

Regular Rings, V-Rings and their Generalizations

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(Received September 5, 1978)

In this paper, the notions of right *p.p.* rings, right *CPP*-rings and right *CPF*-rings, introduced primarily for rings with identity, will be defined for *s*-unital rings. One of the purposes of this paper is to extend the principal results in [19] to *s*-unital rings so as to improve several previous results obtained in [15-18] (Theorems 1-5). Furthermore, we shall present a characterization of an *s*-unital right *CPP*-ring (Theorem 6), which will deduce the main theorem in [6].

Throughout A will represent a ring (possibly without identity). Given a right (resp. left) ideal I of A , I^* will denote the intersection of all maximal right (resp. left) ideals of A containing I . If M is a right (resp. left) A -module and S is a subset of A , then we set $\ell_M(S) = \{u \in M \mid uS = 0\}$ (resp. $r_M(S) = \{u \in M \mid Su = 0\}$). As usual, we write $\ell(S) = \ell_A(S)$ and $r(S) = r_A(S)$. As for other notations and terminologies used in this paper, we follow the previous ones [15] and [16].

1. Preliminaries. Following [15], a non-zero right (resp. left) A -module M is said to be *s-unital* if $u \in uA$ (resp. $u \in Au$) for each $u \in M$. If A_A (resp. ${}_A A$) is *s-unital*, A is called a *right* (resp. *left*) *s-unital ring*. In case A is right and left *s-unital*, we merely say *s-unital*. We begin by stating a lemma which will be used repeatedly in what follows.

LEMMA 1 ([15, Theorem 1] and [11, Lemma 1 (a)]). *If F is a finite subset of a right s-unital ring (resp. an s-unital ring) A , then there exists an element $e \in A$ such that $ae = a$ (resp. $ea = ae = a$) for all $a \in F$.*

A right A -module M is said to be *p-injective* if for any principal right ideal $|a|$ of A and $f: |a|_A \rightarrow M_A$ there exists an element $u \in M$ such that $f(x) = ux$ for all $x \in |a|$. As is well known, A is a regular ring if and only if every right A -module is *p-injective*.

LEMMA 2 (cf. [4, Proposition 1.7 and Corollary 1.9]). *Let A be a right s-unital ring, and M_A an s-unital module. If M_A is p-injective then, for each $a \in A$, there holds $\ell_M(r(a)) = Ma$, and conversely. In particular, for a domain A with 1, a unital module M_A is p-injective if and only if M_A is divisible.*

PROOF. Assume that M_A is *p-injective*. Given $u \in \ell_M(r(a))$, there exists

an element $e' \in A$ such that $ue' = u$. Then, by Lemma 1, there exists an element $e \in A$ such that $ae = a$ and $e'e = e'$. Consider $f: aA_A \rightarrow M_A$ defined by $ax \rightarrow ux$. Since M_A is p -injective, we can find an element $v \in M$ with $ux = vax$ for all $x \in A$. We therefore obtain $u = ue' = ue = vae = va$, which means $\ell_M(r(a)) \subseteq Ma$. The converse inclusion is evident, so that $\ell_M(r(a)) = Ma$. Conversely, assume that $\ell_M(r(a)) = Ma$ for each $a \in A$. If $g: aA_A \rightarrow M_A$ is given, then it is obvious that $g(a) \in \ell_M(r(a)) = Ma$. Hence $g(ax) = vax$ with some $v \in M$.

COROLLARY 1 ([8, Theorem 1 (i)]). *If A is a right s -unital ring, then the following are equivalent:*

- 1) A_A is p -injective.
- 2) For each $a \in A$, $\ell(r(a)) = Aa$.
- 3) A is left s -unital, and every principal left ideal of A is a left annihilator.

COROLLARY 2. *Let A be a right s -unital ring. If aA_A is p -injective then a is von Neumann regular, and so both aA and $r(a)$ are direct summands of A_A .*

A right s -unital ring A is defined to be a right p - V -ring (resp. p - V' -ring) if every irreducible (resp. irreducible, singular) right A -module is p -injective (see [16]). A right A -module M is said to be *regular* if for each $u \in M$ there exists some $f: M_A \rightarrow A_A$ such that $uf(u) = u$. It is easy to see that A is a regular ring if and only if A is a left s -unital ring and A_A is regular.

We shall conclude this section with the following which includes [1, Theorem] and [17, Proposition 6, Theorem 7 and Corollary 8].

THEOREM 1. *The following are equivalent:*

- 1) A is Artinian, semi-primitive.
- 2) A is a semi-prime, right s -unital, right p -injective ring with maximum condition for right annihilators.
- 3) A is a semi-prime, left perfect ring.
- 4) A is a semi-prime ring satisfying the minimum condition and the maximum condition for principal right ideals.
- 5) A is a semi-prime ring satisfying the minimum condition for principal right ideals and the maximum condition for right annihilators.
- 6) A is a right p - V -ring with 1 all of whose maximal right ideals are right annihilators.
- 7) A is a semi-prime ring with 1 all of whose maximal right ideals are right annihilators.
- 8) A is a semi-prime right Goldie, right p - V' -ring with 1 and indecomposable right A -modules with the same associated prime ideal are isomorphic.
- 9) A is an s -unital ring such that A_A is finitely generated and every s -unital right A -module is regular.

10) A is an s -unital ring such that A_A is finitely generated and every irreducible right A -module is regular.

PROOF. 1) implies 2) through 8), and 9) implies 10). Moreover, 6) \Rightarrow 7) \Rightarrow 1) by [15, Propositions 6 and 7] and [9, Corollary 2]. Now, let A be a semi-prime ring with minimum condition for principal right (or left) ideals. Then, from the proof of [14, Satz 2], one can easily see that A coincides with its socle, and therefore, A is a regular ring. Hence, each of 3) and 4) implies 1). Noting that $r(\mathcal{L}(eA))=eA$ for any idempotent e , we obtain 5) \Rightarrow 1). (The implication 3) \Rightarrow 1) is also evident from the fact that any right T -nilpotent ideal of A is contained in its prime radical.)

2) \Rightarrow 1) By Corollary 1, every principal left ideal of A is a left annihilator. Since A satisfies the minimum condition for left annihilators, A does the minimum condition for principal left ideals. Hence, A coincides with its socle, and then A is a regular ring. Since $r(\mathcal{L}(eA))=eA$ for any idempotent e , A is Artinian.

8) \Rightarrow 1) According to [16, Corollary 3], the proof proceeds in the same way as in [17, Proposition 6].

1) \Rightarrow 9) Let M be an arbitrary unital right A -module. Given $u \in M$, uA is a direct summand of the projective module M_A . Since uA is A -homomorphic to A by $a \rightarrow ua$, it is almost evident that $uf(u)=u$ with some $f: M_A \rightarrow A_A$.

10) \Rightarrow 1) Since A is (left) s -unital and A_A is finitely generated, A contains a left identity e (Lemma 1). Let R be an arbitrary maximal right ideal of A , and M the irreducible right A -module $A/R = \bar{e}A$, where $\bar{e} = e + R$. By hypothesis, there exists $f: M_A \rightarrow A_A$ such that $\bar{e}f(\bar{e}) = \bar{e}$. If $g: A_A \rightarrow M_A$ is defined by $a \rightarrow \bar{e}a$ then $\text{Ker } g = r(\bar{e}) = R$. For each $a \in A$ we have then $gf(\bar{e}a) = gf(\bar{e})a = \bar{e}f(\bar{e})a = \bar{e}a$. Hence, $A = f(M) \oplus \text{Ker } g = f(M) \oplus R$, which means that A is a completely reducible module (of finite length). Recalling that A contains a left identity e , we readily see that A is Artinian, semi-primitive.

2. Regular rings. A right s -unital ring A is called a *right p.p. ring* if every $r(a)$ is a direct summand of A_A . A right s -unital ring A is called a *right CPP-ring* (resp. *CPF-ring*) if for each non-zero right ideal R of A either R is a direct summand of A_A (resp. R is a left s -unital ring) or A/R_A is p -injective (see [19]). As was noted in [15, Proposition 1], it is well known that a non-zero right ideal R of A with 1 is a left s -unital ring if and only if A/R_A is flat. It is easy to see that every regular ring is a right *CPP-ring* and every s -unital, right *CPP-ring* is a right *CPF-ring*. Moreover, every homomorphic image of a right *CPP-ring* (resp. *CPF-ring*) is also a right *CPP-ring* (resp. *CPF-ring*).

LEMMA 3. Let R_1 and R_2 be right ideals of a ring A such that $R_1 \cap R_2 = 0$.

(1) If $R_1 + R_2$ is a left s -unital ring then non-zero one of R_i 's is a left s -unital ring.

(2) Assume that A is right s -unital and A_A is p -injective. If the right A -module $A/(R_1 + R_2)$ is p -injective then each A/R_i is p -injective.

PROOF. (1) According to [15, Proposition 1], it suffices to prove that $R_1 \cap (a| = R_1(a|$ for each $a \in A$. Let b be an arbitrary element of $R_1 \cap (a|$. Since $R_1 \cap (a| \subseteq (R_1 + R_2) \cap (a| = (R_1 + R_2)a$, we have $b = (r_1 + r_2)a$ with some $r_i \in R_i$. Then $b - r_1a = r_2a \in R_1 \cap R_2 = 0$. Namely, $b \in R_1a$. This means $R_1 \cap (a| = R_1(a|$.

(2) According to Lemma 2, it suffices to prove that $\not\subset_{A/R_1}(r(a)) \subseteq (A/R_1)a$ for each $a \in A$. Let $b + R_1$ be an arbitrary element of $\not\subset_{A/R_1}(r(a))$. Since $\not\subset_{A/(R_1+R_2)}(r(a)) = (A/(R_1 + R_2))a$, we have $b = xa + r_1 + r_2$ with some $x \in A$ and $r_i \in R_i$. Then $r_2r(a) = (b - xa - r_1)r(a) \subseteq R_2 \cap (br(a) + r_1r(a)) \subseteq R_2 \cap R_1 = 0$, whence it follows that $r_2 \in \not\subset(r(a)) = Aa$ (Corollary 1). Hence, b is in $Aa + R_1$.

LEMMA 4. Let a be an element of a ring A .

(1) If $r(a)$ is a direct summand of A_A and A/aA_A is p -injective, then aA is idempotent.

(2) If $|a$ is idempotent and $(a|$ is a direct summand of an ideal I of A as a left A -module, then a is von Neumann regular.

PROOF. (1) There exists an element $b \in A$ such that $ab = a$ and $r(a) = r(b)$. Consider the A -homomorphisms $f: aA \rightarrow A/aA$ defined by $ax \rightarrow bx + aA$ and $g: A/aA \rightarrow A/(aA)^2$ defined by $y + aA \rightarrow ay + (aA)^2$. Since A/aA_A is p -injective, there exists an element $c \in A$ such that $bx + aA = cax + aA$. Then $ax + (aA)^2 = g(bx + aA) = g(cax + aA) = acax + (aA)^2 = (aA)^2$, which implies $aA = (aA)^2$.

(2) Let $I = (a| \oplus L$ with a left ideal L . Since $a \in |a)^4 \subseteq a(AaA) \subseteq aAI$, we have $a = a(ba + \not\subset)$ with some $b \in A$ and $\not\subset \in L$. Then $a - aba = a\not\subset \in (a| \cap L = 0$. Namely, $a = aba$.

COROLLARY 3. (1) If A is a right CPP-ring then A is a right $p.p.$ ring.

(2) If A is an s -unital, right CPP-ring then A is a fully right idempotent, right $p.p.$ ring.

PROOF. Let a be an arbitrary element of A . First, $r(a)$ is a direct summand of A_A . In fact, if $A/r(a)_A (\cong aA_A)$ is p -injective then a is von Neumann regular by Corollary 2, and so aA is idempotent. If aA is a direct summand of A_A and A is s -unital, then a is von Neumann regular by [16, Lemma 1 (3)]. Finally, if A/aA_A is p -injective then aA is idempotent by Lemma 4 (1).

COROLLARY 4. Let A be a right CPF-ring all of whose essential left ideals are two-sided, and a an element of A . If $r(a)$ is a direct summand of A_A then a is von Neumann regular.

PROOF. If aA is a left s -unital ring then a is evidently von Neumann regular.

If A/aA_A is p -injective then $(aA)^2 = aA$ by Lemma 4 (1). Choose a left ideal L such that $I = (a| \oplus L$ is essential in ${}_A A$. Since I is an ideal, a is von Neumann regular by Lemma 4 (2).

Now, we shall prove the principal theorem of this section.

THEOREM 2. *The following are equivalent:*

- 1) A is a regular ring.
 - 2) A is a right CPP-ring and ${}_A A$ is p -injective.
 - 3) A is a right $p.p.$ ring and A_A is p -injective.
 - 4) A is a right s -unital ring such that A_A and every singular homomorphic image of A_A are p -injective.
 - 5) Every essential right ideal of A is a left s -unital ring.
 - 6) A is an s -unital, right CPF-ring such that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.
 - 7) A is an s -unital ring such that for each essential right ideal R either R is a left s -unital ring or A/R_A is p -injective, and that every principal right ideal is either a direct summand of A_A or the right annihilator of an element.
- 2')-7') *The left-right analogues of 2)-7).*

PROOF. Evidently, 1) implies 2) through 6), and 6) does 7). Now, let a be an arbitrary non-zero element of A .

2) \Rightarrow 3) \Rightarrow 1) By Corollary 3 (1), 2) implies 3). Now, assume 3). Let $A = r(a) \oplus R$ with a right ideal R . Since $aA_A \simeq R_A$, there exists an element $b \in R$ such that $R = bA$ and $r(a) = r(b)$. Recalling that A_A is p -injective, we see that $bA_A (= R_A)$ is p -injective. Hence, by Corollary 2, $b = bcb$ with some c . Setting $e = cb$, Corollary 1 enables us to see that $(a| = Aa = \sphericalangle(r(a)) = \sphericalangle(r(b)) = \sphericalangle(r(e)) = Ae$, which means that A is a regular ring.

4) \Rightarrow 1) There exists a right ideal R such that $r(a) \cap R = 0$ and $r(a) + R$ is an essential right ideal. Since $A/(r(a) + R)_A$ is p -injective by 4), $aA_A \simeq A/r(a)_A$ is p -injective (Lemma 3 (2)). Hence, A is a regular ring by [16, Theorem 2].

5) \Rightarrow 1) There exists a right ideal R such that $|a) \cap R = 0$ and $|a) + R$ is an essential right ideal. Since $|a) + R$ is a left s -unital ring by 5), $|a)$ is a left s -unital ring (Lemma 3 (1)). Then, by [15, Proposition 1], $|a) \cap (a| = |a)(a|$, whence it follows that a is von Neumann regular.

7) \Rightarrow 1) If aA is a direct summand of A_A then a is von Neumann regular by [16, Lemma 1 (3)]. Henceforth we assume that $aA = r(t)$ with some $t \in A$. There exists a right ideal K such that $R = aA \oplus K$ is essential in A_A . If R is a left s -unital ring then aA is a left s -unital ring by Lemma 3 (1), and therefore a is von Neumann regular. In what follows, we consider the case A/R_A is p -injective. We can define $f: tA \rightarrow A/R_A$ by $tx \rightarrow x + R$. Then there exists an element $b \in A$ such that $btA + R = x + R$ for all $x \in A$. Since A is left s -unital, we can find an element e such that $ea = a$. We write $e - bte = ac + k$ ($c \in A, k \in K$). Then

$a = (e - bte)a = aca + ka$ ($\in aA \oplus K$), whence it follows that $a = aca$.

COROLLARY 5 ([19, Theorem 5]). *If A contains 1 then the following are equivalent:*

- 1) A is a regular ring.
- 2) A is a right CPP-ring and A_A is p -injective.
- 3) A is a right p . p . ring and A_A is p -injective.
- 4) A_A and every singular homomorphic image of A_A are p -injective.
- 5) Every singular homomorphic image of A_A is flat.
- 6) A is a right CPF-ring such that every principal right ideal is the right annihilator of an element.

7) Every singular homomorphic image of A_A is either p -injective or flat, and every principal right ideal is the right annihilator of an element.

2')-7') The left-right analogues of 2)-7).

3. V -rings. A right s -unital ring A is called a *right V -ring* if every irreducible right A -module is s -injective. A right s -unital ring A is a right V -ring if and only if every right ideal of A is an intersection of maximal right ideals ([15, Theorem 4]). If A is an s -unital, right V -ring then it is a right p - V -ring ([15, Proposition 4]). Now, we begin by improving [16, Corollary 6].

THEOREM 3 (cf. [19, Theorem 8]). *If A is s -unital, then the following are equivalent:*

- 1) A is a right V -ring.
- 2) A is a right p - V -ring and every right ideal R of A is an ideal of R^* .
- 3) Every minimal right ideal of A is s -injective and every singular homomorphic image of A_A is semi-simple.

PROOF. By [16, Proposition 1 (1) and Theorem 1], 1) implies 2) and 3).

2) \Rightarrow 1) Let R be an arbitrary right ideal of A . Suppose there exists some $b \in R^* \setminus R$, and set $T = R + bA$. There exists a right ideal S which is maximal with respect to the property that $R \subseteq S \subseteq T$. We consider the non-zero homomorphism $g: bA_A \rightarrow T/S_A$ defined by $bx \rightarrow bx + S$. Since T/S_A is p -injective, we can find an element $d \in T$ such that $bx + S = dbx + S$ for all $x \in A$. We define $h: A_A \rightarrow T/S_A$ by $x \rightarrow dx + S$. Since $dR \subseteq R^*R \subseteq R \subseteq S$, R is contained in the maximal right ideal $\text{Ker } h$, so that $R^* \subseteq \text{Ker } h$. But, this implies $bA = R^* \cap bA \subseteq \text{Ker } g$, a contradiction. We have therefore seen that $R^* = R$, proving 1).

3) \Rightarrow 1) Let M_A be irreducible, R an essential right ideal of A , and $h: R_A \rightarrow M_A$ a non-zero homomorphism. Obviously, $H = \text{Ker } h$ is a maximal right subideal of R . If H is not essential in R_A then M_A is isomorphic to some minimal right ideal which is s -injective by hypothesis. If H is essential in R_A , then so is it in A_A and there holds that $H^* = H$ and $R^* = R$. Since $H \subset R$, there exists a

maximal right ideal S of A such that $H \subseteq S$ but $R \not\subseteq S$. Now, it is easy to see that $S \cap R = H$ and $S + R = A$. Hence, in either case, h can be extended to some $g: A_A \rightarrow M_A$, proving 1) ([15, Proposition 3]).

COROLLARY 6. *If every essential left ideal of A is two-sided, then the following are equivalent:*

- 1) A is a regular ring.
- 2) A is an s -unital, right CPP-ring.
- 3) A is a right p - V -ring.
- 4) A is fully right idempotent.
- 5) A is a left s -unital ring such that if L is an essential left ideal with $L^* = L$ then L is a right s -unital ring and ${}_A A/L$ is p -injective.

PROOF. Obviously, 1) implies 2) through 5). By [15, Proposition 6], 3) implies 4).

2) \Rightarrow 4) \Rightarrow 1) By Corollary 3 (2) and Lemma 4 (2).

5) \Rightarrow 1) Let L be an essential left ideal of A . Then, the argument employed in the proof 2) \Rightarrow 1) of Theorem 3 enables us to see that $L^* = L$, and therefore L is a right s -unital ring. Hence, A is a regular ring by Theorem 2.

Combining Theorem 3 with Corollary 6, we readily see that any s -unital right V -ring all of whose essential left ideals are two-sided is a regular ring ([19, Corollary 11]). Furthermore, by Theorem 3 and Corollary 6 (and its proof), we can prove that an s -unital ring A all of whose essential one-sided ideals are two-sided is a right V -ring if and only if A is a regular ring whose minimal right ideals are s -injective ([19, Corollary 12]).

A right s -unital ring A is called a *right V_n -ring* if every irreducible right A -module U has the following property: For any right ideal R of A generated by n elements and $f: R_A \rightarrow U_A$ there exists an element $u \in U$ such that $f(x) = ux$ for all $x \in R$. The notion of a right V_1 -ring coincides with that of a right p - V -ring. If A is a right V_n -ring for all positive integers n then A is called a *right f - V -ring* (see [19]). It is easy to see that every non-zero right ideal of a right p - V -ring contains a maximal right subideal.

THEOREM 4. *Let A be an s -unital ring.*

(1) *A is a right V_n -ring if and only if every maximal right subideal I of any non-zero right ideal R of A generated by n elements is the intersection of I^* and R .*

(2) *If A is a right V_{n+1} -ring then any right ideal of A generated by n elements and its maximal right subideals are intersections of maximal right ideals. In particular, if A is a regular ring then any finitely generated right ideal of A and its maximal right subideals are intersections of maximal right ideals.*

(3) A is a right f - V -ring if and only if all the maximal right subideals of any non-zero finitely generated right ideal of A are intersections of maximal right ideals.

PROOF. (3) is only a combination of (1) and (2).

(1) First, we shall prove the if part. Let U be an irreducible right A -module, $R = a_1A + \cdots + a_nA$, and $f: R_A \rightarrow U_A$ a non-zero homomorphism. Since $F = \text{Ker } f$ is a maximal right subideal of R , by hypothesis there exists a maximal right ideal M such that $F \subseteq M$ but $R \not\subseteq M$. Then there holds $A/F = R/F \oplus M/F$, and it is easy to see that f can be extended to some $g: A_A \rightarrow U_A$. Now, by Lemma 1, there exists an element e such that $ea_i = a_i$. Setting $u = g(e)$, we have $f(x) = ux$ for all $x \in R$. Conversely, suppose that A is a right V_n -ring. Let R be a right ideal of A generated by n elements, and I a maximal right subideal of R . Taking $a \in R \setminus I$, we see that $I + aA = R$. Now, the natural homomorphism $R \rightarrow R/I$ can be extended to some $h: A_A \rightarrow R/I_A$. Then $\text{Ker } h$ is a maximal right ideal containing I and $a \notin \text{Ker } h$. This means $I = I^* \cap R$.

(2) Let R be a right ideal of A generated by n elements. In virtue of (1), it suffices to prove that $R^* = R$. Suppose, on the contrary, there exists some $b \in R^* \setminus R$. Then $T = R + aA$ is a right subideal of R^* generated by $n+1$ elements, and there exists a right ideal S which is maximal with respect to $R \subseteq S \subset T$. Now, the natural homomorphism $T \rightarrow T/S$ can be extended to some $h: A_A \rightarrow T/S_A$. Then $\text{Ker } h$ is a maximal right ideal containing R . Hence $R^* \subseteq \text{Ker } h$ and follows the contradiction $T = T \cap R^* \subseteq T \cap \text{Ker } h = S$.

4. Strongly regular rings. Many authors have given various conditions for a ring to be strongly regular (see, e.g., [2], [15] and [16]). In this section, to the list of equivalent conditions we shall add several news.

A ring A is said to be *semi-commutative* [12] if $xy=0$ implies $xAy=0$ ($x, y \in A$). Evidently, every reduced ring is semi-commutative and every semi-commutative ring is an N -ring in the sense of [7]. It is easy to see that A is semi-commutative if and only if every left annihilator in A is a two-sided ideal, or equivalently, if and only if every right annihilator in A is a two-sided ideal. As a consequence, the class of semi-commutative rings contains left duo rings and right duo rings.

LEMMA 5. *If A is s -unital and semi-commutative, then every idempotent of A is central.*

PROOF. Let e be an idempotent of A . Then $eAr(e) \subseteq er(e) = 0$ and $\ell(e)Ae \subseteq \ell(e)e = 0$. Hence, $eA \subseteq \ell(r(e)) = Ae$ and $Ae \subseteq r(\ell(e)) = eA$, whence it follows that $Ae = eA$ and e is central.

A right s -unital ring A will be called a *right CP^*F^* -ring* if A has the following

property: For each maximal essential right ideal R of A one of the following conditions is verified: (a) A/R_A is p -injective; (b) R is a left s -unital ring; (c) R is the right annihilator of a non-nilpotent element of A .

LEMMA 6. *Let A be a right CP^*F^* -ring. If a right ideal R of A contains $(a) + r(a) + \ell(a)$ with some $a \in A$, then R is a direct summand of A_A .*

PROOF. There exists a right ideal I such that $R \oplus I$ is essential in A_A . Suppose $R \oplus I \neq A$. Then, by [16, Lemma 1 (4)], $R \oplus I$ is contained in a maximal essential right ideal M . If A/M_A is p -injective, considering the homomorphism $aA_A \rightarrow A/M_A$ defined by $ax \rightarrow x + M$, we can find an element $b \in A$ such that $x + M = bax + M$ for all $x \in A$. This implies a contradiction $A = M$. Next, if M is a left s -unital ring, for each $x \in A$ there exists an element $y \in M$ such that $xa = yax$. Then $x - yx \in \ell(a) \subseteq M$, so that $x \in M$. This means $A = M$, a contradiction. Finally, if $M = r(t)$ with a non-nilpotent element t then $ta = 0$ yields $t \in \ell(a) \subseteq M = r(t)$, and therefore $t^2 = 0$. This is a contradiction.

COROLLARY 7. *If A is a right CP^*F^* -ring with 1, then $AaA + r(a) + \ell(a)A = A$.*

PROOF. If $R = AaA + r(a) + \ell(a)A$ then $R = eA$ with some idempotent e (Lemma 6). Since $a - ea \in R \cap r(e) = 0$, we obtain $1 - e \in \ell(a) \subseteq R$, which implies $1 \in R$.

THEOREM 5 (cf. [19, Theorem 13]). *The following are equivalent:*

- 1) A is a strongly regular ring.
 - 2) A is a reduced, right p - V -ring all of whose essential right ideals are two-sided.
 - 3) A is a left duo, right p - V -ring.
 - 4) A is an s -unital, semi-commutative, right V -ring, and $Aa \subseteq aA + r(a)$ for any $a \in A$.
 - 5) A is an s -unital, semi-commutative, right p - V -ring, and $Aa \subseteq aA + r(a)$ for any $a \in A$.
 - 6) A is an s -unital, semi-commutative, right V' -ring, and $Aa \subseteq aA + r(a)$ for any $a \in A$.
 - 7) A is an s -unital, semi-commutative, right p - V' -ring, and $Aa \subseteq aA + r(a)$ for any $a \in A$.
 - 8) A is a reduced, right CPF -ring all of whose maximal essential right ideals are two-sided.
 - 9) A is a reduced, right CP^*F^* -ring all of whose maximal essential right ideals are two sided.
 - 10) A is a right duo, right CP^*F^* -ring such that $r(a) \subseteq \ell(a)$ for each $a \in A$.
- 2')-10') The left-right analogues of 2)-10).

PROOF. Obviously, 1) \Rightarrow 8) \Rightarrow 9), 2) \Rightarrow 9), 4) \Rightarrow 6), and 5) \Rightarrow 7). By [16, Theorem 5], 1) implies 2) through 7) and 10). Moreover, 4) \Rightarrow 5) and 6) \Rightarrow 7) by [16, Proposition 1].

3) \Rightarrow 1) By [15, Proposition 6], A is fully right idempotent. For any $a \in A$, noting that $(a|$ is an ideal, we have $a \in (AaA) \subseteq aA(a|$. Hence, $a = aba$ with some $b \in A$, and A is strongly regular by [2, Theorem].

7) \Rightarrow 1) It suffices to prove that $A = aA + r(a)$ for any $a \in A$. Suppose $aA + r(a) \neq A$ for some a . Let R be a maximal right ideal containing $aA + r(a)$ (see [16, Lemma 1 (4)]). Then $A = R \oplus I$ with a minimal right ideal I ([16, Lemma 2]), and $I = eA$ with some idempotent e ([16, Lemma 1 (3)]). Hence, $I = eA$ is an ideal by Lemma 5. This enables us to see that $aI \subseteq aA \cap I = 0$. But we have then a contradiction $I \subseteq r(a) \subseteq R$.

9) \Rightarrow 1) Again, it suffices to prove that $A = aA + r(a)$ for every $a \in A$. Suppose $aA + r(a) \neq A$ for some a . Let R be a maximal right ideal containing $aA + r(a)$. First we claim that R is essential in A_A . In fact, if not, $A = R \oplus I$ with a minimal right ideal I . Since I is generated by some central idempotent, it follows that $I \subseteq r(a) \subseteq R$. Hence, R is essential in A_A , and therefore an ideal. Accordingly, R is a direct summand of A_A by Lemma 6. But, it is impossible.

10) \Rightarrow 1) Let a be an arbitrary element of A . Then, by Lemma 6, A is the direct sum of $aA + r(a)$ and some ideal I . Hence, we obtain $I \subseteq r(a) \cap I = 0$. Namely, $A = aA + r(a)$.

5. CPP-rings. In [5], a ring A with 1 is called a *right PCI-ring* if each proper cyclic right A -module is injective. A right *PCI-ring* is either Artinian, semi-primitive or a right semi-hereditary simple domain ([5, Theorem 14]). The existence of right *PCI*-domains which are not division rings is shown by Cozzens [3]. Recently, in [6], a ring $A \ni 1$ all of whose cyclic right A -modules are injective or projective has been characterized as $A = S \oplus T$ where S is Artinian, semi-primitive and T is 0 or a right *PCI*-domain. This is the primary motivation for the present section.

LEMMA 7. *Let A be a non-regular, right CPP-ring. If $A = R_1 \oplus R_2$ with right ideals R_1 and R_2 , then R_1 or R_2 is completely reducible.*

PROOF. Suppose neither R_1 nor R_2 is completely reducible. Then there exists a proper essential right subideal R'_i of R_i . Since R'_i cannot be a direct summand of A_A , A/R'_i is a p -injective right A -module. Obviously, $A/R'_i \simeq R_1/R'_1 \oplus R_2$ and $A/R'_2 \simeq R_1 \oplus R_2/R'_2$ as right A -modules. Hence, both R_{iA} are p -injective, so that A_A is p -injective. Combining this with the fact that A is a right $p.p.$ ring (Corollary 3 (1)), Theorem 2 shows that A is a regular ring, a contradiction.

THEOREM 6. *The following are equivalent:*

- 1) A is an s -unital, right CPP-ring.
- 2) A is a regular ring or $A = S \oplus T$ where S is a right (and left) completely reducible, semi-prime ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.

PROOF. It suffices to prove that 1) implies 2). Let S be the unique maximal regular ideal of A (see [9]). If A/S_A is p -injective then $A/S_{A/S}$ is p -injective, and hence A/S is a regular ring by Theorem 2, which means that A itself is regular. In what follows, we assume that $A = S \oplus T$ with a non-zero right ideal T . We claim here that T coincides with the ideal $\mathcal{L}(S)$. In fact, $T \subseteq \mathcal{L}(S)$. Since $(\mathcal{L}(S) \cap S)^2 = 0$ implies $\mathcal{L}(S) \cap S = 0$, we obtain $A = S \oplus \mathcal{L}(S)$ and $T = \mathcal{L}(S)$. If e is a non-zero idempotent of T with $eT \neq T$, then $A = S \oplus eT \oplus r_T(e)$. By Lemma 7, eT or $S \oplus r_T(e)$ is completely reducible, and so TeT or $Tr_T(e)$ is a completely reducible right T -module. Since T is an s -unital semi-prime ring by Corollary 3 (2), TeT or $Tr_T(e)$ is a non-zero regular ideal, a contradiction. Hence, T cannot be completely reducible. Consequently, S is a completely reducible semi-prime ring by Lemma 7. Now, let a be an arbitrary non-zero element of T . Then $T = r_T(a) \oplus bT$ with some $b \in T$ such that $ab = a$ and $r_T(a) = r_T(b)$. Since b is von Neumann regular by [16, Lemma 1 (3)], $r_T(a)$ must be 0. This means that T is a domain with identity (see [16, Lemma 1 (1)]). Since T is fully right idempotent by Corollary 3 (2), T is simple by [15, Proposition 7 (3)]. Finally, if R is a non-zero proper right ideal of T then R cannot be a direct summand of T_T , and so T/R_T is p -injective. Hence T/R_T is divisible by Lemma 2.

COROLLARY 8. *If A is a reduced ring then the following are equivalent:*

- 1) A is an s -unital, right CPP-ring.
- 2) A is a strongly regular ring or $A = S \oplus T$ where S is a direct sum of division rings and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.

COROLLARY 9. *If A contains 1, then the following are equivalent:*

- 1) A is a right CPP-ring.
- 2) A is a regular ring or $A = S \oplus T$ where S is an Artinian, semi-primitive ring and T is a simple domain (not a division ring) all of whose proper cyclic right modules are divisible.

COROLLARY 10 (see [6]). *If A contains 1, then the following are equivalent:*

- 1) Every cyclic right (unital) A -module is injective or projective.
- 2) $A = S \oplus T$ where S is an Artinian, semi-primitive ring and T is 0 or a simple, right semi-hereditary, right Ore domain (not a division ring) all of whose proper cyclic right modules are injective.

PROOF. Obviously, it suffices to show that 1) implies 2). By Corollary 9,

if A is not regular then $A = S \oplus T$ where S is Artinian, semi-primitive and T is a simple domain. Since T is a right semi-hereditary, right Ore domain by [5, Proposition 5 and Theorem 17], it remains only to prove that if A is regular then A is Artinian. Thus we assume henceforth A is a regular ring. (The subsequent proof will provide a shorter proof of [5, Proposition 12].) In the rest of our proof, we shall use freely the following result of Osofsky [13]: *Any right injective right PCI-ring is Artinian, semi-primitive.* We also claim that if $A = R_1 \oplus R_2$ with some right ideals R_i then R_1 or R_2 is completely reducible. In fact, this is obvious by the proof of Lemma 7 and the above result of Osofsky. Now, let S be the right socle of A , and assume that $S \neq A$. Then it is easy to see that S cannot be a direct summand of A_A . Hence, A/S is a division ring and S is essential in A_A . Since S_A cannot be finitely generated, we may distinguish between two cases.

Case I: There exists a homogeneous component of infinite length. Obviously, $S = S_1 \oplus S_2$ with some infinitely generated right ideals S_i such that $S_2 \simeq S$. Since $A/S_1 \simeq (S_1 \oplus S_2)/S_1 \simeq S_2 \simeq S$ as right A -modules and A/S_1 is injective, the injective hull $\hat{A}_A = \hat{S}_A$ is imbedded in A/S_1 . This implies $\hat{A} = \hat{x}A$ with some $\hat{x} \in \hat{A}$. We therefore obtain $\hat{A}/S = A/S$, and so $\hat{A} = A$. But then A is Artinian, a contradiction.

Case II: There exists no homogeneous component of infinite length. In this case, $S = S_1 \oplus S_2$ with some ideals S_i which are infinitely generated right ideals. Since A/S_1 is seen to be Artinian, this is impossible.

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