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S. Rottey and G. Van de Voorde characterized regular pseudo-ovals of $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Here an alternative proof is given and slightly stronger results are obtained.

1. Introduction

Pseudo-ovals and pseudo-hyperovals were introduced in [Thas 1971]; see also [Thas et al. 2006]. These objects play a key role in the theory of translation generalized quadrangles [Payne and Thas 2009; Thas et al. 2006]. Pseudo-hyperovals only exist in even characteristic. A characterization of regular pseudo-ovals in odd characteristic was given in [Casse et al. 1985]; see also [Thas et al. 2006]. In [Rottey and Van de Voorde 2015] a characterization of regular pseudo-ovals and regular pseudo-hyperovals in $\boldsymbol{PG}(3n-1, q)$, q even, $q \neq 2$ and n prime, is obtained. Here a shorter proof is given and slightly stronger results are obtained.

2. Ovals and hyperovals

A *k-arc* in $\boldsymbol{PG}(2, q)$ is a set of k points, $k \geq 3$, no three of which are collinear. Any nonsingular conic of $\boldsymbol{PG}(2, q)$ is a $(q+1)$-arc. If $\mathcal{K}$ is any k-arc of $\boldsymbol{PG}(2, q)$, then $k \leq q+2$. For q odd $k \leq q+1$, and for q even a $(q+1)$-arc extends to a $(q+2)$-arc; see [Hirschfeld 1998]. A $(q+1)$-arc is an *oval*; a $(q+2)$-arc, q even, is a *complete oval* or *hyperoval*.

A famous theorem of B. Segre [1954] tells us that for q odd every oval of $\boldsymbol{PG}(2, q)$ is a nonsingular conic. For q even, there are many ovals that are not conics [Hirschfeld 1998]; also, there are many hyperovals that do not contain a conic [loc. cit.].

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3. Generalized ovals and hyperovals

Arcs, ovals and hyperovals can be generalized by replacing their points with m-dimensional subspaces to obtain generalized k-arcs, generalized ovals and generalized hyperovals. These objects have strong connections to generalized quadrangles, projective planes, circle geometries, flocks and other structures. See [Payne and Thas 2009; Thas et al. 2006; Thas 1971; 2011; Casse et al. 1985; Penttila and Van de Voorde 2013]. Below, some basic definitions and results are formulated; for an extensive study, many applications and open problems, see [Thas et al. 2006].

A *generalized k-arc* of $\boldsymbol{PG}(3n-1,q)$, $n \geq 1$, is a set of k $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1,q)$, every three of which generate $\boldsymbol{PG}(3n-1,q)$. If q is odd, then $k \leq q^n+1$; if q is even, then $k \leq q^n+2$. Every generalized (q^n+1)-arc of $\boldsymbol{PG}(3n-1,q)$, q even, can be extended to a generalized (q^n+2)-arc.

If $\mathcal{O}$ is a generalized (q^n+1)-arc in $\boldsymbol{PG}(3n-1,q)$, then it is a *pseudo-oval* or *generalized oval* or $[n-1]$-*oval* of $\boldsymbol{PG}(3n-1,q)$. For $n=1$, a [0]-oval is just an oval of $\boldsymbol{PG}(2,q)$. If $\mathcal{O}$ is a generalized (q^n+2)-arc in $\boldsymbol{PG}(3n-1,q)$, q even, then it is a *pseudo-hyperoval* or *generalized hyperoval* or $[n-1]$-*hyperoval* of $\boldsymbol{PG}(3n-1,q)$. For $n=1$, a [0]-hyperoval is just a hyperoval of $\boldsymbol{PG}(2,q)$.

If $\mathcal{O}=\{\pi_0,\pi_1,\ldots,\pi_{q^n}\}$ is a pseudo-oval of $\boldsymbol{PG}(3n-1,q)$, then π_i is contained in exactly one $(2n-1)$-dimensional subspace τ_i of $\boldsymbol{PG}(3n-1,q)$ which has no point in common with $(\pi_0\cup\pi_1\cup\cdots\cup\pi_{q^n})\setminus\pi_i$, with $i=0,1,\ldots,q^n$; the space τ_i is the *tangent space* of $\mathcal{O}$ at π_i. For q even the q^n+1 tangent spaces of $\mathcal{O}$ contain a common $(n-1)$-dimensional space π_{q^n+1}, the *nucleus* of $\mathcal{O}$; also, $\mathcal{O}\cup\{\pi_{q^n+1}\}$ is a pseudo-hyperoval of $\boldsymbol{PG}(3n-1,q)$. For q odd, the tangent spaces of a pseudo-oval $\mathcal{O}$ are the elements of a pseudo-oval $\mathcal{O}^*$ in the dual space of $\boldsymbol{PG}(3n-1,q)$.

4. Regular pseudo-ovals and pseudo-hyperovals

In the extension $\boldsymbol{PG}(3n-1,q^n)$ of $\boldsymbol{PG}(3n-1,q)$, we consider n planes ξ_i, $i=1,2,\ldots,n$, that are conjugate in the extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$ and which span $\boldsymbol{PG}(3n-1,q^n)$. This means that they form an orbit of the Galois group corresponding to this extension and span $\boldsymbol{PG}(3n-1,q^n)$.

In ξ_1 consider an oval $\mathcal{O}_1=\{x_0^{(1)},x_1^{(1)},\ldots,x_{q^n}^{(1)}\}$. Further, let $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$, with $i=0,1,\ldots,q^n$, be conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The points $x_i^{(1)},x_i^{(2)},\ldots,x_i^{(n)}$ define an $(n-1)$-dimensional subspace π_i over $\mathbb{F}_q$ for $i=0,1,\ldots,q^n$. Then, $\mathcal{O}=\{\pi_0,\pi_1,\ldots,\pi_{q^n}\}$ is a generalized oval of $\boldsymbol{PG}(3n-1,q)$. These objects are the *regular* or *elementary pseudo-ovals*. If $\mathcal{O}_1$ is replaced by a hyperoval, and so q is even, then the corresponding $\mathcal{O}$ is a *regular* or *elementary pseudo-hyperoval*.

All known pseudo-ovals and pseudo-hyperovals are regular.

5. Characterizations

Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ be a pseudo-oval in $\boldsymbol{PG}(3n-1, q)$. The tangent space of $\mathcal{O}$ at π_i will be denoted by τ_i, with $i = 0, 1, \ldots, q^n$. Choose π_i, $i \in \{0, 1, \ldots, q^n\}$, and let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π_i. Let $\tau_i \cap \boldsymbol{PG}(2n-1, q) = \eta_i$ and $\langle \pi_i, \pi_j \rangle \cap \boldsymbol{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{q^n}\} = \Delta_i$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Now, let q be even and π the nucleus of $\mathcal{O}$. Let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π. If $\zeta_j = \boldsymbol{PG}(2n-1, q) \cap \langle \pi, \pi_j \rangle$, then $\{\zeta_0, \zeta_1, \ldots, \zeta_{q^n}\} = \Delta$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Next, let q be odd. Choose τ_i, with $i \in \{0, 1, \ldots, q^n\}$. If $\tau_i \cap \tau_j = \delta_j$, with $j \neq i$, then $\{\delta_0, \delta_1, \ldots, \delta_{i-1}, \pi_i, \delta_{i+1}, \ldots, \delta_{q^n}\} = \Delta_i^\star$ is an $(n-1)$-spread of τ_i.

Finally, let q be even and let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$ be a pseudo-hyperoval in $\boldsymbol{PG}(3n-1, q)$. Choose π_i, with $i \in \{0, 1, \ldots, q^n+1\}$, and let $\boldsymbol{PG}(2n-1, q) \subseteq \boldsymbol{PG}(3n-1, q)$ be skew to π_i. Let $\langle \pi_i, \pi_j \rangle \cap \boldsymbol{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{q^n+1}\} = \Delta_i$ is an $(n-1)$-spread of $\boldsymbol{PG}(2n-1, q)$.

Theorem 5.1 [Casse et al. 1985]. *Consider a pseudo-oval $\mathcal{O}$ with q odd. Then at least one of the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}, \Delta_0^\star, \Delta_1^\star, \ldots, \Delta_{q^n}^\star$ is regular if and only if they all are regular if and only if the pseudo-oval $\mathcal{O}$ is regular. In such a case $\mathcal{O}$ is essentially a conic over $\mathbb{F}_{q^n}$.*

Theorem 5.2 [Rottey and Van de Voorde 2015]. *Consider a pseudo-oval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$ with $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.*

6. Alternative proof and improvements

Theorem 6.1. *Consider a pseudo-hyperoval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads Δ_i, with $i = 0, 1, \ldots, q^n+1$, are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads Δ_i, with $i = 0, 1, \ldots, q^n+1$, are regular.

Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n+1}\}$, and let $\widehat{\mathcal{O}} = \{\beta_0, \beta_1, \ldots, \beta_{q^n+1}\}$ be the dual of $\mathcal{O}$, with β_i being the dual of π_i.

Choose β_i, $i \in \{0, 1, \ldots, q^n+1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}$, $j \neq i$. Then

$$\{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+1}, \ldots, \alpha_{i,q^n+1}\} = \Gamma_i \tag{1}$$

is an $(n-1)$-spread of β_i.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, $j \neq i$. In Γ_j we next consider an $(n-1)$-regulus γ_j containing α_{ij}. The $(n-1)$-regulus γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$; see Section 4.5 in [Hirschfeld and Thas 2016]. The $(n-1)$-regulus γ_j and the $(n-1)$-spread Γ_i of β_i generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $\boldsymbol{PG}(3n-1, q)$. This can be seen as follows. The elements of Γ_i intersect n lines $U_1, U_2, \ldots, U_n$ which are conjugate in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$; that is, they form an orbit of the Galois group corresponding to this extension. Let $\alpha_{ij} \cap U_l = \{u_l\}$, with $l = 1, 2, \ldots, n$. Now consider the transversals $T_1, T_2, \ldots, T_n$ of the elements of γ_j, with T_l containing u_l. The n planes $T_l U_l = \theta_l$ intersect all elements of γ_j and Γ_i. The $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1, q)$ intersecting $\theta_1, \theta_2, \ldots, \theta_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}(2, q^n)$, with its lines corresponding to the $(2n-1)$-dimensional spaces containing at least two (and then $q^n + 1$) elements of the spread. Hence, the $q+2$ elements of $\widehat{\mathcal{O}}$ containing an element of γ_j, say $\beta_i = \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+1}}, \beta_{i_{q+2}} = \beta_j$, correspond to lines of $\boldsymbol{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ correspond to points of $\boldsymbol{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j, and repeat the argument. Then there arise n planes θ'_l intersecting all elements of γ_j and Γ_{i_2}. The $(n-1)$-dimensional subspaces of $\boldsymbol{PG}(3n-1, q)$ intersecting $\theta'_1, \theta'_2, \ldots, \theta'_n$ are the elements of the regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_{i_2})$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}'(2, q^n)$, and the lines of this plane correspond to the $(2n-1)$-dimensional spaces containing $q^n + 1$ elements of the spread. Hence, $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+2}}$ correspond to lines of $\boldsymbol{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ correspond to points of $\boldsymbol{PG}'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\boldsymbol{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set $\mathcal{M}$ of maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$ [Burau 1961]. The planes $\theta_1, \theta_2, \ldots, \theta_n, \theta'_1, \theta'_2, \ldots, \theta'_n$ are elements of $\mathcal{M}$. It follows that $(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n) = \varnothing$.

Now consider any $(n-1)$-dimensional subspace $\pi \in \{\pi_{i_5}, \pi_{i_6}, \ldots, \pi_{i_{q+2}}\}$ of $\boldsymbol{PG}(3n-1, q)$. We will show that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. Let $\theta_i \cap \pi_j = \{t_{ij}\}$, $\theta'_i \cap \pi_j = \{t'_{ij}\}$, $i = 1, 2, \ldots, n$ and $j = i_1, i_2, \ldots, i_{q+2}$. If $t_{ij_1} t_{ij_2} \cap t_{ij_3} t_{ij_4} = \{v_i\}$ and $t'_{ij_1} t'_{ij_2} \cap t'_{ij_3} t'_{ij_4} = \{v'_i\}$, with j_1, j_2, j_3, j_4 distinct, then $v_1, v_2, \ldots, v_n$ are conjugate and similarly $v'_1, v'_2, \ldots, v'_n$ are conjugate. Hence, $\langle v_1, v_2, \ldots, v_n \rangle = \langle v'_1, v'_2, \ldots, v'_n \rangle$ defines an $(n-1)$-dimensional space over $\mathbb{F}_q$ which intersects $\theta_1, \theta_2, \ldots, \theta'_n$ (over $\mathbb{F}_{q^n}$). The points t_{ij}, with $j = i_1, i_2, \ldots, i_{q+2}$, generate a subplane of θ_i, and the points t'_{ij}, with $j = i_1, i_2, \ldots, i_{q+2}$, generate a subplane of θ'_i, with $i = 1, 2, \ldots, n$. Let $q = 2^h$, and let $\mathbb{F}_{2^v}$ be the subfield of $\mathbb{F}_{q^n} = \mathbb{F}_{2^{hn}}$ over which these subplanes are defined, so $v \mid hn$. Then $v < hn$ as otherwise the spreads of $\boldsymbol{PG}(3n-1, q)$ defined by $\theta_1, \theta_2, \ldots, \theta_n$ and $\theta'_1, \theta'_2, \ldots, \theta'_n$ coincide, which is clearly not possible. The $(n-1)$-regulus γ_j implies that the subplanes contain a line over $\mathbb{F}_q$, so $h \mid v$. As n is prime we have $v = h$, so $2^v = q$.

Hence, the $2n$ subplanes are defined over $\mathbb{F}_q$. It follows that the $q+2$ elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ are maximal subspaces of the Segre variety $\mathcal{S}_{2;n-1}$. Hence, π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_1, \pi_2, \ldots, \pi_{q+2}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Now consider a $\boldsymbol{PG}(2,q)$ intersecting $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+2}}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ intersecting $\boldsymbol{PG}(2,q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \boldsymbol{PG}(3n-1,q)$ of $\boldsymbol{PG}(3n-1,q)$.

Consider π_{i_1} and also a $\boldsymbol{PG}(2n-1,q)$ skew to π_{i_1}. If we project $\pi_{i_2}, \pi_{i_3}, \ldots, \pi_{i_{q+2}}$ from π_{i_1} onto $\boldsymbol{PG}(2n-1,q)$, then by the foregoing paragraph the $q+1$ projections constitute an $(n-1)$-regulus of $\boldsymbol{PG}(2n-1,q)$. We arrive at a similar conclusion if we project from π_{i_s}, s any element of $\{1,2,\ldots,q+2\}$. Equivalently, if $s \in \{1,2,\ldots,q+2\}$, then the spaces $\beta_{i_s} \cap \beta_{i_t}$, with $t = 1,2,\ldots,s-1,s+1,\ldots,q+2$, form an $(n-1)$-regulus of β_{i_s}.

Now assume that the condition $\{\theta_1,\theta_2,\ldots,\theta_n\} \cap \{\theta_1',\theta_2',\ldots,\theta_n'\} = \varnothing$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$. In such a case every $(n-1)$-regulus contained in a spread Γ_s defines a Segre variety $\mathcal{S}_{2;n-1}$ over $\mathbb{F}_q$. Let us define the following design $\mathcal{D}$. Points of $\mathcal{D}$ are the elements of $\widehat{\mathcal{O}}$, a block of $\mathcal{D}$ is a set of $q+2$ elements of $\widehat{\mathcal{O}}$, containing at least one space of an $(n-1)$-regulus contained in some regular spread Γ_s, and incidence is containment. Then $\mathcal{D}$ is a $4-(q^n+2, q+2, 1)$ design. By Kantor [1974] this implies that $q=2$, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$,

$$\{\theta_1,\theta_2,\ldots,\theta_n\} = \{\theta_1',\theta_2',\ldots,\theta_n'\}. \tag{2}$$

In such a case the q^n+2 elements of $\widehat{\mathcal{O}}$ correspond to lines of the plane $\boldsymbol{PG}(2,q^n)$. It follows that $\mathcal{O}$ is regular. □

Theorem 6.2. *Consider a pseudo-oval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1,q)$, with $q = 2^h, h > 1$ and n prime. Then $\mathcal{O}$ is regular if and only if all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular.

Conversely, assume that the $(n-1)$-spreads $\Delta_0, \Delta_1, \ldots, \Delta_{q^n}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$, let π_{q^n+1} be the nucleus of $\mathcal{O}$, let $\overline{\mathcal{O}} = \mathcal{O} \cup \{\pi_{q^n+1}\}$, let $\widehat{\mathcal{O}}$ be the dual of $\mathcal{O}$, let $\widehat{\overline{\mathcal{O}}}$ be the dual of $\overline{\mathcal{O}}$, and let β_i be the dual of π_i.

Choose β_i, $i \in \{0,1,\ldots,q^n+1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}$, $j \neq i$. Then

$$\{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+1}, \ldots, \alpha_{i,q^n+1}\} = \Gamma_i \tag{3}$$

is an $(n-1)$-spread of β_i.

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, with $j \neq i$ and $i, j \in \{0,1,\ldots,q^n\}$. In Γ_j we next consider an $(n-1)$-regulus γ_j containing α_{ij} and α_{j,q^n+1}. The $(n-1)$-regulus

γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$. The $(n-1)$-regulus γ_j and the $(n-1)$-spread Γ_i of β_i generate a regular $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_i)$ of $\boldsymbol{PG}(3n-1, q)$. Such as in the proof of Theorem 6.1 we introduce the elements $\underset{\frown}{U}_l, u_l, T_l, \theta_l$, $l = 1, 2, \ldots, n$, and the plane $\boldsymbol{PG}(2, q^n)$. The $q+2$ elements of $\overline{\mathcal{O}}$ containing an element of γ_j, say $\beta_i = \beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_q}, \beta_j = \beta_{i_{q+1}}, \beta_{q^n+1}$, correspond to lines of $\boldsymbol{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\boldsymbol{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j, and repeat the argument. Then there arise n planes θ'_l of $\boldsymbol{PG}(3n-1, q^n)$ intersecting all elements of γ_j and Γ_{i_2}, and an $(n-1)$-spread $\Sigma(\gamma_j, \Gamma_{i_2})$ of $\boldsymbol{PG}(3n-1, q)$. The elements of this spread correspond to the points of a plane $\boldsymbol{PG}'(2, q^n)$. The spaces $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{q+1}}, \beta_{q^n+1}$ correspond to lines of $\boldsymbol{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\boldsymbol{PG}'(2, q^n)$.

First, assume $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\boldsymbol{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set $\mathcal{M}$ of maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$. The planes $\theta_1, \theta_2, \ldots, \theta_n, \theta'_1, \theta'_2, \ldots, \theta'_n$ are elements of $\mathcal{M}$. It follows that $(\theta_1 \cup \theta_2 \cup \cdots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \cdots \cup \theta'_n) = \varnothing$. Let $\pi \in \{\pi_{i_5}, \pi_{i_6}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}\}$. As in the proof of Theorem 6.1 one shows that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Next consider a $\boldsymbol{PG}(2, q)$ that intersects $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$-dimensional spaces $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ which intersect the plane $\boldsymbol{PG}(2, q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \boldsymbol{PG}(3n-1, q)$ of $\boldsymbol{PG}(3n-1, q)$. As in the proof of Theorem 6.1 it follows that the spaces $\beta_{q^n+1} \cap \beta_{i_t}$, with $t = 1, 2, \ldots, q+1$, form an $(n-1)$-regulus of β_{q^n+1}.

Now assume that the condition $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$, $j \neq i$ and $i, j \in \{0, 1, \ldots, q^n\}$. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct elements of Γ_{q^n+1}. Then $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ can be chosen in such a way that $\alpha_1 \in \beta_i$, $\alpha_2 \in \beta_j$, $\alpha_2 \in \gamma_j$ and $\beta_{i_2} \cap \beta_j \in \gamma_j$ with $\alpha_3 \in \beta_{i_2}$. Hence, the $(n-1)$-regulus in β_{q^n+1} defined by $\alpha_1, \alpha_2, \alpha_3$ is a subset of Γ_{q^n+1}. From [Hirschfeld and Thas 2016, Theorem 4.123] now follows that the $(n-1)$-spread Γ_{q^n+1} of β_{q^n+1} is regular. By Theorem 6.1 the pseudo-hyperoval $\overline{\mathcal{O}}$ is regular, and so $\mathcal{O}$ is regular. But in such a case the condition $\{\theta_1, \theta_2, \ldots, \theta_n\} \cap \{\theta'_1, \theta'_2, \ldots, \theta'_n\} = \varnothing$ is never satisfied, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ we have $\{\theta_1, \theta_2, \ldots, \theta_n\} = \{\theta'_1, \theta'_2, \ldots, \theta'_n\}$. In such a case the q^n+2 elements of $\overline{\mathcal{O}}$ correspond to lines of the plane $\boldsymbol{PG}(2, q^n)$. It follows that $\overline{\mathcal{O}}$, and hence also $\mathcal{O}$, is regular. □

Theorem 6.3. *A pseudo-hyperoval $\mathcal{O}$ in $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime, is regular if and only if at least $q^n - 1$ elements of $\{\Delta_0, \Delta_1, \ldots, \Delta_{q^n+1}\}$ are regular.*

Proof. If $\mathcal{O}$ is regular, then clearly all $(n-1)$-spreads Δ_i, with $i = 0, 1, \dots, q^n + 1$, are regular.

Conversely, assume that ρ, with $\rho \geq q^n - 1$, elements of $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$ are regular.

If $\rho = q^n + 2$, then $\mathcal{O}$ is regular by Theorem 6.1; if $\rho = q^n + 1$, then $\mathcal{O}$ is regular by Theorem 6.2.

Now assume that $\rho = q^n$ and that $\Delta_2, \Delta_3, \dots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1, in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$, then let γ_j contain $\beta_j \cap \beta_i$, $\beta_j \cap \beta_0$, $\beta_j \cap \beta_1$ and let $\beta_{i_2} \neq \beta_1$, with $i, j \in \{2, 3, \dots, q^n + 1\}$. Now see the proof of the preceding theorem.

Finally, assume that $\rho = q^n - 1$ and that $\Delta_3, \Delta_4, \dots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If exactly one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1, in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$ or $\beta_0 \cap \beta_2$, then proceed as in the preceding paragraph with $\beta_{i_2} \neq \beta_1, \beta_2$. Now assume that two of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1 and α_2, are $\beta_0 \cap \beta_1$ and $\beta_0 \cap \beta_2$. Now consider all $(n-1)$-reguli in Δ_0 containing α_1 and α_3, and assume, by way of contradiction, that no one of these $(n-1)$-reguli contains α_2. The number of these $(n-1)$-reguli is $(q^n - 2)/(q - 1)$, and so $q = 2$, a contradiction. It follows that the $(n-1)$-regulus in β_0 defined by $\alpha_1, \alpha_2, \alpha_3$ is contained in Δ_0. Now we proceed as in the proof of Theorem 6.2. □

7. Final remarks

The cases q = 2 and n not prime. For $q = 2$ or n not prime other arguments have to be developed.

Improvement of Theorem 6.3. Let $\mathcal{D} = (P, B, \in)$ be an incidence structure satisfying the following conditions:

(i) $|P| = q^n + 1$, q even, $q \neq 2$,

(ii) the elements of B are subsets of size $q + 1$ of P and every three distinct elements of P are contained in at most one element of B, and

(iii) Q is a subset of size δ of P such that any triple of elements in P with at most one element in Q is contained in exactly one element of B.

Assumption. Any such $\mathcal{D}$ is a $3-(q^n + 1, q + 1, 1)$ design whenever $\delta \leq \delta_0$ with $\delta_0 \leq q - 2$.

Theorem 7.1. *Consider a pseudo-hyperoval $\mathcal{O}$ in* $\boldsymbol{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ *and n prime. Then $\mathcal{O}$ is regular if and only if at least $q^n + 1 - \delta_0$ elements of* $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$ *are regular.*

Proof. Similar to the proof of Theorem 6.3. □

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