on this space of functions by right translation, then $\Pi \cong \mathbf{W}^{\Lambda,\psi}$. For $r \in R$, $g \in GSp_4(k)$ and $v \in V_\Pi$ we have

$$W_v(rg) = \alpha_{\Lambda,\psi}(r)W_v(g).$$

Also define $h_x := \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix}$ and $\varphi_v(x) := W_v(h_x)$. The next 2 theorems provide the definitions of L-functions and L-factors.

Theorem 2.2 *[6] Let $\Phi \in C_c^\infty(K^2)$ and μ be a character of k^*. Then for $s \in \mathbb{C}$, the integral*

$$L(s; W_v, \Phi, \mu) = \int_{N\backslash G} W_v(g)\Phi[(0,1)g]\mu(detg)|detg|_k^{s+\frac{1}{2}}dg$$

converges absolutely for $Re(s)$ large enough and has a meromorphic continuation to the whole plane.

Theorem 2.3 *[6] The integrals $\{L(s; W_v, \Phi, \mu) : W_v \in \mathbf{W}^{\Lambda,\psi}, \Phi \in C_c^\infty(K^2)\}$ form a fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$ of the form $L(s; \Pi, \mu)\mathbb{C}[q^s, q^{-s}]$. The factor $L(s; \Pi, \mu)$ is of the form $P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$, $P(0) = 1$ and is called the L-factor of Π twisted by μ.*

Definition 2.4 *A pole of $L(s; \Pi, \mu)$ is called a regular pole if it is a pole of some $L(s; W_v, \Phi, \mu)$ with $\Phi(0,0) = 0$. Any other pole is called an exceptional pole.*

We compute the regular poles in terms of the Tate L-functions, which are defined for a character χ of k^* as

$$L(s, \chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified} \\ (1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified.} \end{cases}$$

Proposition 2.5 below gives the characterization of the regular poles.

Proposition 2.5 *Regular poles of $L(s; \Pi, \mu)$ are the poles of the integrals*

$$\int_{k*} \varphi_v(x)\mu(x)|x|^{s-3/2}d^*x, \qquad v \in V_\Pi.$$

In particular, if φ_v has compact support in k^* for every $v \in V_\Pi$, then $L(s; \Pi, \mu)$ does not have regular poles.

Proof $L(s; W_v, \Phi, \mu)$ is absolutely convergent for sufficiently large s, so by Iwasawa decomposition,

$$L(s; W_v, \Phi, \mu) = \int_{K_G} \mu(\det k) \int_{k^*} W_v\left[\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} k\right] \mu(x)|x|_k^{s-\frac{3}{2}}d^*x$$

$$\times \int_{K^*} \Phi[(0,t)k]\Lambda(t)\mu(t\bar{t})|t\bar{t}|_k^{s+\frac{1}{2}}d^*tdk,$$

where K_G is the maximal compact subgroup of G. Take $K_G^o \subset K_G$, a compact open subgroup, which stabilizes W_v, μ and Φ. Write $K_G = \cup_i^N k_i K_G^o$ for $N = (K_G : K_G^o)$ and let $\Phi_i = R_{k_i}\Phi, W_{v,i} = \pi(k_i)W_v$, where R_{k_i} is the right translation by k_i. Then

$$L(s; W_v, \Phi, \mu) = \left[\mathrm{Vol}(K_G^o) \sum_{i=1}^{N} \mu(\det k_i)\right] \times \tag{1}$$

$$\int_{k^*} W_{v,i}\left[\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix}\right]\mu(x)|x|_k^{s-\frac{3}{2}}d^*x \int_{K^*}\Phi_i[(0,t)]\Lambda(t)\mu(t\bar{t})|t\bar{t}|_k^{s+\frac{1}{2}}d^*t.$$

If $\Phi(0,0)=0$, then $\Phi_i(0,0)=0$ and $\Phi_i(0,\cdot)$ has compact support in K^* for each i. Let U^o be the open compact subgroup of K^* such that $\Phi_i(0,\cdot),\Lambda,\mu\circ N_{K/k}$, and ν are invariant under U^o for $i=1,..,N$. Let $\mathfrak{S}=\cup_{i=1}^N\text{Support}[\Phi_i(0,t)]$. Since $\mathfrak{S}$ is a finite union of compact sets, it is compact and we have $\mathfrak{S}=\cup_{j=1}^M t_jU^o$. Then (1) becomes

$$\text{Vol}(K_G^o)\text{Vol}(U^o)\sum_{i=1}^N\sum_{j=1}^M\mu(\det k_i)\Phi_i[(0,t_j)]\Lambda(t_j)\mu(t_j\bar{t_j})|t_j\bar{t_j}|_k^{s+\frac{1}{2}}$$

$$\times\int_{k^*} W_{v,i}\left[\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix}\right]\mu(x)|x|_k^{s-\frac{3}{2}}d^*x.$$

Therefore, if s_o is a regular pole, then it is a pole of the integral above for some $W_{v,i}$. Conversely, for a suitable choice of Φ, each of the integrals above individually contributes to $L(s;\Pi,\mu)$. □

The following theorem shows that for generic representations, finding the regular poles is enough to determine the L-factor.

Theorem 2.6 *[6] If Π is generic, then its L-factor has only regular poles.*

2.3. Parabolic induction and Jacquet module

Let P_o be a maximal standard parabolic subgroup of $GSp_4(k)$ with Levi decomposition $P_o=M_oU_o$. Let (τ,V_τ) be a representation of M_o and let δ_{P_o} be the modular character of P_o. The normalized parabolic induction from P_o to $GSp_4(k)$ is defined as $\text{Ind}_{P_o}^{GSp_4}\tau=\{f: GSp_4(k)\to V_\tau : f(mug)=\delta_{P_o}(m)^{1/2}\tau(m)f(g),\quad \text{for } m\in M_o \text{ and } u\in U_o\}$. The action of $GSp_4(k)$ on $\text{Ind}_{P_o}^{GSp_4}\tau$ is by right translation.

Let B denote the Borel subgroup $GL_2(k)$. For the characters χ_1,χ_2 of k^*, similarly define $B(\chi_1,\chi_2)$ to be the induction from B to $GL_2(k)$.

Let (Π,V_Π) be an admissible and irreducible representation of $GSp_4(k)$. Define

$$V_S(\Pi):=\text{span}\{v-\Pi(s)v : s\in S, v\in V_\Pi\}.$$

The normalized Jacquet module with respect to S is the smooth representation of M defined by

$$R_S(\Pi)=\Pi_S\otimes\delta_P^{-1/2},$$

where $(\Pi_S,V_\Pi/V_S(\Pi))$ is the regular Jacquet module.

If $p=\begin{pmatrix} A & * \\ & \lambda(A')^{-1}\end{pmatrix}\in P$ for $A\in GL_2(k)$, then $\delta_P(p)=|\det(A)^3\lambda^{-3}|$.

3. Asymptotic behavior of φ_v

In this section, we determine the behavior of $\varphi_v(x)$ for small enough $|x|$. We also compute the poles of the integrals in Proposition 2.5. We begin by showing that the Jacquet module controls the asymptotic behavior of φ_v.

Proposition 3.1 *Let* (Π, V_Π) *be a smooth representation of* $GSp_4(k)$*.*
1) If $v \in V_\Pi$*, then there exists a constant* C*, depending on* v*, such that* $\varphi_v(x) = 0$ *for* $|x| > C$*.*
2) If $v \in V_S(\Pi)$*, then there exists a constant* $\epsilon > 0$*, depending on* v*, such that* $\varphi_v(x) = 0$ *for* $|x| < \epsilon$*. Therefore,* φ_v *has compact support in* k^**.*

Proof 1) Since Π is a smooth representation, W_v is stabilized by the subgroup

$$\left\{ \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} : Y = \begin{pmatrix} & 0 \\ u & \end{pmatrix}, u \in \mathcal{P}^n \right\}$$

for sufficiently large n. Let $\begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix}$ be an element of the above subgroup with $Y = \begin{pmatrix} & 0 \\ u & \end{pmatrix}$ and $u \in \mathcal{P}^n$.

We have

$$\begin{aligned}
\varphi_v(x) &= W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= W_v \left[\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} \right] \\
&= W_v \left[\begin{pmatrix} I_2 & xY \\ & I_2 \end{pmatrix} \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \right] \\
&= \psi \left[\mathrm{tr} \begin{pmatrix} & 1 \\ -\rho & \end{pmatrix} \begin{pmatrix} & 0 \\ xu & \end{pmatrix} \right] W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= \psi(xu) W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= \psi(xu)\varphi_v(x).
\end{aligned}$$

Hence, $[1 - \psi(xu)]\varphi_v(x) = 0$ and $\varphi_v(x) = 0$ for $x \notin \mathcal{P}^{-n}$.

2) Fix $u_1, u_2, u_3 \in k$ and let $Y = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_1 \end{pmatrix}$ and $\mathcal{S} = \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix}$. If $|x|$ is small enough then $x(u_3 - u_2\rho) \in \mathcal{O}$. Hence,

$$\begin{aligned}
\varphi_{\Pi(\mathcal{S})v - v} &= W_{\pi(\mathcal{S})v - v} \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= W_v \left[\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \begin{pmatrix} I_2 & Y \\ & I_2 \end{pmatrix} \right] - W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= W_v \left[\begin{pmatrix} I_2 & xY \\ & I_2 \end{pmatrix} \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \right] - W_v \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} \\
&= \psi \left[tr \begin{pmatrix} & 1 \\ -\rho & \end{pmatrix} \begin{pmatrix} u_1x & u_2x \\ u_3x & u_1x \end{pmatrix} \right] \varphi_v(x) - \varphi_v(x) \\
&= [\psi(x(u_3 - u_2\rho)) - 1]\varphi_v(x) \\
&= 0.
\end{aligned}$$

□

Proposition 3.2 *Let (Π, V_Π) be an irreducible and admissible representation of $GSp_4(k)$ and let χ be a character of k^*. If $\Pi(h_x)u - \chi(x)u \in V_S(\Pi)$ for every $x \in k^*$, then there exists a constant C and positive integer j_o such that*

$$\varphi_u(x) = C\chi(x)$$

for $|x| \leq q^{-j_o}$.

Proof Let $x_o \in \varpi\mathcal{O}^*$, and then we have $\Pi(h_{x_o})u - \chi(x_0)u \in V_S(\Pi)$. By Proposition 3.1 $\varphi_{\Pi(h_{x_o})u-\chi(x_0)u}$ vanishes near zero, so there exists a constant $\epsilon(x_o)$ such that

$$0 = \varphi_{\Pi(h_x)u-\chi(x)u}(t) = \varphi_u(xt) - \chi(x)\varphi_u(t) \tag{2}$$

for $x = x_o$ and $|t| \leq \epsilon(x_o)$. Since Π and χ are smooth, this is also valid when x is near x_o and $|t| \leq \epsilon(x_o)$. By compactness of $\varpi\mathcal{O}^*$, this is also true for $x \in \varpi\mathcal{O}^*$ and $|t| \leq \epsilon = q^{-(j_o-1)}$ for some constant j_o. □

Lemma 3.3

$$\varphi_u(\varpi^i z) = \chi(\varpi^i z)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1})$$

for $i \geq j_o$ and $z \in \mathcal{O}^*$.

Proof Proof is by induction. By (2),

$$\varphi_u(\varpi^{j_o} z) = \varphi_u(\varpi z \varpi^{j_o-1}) = \chi(\varpi^{j_o} z)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}).$$

Now assume the result for some $i \geq j_0$ and prove it for $i+1$ using (2):

$$\begin{aligned} \varphi_u(\varpi^{i+1} z) &= \varphi_u(\varpi z \varpi^i) \\ &= \chi(\varpi z)\varphi_u(\varpi^i) \\ &= \chi(\varpi z)\chi(\varpi^i)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}) \\ &= \chi(\varpi^{i+1} z)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}). \end{aligned}$$

Let $C = \chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1})$; then the proposition follows from the lemma above. □

Lemma 3.4 *If $\varphi_u(x) = C|x|^{3/2}\chi(x)$ for some character χ of k^* and $|x| \leq q^{-j_o}$, then the pole of $\int_{k^*} \varphi_u(x)|x|^{s-3/2}\, d^*x$ is the pole of $CL(s,\chi)$.*

Proof

$$\begin{aligned}\int_{|x|\le q^{-j_o}} \varphi_u(x)|x|^{s-3/2}d^*x &= C\int_{|x|\le q^{-j_o}} |x|^{3/2}\chi(x)|x|^{s-3/2}\,d^*x\\ &= C\int_{|x|\le q^{-j_o}} \chi(x)|x|^s\,d^*x\\ &= C\sum_{i=j_o}^{\infty}\int_{|x|=q^{-i}}\chi(x)|x|^s\,d^*x\\ &= C\sum_{i=j_o}^{\infty} q^{-is}\chi(\varpi^i)\int_{\mathcal{O}^*}\chi(u)\,du\\ &= \begin{cases}0 & \chi \text{ is ramified}\\ \frac{C[q^{-s}\chi(\varpi)]^{j_o}}{1-q^{-s}\chi(\varpi)}(1-\frac{1}{q}) & \text{otherwise.}\end{cases}\end{aligned}$$

□

Proposition 3.5 *Let U be a subrepresentation of $R_S(\Pi)$, W a subrepresentation of U and u, w elements of V_Π such that $\bar{w}\in W$ and $\bar{u}\in U$ are images of w and u in $V_\Pi/V_S(\Pi)$, respectively. If $\Pi(h_x)u-\chi(x)u-\chi(x)v_k(x)w\in V_S(\Pi)$ and $\Pi(h_x)w-\chi(x)w\in V_S(\Pi)$ for sufficiently small $|x|$, then we have*

$$\varphi_u(x)=C_1\chi(x)+C_2\chi(x)v_k(x).$$

Proof By Proposition 3.2, $\varphi_w(x)=C_2\chi(x)$ for small $|x|$. Let $x_o\in\varpi\mathcal{O}^*$; then we have $\Pi(h_{x_o})u-\chi(x_0)u-\chi(x_o)v_k(x_o)w\in V_S(\Pi)$. By Proposition 3.1, $\varphi_{\Pi(h_{x_o})u-\chi(x_0)u-\chi(x_o)v(x_o)w}$ vanishes near zero. Thus, there exists a constant $\epsilon(x_o)$ such that

$$\begin{aligned}0 &= \varphi_{\Pi(h_x)u-\chi(x)u-\chi(x)v_k(x)w}(t)\\ &= \varphi_u(xt)-\chi(x)\varphi_u(t)-\chi(x)v_k(x)\varphi_w(t)\\ &= \varphi_u(xt)-\chi(x)\varphi_u(t)-C_2\chi(xt)v_k(x) \qquad (3)\end{aligned}$$

for $x=x_o$ and $|t|\le\epsilon(x_o)$. Since Π and χ are smooth, this is also valid when x is near x_o and $|t|\le\epsilon(x_o)$, so by compactness of $\varpi\mathcal{O}^*$, this is also true for $x\in\varpi\mathcal{O}^*$ and $|t|\le\epsilon=q^{-(j_o-1)}$ for some constant j_o. □

Lemma 3.6

$$\varphi_u(\varpi^i z)=\chi(\varpi^i z)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1})+C_2[i-(j_o-1)]\chi(\varpi^i z)$$

for $i\ge j_o$ and $z\in\mathcal{O}^*$.

Proof Proof is by induction. The base case follows from (3). Now assume the result for some $i\ge j_0$ and prove it for $i+1$:

$\varphi_u(\varpi^{i+1}z)$

$$\begin{aligned}
&= \varphi_u(\varpi z\varpi^i) \\
&= \chi(\varpi z)\varphi_u(\varpi^i) + C_2\chi(\varpi^{i+1}z)v_k(\varpi z) \\
&= \chi(\varpi z)[\chi(\varpi^i)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}) + C_2[i-(j_o-1)]\chi(\varpi^i)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + C_2\chi(\varpi^{i+1}z) \\
&= \chi(\varpi^{i+1}z)\chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}) + C_2[(i+1)-(j_o-1)]\chi(\varpi^{i+1}z).
\end{aligned}$$

Let $C_1 = \chi(\varpi^{-(j_o-1)})\varphi_u(\varpi^{j_o-1}) - C_2(j_o-1)$ and $|x| \le q^{-j_o}$, and then $x = \varpi^i z$ for some $i \ge j_o$ and $z \in \mathcal{O}^*$, so the proposition follows from the previous lemma. □

Lemma 3.7 *If $\varphi_u(x) = C_1|x|^{3/2}\chi(x) + C_2|x|^{3/2}\chi(x)v_k(x)$ for some character χ of k^* and $|x| \le q^{-j_o}$, then the poles of $\int_{k^*}\varphi_u(x)|x|^{s-3/2}d^*x$ are the poles of the least common multiple of $C_1L(s,\chi)$ and $C_2L(s,\chi)^2$.*

Proof

$$\int_{|x|\le q^{-j_o}} C_2|x|^{3/2}\chi(x)v_k(x)|x|^{s-3/2}d^*x$$

$$\begin{aligned}
&= C_2\int_{|x|\le q^{-j_o}}\chi(x)v_k(x)|x|^s d^*x \\
&= C_2\sum_{i=j_o}^{\infty}\int_{|x|=q^{-i}}\chi(x)v_k(x)|x|^s d^*x \\
&= C_2\sum_{i=j_o}^{\infty} iq^{-is}\chi(\varpi^i)\int_{\mathcal{O}^*}\chi(u)du \\
&= \begin{cases} 0 & \chi \text{ is ramified} \\ C_2\frac{j_o[q^{-s}\chi(\varpi)]^{j_o}(1-q^{-s}\chi(\varpi))+[q^{-s}\chi(\varpi)]^{j_o+1}}{[1-q^{-s}\chi(\varpi)]^2}(1-\frac{1}{q}) & \text{otherwise.}\end{cases}
\end{aligned}$$

Now the result follows from Lemma 3.4. □

4. Compactly supported φ_v in k^*

In this section, we find the $v \in V_\Pi$ for which $\varphi_v(x)$ has compact support in k^*. We use similar methods that are used for $GL_2(k)$. There are 2 steps. The first is showing that $\{\varphi_v : v \in V_S(\Pi)\} = C_c^\infty(k^*)$ and the second is determining the space $\{v \in V : \varphi_v = 0\}$. We need 2 subgroups of $GL_2(k)$, which can be embedded into $GSp_4(k)$. Let B_1 be the subgroup of B of the elements of the form $\begin{pmatrix} a & b \\ & 1\end{pmatrix}$. Define

$$B^G := \left\{ \left(\begin{array}{cc|cc} a & & & \\ & a & b & \\ \hline & & d & \\ & & & d \end{array}\right) \in GSp_4(k) \right\} \cong B$$

and

$$B_1^G := \left\{ \left(\begin{array}{cc|cc} a & & & \\ & a & b & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \in GSp_4(k) \right\} \cong B_1.$$

Lemma 4.1

$$\{\varphi_v : v \in V_S(\Pi)\} = C_c^\infty(k^*).$$

Proof First note that by Proposition 3.1, $\{\varphi_v : v \in V_S(\Pi)\} \subset C_c^\infty(k^*)$. Define an action of B_1^G on $C_c^\infty(k^*)$ by

$$\begin{pmatrix} a & & & \\ & a & b & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot f(x) = \psi(bx) f(ax).$$

Under this action, $\{\varphi_v : v \in V_S(\Pi)\}$ is a nonzero invariant subspace of $C_c^\infty(k^*)$ because, if it is zero, then for every $v \in V_\Pi$ and $s \in S$, we have

$$0 = \varphi_{\Pi(s)v - v}(1) = l(\Pi(s)v) - l(v) = \psi(s)l(v) - l(v) = (\psi(s) - 1)l(v).$$

Thus, either $l = 0$ or $\psi = 1$, contradicting our assumptions. By Proposition 4.7.3 of [1], $C_c^\infty(k^*)$ is an irreducible representation of B_1^G, so the proposition follows. □

Now we will find the space $\{v \in V_\Pi : \varphi_v = 0\}$. Let

$$V_R(\alpha_{\psi,\Lambda}, \Pi) = span\{\Pi(r)v - \alpha_{\psi,\Lambda}(r)v : v \in V_\Pi, r \in R\}.$$

Lemma 4.2 *If $l \in Hom_R(\Pi, \alpha_{\Lambda,\psi})$ and is nonzero, then the kernel of l is $V_R(\alpha_{\psi,\Lambda}, \Pi)$.*

Proof First note that

$$\mathrm{Hom}_R(\Pi, \alpha_{\Lambda,\psi}) \cong \mathrm{Hom}_{\mathbb{C}}(V_\Pi / V_R(\alpha_{\psi,\Lambda}, \Pi), 1).$$

By uniqueness of the Bessel model, dimensions of the above spaces are one. Hence, $dim_{\mathbb{C}}\,(V_\Pi / V_R(\alpha_{\psi,\Lambda}, \Pi)) = 1$. Since l is nonzero and its kernel contains $V_R(\alpha_{\psi,\Lambda}, \Pi)$, the kernel should be $V_R(\alpha_{\psi,\Lambda}, \Pi)$. □

Define $S_n' := \left\{ \left(\begin{array}{cc|cc} 1 & & & 0 \\ & 1 & u & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) : u \in \mathcal{P}^{-n} \right\}$,

$$S_n := \left\{ \left(\begin{array}{cc|cc} 1 & & u_1 & u_2 \\ & 1 & u_3 & u_1 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) : u_1, u_2, u_3 \in \mathcal{P}^{-n} \right\}$$ and $R_n^S := TS_n$. Similarly define N_n and $R_n^N := TN_n$.

We now find a characterization of the elements of $V_R(\alpha_{\psi,\Lambda}, \Pi)$.

Proposition 4.3 *There is an increasing sequence of R_n^S/Z of open compact subgroups of R/Z, which exhausts R/Z.*

Proof $\left(\begin{array}{cc|cc} a & b & & \\ b\rho & a & & \\ \hline & & a & -b \\ & & -b\rho & a \end{array}\right)\left(\begin{array}{cc|cc} 1 & & u_1 & u_2 \\ & 1 & u_3 & u_1 \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} a & b & & \\ b\rho & a & & \\ \hline & & a & -b \\ & & -b\rho & a \end{array}\right)^{-1} = \left(\begin{array}{cc|cc} 1 & & u_1' & u_2' \\ & 1 & u_3' & u_1' \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ where

$u_1' = \frac{1}{a^2-b^2\rho}[(a^2+b^2\rho)u_1 + (ab\rho)u_2 + (ab)u_3]$

$u_2' = \frac{1}{a^2-b^2\rho}[(2ab)u_1 + (a^2)u_2 + (b^2)u_3]$

$u_3' = \frac{1}{a^2-b^2\rho}[(2ab\rho)u_1 + (b^2\rho^2)u_2 + (a^2)u_3]$.

Since by Lemma 5.1 of [2], the coefficients of u_1, u_2, u_3 above are all in $\mathcal{O}$, we have $u_1', u_2', u_3' \in \mathcal{P}^{-n}$. Hence, T normalizes S_n and R_n^S is a subgroup of R/Z.

Now consider the multiplication on the group R/Z. The image of $(T/Z) \times (ZS_n/Z)$ in R/Z is R_n^S/Z. Since we are in the field case, T/Z is compact. Also, $ZS_n/Z \cong S_n$, and therefore (ZS_n/Z) is open and compact. Hence, R_n^S/Z is an open and compact subgroup of R/Z. □

Proposition 4.4 $v \in V_R(\alpha_{\psi,\Lambda}, \Pi)$ *if and only if for sufficiently large* n

$$\int_{R_n^S/Z} \alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)v\,dr = 0. \tag{4}$$

Proof Let $v = \Pi(r_0)w - \alpha_{\psi,\Lambda}(r_0)w$ be a typical generator of $V_R(\alpha_{\psi,\Lambda}, \Pi)$. If the image of r_o in R/Z is in R_n^S, then

$$\int_{R_n^S/Z} \alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)\Pi(r_0)w\,dr = \alpha_{\Lambda,\psi}(r_0)\int_{R_n^S/Z} \alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)w\,dr.$$

Hence, the integral (4) vanishes and this shows that $V_R(\alpha_{\psi,\Lambda}, \Pi)$ is contained in the space of all v that satisfy (4).

Conversely, suppose that v satisfies (4) for some n. $\alpha_{\Lambda,\psi}^{-1}\Pi$ is a smooth representation of R/Z and therefore there exists $m \in \mathbb{Z}$ such that $m < n$ and $\alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)v = v$ for every $v \in R_m^S/Z$. Hence,

$$\begin{aligned} 0 &= \int_{R_n^S/Z} \alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)v\,dr \\ &= Vol(R_m^S/Z) \sum_{x \in (R_n^S/Z)/(R_m^S/Z)} \alpha_{\Lambda,\psi}^{-1}(x)\Pi(x) \end{aligned}$$

and

$$\begin{aligned} v &= v - Vol(R_n^S/R_m^S)^{-1} \sum_{x \in (R_n^S/Z)/(R_m^S/Z)} \alpha_{\Lambda,\psi}^{-1}(x)\Pi(x) \\ &= Vol(R_n^S/R_m^S)^{-1} \sum_{x \in (R_n^S/Z)/(R_m^S/Z)} [v - \alpha_{\Lambda,\psi}^{-1}(x)\Pi(x)v]. \end{aligned}$$

This shows that $v \in V_R(\alpha_{\psi,\Lambda}, \Pi)$. □

Let

$$V_{T,N}(\Lambda,\Pi) := \{\Pi(tn)v - \Lambda(t)v : v \in V_\Pi\}$$

and

$$V_{T,S}(\Lambda,\Pi) := \{\Pi(ts)v - \Lambda(t)v : v \in V_\Pi\}.$$

Proposition 4.5 *$v \in V_{T,N}(\Lambda,\Psi)$ if and only if for sufficiently large n*

$$\int_{R_n^N/Z} \Lambda^{-1}(r)\Pi(r)v\,dr = 0.$$

Proof Similar to the proof of the previous proposition. □

Proposition 4.6 *If (Π, V_Π) is an infinite dimensional, irreducible, and admissible representation of $GSp_4(k)$, then there is no nonzero $v \in V_\Pi$ that is invariant under S'.*

Proof Assume that v is invariant under S'. The stabilizer of v is open and so there exists $g \in SL_2 - B$ such that $\begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix}$ is in the stabilizer of v. Since SL_2 is generated by its unipotent part and an element in $SL_2 - B$,

$$\Pi\begin{pmatrix} 1 & & \\ & SL_2 & \\ & & 1 \end{pmatrix} v = v. \tag{5}$$

Let $x \in k$ and choose n large enough so that $\left(\begin{array}{cc|cc} 1 & & \varpi^n x & \\ & 1 & & \varpi^n x \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ is in the stabilizer of v. Define

$$\alpha := \left(\begin{array}{cc|cc} 1 & & & \\ & \varpi^n & & \\ \hline & & \varpi^{-n} & \\ & & & 1 \end{array}\right) \in \begin{pmatrix} 1 & & \\ & SL_2 & \\ & & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \Pi\left(\begin{array}{cc|cc} 1 & & x & \\ & 1 & & x \\ \hline & & 1 & \\ & & & 1 \end{array}\right) v &= \Pi\left(\alpha^{-1}\alpha\left(\begin{array}{cc|cc} 1 & & x & \\ & 1 & & x \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\alpha^{-1}\right) v = \Pi\left(\alpha^{-1}\left(\left(\begin{array}{cc|cc} 1 & & \varpi^n x & \\ & 1 & & \varpi^n x \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\right)\right) v \\ &= v \end{aligned}$$

and

$$\Pi\left(\begin{array}{cc|cc} 1 & x & & \\ & 1 & & \\ \hline & & 1 & -x \\ & & & 1 \end{array}\right) v = \Pi\left(\left(\begin{array}{cc|cc} 1 & & & \\ & & & 1 \\ \hline & -1 & & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & x & \\ & 1 & & x \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & -1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right)\right) v = v.$$

Since Π is smooth, there exists $g \in SL_2 - B$ such that v is fixed by $\Pi\begin{pmatrix} g & \\ & (g')^{-1}\end{pmatrix}$. Since $SL_2(k)$ is generated by its unipotent part and an element in $SL_2 - B$, we have

$$\Pi\begin{pmatrix} g & \\ & (g')^{-1}\end{pmatrix} v = v \tag{6}$$

for every $g \in SL_2(k)$. Additionally, by (5),

$$\Pi\left(\begin{array}{cc|cc} 1 & & & \\ & x & & \\ \hline & & x^{-1} & \\ & & & 1 \end{array}\right) v = v.$$

Therefore, (6) is also valid for every $g \in GL_2(k)$. Now for $x \in k$ choose m large enough so that

$$\Pi\left(\begin{array}{cc|cc} 1 & & & \varpi^{2m}x \\ & 1 & 0 & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) v = v.$$

Hence, v is fixed by

$$\Pi\left(\begin{array}{cc|cc} 1 & & & x \\ & 1 & 0 & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) = \Pi\left(\left(\begin{array}{cc|cc} \varpi^{-m} & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & \varpi^{m} \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \varpi^{2m}x \\ & 1 & 0 & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\left(\begin{array}{cc|cc} \varpi^{m} & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & \varpi^{-m} \end{array}\right)\right).$$

Additionally, by (5) and (6), v is fixed by

$$\Pi\left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right) = \Pi\left(\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & -1 & & \\ & & & 1 \end{array}\right)\right)^2.$$

Hence, v is invariant under all generators of $Sp_4(k)$ and thus invariant under $Sp_4(k)$. Since the center Z of $GSp_4(k)$ acts on v by a character, the one-dimensional subspace spanned by v is fixed by $< Sp_4(k), Z >$. The characteristic of k is not 2, so the coset of $g \in GSp_4(k)$ in $GSp_4(k)/Sp_4(k)Z$ is determined by the class of $\lambda(g)$ in $k^*/(k^*)^2$. Hence, there are only finitely many classes, and so the space spanned by $\Pi(g)v$ is finite-dimensional, which contradicts the infinite dimensionality and irreducibility of Π. □

Proposition 4.7 $\varphi_v(x) = 0 \Longleftrightarrow v \in V_{T,N}(\Lambda, \Pi)$.

Proof If $\varphi_v(x) = 0$ for all $x \in k^*$, then $\Pi\begin{pmatrix} xI_2 & \\ & I_2\end{pmatrix} v \in V_R(\alpha_{\Lambda,\psi}, \Pi)$ for all $x \in k^*$. Hence, by Proposition 4.4 there exists $n \in \mathbb{Z}$ such that

$$
\begin{aligned}
0 &= \int_{R_n^S/Z} \alpha_{\Lambda,\psi}^{-1}(r)\Pi(r)\Pi\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} v\,dr \\
&= \int_{S'_n}\int_{N_n}\int_{T/Z} \Lambda^{-1}(t)\psi^{-1}(s')\Pi(s'nt)\Pi\begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} v\,dtdnds' \\
&= \int_{S'_n}\int_{N_n}\int_{T/Z} \Lambda^{-1}(t)\psi^{-1}(xs')\Pi(s'nt)v\,dtdnds' \\
&= \int_{S'_n} \psi^{-1}(xs')\Pi(s')\left(\int_{R_n^N/Z} \Lambda^{-1}(t)\Pi(nt)v\,dr'\right) ds'.
\end{aligned}
$$

By similar computations to that of Lemma 3 of [4],

$$
\int_{R_n^N/Z} \Lambda^{-1}(t)\Pi(nt)v\,dr' \tag{7}
$$

is fixed by S'. Therefore, by Proposition 4.6 integral (7) vanishes and v is in $V_{T,N}(\Lambda,\Pi)$ by Proposition 4.5. □

Lemma 4.8 $V_{T,N}(\Lambda,\Pi)+V_S(\Pi)=V_{T,S}(\Lambda,\Pi)$.

Proof $V_{T,N}(\Lambda,\Pi)$ and $V_S(\Pi)$ are obviously subspaces of $V_{T,S}(\Lambda,\Pi)$. Conversely, for a typical generator $\Pi(ts)v-\Lambda(t)v$ of $V_{T,S}(\Lambda,\Pi)$, we have

$$
\begin{aligned}
\Pi(ts)v-\Lambda(t)v &= \Pi(tst^{-1})\Pi(t)v-\Lambda(t)v \\
&= \Pi(s')\Pi(t)v-\Lambda(t)v \\
&= \{\Pi(s')[\Pi(t)v]-[\Pi(t)v]\}+[\Pi(t)v-\Lambda(t)v]
\end{aligned}
$$

and this is an element of $V_S(\Pi)+V_{T,N}(\Lambda,\Pi)$. □

Theorem 4.9

$$
\varphi_v \in C_c^\infty(k^*) \Longleftrightarrow v \in V_{T,S}(\Lambda,\Pi).
$$

Proof

$$
\begin{aligned}
\varphi_v \in C_c^\infty(k^*) &\Longleftrightarrow \exists\, v' \in V_S(\Pi) \text{ such that } \varphi_v = \varphi_{v'} \\
&\Longleftrightarrow \exists\, v' \in V_S(\Pi) \text{ such that } \varphi_{v-v'} = 0 \\
&\Longleftrightarrow \exists\, v' \in V_S(\Pi) \text{ such that } v-v' \in V_{T,N}(\Lambda,\Pi) \\
&\Longleftrightarrow v \in V_S(\Pi)+V_{T,N}(\Lambda,\Pi) \\
&\Longleftrightarrow v \in V_{T,S}(\Lambda,\Pi).
\end{aligned}
$$

The first equality follows from Lemma 4.1, the third from Proposition 4.7, and the last from Lemma 4.8. □

5. Computation of regular poles

The following lemmas are required to determine whether constants in Proposition 3.2 and Proposition 3.5 are nonzero or not.

Lemma 5.1 *Let K be a quadratic extension of k and $T \cong K^*$; then $Hom_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero for a character of K^*, which satisfies $\sigma^2 = \Lambda|_{k^*}$ if and only if $\Lambda \neq \sigma \circ N_{K/k}$. If $Hom_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero then it is one-dimensional.*

Proof Follows from Proposition 1.7 in [13]. □

Lemma 5.2 *Let K be a quadratic extension of k and $T \cong K^*$. If π is an irreducible representation of $GL_2(k)$, which is induced from a character of the torus of $GL_2(k)$, then $Hom_T(\pi, \Lambda)$ is nonzero for every character Λ of K^* such that $\omega_\pi = \Lambda|_{k^*}$ and $Hom_T(\pi, \Lambda)$ is one-dimensional.*

Proof Follows from Proposition 1.6 in [13]. □

Lemma 5.3 *Let*

$$0 \longrightarrow W \longrightarrow U \longrightarrow U/W \longrightarrow 0$$

be an exact sequence of T modules. If $Hom_T(U/W, \Lambda)$ is nonzero, then there exists a nonzero $f \in Hom(U, \Lambda)$ such that $f|W = 0$.

Proof Take $f' \in \mathrm{Hom}_T(U/W, \Lambda)$ and $f' \neq 0$, and then define $f(u) = f'(\bar{u})$. □

From now on, we assume that (Π, V_Π) has a Bessel model with respect to ψ and Λ. Also, for simplicity, we take $\mu = 1$. Irreducible and admissible representations of $GSp_4(k)$, which has Jacquet module length of less than or equal to 2, are given in Table 1 due to the Sally–Tadic classification in [10]. In this table, nonsupercuspidal representations are named as IIIa, IIIb, IVa, IVb, IVc, IVd, Va, Vb, Vc, Vd, VIb, VIc, VII, VIIIa, VIIIb, IXa,IXb, X, XIa, and XIb. Additionally, semisimplifications of these representations are given in Table 2.

5.1. Representations with Jacquet module length 0

Proposition 5.4 *If (Π, V_Π) is an irreducible and admissible representation of $GSp_4(k)$ and $R_S(\Pi)$ is a zero space, then the L-factor of Π does not have regular poles.*

Proof If $R_S(\Pi)$ is a zero space, then the spaces V_Π and $V_S(\Pi)$ are equal. Therefore, by Proposition 2.5 and Proposition 3.1, there is no regular pole. □

Theorem 5.5 *Nongeneric supercuspidal representations of $GSp_4(k)$ and representations VIIIb and IXb do not have regular poles.*

Proof Follows from Proposition 5.4. □

Theorem 5.6 *The L-factors of generic supercuspidal representations of $GSp_4(k)$ and representations VII, VIIIa, and IXa are equal to one.*

Proof Follows from Proposition 5.4 and Theorem 2.6. □

Table 1. Regular poles and expected exceptional poles.

		Representation	Regular	Exceptional
III	a	$\chi \rtimes \sigma St_{GL(2)}$	$L(s,\nu^{1/2}\chi\sigma)$ $L(s,\nu^{1/2}\sigma)$	-
III	b	$\chi \rtimes \sigma 1_{GL(2)}$	$L(s,\nu^{-1/2}\chi\sigma)$ $L(s,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\chi\sigma)$ $L(s,\nu^{1/2}\sigma)$
IV	a	$\sigma St_{GSp(4)}$	$L(s,\nu^{3/2}\sigma)$	-
IV	b	$L(\nu^2,\nu^{-1}\sigma St_{GL(2)})$	$L(s,\nu^{3/2}\sigma)$ $L(s,\nu^{-1/2}\sigma)$	-
IV	c	$L(\nu^{3/2}St_{GL(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^{-3/2}\sigma)$ $L(s,\nu^{1/2}\sigma)$	$L(s,\nu^{3/2}\sigma)$
IV	d	$\sigma 1_{GSp(4)}$	-	-
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$ $L(s,\nu^{1/2}\xi\sigma)$	-
V	b	$L(\nu^{1/2}\xi St_{GL(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$ $L(s,\nu^{1/2}\xi\sigma)$	$L(s,\nu^{1/2}\sigma)$
V	c	$L(\nu^{1/2}\xi St_{GL(2)},\xi\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\xi\sigma)$ $L(s,\nu^{1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma)$
V	d	$L(\nu\xi,\xi \rtimes \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\xi\sigma)$ $L(s,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma)$ $L(s,\nu^{1/2}\sigma)$
VI	b	$\tau(T,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$	-
VI	c	$L(\nu^{1/2}St_{GL(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)^2$
VII		$\chi \rtimes \pi$	-	-
VIII	a	$\tau(S,\pi)$	-	-
VIII	b	$\tau(T,\pi)$	-	-
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	-	-
IX	b	$L(\nu\xi,\nu^{-1/2}\pi)$	-	-
X		$\pi \rtimes \sigma$	$L(s,\sigma)L(s,\omega_\pi\sigma)$	-
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$	-
XI	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma)$

5.2. Representations with Jacquet module length 1

First we determine the asymptotic behavior of φ_v.

Lemma 5.7 *Let $\eta \cong \tau \otimes \rho$ be a subrepresentation of $R_S(\Pi)$ and v an element of V_Π such that the image of v in $R_S(\Pi)$ is $\bar{v}$ and $\bar{v}$ is in V_η. Then for small enough $|x|$ we have*

$$\varphi_v(x) = C|x|^{3/2}\omega_\tau(x)\rho(x).$$

Proof Let ϕ be the isomorphism between η and $\tau \otimes \rho$; then

$$\phi(\delta_P^{-1/2}\Pi_S(m)\bar{v}) = \tau \otimes \rho(m)\phi(\bar{v})$$

for every $\bar{v} \in V_\eta$. In particular, if

$$h_x = \begin{pmatrix} xI_2 & \\ & I_2 \end{pmatrix} = \begin{pmatrix} xI_2 & \\ & x((xI_2)^t)^{-1} \end{pmatrix},$$

Table 2. Jacquet modules: the Siegel parabolic.

		Representation	Semisimplification	#	g
III	a	$\chi \rtimes \sigma St_{GL(2)}$	$B(\chi,\nu)\otimes\sigma\nu^{-1/2}+$ $B(\nu,\chi^{-1})\otimes\chi\sigma\nu^{-1/2}$	2	•
III	b	$\chi \rtimes \sigma 1_{GL(2)}$	$B(\chi,\nu^{-1})\otimes\sigma\nu^{1/2}+$ $B(\nu^{-1},\chi^{-1})\otimes\chi\sigma\nu^{1/2}$	2	
IV	a	$\sigma St_{GSp(4)}$	$\nu^{3/2}St_{GL(2)}\otimes\nu^{-3/2}\sigma$	1	•
IV	b	$L(\nu^2,\nu^{-1}\sigma St_{GL(2)})$	$\nu^{3/2}1_{GL(2)}\otimes\sigma\nu^{-3/2}+$ $B(\nu,\nu^{-2})\otimes\sigma\nu^{1/2}$	2	
IV	c	$L(\nu^{3/2}St_{GL(2)},\nu^{-3/2}\sigma)$	$\nu^{-3/2}St_{GL(2)}\otimes\sigma\nu^{3/2}+$ $B(\nu^2,\nu^{-1})\otimes\sigma\nu^{-1/2}$	2	
IV	d	$\sigma 1_{GSp(4)}$	$\nu^{3/2}St_{GL(2)}\otimes\nu^{-3/2}\sigma$	1	
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\nu^{1/2}\xi St_{GL(2)}\otimes\sigma\nu^{-1/2}+$ $\nu^{1/2}\xi St_{GL(2)}\otimes\sigma\xi\nu^{-1/2}$	2	•
V	b	$L(\nu^{1/2}\xi St_{GL(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi St_{GL(2)}\otimes\sigma\nu^{1/2}+$ $\nu^{1/2}\xi 1_{GL(2)}\otimes\sigma\xi\nu^{-1/2}$	2	
V	c	$L(\nu^{1/2}\xi St_{GL(2)},\xi\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi St_{GL(2)}\otimes\xi\sigma\nu^{1/2}+$ $\nu^{1/2}\xi 1_{GL(2)}\otimes\sigma\nu^{-1/2}$	2	
V	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\nu^{-1/2}\xi 1_{GL(2)}\otimes\xi\sigma\nu^{1/2}+$ $\nu^{-1/2}\xi 1_{GL(2)}\otimes\sigma\nu^{1/2}$	2	
VI	b	$\tau(T,\nu^{-1/2}\sigma)$	$\nu^{1/2}1_{GL(2)}\otimes\nu^{-1/2}\sigma$	1	•
VI	c	$L(\nu^{1/2}St_{GL(2)},\nu^{-1/2}\sigma)$	$\nu^{-1/2}St_{GL(2)}\otimes\nu^{1/2}\sigma$	1	
VII		$\chi\rtimes\pi$	0	0	•
VIII	a	$\tau(S,\pi)$	0	0	•
VIII	b	$\tau(T,\pi)$	0	0	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	0	0	•
IX	b	$L(\nu\xi,\nu^{-1/2}\pi)$	0	0	
X		$\pi\rtimes\sigma$	$\pi\otimes\sigma+\tilde{\pi}\otimes\omega_\pi\sigma$	2	•
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{1/2}\pi\otimes\nu^{-1/2}\sigma$	1	•
XI	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{-1/2}\pi\otimes\nu^{1/2}\sigma$	1	

then $\lambda(h_x)=x$ and $\delta_P(h_x)=|x^2x^{-1}|^3=|x|^3$. Therefore, $\tau\otimes\rho(h_x)\phi(\bar{v})=\omega_\tau(x)\rho(x)\phi(\bar{v})$ and $|x|^{-3/2}\Pi_S(h_x)\bar{v}-\omega_\tau(x)\rho(x)\bar{v}$ is in the kernel of the ϕ, which is zero. Hence

$$\Pi(h_x)v-|x|^{3/2}\omega_\tau(x)\rho(x)v\in V_S(\Pi).$$

Now use Proposition 3.2. □

Next, we find a condition that guarantees that the constant in Lemma 5.7 is nonzero.

Proposition 5.8 *If $R_S(\Pi)\cong\tau\otimes\rho$ and $Hom_T(R_S(\Pi),\Lambda)$ is nonzero, then the regular pole of L-factor of Π is the pole of $L(s,\omega_\tau\rho)$.*

Proof By Lemma 5.7, $\varphi_v(x)=C|x|^{3/2}\omega_\tau(x)\rho(x)$ for small enough $|x|$. If C is zero for every v in V_Π, then φ_v has compact support in k^*. Therefore, by Theorem 4.9, $v\in V_{T,S}(\Lambda,\Pi)$ and $\bar{v}\in\text{span}\{\Pi_S(t)\bar{w}-\Lambda(t)\bar{w}:\bar{w}\in V_\Pi/V_S(\Pi),t\in T\}$. Since $\delta_P(t)=1$ for every $t\in T$ we have $\Pi_S=R_S(\Pi)$, which implies that $\text{Hom}_T(R_S(\Pi),\Lambda)$ is zero and contradicts our assumption. Hence, C is nonzero for some v in V_Π and the result follows from Lemma 3.4. □

Now we determine the regular poles of the representations in Table 2, which has Jacquet module length one. The condition of Proposition 5.8 will be satisfied by the Bessel model existence condition. For more details about Bessel model existence conditions, one can look at the unpublished notes, "Bessel models for GSp(4) over a p-adic field", of R Schmidt).

Theorem 5.9 *For the representations of the group $GSp_4(k)$, which have a Jacquet module length of one, we have:*
i) L-factor of IV-a is $L(s, \nu^{3/2}\sigma)$.
ii) Regular pole of L-factor of VI-b is pole of $L(s, \nu^{1/2}\sigma)$.
iii) Regular pole of L-factor of VI-c is pole of $L(s, \nu^{-1/2}\sigma)$.
iv) L-factor of XI-a is $L(s, \nu^{1/2}\sigma)$.
v) Regular pole of L-factor of XI-b is pole of $L(s, \nu^{-1/2}\sigma)$.

Proof i) By Proposition 5.1 of [8] if IV-a has a Bessel model then $\Lambda \neq \sigma \circ N_{K/k}$. Hence, by Lemma 5.1, $\mathrm{Hom}_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero. Since we have $R_S(\Pi) \cong \nu^{3/2} St_{GL_2(k)} \otimes \nu^{-3/2}\sigma$, as a representation of T, $R_S(\Pi) \cong \sigma St_{GL_2(k)}$. Hence the condition of Proposition 5.8 is satisfied. Also, $\omega_{\nu^{3/2} St_{GL_2(k)}} = \nu^3$, and so by Proposition 5.8 the regular pole of the L-factor of IV-a is the pole of $L(s, \nu^{3/2}\sigma)$. Since we are in the generic case, the result follows from Theorem 2.6.

ii) By Sally–Tadic classification and exactness of the twisted Jacquet module, we have

$$0 \longrightarrow (VI-b)_\psi \longrightarrow (\nu^{1/2} 1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (VI-d)_\psi \longrightarrow 0,$$

where the subscript ψ denotes the largest quotient on which S operates by ψ. By Proposition 2.1 of [8], $(\nu^{1/2} 1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi = \sigma \circ N_{K/k}$. Hence, if VI-b has a Bessel model, then $\Lambda = \sigma \circ N_{K/k}$. Since $R_S(\Pi) \cong \nu^{1/2} 1_{GL_2(k)} \otimes \nu^{-1/2}\sigma$, as a representation of T we have $R_S(\Pi) \cong \sigma 1_{GL_2}$. Hence, the condition of Proposition 5.8 is satisfied. Also, $\omega_{\nu^{1/2} 1_{GL_2(k)}} = \nu$, and so by Proposition 5.8 the regular pole of the L-factor of VI-b is the pole of $L(s, \nu^{1/2}\sigma)$.

iii) By Sally–Tadic classification and exactness of the twisted Jacquet module, we have

$$0 \longrightarrow (VI-a)_\psi \longrightarrow (\nu^{1/2} St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (VI-c)_\psi \longrightarrow 0.$$

By Proposition 2.1 of [8], $(\nu^{1/2} St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi = \sigma St_{GL_2(k)}$. Hence, if VI-c has a Bessel model, then $\mathrm{Hom}_T(\sigma St_{GL_2(k)}, \Lambda)$ is nonzero. Since $R_S(\Pi) \cong \nu^{-1/2} St_{GL_2(k)} \otimes \nu^{1/2}\sigma$, as a representation of T we have $R_S(\Pi) \cong \sigma St_{GL_2(k)}$. Hence, the condition of Proposition 5.8 is satisfied. Also, $\omega_{\nu^{-1/2} St_{GL_2(k)}} = \nu^{-1}$, and so by Proposition 5.8 the regular pole of the L-factor of VI-c is the pole of $L(s, \nu^{-1/2}\sigma)$.

iv) Since K is a field, the sequence

$$0 \longrightarrow (XI-a)_\psi \longrightarrow (\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (XI-b)_\psi \longrightarrow 0$$

splits, and by Proposition 2.1 of [8] we have

$$\mathrm{Hom}_T(\sigma\pi, \Lambda) = \mathrm{Hom}_T((XI-a)_\psi, \Lambda) \oplus \mathrm{Hom}_T((XI-b)_\psi, \Lambda).$$

Hence, if XI-a or XI-b has a Bessel model, then $\mathrm{Hom}_T(\sigma\pi, \Lambda)$ is nonzero. Since $R_S(\Pi) \cong \nu^{1/2}\pi \otimes \nu^{-1/2}\sigma$, as a representation of T we have $R_S(\Pi) \cong \sigma\pi$. Hence, the condition of Proposition 5.8 is satisfied. Also, $\omega_{\nu^{1/2}\pi} \otimes \nu^{-1/2}\sigma = \nu^{1/2}\sigma$, and so by Proposition 5.8, the regular pole of the L-factor of XI-a is the pole of $L(s, \nu^{1/2}\sigma)$. Therefore, the result follows from Theorem 2.6.

v) Follows from the proof of (iv). □

5.3. Representations with Jacquet module length 2

For these representations, we show that regular poles are either 2 simple poles or a double pole depending on the structure of the Jacquet module. First, we find the asymptotic behavior of φ_v up to 2 constants.

Lemma 5.10 *Let U be a subrepresentation of $R_S(\Pi)$ and*

$$0 \longrightarrow W \longrightarrow U \longrightarrow U/W \longrightarrow 0$$

an exact sequence of representations of M, where $W \cong \tau_1 \otimes \rho_1$ and $U/W \cong \tau_2 \otimes \rho_2$ as a representation of $GL_2(k) \times k^*$. Let χ_i be $\omega_{\tau_i}\rho_i$ for $i = 1, 2$ and $\bar{u} \in U$ the image of $u \in V_\Pi$ in $R_S(\Pi)$. Then we have:

1) If $\chi_1 \neq \chi_2$ then $U = \bigoplus \nu^{3/2}\chi_1 \oplus \bigoplus \nu^{3/2}\chi_2$ as a representation of H. For small enough $|x|$ and constants C_1 and C_2, we have

$$\varphi_u(x) = C_1|x|^{3/2}\chi_1(x) + C_2|x|^{3/2}\chi_2(x).$$

2) If $\chi_1 = \chi_2 = \chi$, then for some $\overline{w}_u \in W$ we have

$$\Pi_S(h_x)\bar{u} = |x|^{3/2}\chi(x)\bar{u} + |x|^{3/2}\chi(x)v_k(x)\overline{w}_u.$$

Proof It is similar to the proof of Proposition 4.5.8 of [1]. Since $U/W \cong \chi_2$ as an H module, for every $\bar{u} \in U$ we have

$$R_S(\Pi)(h_x)\bar{u} = \chi_2(x)\bar{u} + \overline{w}_u(x) \tag{8}$$

for some $\overline{w}_u(x) \in W$. Also, by the proof of Lemma 5.7,

$$R_S(\Pi)(h_x)\overline{w} = \chi_1(x)\overline{w} \tag{9}$$

for every $\overline{w} \in W$. From (8),

$$R_S(\Pi)(h_{xy})\bar{u} = \chi_2(xy)\bar{u} + \overline{w}_u(xy). \tag{10}$$

By (8) and (9),

$$\begin{aligned} R_S(\Pi)(h_{xy})\bar{u} &= R_S(\Pi)(h_y)R_S(\Pi)(h_x)\bar{u} \\ &= R_S(\Pi)(h_y)[\chi_2(x)\bar{u} + \overline{w}_u(x)] \\ &= \chi_2(x)R_S(\Pi)(h_y)\bar{u} + R_S(\Pi)(h_y)\overline{w}_u(x) \\ &= \chi_2(x)[\chi_2(y)\bar{u} + \overline{w}_u(y)] + R_S(\Pi)(h_y)\overline{w}_u(x) \\ &= \chi_2(x)[\chi_2(y)\bar{u} + \overline{w}_u(y)] + \chi_1(y)\overline{w}_u(x) \\ &= \chi_2(xy)\bar{u} + \chi_2(x)\overline{w}_u(y) + \chi_1(y)\overline{w}_u(x). \end{aligned}$$

Thus, by (10), we have

$$\overline{w}_u(xy) = \chi_2(x)\overline{w}_u(y) + \chi_1(y)\overline{w}_u(x). \tag{11}$$

CASE 1: $\chi_1 \neq \chi_2$. There exists $y_o \in k^*$ such that $\chi_1(y_o) \neq \chi_2(y_o)$ and also by symmetry in (11)

$$\chi_2(x)\overline{w}_u(y_o) + \chi_1(y_o)\overline{w}_u(x) = \chi_2(y_o)\overline{w}_u(x) + \chi_1(x)\overline{w}_u(y_o).$$

Hence

$$\overline{w}_u(x) = \frac{\overline{w}_u(y_o)}{\chi_1(y_o) - \chi_2(y_o)}[\chi_1(x) - \chi_2(x)].$$

Let $\overline{w}_u = \frac{\overline{w}_u(y_o)}{\chi_1(y_o)-\chi_2(y_o)}$; then by (8),

$$R_S(\Pi)(h_x)\bar{u} = \chi_2(x)\bar{u} + \overline{w}_u[\chi_1(x) - \chi_2(x)]$$

and by (9),

$$R_S(\Pi)(h_x)[\bar{u} - \overline{w}_u] = \chi_2(x)[\bar{u} - \overline{w}_u].$$

Hence, the exact sequence in the statement of the lemma splits as an H module. Since $\delta_P(h_x)^{-1/2} = |x|^{-3/2}$, the result follows from Proposition 3.2.

CASE 2: $\chi_1 = \chi_2 = \chi$.
By (11)

$$\overline{w}_u(xy) = \chi(x)\overline{w}_u(y) + \chi(y)\overline{w}_u(x).$$

Hence

$$\frac{\overline{w}_u(xy)}{\chi(xy)} = \frac{\overline{w}_u(y)}{\chi(y)} + \frac{\overline{w}_u(x)}{\chi(x)},$$

and so there is a homomorphism from k^* to W given by

$$x \longrightarrow \frac{\overline{w}_u(x)}{\chi(x)}.$$

As an additive group only the compact subgroup of $V_S(\Pi)$ is zero, and so the kernel of this homomorphism contains $\mathcal{O}^*$. Hence, if $x = x'\varpi^i$ for $x' \in \mathcal{O}^*$, then

$$\frac{\overline{w}_u(x)}{\chi(x)} = \frac{\overline{w}_u(x')}{\chi(x')} + \frac{\overline{w}_u(\varpi^i)}{\chi(\varpi^i)} = 0 + \frac{\overline{w}_u(\varpi^i)}{\chi(\varpi^i)} = i\frac{\overline{w}_u(\varpi)}{\chi(\varpi)}.$$

Let $\overline{w}_u = \frac{\bar{w}_u(\varpi)}{\chi(\varpi)}$, and then $\overline{w}_u(x) = \chi(x)v_k(x)\overline{w}_u$. Hence, by (8),

$$R_S(\Pi)(h_x)\bar{u} = \chi(x)\bar{u} + \chi(x)v_k(x)\overline{w}_u.$$

Since $\delta_P(h_x)^{-1/2} = |x|^{-3/2}$, the result follows. □

Next, we find 2 conditions that guarantee that the constants in the first part of the previous lemma are nonzero.

Proposition 5.11 *Let*

$$0 \longrightarrow W \longrightarrow R_S(\Pi) \longrightarrow R_S(\Pi)/W \longrightarrow 0$$

be an exact sequence of representations of M*, where* $W \cong \tau_1 \otimes \rho_1$ *and* $R_S(\Pi)/W \cong \tau_2 \otimes \rho_2$*. Also let* $\chi_i = \omega_{\tau_i}\rho_i$ *for* $i = 1, 2$ *and* $\chi_1 \neq \chi_2$*. If the spaces* $Hom_T(W, \Lambda)$ *and* $Hom_T(R_S(\Pi)/W, \Lambda)$ *are nonzero, then the regular poles of* L*-factor of* Π *are poles of* $L(s, \chi_1)L(s, \chi_2)$*.*

Proof By the previous proposition, we have

$$\varphi_u(x) = C_1|x|^{3/2}\chi_1(x) + C_2|x|^{3/2}\chi_2(x)$$

for some constants C_1, C_2 and sufficiently small $|x|$.

Assume that $C_1 = 0$ for every $u \in V_\Pi$. If $\bar{u} \in W$, then by Theorem 4.9, $u \in V_{T,S}(\Lambda, \Pi)$. Therefore, $\bar{u} \in \text{span}\{\Pi_S(t)\bar{v} - \Lambda(t)\bar{v} : \bar{v} \in V_\Pi/V_S(\Pi), t \in T\}$ and there exist $\bar{u}_i \in W$ and $\overline{w}_j \notin W$ such that

$$\bar{u} = \sum_i a_i[\Pi_S(t_i)\bar{u}_i - \Lambda(t_i)\bar{u}_i] + \sum_j b_j[\Pi_S(t_j)\overline{w}_j - \Lambda(t_j)\overline{w}_j]$$

for some $a_i, b_j \in k$. Hence,

$$\bar{u} - \sum_i a_i[\Pi_S(t_i)\bar{u}_i - \Lambda(t_i)\bar{u}_i] = \sum_j b_j[\Pi_S(t_j)\overline{w}_j - \Lambda(t_j)\overline{w}_j].$$

Now apply $\Pi_S(h_x)$ to both sides to get

$$|x|^{3/2}\chi_1(x)(\bar{u} - \sum_i a_i[\Pi_S(t_i)\bar{u}_i - \Lambda(t_i)\bar{u}_i])$$

$$= |x|^{3/2}\chi_2(x)\sum_j b_j[\Pi_S(t_j)\overline{w}_j - \Lambda(t_j)\overline{w}_j].$$

Hence, it follows that

$$\chi_1(x)\sum_j b_j[\Pi_S(t_j)\bar{w}_j - \Lambda(t_j)\overline{w}_j] = \chi_2(x)\sum_j b_j[\Pi_S(t_j)\bar{w}_j - \Lambda(t_j)\overline{w}_j].$$

Therefore, $\sum_j b_j[\Pi_S(t_j)\overline{w}_j - \Lambda(t_j)\overline{w}_j]$ is zero and $\bar{u} = \sum_i a_i[\Pi_S(t_i)\bar{u}_i - \Lambda(t_i)\bar{u}_i]$, which implies that $\text{Hom}_T(W, \Lambda) = 0$ and gives a contradiction.

Now assume that $C_2 = 0$ for every $u \in V_\Pi$. For every u such that $\bar{u} \notin W$, Theorem 4.9 implies that $u \in V_{T,S}(\Lambda, \Pi)$, in which case $\bar{u} \in \text{span}\{\Pi_S(t)\bar{v} - \Lambda(t)\bar{v} : \bar{v} \in V_\Pi/V_S(\Pi), t \in T\}$ and $\text{Hom}_T(R_S(\Pi)/W, \Lambda)$ is zero, which is a contradiction. □

Next, we compute the regular poles of the representations in Table 2, which have a Jacquet module length of 2. In most of the cases the conditions of Proposition 5.11 are provided by the Bessel existence conditions.

We need the following remarks to determine the Bessel existence conditions.

Remark 5.12 *By Sally–Tadic classification and exactness of the twisted Jacquet module, we have*

$$0 \longrightarrow (V-b)_\psi \longrightarrow (\nu^{1/2}\xi 1_{GL_2(k)} \rtimes \xi\nu^{-1/2}\sigma)_\psi \longrightarrow (V-d)_\psi \longrightarrow 0.$$

Since T *is nonsplit, this sequence splits, and by Proposition 2.1 of [8],* $(\nu^{1/2}\xi 1_{GL_2(k)} \rtimes \xi\nu^{-1/2}\sigma)_\psi = \sigma \circ N_{K/k}$*. Thus, if V-b or V-d has a Bessel model with respect to* ψ *and* Λ *then* $\Lambda = \sigma \circ N_{K/k}$*.*

Remark 5.13 *By Sally–Tadic classification and exactness of the twisted Jacquet module, we have*

$$0 \longrightarrow (V-c)_\psi \longrightarrow (\nu^{1/2}\xi 1_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (V-d)_\psi \longrightarrow 0.$$

Similar to the previous remark, if V-c or V-d has a Bessel model with respect to ψ *and* Λ, *then* $\Lambda = \xi\sigma \circ N_{K/k}$.

Remark 5.14 *By Sally–Tadic classification and exactness of the twisted Jacquet module, we have*

$$0 \longrightarrow (V-a)_\psi \longrightarrow (\nu^{1/2}\xi St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi \longrightarrow (V-b)_\psi \longrightarrow 0.$$

Since T *is nonsplit, this sequence splits, and by Proposition 2.1 of [8]* $(\nu^{1/2}\xi St_{GL_2(k)} \rtimes \nu^{-1/2}\sigma)_\psi = \xi\sigma St_{GL_2(k)}$. *Hence, if V-a or V-b has a Bessel model with respect to* ψ *and* Λ, *then* $Hom_T(\xi\sigma St_{GL_2(k)}, \Lambda)$ *is nonzero.*

Remark 5.15 *By Sally–Tadic classification and exactness of the twisted Jacquet module, we have*

$$0 \longrightarrow (V-a)_\psi \longrightarrow (\nu^{1/2}\xi St_{GL_2(k)} \rtimes \xi\nu^{-1/2}\sigma)_\psi \longrightarrow (V-c)_\psi \longrightarrow 0.$$

Similar to the previous remark, if V-a or V-c has a Bessel model with respect to ψ *and* Λ, *then* $Hom_T(\sigma St_{GL_2(k)}, \Lambda)$ *is nonzero.*

Theorem 5.16 *For the representations of the group* $GSp_4(k)$, *which have a Jacquet module length of 2, we have:*

i) L*-factor of III-a is* $L(s, \nu^{1/2}\chi\sigma)\, L(s, \nu^{1/2}\sigma)$.

ii) Regular poles of L*-factor of III-b are poles of* $L(s, \nu^{-1/2}\chi\sigma)L(s, \nu^{-1/2}\sigma)$.

iii) Regular poles of L*-factor of IV-b are poles of* $L(s, \nu^{3/2}\sigma)L(s, \nu^{-1/2}\sigma)$.

iv) IV-c has no Bessel function for nonsplit cases.

v) L*-factor of V-a is* $L(s, \nu^{1/2}\sigma)L(s, \nu^{1/2}\xi\sigma)$.

vi) Regular poles of L*-factor of V-b are poles of* $L(s, \nu^{-1/2}\sigma)L(s, \nu^{1/2}\xi\sigma)$.

vii) Regular poles of L*-factor of V-c are poles of* $L(s, \nu^{-1/2}\xi\sigma)L(s, \nu^{1/2}\sigma)$.

viii) Regular poles of L*-factor of V-d are poles of* $L(s, \nu^{-1/2}\xi\sigma)L(s, \nu^{-1/2}\sigma)$.

ix) L*-factor of X is* $L(s, \sigma)L(s, \omega_\pi\sigma)$.

Proof i) In this case, the constituents of $R_S(\Pi)$ are $B(\chi, \nu) \otimes \sigma\nu^{-1/2}$ and $B(\nu, \chi^{-1}) \otimes \chi\sigma\nu^{-1/2}$, where $\chi \neq \{1, \nu^{-2}, \nu^2\}$. As a representation of T, $B(\chi, \nu) \otimes \sigma\nu^{-1/2} \cong \sigma\nu^{-1/2}B(\chi, \nu)$, which is irreducible and infinite-dimensional. Therefore, by Lemma 5.2 we have

$$\mathrm{Hom}_T(B(\chi, \nu) \otimes \sigma\nu^{-1/2}, \Lambda) \cong \mathrm{Hom}_T(\sigma\nu^{-1/2}B(\chi, \nu), \Lambda) \neq 0.$$

Similarly,

$$\mathrm{Hom}_T(B(\nu, \chi^{-1}) \otimes \chi\sigma\nu^{-1/2}, \Lambda) \neq 0.$$

Also,

$$\{\chi_1, \chi_2\} = \{\omega_{B(\chi,\nu)}\sigma\nu^{-1/2}, \omega_{B(\nu,\chi^{-1})}\chi\sigma\nu^{-1/2}\} = \{\chi\sigma\nu^{1/2}, \sigma\nu^{1/2}\}.$$

Now apply Proposition 5.11 and Theorem 2.6.

ii) Similar to the proof of (i).

iii) By Sally–Tadic classification and exactness of the twisted Jacquet module,

$$0 \longrightarrow (IV-b)_\psi \longrightarrow (\nu^{3/2}1_{GL_2(k)} \rtimes \nu^{-3/2}\sigma)_\psi \longrightarrow (\sigma 1_{GSp_4(k)})_\psi \longrightarrow 0$$

, and since $(\sigma 1_{GSp_4(k)})_\psi$ is one-dimensional, we have $(\sigma 1_{GSp_4(k)})_\psi = 0$. Hence, by Proposition 2.1 of [8], $(IV-b)_\psi = \sigma 1_{GL_2(k)}$, and if IV-b has a Bessel model, then $\mathrm{Hom}_T(\sigma 1_{GL_2(k)}, \Lambda)$ is nonzero. Since the constituent $\nu^{3/2}1_{GL_2(k)} \otimes \sigma\nu^{-3/2}$ of $R_S(\Pi)$ is isomorphic to $\sigma 1_{GL_2(k)}$ as a T module, we have

$$\mathrm{Hom}_T(\nu^{3/2}1_{GL_2(k)} \otimes \sigma\nu^{-3/2}, \Lambda) \cong \mathrm{Hom}_T(\sigma 1_{GL_2(k)}, \Lambda) \neq 0.$$

The other constituent $\sigma\nu^{1/2}B(\nu,\nu^{-2})$ is irreducible and infinite-dimensional, so by Lemma 5.2,

$$\mathrm{Hom}_T(B(\nu,\nu^{-2}) \otimes \sigma\nu^{1/2}, \Lambda) \cong \mathrm{Hom}_T(\sigma\nu^{1/2}B(\nu,\nu^{-2}), \Lambda) \neq 0.$$

Also,

$$\{\chi_1, \chi_2\} = \{\omega_{\nu^{3/2}1_{GL_2(k)}}\sigma\nu^{-3/2}, \omega_{B(\nu,\nu^{-2})}\sigma\nu^{1/2}\} = \{\sigma\nu^{3/2}, \sigma\nu^{-1/2}\}.$$

Now apply Proposition 5.11.

iv) By Sally–Tadic classification and exactness of the twisted Jacquet module,

$$0 \longrightarrow (IV-c)_\psi \longrightarrow (\nu^2 \rtimes \nu^{-1}\sigma 1_{GL_2(k)})_\psi \longrightarrow (\sigma 1_{GSp_4(k)})_\psi \longrightarrow 0$$

, and by Proposition 2.1 of [8] $(IV-c)_\psi = (\nu^2 \rtimes \nu^{-1}\sigma 1_{GL_2(k)})_\psi$, which is zero by the remarks before Proposition 2.4 of [8]. Hence, there is no Bessel function for nonsplit cases.

v)Since T is nonsplit, if V-a has a Bessel model with respect to Λ and ψ, then by Remark 5.14 and Remark 5.15 $\mathrm{Hom}_T(\xi\sigma St_{GL_2(k)}, \Lambda)$ and $\mathrm{Hom}_T(\sigma St_{GL_2(k)}, \Lambda)$ are nonzero. Since the constituents $\nu^{1/2}\xi St_{GL_2(k)} \otimes \sigma\nu^{-1/2}$ and $\nu^{1/2}\xi St_{GL_2(k)} \otimes \sigma\xi\nu^{-1/2}$ of $R_S(\Pi)$ are isomorphic to $\xi\sigma St_{GL_2(k)}$ and $\sigma St_{GL_2(k)}$, respectively, as a T module we have

$$\mathrm{Hom}_T(\nu^{1/2}\xi St_{GL_2(k)} \otimes \sigma\nu^{-1/2}, \Lambda) \cong \mathrm{Hom}_T(\xi\sigma St_{GL_2(k)}, \Lambda) \neq 0$$

and

$$\mathrm{Hom}_T(\nu^{1/2}\xi St_{GL_2(k)} \otimes \sigma\xi\nu^{-1/2}, \Lambda) \cong \mathrm{Hom}_T(\sigma St_{GL_2(k)}, \Lambda) \neq 0.$$

Also,

$$\{\chi_1, \chi_2\} = \{\omega_{\nu^{1/2}\xi St_{GL_2(k)}}\sigma\nu^{-1/2}, \omega_{\nu^{1/2}\xi St_{GL_2(k)}}\sigma\xi\nu^{-1/2}\} = \{\sigma\nu^{1/2}, \sigma\xi\nu^{1/2}\}.$$

Now apply Proposition 5.11 and Theorem 2.6.

vi) Similar to the proof of (v) but use Remark 5.12 and Remark 5.14.

vii) Similar to the proof of (v) but use Remark 5.13 and Remark 5.15.

viii) Similar to the proof of (v) but use Remark 5.12 and Remark 5.13.

ix) In this case, the constituents of $R_S(\Pi)$ are $\pi \otimes \sigma$ and $\tilde{\pi} \otimes \omega_\pi \sigma$, where π is a supercuspidal representation of $GL_2(k)$. Also note that by Proposition 2.1 of [8], X has a Bessel model with respect to ψ and Λ if and only if $\mathrm{Hom}_T(\sigma\pi, \Lambda)$ is nonzero.

Case 1: $\omega_\pi \neq \mathbf{1}(\pi \ncong \tilde{\pi})$

By Theorem 4.2.2 of [1], $\tilde{\pi}\omega_\pi \cong \pi$, and so by the Bessel model existing condition,

$$\mathrm{Hom}_T(\tilde{\pi} \otimes \omega_\pi \sigma, \Lambda) \cong \mathrm{Hom}_T(\tilde{\pi}\omega_\pi \sigma, \Lambda) \cong \mathrm{Hom}_T(\pi\sigma, \Lambda) \neq 0.$$

Also,

$$\{\chi_1, \chi_2\} = \{\omega_\pi \sigma, \omega_{\tilde{\pi}} \omega_\pi \sigma\} = \{\omega_\pi \sigma, \sigma\}.$$

Now apply Proposition 5.11 and Theorem 2.6.

Case 2: $\omega_\pi = \mathbf{1}(\pi \cong \tilde{\pi})$ In this case we have

$$0 \longrightarrow \pi \otimes \sigma \longrightarrow R_S(\Pi) \longrightarrow \pi \otimes \sigma \longrightarrow 0$$

and $\chi = \chi_1 = \chi_2 = \sigma$. By Lemma 5.10, for every $\bar{u} \in V_\Pi / V_S(\Pi)$ there exists $\overline{w}_u \in V_{\pi \otimes \sigma}$ such that

$$R_S(\Pi)(h_x)\bar{u} = \chi(x)\bar{u} + \chi(x) v_k(x) \overline{w}_u. \tag{12}$$

Assume that $\overline{w}_u = 0$ for every $\bar{u} \in V_\Pi / V_S(\Pi)$, and then $R_S(\Pi)(h_x)\bar{u} = \chi(x)\bar{u}$. Hence we have

$$0 \longrightarrow \pi \longrightarrow R_S(\Pi) \longrightarrow \pi \longrightarrow 0$$

as a $GL_2(k)$ module. The center of $GL_2(k)$ acts trivially on $R_S(\Pi)$ and π, and so by theorem 5.4.1 of "Introduction to the theory of admissible representations of p-adic reductive groups (unpublished notes of W Casselman)", this exact sequence splits and we have $R_S(\Pi) = \pi \oplus \pi$ as a $GL_2(k)$ module. Since $R_S(\Pi)(h_x)\bar{u} = \sigma(x)\bar{u}$ for every $\bar{u}$, we also have $R_S(\Pi) = \pi \otimes \sigma \bigoplus \pi \otimes \sigma$ as an M module. Hence, by Frobenius reciprocity,

$$\begin{aligned} 2 = \dim[\mathrm{Hom}_M(R_S(\Pi), \pi \otimes \sigma)] &= \dim[\mathrm{Hom}_P(\Pi, \pi \otimes \sigma)] \\ &= \dim[\mathrm{Hom}_{GSp_4(k)}(\Pi, \pi \rtimes \sigma)], \end{aligned}$$

which is a contradiction. Hence, there exists $\bar{u}$ such that $\overline{w}_u \neq 0$, and by (12)

$$\Pi(S)(h_x)\bar{u} = |x|^{3/2}\chi(x)\bar{u} + |x|^{3/2}\chi(x) v_k(x) \overline{w}_u.$$

Now apply Lemma 3.5, Lemma 3.7 and Theorem 2.6. □

A. Tables

Table 1 displays the regular poles of the nonsupercuspidal representations [10], which have Jacquet module length of at most 2, in terms of the poles of Tate L-functions. The last column shows the expected exceptional poles from the local Langlands conjecture. Table 2 lists all irreducible, admissible, and nonsupercuspidal representations of $GSp_4(k)$, which have Jacquet module length of at most 2. The fourth column shows the semisimplifications of the Jacquet modules with respect to the Siegel parabolic given in the appendix of [9]. The $'\#'$ and $'g'$ columns indicate the number of constituents of the Jacquet module and generic representations, respectively.

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