Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The CWI is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

CWI Tract 40

Regular variation, extensions and Tauberian theorems

J.L. Geluk, L. de Haan

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

1980 Mathematics Subject Classification: 26A12, 40E05.
ISBN 90 6196 324 9
NUGI-code: 811

Printed in the Netherlands

Contents

	page
Introduction	1
I. Regular variation and the class Π	2
I.1 Regularly varying functions	3
I.2 The class Π	15
I.3 The class Γ	34
I.4 Beurling slowly varying functions	48
I.5 Sequential versions of regular variation	54
I.6 Discussion	57
II. Transforms of regularly varying functions	59
II.1 The complementary function	59
II.2 The Laplace transform	66
II.3 General kernel transforms	92
II.4 Discussion	96
III. O-regular variation and O-versions of the class Π	97
III.1 O-regular variation	97
III.2 O-versions of the class Π: asymptotically balanced functions	108
III.3 Discussion	119
IV. Tauberian theorems for O-varying functions	120
IV.1 The Laplace transform	120
IV.2 General kernel transforms	125
IV.3 Discussion	128
References	129

Introduction

Functions of regular variation were invented by Karamata in 1930 as a suitable class of functions in connection with a Tauberian theorem for Laplace transforms. Many other applications are known. The present text intends to give a self-contained, smooth and coherent introduction to the theory of regular variation and its main extensions. Disregarding the possible applications we show how these classes of functions are a natural setting for Tauberian theorems of the Laplace type. Also some results are given for general kernel transforms.

In the text there is a clear separation between the various classes of functions. We have tried to stick to the main line of the theory putting little emphasis on various refinements, minimality of conditions and other specialized topics. The theory is built in circles. After a full treatment of regularly varying (RV) functions sections on the function classes Π and Γ follow. The theory of these function classes parallels closely the theory of regular variation.

Next (chapter 2) Tauberian theorems for Laplace transforms are treated in which these function classes (RV, Π and Γ) play a central role.

Finally (chapter 3 and 4) the theory is further extended. Here limits are replaced by upper and lower bounds. Chapter 3 gives the theory of these further generalizations of regular variation and in chapter 4 Tauberian theorems are given in which these generalizations play a central role.

Ideas and proofs from A.A. Balkema have been used in many places. We thank him for these contributions.

Elli Hoek van Dijke gave our poorly handwritten text the present form.

I. Regular variation and the class Π

One way to think about regular variation is as a derivative at infinity. For a real measurable function g write the differential quotient

$$\frac{g(y+h) - g(y)}{h} \tag{1.1}$$

where $h \neq 0$. Now we do not take the limit $h \to 0$ for fixed y as usual but take the limit $y \to \infty$ for fixed h. If this limit exists for all $h \neq 0$, then it follows (theorem 1.2 below) that the limit does not depend on h and we can write (see prop. 1.7.3) $g(y) = g_0(y) + o(1)$ $(y \to \infty)$ where g_0 is differentiable and

$$\lim_{y\to\infty} g_0'(y) = \lim_{y\to\infty} \frac{g(y+h) - g(y)}{h}.$$

If the limit in (1.1) as $y \to \infty$ exists, the function $f: \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(t) = \exp g(\log t)$ satisfies

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha \text{ for all } x \in \mathbb{R}^+ \tag{1.2}$$

for some $\alpha \in \mathbb{R}$. Then f is called a regularly varying function.
In this chapter these functions are studied thoroughly. Moreover we study the more general class of functions $f: \mathbb{R}^+ \to \mathbb{R}$ for which

$$\lim_{t\to\infty} \frac{f(tx) - b(t)}{a(t)} \text{ exists for all } x \in \mathbb{R}^+ \tag{1.3}$$

where $a > 0$ and b are suitable chosen auxiliary functions. The results for functions satisfying (1.3) are surprisingly similar to those for functions satisfying (1.2).
Finally a different variant of (1.2) is studied, namely non-decreasing functions $f: \mathbb{R} \to \mathbb{R}^+$ for which

$$\lim_{t\to\infty} \frac{f(t + x\,c(t))}{f(t)} \text{ exists and is positive for all } x \in \mathbb{R}, \tag{1.4}$$

where $c > 0$ is a suitable auxiliary function. Here again analogous properties are obtained. We shall see that the functions satisfying (1.4) are essentially inverses of the functions satisfying (1.3)
The chapter closes with a discussion of regularly varying sequences.

I.1. Regularly varying functions

Definition 1.1

A Lebesgue measurable function $f\colon \mathbb{R}^+ \to \mathbb{R}$ which is eventually positive is regularly varying (at infinity) if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha \quad (x > 0) \text{ for some } \alpha \in \mathbb{R}. \tag{1.5}$$

Notation: $f \in RV_\alpha^\infty$ or $f \in RV_\alpha$. ◇

We use the notation $f \in RV_\alpha^0$ if $g \in RV_{-\alpha}^\infty$ where $g(t) := f(1/t)$.
The number α in the above definition is called the index of regular variation.
A function satisfying (1.5) with $\alpha = 0$ is called slowly varying.

Examples

For α, $\beta \in \mathbb{R}$ the functions x^α, $x^\alpha(\log x)^\beta$, $x^\alpha(\log\log x)^\beta$ are elements of RV_α. The functions $2 + \sin\log\log x$, $\exp\{(\log x)^\alpha\}$ $(0 < \alpha < 1)$, $x^{-1}\log\Gamma(x)$, $\Sigma_{k\leq x} 1/k$, $(\log t)^{\sin\log\log t}$ are slowly varying. The functions $2 + \sin x$, $\exp[\log x]$, $2 + \sin\log x$, $x\exp\sin\log x$ are not regularly varying.

Our next result shows that it is possible to weaken the conditions in definition 1.1.

Theorem 1.2

Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable, eventually positive and

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} \tag{1.6}$$

exists, is finite and positive for all x in a set of positive Lebesgue measure, then $f \in RV_\alpha^\infty$ for some $\alpha \in \mathbb{R}$.

Proof

Define $F(t) := \log f(e^t)$. Then $\{\lim_{t\to\infty} F(t+x) - F(t)\}$ exists for all x in a set K of positive Lebesgue measure. Define $\Phi\colon K \to \mathbb{R}$ by $\Phi(x) := \lim_{t\to\infty}\{F(t+x) - F(t)\}$.
By Steinhaus' theorem (cf. Hewitt, Stromberg p. 143) the set $K-K := \{x-y;\ x, y \in K\}$ contains a neighbourbood of zero. Since K is an additive subgroup of $\mathbb{R}$, we have $K = \mathbb{R}$ and thus $\Phi(x)$ is defined for all $x \in \mathbb{R}$ and

$$\Phi(x+y) = \Phi(x) + \Phi(y) \text{ for all } x, y \in \mathbb{R}. \tag{1.7}$$

It remains to solve the equation (1.7) for measurable Φ:
Consider the restriction of Φ to an interval $L \subset \mathbb{R}$. By Lusin's theorem (cf. Halmos p. 242) there exists a compact set $M \subset L$ with positive Lebesgue measure λM such that the restriction of Φ to M is continuous. Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\Phi(y) - \Phi(x) \in (-\varepsilon, \varepsilon)$ whenever $x, y \in M$ and $|x-y| < \delta$ (since the restriction of Φ to M is uniformly continuous) and also such that M-M contains the interval $(-\delta, \delta)$ (by Steinhaus' theorem).
For each $s \in (-\delta, \delta) \subset M-M$ there exists $x_0 \in M$ such that also $x_0 + s \in M$. Then $\Phi(x+s) - \Phi(x) = \Phi(s) = \Phi(x_0 + s) - \Phi(x_0) \in (-\varepsilon, \varepsilon)$ for all $x \in \mathbb{R}$, hence Φ is uniformly continuous on $\mathbb{R}$.
Since $\Phi(n/m) = n\,\Phi(1/m) = n\,\Phi(1)/m$ for $n, m \in \mathbb{Z}$, $m \neq 0$, we have by the continuity of Φ, $\Phi(x) = \Phi(1)\,x$ for $x \in \mathbb{R}$. Now (1.5) follows. ◇

Theorem 1.3 (uniform convergence theorem)
If $f \in RV_\alpha^\infty$, then relation (1.5) holds uniformly for $x \in [a,b]$ with $0 < a < b < \infty$.

Proof
Without loss of generality we may suppose $\alpha = 0$ (if not, replace f(t) by $f(t)/t^\alpha$).
We define the function F by $F(x) := \ln f(e^x)$. It is sufficient to deduce a contradiction from the following assumption:
Suppose there exist $\delta > 0$ and sequences $t_n \to \infty$, $x_n \to 0$ $(n\to\infty)$ such that

$$|F(t_n + x_n) - F(t_n)| > \delta \text{ for } n = 1, 2, \ldots$$

For an arbitrary finite interval $J \subset \mathbb{R}$ we consider the sets

$$Y_{1,n} = \{y \in J; |F(t_n + y) - F(t_n)| > \tfrac{\delta}{2}\}$$

and

$$Y_{2,n} = \{y \in J; |F(t_n + x_n) - F(t_n + y)| > \tfrac{\delta}{2}\}.$$

The above sets are measurable for each n and $Y_{1,n} \cup Y_{2,n} = J$, hence either $\lambda(Y_{1,n}) \geq \frac{1}{2}\lambda(J)$ or $\lambda(Y_{2,n}) \geq \frac{1}{2}\lambda(J)$ (or both) where λ denotes Lebesgue measure.

Now we define

$$Z_n = \{z; |F(t_n + x_n) - F(t_n + x_n - z)| > \frac{\delta}{2}, x_n - z \in J\} = \{z; x_n - z \in Y_{2,n}\}.$$

Then $\lambda(Z_n) = \lambda(Y_{2,n})$ and thus we have either $\lambda(Y_{1,n}) \geq \frac{1}{2}\lambda(J)$ infinitely often or $\lambda(Z_n) \geq \frac{1}{2}\lambda(J)$ infinitely often (or both).
Since all the $Y_{1,n}$'s are subsets of a fixed finite interval we have $\lambda(\limsup_{n\to\infty} Y_{1,n}) = \lim_{k\to\infty} \lambda(\bigcup_{n=k}^{\infty} Y_{1,n}) \geq \frac{1}{2}\lambda(J)$ or a similar statement for the Z_n's (or both). This implies the existence of a real number x_0 contained in infinitely many $Y_{1,n}$ or infinitely many Z_n, which contradicts the assumption $\lim_{t\to\infty} F(t + x_0) - F(t) = 0$. ◇

Theorem 1.4 (Karamata's theorem)
Suppose $f \in RV_\alpha^\infty$.
There exists $t_0 > 0$ such that $f(t)$ is positive and locally bounded for $t \geq t_0$.
If $\alpha \geq -1$, then

$$\lim_{t\to\infty} \frac{t\, f(t)}{\int_{t_0}^{t} f(s)ds} = \alpha + 1. \tag{1.8}$$

If $\alpha < -1$, or $\alpha = -1$ and $\int_0^\infty f(s)ds < \infty$, then

$$\lim_{t\to\infty} \frac{t\, f(t)}{\int_{t}^{\infty} f(s)ds} = -\alpha - 1. \tag{1.9}$$

Conversely: if (1.8) holds with $-1 < \alpha < \infty$, then $f \in RV_\alpha^\infty$; if (1.9) holds with $-\infty < \alpha < -1$, then $f \in RV_\alpha^\infty$. ◇

Proof
Suppose $f \in RV_\alpha$.
By theorem 1.3, there exist t_0, c such that $f(tx)/f(t) < c$ for $t \geq t_0$, $x \in [1,2]$. Then for $t \in [2^n t_0, 2^{n+1} t_0]$ we have

$$\frac{f(t)}{f(t_0)} = \frac{f(t)}{f(2^{-1}t)} \cdot \frac{f(2^{-1}t)}{f(2^{-2}t)} \cdots \frac{f(2^{-n}t)}{f(t_0)} < c^{n+1}.$$

Hence $f(t)$ is locally bounded for $t \geq t_0$ and $\int_{t_0}^{t} f(s)ds < \infty$ for $t \geq t_0$.

In order to prove (1.8), we first show $\int_{t_0}^{\infty} f(s)\,ds = \infty$ for $\alpha > -1$.

Since $f(2s) \geq 2^{-1} f(s)$ for s sufficiently large, we have for $n \geq n_0$

$$\int_{2^n}^{2^{n+1}} f(s)ds = 2 \int_{2^{n-1}}^{2^n} f(2s)ds \geq \int_{2^{n-1}}^{2^n} f(s)ds. \text{ Hence}$$

$$\int_{2^{n_0}}^{\infty} f(s)\,ds = \sum_{n=n_0}^{\infty} \int_{2^n}^{2^{n+1}} f(s)ds \geq \sum_{n=n_0}^{\infty} \int_{2^{n_0}}^{2^{n_0+1}} f(s)ds = \infty.$$

Next we prove $F(t) := \int_{t_0}^{t} f(s)\,ds \in RV_{\alpha+1}$ for $\alpha > -1$. Fix $x > 0$. For arbitrary $\varepsilon > 0$ there exists $t_1 = t_1(\varepsilon)$ such that $f(xt) < (1+\varepsilon)\, x^{\alpha} f(t)$ for $t > t_1$. Since $\lim_{t\to\infty} F(t) = \infty$,

$$\frac{F(tx)}{F(t)} = \frac{\int_{t_0}^{tx} f(s)ds}{\int_{t_0}^{t} f(s)ds} \sim \frac{\int_{t_1 x}^{tx} f(s)ds}{\int_{t_1}^{t} f(s)ds} = \frac{x\int_{t_1}^{t} f(xs)ds}{\int_{t_1}^{t} f(s)ds} \quad (t\to\infty)$$

and hence

$$F(tx)/F(t) < (1+2\varepsilon)x^{\alpha+1} \tag{1.10}$$

for t sufficiently large. A similar lower inequality is easily derived and we obtain $F \in RV_{\alpha+1}$ for $\alpha > -1$.
In case $\alpha = -1$ and $F(t) \to \infty$ the same proof applies. If $\alpha = -1$ and $F(t)$ has a finite limit, obviously $F \in RV_0$.
Now

$$\frac{F(tx) - F(t)}{t\, f(t)} = \int_1^x \frac{f(tu)}{f(t)}\,du \to \frac{x^{\alpha+1} - 1}{\alpha + 1} \quad (t\to\infty) \tag{1.11}$$

by the uniform convergence theorem (theorem 1.3). Since $F \in RV_{\alpha+1}$, (1.8) follows. For the proof of (1.9) we first show the finiteness of the function G defined by

$$G(t): = \int_t^\infty f(s)ds.$$

Since, in case $\alpha < -1$, there exists $\delta > 0$ such that $f(2s) \leq 2^{-1-\delta} f(s)$ for s sufficiently large, we have for n_1 sufficiently large

$$\int_{2^{n_1}}^\infty f(s)\,ds = \sum_{n=n_1}^\infty \int_{2^n}^{2^{n+1}} f(s)\,ds \leq \sum_{n=n_1}^\infty 2^{-\delta(n-n_1)} \int_{2^{n_1}}^{2^{n_1+1}} f(s)ds < \infty.$$

The rest of the proof is analogous.

Conversely suppose (1.8) holds. Define

$$b(t): = t\, f(t)/F(t). \tag{1.12}$$

Without loss of generality we suppose $f(t) > 0$ $(t > 0)$.
Integrating both sides of $b(t)/t = f(t)/F(t)$ we find for some real c_1 and all $x > 0$ (note that log F is indeed an absolutely continuous function)

$$\int_1^x \frac{b(t)}{t}\,dt = \log F(x) + c_1 \tag{1.13}$$

(since the derivatives of the two parts exist and are equal a.e.). Using the definition of b again we find from (1.13)

$$f(x) = cb(x) \exp \left\{\int_1^x \frac{b(t) - 1}{t}\,dt\right\} \text{ for all } x > 0, \tag{1.14}$$

with $c = e^{-c_1} > 0$, hence for all $x, t > 0$

$$\frac{f(tx)}{f(t)} = \frac{b(tx)}{b(t)} \exp\{\int_1^x \frac{b(ts) - 1}{s}\,ds\}.$$

Now for arbitrary $\varepsilon > 0$ there is a t_0 such that $|b(ts) - \alpha - 1| < \varepsilon$ for $t \geq t_0$ and $s \geq \min(1, x)$. Hence the function f satisfies (1.5).
The last statement of the theorem ((1.9) implies $f \in RV_\alpha^\infty$) can be proved in a similar way. ◇

<u>Theorem 1.5</u> (representation theorem)
If $f \in RV_\alpha^\infty$, there exist measurable functions a: $\mathbb{R}^+ \to \mathbb{R}$ and c: $\mathbb{R}^+ \to \mathbb{R}$ with

$$\lim_{t\to\infty} c(t) = c_0 \quad (0 < c_0 < \infty) \text{ and } \lim_{t\to\infty} a(t) = \alpha \tag{1.15}$$

and $t_0 \in \mathbb{R}^+$ such that for $t > t_0$

$$f(t) = c(t) \exp \{\int_{t_0}^{t} \frac{a(s)}{s} ds\}. \tag{1.16}$$

Conversely if (1.16) holds with a and c satisfying (1.15), then $f \in RV_\alpha^\infty$.

<u>Proof</u>
Suppose $f \in RV_\alpha^\infty$.
The function $t^{-\alpha} f(t)$ is slowly varying and hence has a representation as in (1.16) by (1.14). Then f has such a representation with a(s) replaced by a(s) + α and c(t) replaced by t_0^α c(t). Now the result follows. Conversely one verifies directly that (1.5) follows from (1.16). ◇

<u>Remarks</u>
1) In formula (1.16) we may take $t_0 \in [0,\infty)$ arbitrarily by changing the functions c(t) and a(t) suitably on the interval $[0, t_0]$.
2) The functions a(t) and c(t) (given (1.16)) are not uniquely determined. It can easily be seen that it is possible to choose a(t) continuous: define

$$f_0(t): = \exp \{\int_{t_0}^{t} a(v)dv/v\} \text{ and } b_0(t): = t\, f_0(t) \Big/ \int_{t_0}^{t} f_0(s)ds.$$

Since $f_0 \in RV_\alpha$ we get (1.14) with f and b replaced by f_0 and b_0 respectively, i.e.

$$f(x) = c(x)\ c\ b_0(x) \exp[\int_1^x (b_0(t) -1)\ dt/t]$$

for all $x > 0$ with $b_0(t) - 1$ continuous.
It is possible to put all the undesirable behaviour of the function f into the function c(t). We will prove (cor. 2.16) that it is possible to construct a representation with $a \in C^\infty$.

We are going to list of number of consequences of the above theorems.

We need the following definition.

Definition 1.6

Suppose $f: (t_0, \infty) \to \mathbb{R}$ for some $t_0 \geq -\infty$ is bounded on intervals of the form (t_0, a) with $a < \infty$ and $\lim_{t\to\infty} f(t) = \infty$.
Since $\lim_{t\to\infty} f(t) = \infty$, the set $\{y;\ f(y) \geq x\}$ is non-empty for all $x \in \mathbb{R}$.
Hence $-\infty \leq \inf\{y;\ f(y) \geq x\} < \infty$ for $x \in \mathbb{R}$. Note that this infimum is non-decreasing in x. Since f is bounded on intervals of the form (t_0, a), $\lim_{x\to\infty} \inf\{y;\ f(y) \geq x\} = \infty$.
Hence there exists $x_0 \in \mathbb{R}$ such that $\inf\{y;\ f(y) \geq x\} > -\infty$ for all $x \geq x_0$. The generalized inverse function $f^{+}: (x_0, \infty) \to \mathbb{R}$ is defined by

$$f^{+}(x) := \inf\{y;\ f(y) \geq x\}. \qquad \diamond$$

Proposition 1.7 (properties of RV functions)

1. If $f \in RV_\alpha^\infty$ then $\log f(t)/\log t \to \alpha\ (t \to \infty)$. (1.17)

   This implies $\lim_{t\to\infty} f(t) = \begin{cases} 0 & \text{if } \alpha < 0 \\ \infty & \text{if } \alpha > 0 \end{cases}$.

2. If $f_1 \in RV_{\alpha_1}^\infty$, $f_2 \in RV_{\alpha_2}^\infty$, then $f_1 + f_2 \in RV_{\max(\alpha_1,\alpha_2)}^\infty$.

   If moreover $\lim_{t\to\infty} f_2(t) = \infty$, then the composition $f_1 \circ f_2 \in RV_{\alpha_1\alpha_2}^\infty$. (1.18)

3. If $f \in RV_\alpha^\infty$ with $\alpha > 0$ ($\alpha < 0$) then f is asymptotically equivalent to a strictly increasing (decreasing) differentiable function g with derivative $g' \in RV_{\alpha-1}$ if $\alpha > 0$ and $-\ g' \in RV_{\alpha-1}$ if $\alpha < 0$.
   As a consequence of this:
   If $f \in RV_\alpha$ ($\alpha > 0$) is bounded on finite intervals of $\mathbb{R}^+$, then

   $$\sup_{0<x\leq t} f(x) \sim f(t)\ (t\to\infty). \qquad (1.19)$$

   If $f \in RV_\alpha$ ($\alpha < 0$), then $\inf_{x\geq t} f(x) \sim f(t)\ (t\to\infty)$.

4. If $f \in RV_\alpha^\infty$ is integrable on finite intervals of $\mathbb{R}^+$ and $\alpha \geq -1$, then $\int_0^t f(s)\, ds$ is regularly varying with exponent $\alpha + 1$.

If $f \in RV_\alpha^\infty$ and $\alpha < -1$, then $\int_t^\infty f(s)\, ds$ exists for t sufficiently large and is regularly varying with exponent $\alpha + 1$. The same is true for $\alpha = -1$ provided $\int_1^\infty f(s)\, ds < \infty$.

5. Suppose $f \in RV_\alpha^\infty$. If $\delta_1, \delta_2 > 0$ are arbitrary, there exists $t_o = t_o(\delta_1, \delta_2)$ such that for $t \geq t_o$, $x \geq 1$

$$(1-\delta_1)\, x^{\alpha-\delta_2} < \frac{f(tx)}{f(t)} < (1 + \delta_1) x^{\alpha+\delta_2}. \tag{1.20}$$

Note that conversely if f satisfies the above property, then $f \in RV_\alpha^\infty$.

6. Suppose $f \in RV_\alpha^\infty$ is bounded on finite intervals of $\mathbb{R}^+$ and $\alpha > 0$. For $\xi > 0$ arbitrary there exist $c > 0$ and t_o such that for $t \geq t_o$ and $0 < x \leq \xi$

$$\frac{f(tx)}{f(t)} \leq c. \tag{1.21}$$

7. If $f \in RV_\alpha^\infty$, $\alpha \leq 0$ is bounded on finite intervals of $\mathbb{R}^+$ and δ, $\xi > 0$ arbitrary, there exist $c > 0$ and t_o such that for $t \geq t_o$ and $0 < x \leq \xi$

$$\frac{f(tx)}{f(t)} < cx^{\alpha-\delta}. \tag{1.22}$$

8. If $f(t) = \exp\left\{\int_0^t a(s)\, ds/s\right\}$ (1.23)

with a continuous $a(s) \to \alpha > 0$ $(s \to \infty)$, then $f^{\leftarrow} \in RV_{1/\alpha}$ where $f^{\leftarrow}$ is the inverse function of f.

9. Suppose $f \in RV_\alpha^\infty$, $\alpha > 0$, is bounded on finite intervals of $\mathbb{R}^+$. Then $f^+ \in RV_{1/\alpha}^\infty$. (Formally f^+ is only defined on a neighbourhood of infinity; we can extend its domain of definition by taking f^+ zero elsewhere).

In particular, if $f \in RV_\alpha$, $\alpha > 0$ and f is increasing, the inverse function $f^{\leftarrow}$ is in $RV_{1/\alpha}$.

10. If $f \in RV_\alpha^\infty$, $\alpha > 0$, there exists an asymptotically unique function h such that $f(h(x)) \sim h(f(x)) \sim x$ $(x \to \infty)$. Moreover $h \sim f^{\leftarrow}$ if f is bounded on finite intervals of $\mathbb{R}^+$.

11. If $f \in RV_\alpha^\infty$ $(\alpha \geq 0)$ and $f(t) = f(t_0) + \int_{t_0}^{t} \psi(s)ds$ for $t \geq t_0$ with ψ monotone, then

$$\lim_{t\to\infty} \frac{t\ \psi(t)}{f(t)} = \alpha.$$

Hence in case $\alpha > 0$ we have $\psi \in RV_{\alpha-1}^\infty$.

Moreover if $f \in RV_\alpha^\infty$ $(\alpha \leq 0)$ and $f(t) = \int_t^\infty \psi(s)ds < \infty$ with ψ nonincreasing, then $t\ \psi(t)/f(t) \to -\alpha$ $(t\to\infty)$. Hence in case $\alpha < 0$ we have $\psi \in RV_{\alpha-1}^\infty$.

12. Any $f \in RV_\alpha^\infty$ with $\alpha + 1 \in \mathbb{N}$ is asymptotic to a function f_1 with the property that the absolute values of all its derivatives are regularly varying.

Proof

ad 1,2,3,4,5. Properties 1, 3 and 5 follow immediately from the representation theorem (thm. 1.5). In order to prove regular variation of $|f'|$ in property 3 one also needs remark 2 following thm. 1.5. Properties 2 and 4 are easy consequences of the uniform convergence theorem (thm. 1.3) and theorem 1.4 respectively.

ad 6. Take $\xi > 0$. By property 5 there exists t_0' such that if $t \geq t_0'$

$$\frac{f(tx)}{f(t)} < 2\ x^{\alpha+1} \quad \text{for } x \geq 1.$$

Also, by property 3, if $t \geq t_0''$

$$\frac{f(tx)}{f(t)} \leq \frac{\sup_{u\leq t} f(u)}{f(t)} < 2 \text{ for } 0 < x < 1.$$

Hence, if $t > t_0 := \max(t_0', t_0'')$,

$$\frac{f(tx)}{f(t)} < \max(2, 2\,\xi^{\alpha+1}) \qquad \text{for } 0 < x \leq \xi.$$

ad 7. Apply property 6 above to the function $t^{-\alpha+\delta} f(t)$.

ad 8. Since $f(t) \to \infty$ $(t\to\infty)$ and f is eventually strictly increasing and differentiable, there exists - for x sufficiently large - a unique differentiable inverse function $g \equiv f^{\leftarrow}$ and

$$f(g(x)) = g(f(x)) = x \text{ for } x > x_0. \tag{1.24}$$

Differentiating the second equality in (1.24) we get using (1.23)

$$\frac{g'(f(x))\ f(x)}{g(f(x))} = \frac{1}{a(x)}. \tag{1.25}$$

Since f is continuous and $f(x) \to \infty$ $(x\to\infty)$, (1.25) implies

$$tg'(t)/g(t) \to \alpha^{-1} \ (t\to\infty).$$

Application of theorem 1.4 gives $g' \in RV_{-1+1/\alpha}$, hence $g = f^{\leftarrow} \in RV^{\infty}_{1/\alpha}$ by property 4 above.

ad 9. Suppose $f \in RV^{\infty}_{\alpha}$, $\alpha > 0$. By theorem 1.5 and the remarks thereafter f has the representation (1.16) with $t_0 = 1$ and a continuous. For arbitrary $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon)$ such that for $x > x_0$

$$(c_0 - \varepsilon)\ g(x) \leq f(x) \leq (c_0 + \varepsilon)\ g(x), \tag{1.26}$$

where $g(x) = \exp\{\int_1^x a(s)\ ds/s\}$.
The inequality (1.26) implies

$$g^{\leftarrow}(x/(c_0 - \varepsilon)) \geq f^{\leftarrow}(x) \geq g^{\leftarrow}(x/(c_0 + \varepsilon)) \tag{1.27}$$

for x sufficiently large. By property 8 above we have $g^{\leftarrow} \in RV^{\infty}_{1/\alpha}$.
Hence $g^{\leftarrow}(x/(c_0 \pm \varepsilon)) \sim (c_0 \pm \varepsilon)^{-1/\alpha} g^{\leftarrow}(x)$.
Since $\varepsilon > 0$ is arbitrary, (1.27) implies $f^{\leftarrow} \sim g^{\leftarrow} \in RV^{\infty}_{1/\alpha}$.

ad 10. Without loss of generality we may and do suppose f bounded on finite intervals of $\mathbb{R}^+$.

Then the proof of property 9 gives the existence of functions g and $g^{\leftarrow}$ such that $f(x) \sim g(x)$, $f^{+}(x) \sim g^{\leftarrow}(x)$ $(x\to\infty)$, $g(g^{\leftarrow}(x)) = g^{\leftarrow}(g(x)) = x$ for x sufficiently large.

This implies $x = g^{\leftarrow}(g(x)) \sim g^{\leftarrow}(f(x)) \sim f^{+}(f(x))$ $(x\to\infty)$, the first asymptotic equivalence follows from $f(x) \sim g(x)$ $(x\to\infty)$, $g^{\leftarrow} \in RV^{\infty}_{1/\alpha}$ and the uniform convergence theorem.

The statement $f(f^{+}(x)) \sim x$ $(x\to\infty)$ follows similarly.

Suppose now

$$f(h_i(x)) \sim h_i(f(x)) \sim x \ (x\to\infty) \text{ for } i = 1, 2.$$

Now $\lim_{n\to\infty} f(h_1(x_n))/f(h_2(x_n)) = \lim_{n\to\infty} \{h_1(x_n)/h_2(x_n)\}^{\alpha}$ for any sequence $x_n \to \infty$ by the uniform convergence theorem, hence $h_1(x) \sim h_2(x)$ $(x\to\infty)$.

ad 11. Suppose first ψ is non-decreasing and $f(t) = f(t_0) + \int_{t_0}^{t} \psi(s)ds$ for $t \geq t_0$. Then for $a > 1$ and $t \geq t_0$ we have

$$\frac{t(a-1)\ \psi(t)}{f(t)} \leq \int_1^a \frac{t\psi(tv)dv}{f(t)} = \frac{f(ta) - f(t)}{f(t)} .$$

Since $f \in RV^{\infty}_{\alpha}$ we find $\overline{\lim_{t\to\infty}} \frac{t\psi(t)}{f(t)} \leq \frac{a^{\alpha} - 1}{a-1}$ for all $a > 1$.

Letting $a \to 1$ we get

$$\overline{\lim_{t\to\infty}} \frac{t\psi(t)}{f(t)} \leq \alpha.$$

Similar inequalities for $0 < a < 1$ lead to $\underline{\lim_{t\to\infty}} \frac{t\ \psi(t)}{f(t)} \geq \alpha$.

The cases ψ non-increasing and $\alpha \leq 0$ can be proved similarly.

ad 12. This property will be proved in chapter 2 (see cor. 2.12). ◇

Remarks

1. There is no analogue of property 3 in case $\alpha = 0$; even if $\lim_{t\to\infty} f(t) = \infty$ with $f \in RV^{\infty}_{0}$, then f is not necessarily asymptotic to a non-decreasing function as the following example (due to Karamata) shows.

Define $f(x) := \exp\left(\int_0^x \varepsilon(s)ds/s\right)$, where

$$\varepsilon(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1 \\ a_n & \text{for } (2n-1)! < s \le (2n)!,\ n = 1, 2, 3, \ldots \\ -a_n/2 & \text{for } (2n)! < s \le (2n+1)!,\ n = 1, 2, 3, \ldots \end{cases}$$

where the sequence a_n is such that $a_n \to 0$ $(n\to\infty)$ and $a_n \log n \to \infty$ $(n\to\infty)$. Then

$$\sup_{0<x\le(2n+1)!} f(x)/f((2n+1)!) = f((2n)!)/f((2n+1)!)$$

$$= \exp\{-\int_{(2n)!}^{(2n+1)!} \varepsilon(s)ds/s\} = \exp\{+\frac{a_n}{2}\log(2n+1)\} \to \infty \ (n\to\infty).$$

Hence (1.13) does not hold.

2. Using the representation theorem for regularly varying functions, it is possible to show that if f is locally bounded and $f \in RV_0^\infty$, then the function $\sup_{0<x\le t} f(x)$ is slowly varying.

3. Note that $\int_0^t f(s)ds \in RV_{\alpha+1}$ with $\alpha > -1$ does not imply $f \in RV_\alpha$. Example: $f(t) = \exp[\log t]$.

4. Note that property 12 strengthens property 3.

The following result is a generalization of theorem 1.4 (the kernel function k below is constant in theorem 1.4). A converse statement (thm. 2.34) will be given in chapter 2.

<u>Theorem 1.8</u>

Let $f \in RV_\alpha^\infty$ and suppose f is (Lebesgue) integrable on finite intervals of $\mathbb{R}^+$.

(i) If $\alpha > -1$ and the function $k: \mathbb{R}^+ \to \mathbb{R}$ is bounded on (0,1), then

$$\lim_{t\to\infty} \int_0^1 k(s)\, f(ts)\, ds/f(t) = \int_0^1 k(s)s^\alpha ds. \tag{1.28}$$

(ij) If $t^{+\varepsilon+\alpha}\, k(t)$ is integrable on $(1,\infty)$ for some $\varepsilon > 0$, then $\int_1^\infty k(s)\, f(ts)\, ds < \infty$, for $t > 0$ and

$$\lim_{t\to\infty} \int_1^\infty k(s)\, f(ts)\, ds/f(t) = \int_1^\infty k(s) s^\alpha ds. \tag{1.29}$$

Proof

(i) Note that for $0 < \varepsilon < \alpha+1$ the function $t^{\alpha-\varepsilon}k(t)$ is integrable on $(0,1)$. Since there exists $c > 1$ and $\varepsilon > 0$ such that $f(tx)/f(t) \leq cx^{\alpha-\varepsilon}$ for $tx \geq t_0$, $0 < x \leq 1$ by Prop. 1.7.5, we can apply Lebesgue's dominated convergence theorem to obtain

$$\int_{t_0/t}^1 k(s)\, \frac{f(ts)}{f(t)}\, ds \to \int_0^1 k(s) s^\alpha ds,\ t\to\infty.$$

Furthermore

$$\left| \int_0^{t_0/t} k(s) \frac{f(ts)}{f(t)}\, ds \right| = (tf(t))^{-1} \int_0^{t_0} |\, k(s/t)\, f(s)| ds \to 0\ (t\to\infty)$$

since k is bounded and $tf(t) \to \infty$ $(t \to \infty)$.

(ij) The second statement is proved in a similar way. ◇

Remark

N.G. de Bruijn (1959) noted that for any slowly varying function L there exists an asymptotically unique slowly varying function L* called the conjugate slowly varying function satisfying $L(x)\, L^*(xL(x)) \to 1$,
$L^*(x)\, L(x\, L^*(x)) \to 1$ $(x\to\infty)$.
Note that one can obtain L* as follows: define $h(x) := xL(x)$. Then $L^*(x) \sim x^{-1}h^{+}(x)$ $(x\to\infty)$. In special cases one has $L^*(x) \sim 1/L(x)$ $(x\to\infty)$.
Example: $L(x) \sim (\log x)^\alpha (\log\log x)^\beta$ $(x\to\infty)$, $\alpha > 0$, $\beta \in \mathbb{R}$, i.e. if $h(x) \sim x(\log x)^\alpha (\log\log x)^\beta$, $(x\to\infty)$, then $h^{+}(x) \sim x(\log x)^{-\alpha} (\log\log x)^{-\beta}$, $x\to\infty$.
If we replace x by x^γ and take $\beta=0$, we find $f(x) \sim x^\gamma (\log x)^\delta$, $\gamma > 0$, $\delta \in \mathbb{R}$ implies $f^{+}(x) \sim \gamma^{\delta/\gamma}\, x^{1/\gamma}\, (\log x)^{-\delta/\gamma}$ $(x\to\infty)$.

I.2. The class Π

By way of introduction for the class Π which is a generalization of the class RV we formulate the RV property somewhat differently. A measurable function

$f: \mathbb{R}^+ \to \mathbb{R}$ is in RV if there exists a positive function a such that for all $x > 0$ the limit

$$\lim_{t\to\infty} \frac{f(tx)}{a(t)}$$

exists and is positive.

An obvious generalization is the following: Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable and there exist real functions $a > 0$ and b such that for all $x > 0$ the limit

$$\lim_{t\to\infty} \frac{f(tx) - b(t)}{a(t)} \tag{1.30}$$

exists and the limit function is not constant (this is to avoid trivialities). First note that (1.30) is equivalent to:

$$\psi(x) := \lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} \tag{1.31}$$

exists for all $x > 0$ with ψ not constant.

Next we identify the class of possible limit functions ψ.

Theorem 1.9

If $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable, a is positive. If (1.31) holds with ψ not constant, then

$$\psi(x) = c \cdot \frac{x^\rho - 1}{\rho} \quad (x > 0) \tag{1.32}$$

for some $\rho \in \mathbb{R}$, $c \neq 0$ (for $\rho = 0$ read $\psi(x) = c \log x$). Moreover (1.31) holds with a function a which is measurable and in RV_ρ.

Proof

Since ψ is not constant, there exists $x_0 > 0$ such that $\psi(x_0) \neq 0$. From (1.31) it follows that we can choose $a(t) = \{f(x_0t) - f(t)\}/\psi(x_0)$. Hence without loss of generality we may assume a to be measurable. For $y > 0$ arbitrary we have

$$\frac{a(ty)}{a(t)} = \left\{\frac{f(tx_0y) - f(t)}{a(t)} - \frac{f(ty) - f(t)}{a(t)}\right\} / \frac{f(tx_0y) - f(ty)}{a(ty)} \to$$

$$\to \frac{\psi(x_0y) - \psi(y)}{\psi(x_0)} \quad (t\to\infty).$$

Hence $A(y) := \lim_{t\to\infty} a(ty)/a(t)$ exists (and is non-negative) for all $y > 0$.

Since $\frac{a(txy)}{a(t)} = \frac{a(txy)}{a(tx)} \frac{a(tx)}{a(t)}$ we have

$$A(xy) = A(x) \,.\, A(y) \text{ for all } x,\ y > 0. \tag{1.33}$$

Since a is measurable the function A is measurable. Moreover the only measurable solutions of Cauchy's functional equation (1.33) are $A(y) = y^\rho$ for some $\rho \in \mathbb{R}$ (see the proof of theorem 1.2) and $A(y) = 0$ for $y > 0$.
However if $A(y) = 0$ for $y > 0$, then since $A(y)\ \psi(x) = \psi(xy) - \psi(y)$ for all x, $y > 0$, we have ψ is constant contrary to our assumption. Hence $a \in RV_\rho$ for some $\rho \in \mathbb{R}$. As a consequence we have

$$y^\rho \psi(x) = \psi(xy) - \psi(y) \text{ for all } x,\ y > 0 \tag{1.34}$$

If $\rho = 0$ we have Cauchy's functional equation again and $\psi(y) = c \log x$ for some $c \neq 0$, $x > 0$.
Next suppose $\rho \neq 0$. Interchanging x and y in (1.34) and subtracting the resulting relations we get

$$\psi(x)\ (1 - y^\rho) = \psi(y)\ (1 - x^\rho) \text{ for } x,\ y > 0.$$

Hence $\psi(x)/(1 - x^\rho)$ is constant, i.e. $\psi(x) = c \,.\, \frac{1 - x^\rho}{\rho}$ for $x > 0$, with $c \neq 0$.

◇

The following theorem states that for $\rho \neq 0$ relation (1.31) defines classes of functions we have met before. Note that it is sufficient to consider (1.32) with $c > 0$ since replacing f by -f in (1.31) changes the sign of c.

Theorem 1.10

Suppose the assumptions of theorem 1.9 are satisfied with $\rho \neq 0$ and $c > 0$. If $\rho > 0$ then $f \in RV_\rho^\infty$. If $\rho < 0$ then $f(\infty) := \lim_{x\to\infty} f(x)$ exists and $f(\infty) - f(x) \in RV_\rho^\infty$.

Proof

The proofs of theorem 1.14 and corollary 1.16 below can easily be adapted to show that if $\rho > 0$ ($\rho < 0$) there is a non-decreasing (non-increasing) function g such that

$$f(t) - g(t) = o(a(t)) \quad (t\to\infty). \tag{1.35}$$

Since we may assume $a \in RV_\rho$ (thm. 1.9) it follows that we also have

$$\lim_{t\to\infty} \frac{g(tx) - g(t)}{a(t)} = c\, \frac{x^\rho - 1}{\rho} \tag{1.36}$$

It will become apparent that it is sufficient to prove the theorem for g. Take $y > 1$ arbitrarily and define $t_1 = 1$ and $t_{n+1} = t_n y$ for $n = 1, 2, \ldots$ We have by (1.36)

$$\lim_{n\to\infty} \frac{g(t_{n+2}) - g(t_{n+1})}{g(t_{n+1}) - g(t_n)} = y^\rho. \tag{1.37}$$

Suppose $\rho > 0$. Then (1.37) immediately implies $g(t_n) \to \infty$ $(n\to\infty)$. Further for any $\varepsilon > 0$ there exists n_0 such that for any $n > n_0$

$$g(t_{n+2}) - g(t_{n_0+1}) = \sum_{k=n_0}^{n} \{g(t_{k+2}) - g(t_{k+1})\} <$$

$$< y^\rho(1+\varepsilon) \sum_{k=n_0}^{n} \{g(t_{k+1}) - g(t_k)\} = y^\rho(1+\varepsilon)\ \{(g(t_{n+1}) - g(t_{n_0})$$

and a similar lower inequality. It follows that

$$\lim_{n\to\infty} \frac{g(t_{n+1})}{g(t_n)} = y^\rho \tag{1.38}$$

and hence

$$a(t_n) \sim \frac{g(t_{n+1}) - g(t_n)}{c(y^\rho - 1)/\rho} \sim \frac{\rho}{c} \cdot g(t_n). \tag{1.39}$$

Further for $x > 1$

$$\frac{g(t_n x)}{g(t_n)} - 1 = \frac{g(t_n x) - g(t_n)}{g(t_n)} \sim \frac{g(t_n x) - g(t_n)}{c\ a(t_n)/\rho} \to x^\rho - 1 \quad (n\to\infty). \tag{1.40}$$

For any $s > 0$ choose $n(s) \in \mathbb{N}$ such that $t_{n(s)} \leq s < t_{n(s)+1}$. Then by (1.38) and (1.40)

$$\frac{g(sx)}{g(s)} \leq \frac{g(t_{n(s)+1}x)}{g(t_{n(s)+1})} \cdot \frac{g(t_{n(s)+1})}{g(t_{n(s)})} \to x^\rho y^\rho \quad (n\to\infty).$$

Similarly

$$\frac{g(sx)}{g(s)} \geq \frac{g(t_{n(s)}x)}{g(t_{n(s)})} \frac{g(t_{n(s)})}{g(t_{n(s)+1})} \to x^{\rho}y^{-\rho} \ (n\to\infty).$$

Since $y > 1$ is arbitrary, we have proved $g \in RV_{\rho}^{\infty}$.
Combination with (1.36) gives $a(t)/g(t) \to \rho/c$ $(t\to\infty)$. With (1.35) this implies $f(t) \sim ca(t)/\rho$ $(t\to\infty)$ hence $f \in RV_{\rho}^{\infty}$.

Suppose next $\rho < 0$. Then (1.37) immediately implies $\lim_{n\to\infty} g(t_n) < \infty$. Write $h(x) := \lim_{t\to\infty} g(t) - g(x)$. We have

$$\frac{h(t_n)}{a(t_n)} = \sum_{k=n}^{\infty} \frac{g(t_{k+1}) - g(t_k)}{a(t_k)} \frac{a(t_k)}{a(t_n)}.$$

Choose $\varepsilon > 0$ and $y > (1+\varepsilon)^{-1/\rho}$. Note that since $a \in RV_{\rho}$ the above expression is bounded above for $n \geq n_0$ by

$$\sum_{k=n}^{\infty} c \frac{y^{\rho} - 1}{\rho} (1+\varepsilon) \{y^{\rho}(1+\varepsilon)\}^{k-n} = \frac{y^{\rho} - 1}{1 - y^{\rho}(1+\varepsilon)} \cdot \frac{1+\varepsilon}{\rho} c,$$

which tends to $-c/\rho$ as $\varepsilon \to 0+$. A similar lower bound is easily obtained and we conclude

$$\lim_{n\to\infty} \frac{h(t_n)}{a(t_n)} = -\frac{c}{\rho}.$$

Further for $x > 1$

$$\frac{h(t_{n+1})}{h(t_n)} - 1 = \frac{a(t_n)}{h(t_n)} \cdot \frac{h(t_{n+1}) - h(t_n)}{a(t_n)} \to y^{\rho} - 1 \ (n\to\infty).$$

The rest of the proof follows closely the case $\rho > 0$. ◊

<u>Definition 1.11</u>

A measurable function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is said to belong to the class Π if there exists a function $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for $x > 0$

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = \log x. \tag{1.41}$$

Notation: $f \in \Pi$ or $f \in \Pi(a)$.

The function a is called an auxiliary function for f.
We say that $f \in \Pi^0$ if $g \in \Pi$ where $g(t) = f(1/t)$. ◇

Remarks

1. Note that any positive function a_1 is an auxiliary function for f if and only if $a_1(t) \sim a(t)$ $(t\to\infty)$.

2. For the definition of Π it is sufficient to require (1.41) for all x in a set A satisfying the following requirements: $\lambda(A) > 0$ and there exists a sequence $x_n \in A$ $(n = 1, 2, \ldots)$ such that $x_n \to 1$ $(n\to\infty)$.

3. We can weaken the definition as follows: there exist functions $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ such that for $x > 0$

$$\lim_{t\to\infty} \frac{f(tx) - g(t)}{a(t)} = \log x.$$

Theorem 1.12

If $f \in \Pi(a)$, then $\lim_{t\to\infty} a(tx)/a(t) = 1$ for all $x > 0$. Moreover (1.41) holds with a function a which is measurable and hence in RV_0^∞.

Proof

This is a special case of theorem 1.9. ◇

Theorem 1.13

If $f \in \Pi(a)$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable and satisfies

$$\lim_{t\to\infty} \frac{f(t) - g(t)}{a(t)} = c \tag{1.42}$$

for some $c \in \mathbb{R}$, then (1.41) is satisfied with f replaced by g, hence $g \in \Pi(a)$. ◇

This follows immediately from (1.24) and (1.25). Obviously for fixed auxiliary function a the relation (1.25) between functions $f, g \in \Pi(a)$ is an equivalence relation. We shall see below (proposition 1.17.3 and 6) that any equivalence class contains a very smooth Π-function.

Theorem 1.14 (uniform convergence theorem)
If $f \in \Pi$, then for $0 < a < b < \infty$ relation (1.41) holds uniformly for $x \in [a, b]$.

Proof
Define $F(t) := f(e^t)$, $A(t) := a(e^t)$. It is sufficient to deduce a contradiction from the following assumption: there exist $\delta > 0$ and sequences $t_n \to \infty$, $x_n \to 0$ $(n \to \infty)$ such that for all n

$$\left| \frac{F(x_n + t_n) - F(t_n)}{A(t_n)} \right| > \delta.$$

Consider the sets

$$J := [-\delta/5, +\delta/5],$$
$$Y_{1,n} = \{y; |(F(t_n + y) - F(t_n))/A(t_n)| > \delta/2, \ y \in J\},$$
$$Y_{2,n} = \{y; |(F(t_n + x_n) - F(t_n + y))/A(t_n)| > \delta/2, \ y \in J\}.$$

The above sets are measurable for each n and $Y_{1,n} \cup Y_{2,n} = J$, hence either $\lambda(Y_{1,n}) \geq \frac{1}{2} \lambda(J)$ or $\lambda(Y_{2,n}) \geq \frac{1}{2} \lambda(J)$ (or both), where λ denotes Lebesgue measure. Define

$$Z_{1,n} = \{z; |(F(t_n + x_n) - F(t_n + x_n - z))/A(t_n)| > \delta/2, \ x_n - z \in J\}.$$

Then $\lambda(Z_{1,n}) = \lambda(Y_{2,n})$.

Since $a \in RV_0$ (theorem 1.9) we have the inequality $A(t_n) \geq \frac{1}{2} A(t_n + x_n - z)$ for $z \in Z_{1,n}$ and $n \geq n_0$ by proposition 1.7.5. As a consequence $Z_{1,n} \subset Z_{2,n}$ for $n \geq n_0$, where $Z_{2,n}$ is defined by

$$Z_{2,n} := \{z; |(F(t_n + x_n) - F(t_n + x_n - z))/ A(t_n + x_n - z)| > \delta/4, \ x_n - z \in J\}$$

$\subset [-\delta/4, +\delta/4]$ for n sufficiently large since $x_n \to 0$.
Hence we find $\lambda(\limsup_{n\to\infty} Z_{2,n}) \geq \lambda(\limsup_{n\to\infty} Z_{1,n}) \geq \frac{1}{2} \lambda(J)$ or $\lambda(\limsup_{n\to\infty} Y_{1,n}) \geq \frac{1}{2} \lambda(J)$.

This implies the existence of a real number x_0 contained in infinitely many $Y_{1,n}$ or infinitely many $Z_{2,n}$ which contradicts the assumption $\lim_{t\to\infty}\{F(t + x_0) - F(t)\}/A(t) = x_0$. ◇

Corollary 1.15

If $f \in \Pi(a)$, for any $\varepsilon > 0$ there exist t_0, $c > 0$ such that for $t \geq t_0$, $x \geq 1$

$$\left|\frac{f(tx) - f(t)}{a(t)}\right| \leq cx^{\varepsilon}. \qquad (1.43)$$

Hence $f(t)$ is locally bounded for $t \geq t_0$.

Proof

By the uniform convergence theorem (theorem 1.14) we have

$$-2 \leq \frac{f(tu) - f(t)}{a(t)} \leq 2 \text{ for } t \geq t_1 \text{ and } 1 \leq u \leq e. \qquad (1.44)$$

For $x > 1$ define $n \in \mathbb{N}$ by $e^n \leq x < e^{n+1}$. Then

$$\frac{f(tx) - f(t)}{a(t)} = \sum_{k=0}^{n-1} \frac{f(e^{k+1}t) - f(e^k t)}{a(e^k t)} \frac{a(e^k t)}{a(t)}$$

$$+ \frac{f(tx) - f(e^n t)}{a(e^n t)} \frac{a(e^n t)}{a(t)}.$$

Using (1.44) and the inequality $a(tx)/a(t) \leq c_1 x^{\varepsilon}$ for some $c_1 > 0$, $t \geq t_2$ (prop. 1.7.5) we find that for $t \geq t_0 =: \max(t_1, t_2)$

$$\left|\frac{f(tx) - f(t)}{a(t)}\right| \leq 2c_1 \sum_{k=0}^{n} e^{\varepsilon k} \leq ce^{n\varepsilon} \leq cx^{\varepsilon}.$$

For the last statement, take $t = t_0$ in (1.43). ◇

Corollary 1.16

If $f \in \Pi(a)$, there exists a non-decreasing function g such that $f(t) - g(t) = o(a(t))$ $(t\to\infty)$. In particular $g \in \Pi(a)$ by theorem 1.13.

Proof

By corollary 1.15 the function f is locally integrable on $[t_0, \infty)$. Note that by theorem 1.14

$$\lim_{t\to\infty} \int_1^e \frac{f(tx) - f(t)}{a(t)} \frac{dx}{x} = \int_1^e \log x \frac{dx}{x} = \frac{1}{2}. \tag{1.45}$$

Now choose $t_1 \geq t_0$ such that $f(ex) - f(x) > 0$ for $x > t_1$.
Then

$$\int_1^e \frac{f(tx)}{x} dx = \int_{t_1}^{te} \frac{f(x)}{x} dx - \int_{t_1}^{t} \frac{f(x)}{x} dx$$

$$= \int_{t_1}^{et_1} \frac{f(x)}{x} dx + \int_{t_1}^{t} \frac{f(ex) - f(x)}{x} dx =: g_0(t).$$

Note that g_0 is non-decreasing and by (1.45)

$$\lim_{t\to\infty} \frac{g_0(t) - f(t)}{a(t)} = \frac{1}{2}.$$

Now $g_0 \in \Pi(a)$ by theorem 1.13. Define $g(t) := g_0(te^{-\frac{1}{2}})$. Then $g \in \Pi(a)$ and $g(t) - f(t) = o(a(t))$ $(t\to\infty)$. ◇

The following theorem gives a characterization of the class Π.

<u>Theorem 1.17</u>

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable.
For $t_0 \geq 0$ let $\psi: (t_0, \infty) \to \mathbb{R}$ be defined by

$$\psi(t) := f(t) - t^{-1} \int_{t_0}^{t} f(s)ds. \tag{1.46}$$

The following statements are equivalent:

a. $f \in \Pi$. (1.47)

b. The function $\psi:(t_0, \infty) \to \mathbb{R}$ is well-defined for some $t_0 \geq 0$, eventually positive and

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{\psi(t)} = \log x \tag{1.48}$$

for $x > 0$.

c. The function $\psi:(t_0, \infty) \to \mathbb{R}$ is well-defined for $t \geq t_0$ and slowly varying at infinity. (1.49)

d. There exists $\rho \in RV_0^\infty$ such that

$$f(t) = \rho(t) + \int_{t_0}^{t} \rho(s)ds/s. \tag{1.50}$$

e. There exist c_1, $c_2 \in \mathbb{R}$, a_1, $a_2 \in RV_0^\infty$ with $a_1(t) \sim a_2(t)$ $(t\to\infty)$ such that

$$f(t) = c_1 + c_2 a_1(t) + \int_1^t a_2(s)\, ds/s. \tag{1.51}$$

If f satisfies (1.50) (or (1.51)) then $f \in \Pi(\rho)$ (or $f \in \Pi(a_2)$ respectively). Hence $\rho(t) \sim a_2(t) \sim \psi(t)$ $(t\to\infty)$.

Proof

$a \to b$

Suppose $f \in \Pi(a)$.
Take t_0 as in cor. 1.15. Then $\psi(t)$ is well-defined for $t \geq t_0$.
Note that for $t \geq t_0$

$$\frac{\psi(t)}{a(t)} = \frac{t_0\, f(t)}{t\, a(t)} + \int_{t_0/t}^{1} \frac{f(t) - f(tu)}{a(t)}\, du. \tag{1.52}$$

From cor. 1.15 it follows that $f(t) = o(t^\beta)$ $(t\to\infty)$ for any $\beta > 0$ (take $t = t_0$ in (1.43)).
Since $ta(t) \in RV_1^\infty$, (thm. 1.12) we have $f(t) = o(ta(t))$ $(t\to\infty)$.
We can apply Lebesgue's theorem on dominated convergence to the second term on the right-hand side in (1.52) since by cor. 1.15 for $tu \geq t_0$, $0 < u \leq 1$

$$\left|\frac{f(tu/u) - f(tu)}{a(tu)}\right| \leq cu^{-\varepsilon}$$

and by prop. 1.7.5 for $tu \geq t_1$, $0 < u \leq 1$

$$0 < a(tu)/a(t) \leq c_1 u^{-\varepsilon}.$$

Hence $\lim_{t\to\infty} \frac{\psi(t)}{a(t)} = -\int_0^1 \log u\; du = 1$, which proves the implication $a \to b$.

$b \to c$

See theorem 1.12.

$c \to d$

By Fubini's theorem we have

$$\int_{t_0}^{t} \frac{\psi(s)}{s}\,ds = \int_{t_0}^{t} \frac{f(s)}{s}\,ds - \int_{t_0}^{t}\int_{t_0}^{s} \frac{f(u)}{s^2}\,du\,ds$$

$$= \frac{1}{t}\int_{t_0}^{t} f(u)\,du = f(t) - \psi(t).$$

Hence (1.50) with $\rho = \psi$.

e → a

By the uniform convergence theorem (thm. 1.3) for functions in RV

$$\frac{f(tx) - f(t)}{a_2(t)} = c_2\left\{\frac{a_1(tx)}{a_1(t)} - 1\right\} \frac{a_1(t)}{a_2(t)} + \int_1^x \frac{a_2(tu)}{a_2(t)} \frac{du}{u} \to \log x \quad (t\to\infty)$$

for all $x > 0$. ◇

Corollary 1.18

If $f \in \Pi$, then $\lim_{t\to\infty} f(t) =: f(\infty) \leq \infty$ exists.

If the limit is infinite, then $f \in RV_0^\infty$. If the limit is finite, $f(\infty) - f(t) \in RV_0^\infty$.

Proof

Consider the representation (1.50). Theorem 1.4 implies that $\rho(t) = o(\int_1^t \rho(s)\,ds/s)$ $(t\to\infty)$. Hence, if $\int_1^\infty \rho(s)\,ds/s < \infty$, $\rho(t) \to 0$ $(t\to\infty)$ and $\lim_{t\to\infty} f(t) = c + \int_1^\infty \rho(s)\,ds/s$. Then $f(\infty) - f(t) = \int_t^\infty \rho(s)\,ds/s \in RV_0^\infty$ (prop. 1.7.4). If $\int_1^\infty \rho(s)\,ds/s = \infty$, then $f(t) \sim \int_1^t \rho(s)\,ds/s \in RV_0^\infty$ (prop. 1.7.4). ◇

Remarks

1. Note that from the proof of cor. 1.18 it follows, using (1.46), $\psi(t) \sim a(t)$ $(t\to\infty)$ and theorem 1.4, that $a(t) = o(f(t))$ $(t\to\infty)$. As a consequence, the limit relation (1.42) above is strictly stronger than $f(t) \sim g(t)$ $(t\to\infty)$.

2. Theorem 1.17 is also true - and the proof not much different - with ψ replaced by ϕ defined as follows:

$$\phi(t) := t\int_t^\infty f(u)\,\frac{du}{u^2} - f(t).$$

3. The result of cor. 1.16 is reobtained from theorem 1.17 by taking $g(t) = \int_{t_0}^{et} \rho(s)\,ds/s$ with ρ as in (1.50).

4. Suppose f is locally integrable on $\mathbb{R}^+$ and $a \in RV_0$. Then

$$\frac{f(tx) - f(t)}{a(t)} \to 0 \qquad (t\to\infty) \qquad \text{for } x > 0 \tag{1.53}$$

and

$$\frac{f(t) - t^{-1}\int_0^t f(s)\,ds}{a(t)} \to 0 \qquad (t\to\infty) \tag{1.54}$$

are equivalent.
The proof follows closely the proof of theorem 1.17.

5. From theorem 1.17e it is clear that for any $a \in RV_0^\infty$, there exists a function f such that $f \in \Pi(a)$.

6. Let $t_1 \geq 0$ be such that f is locally integrable on (t_1, ∞). Then theorem 1.17 holds for any $t_0 \geq t_1$.

We mention some properties of functions which belong to the class Π.

<u>Proposition 1.19</u>

1. If $f, g \in \Pi$ then $f + g \in \Pi$. If $f \in \Pi$, and $h \in RV_\alpha^\infty$, $\alpha > 0$, then $f \circ h \in \Pi$. If $f \in \Pi$, $\lim_{t\to\infty} f(t) = \infty$ and h is differentiable with $h' \in RV_\alpha^\infty$ $(\alpha > -1)$, then $h \circ f \in \Pi$, where $h \circ f$ denotes the composition of the two functions.

2. If $f \in \Pi$ (a) is integrable on finite intervals of $\mathbb{R}^+$ and the function f_1 is defined by

$$f_1(t) := t^{-1} \int_0^t f(s)\,ds \quad (t > 0), \tag{1.55}$$

then $f_1 \in \Pi(a)$.
Conversely if $f_1 \in \Pi(a)$ and f is non-decreasing, then $f \in \Pi(a)$.

3. If $f \in \Pi(a)$, there exists a twice differentiable function $\bar{f}$ with $-\bar{f}'' \in RV_{-2}^\infty$ such that

$$\lim_{t\to\infty} \frac{f(t) - \bar{f}(t)}{a(t)} = 0. \tag{1.56}$$

In particular $\bar{f}$ is eventually concave. As a consequence of this:

If $f \in \Pi$ is bounded on finite intervals of $\mathbb{R}^+$ and $\lim_{t\to\infty} f(t) = \infty$, then $\sup_{0<x\leq t} f(x) - f(t) = o(a(t))$ $(t\to\infty)$.

4. Suppose $f \in \Pi$ (a). For arbitrary δ_1, $\delta_2 > 0$ there exists $t_0 = t_0$ (δ_1, δ_2) such that for $x \geq 1$, $t \geq t_0$

$$(1-\delta_2)\,\frac{1 - x^{-\delta_1}}{\delta_1} - \delta_2 < \frac{f(tx) - f(t)}{a(t)} < (1 + \delta_2)\,\frac{x^{\delta_1} - 1}{\delta_1} + \delta_2. \quad (1.57)$$

Note that conversely if f satisfies the above property, then $f \in \Pi(a)$.

5. Suppose

$$f(t) = f(t_0) + \int_{t_0}^{t} g\,(s)ds,\ t > t_0 \quad (1.58)$$

with $g \in RV^{\infty}_{-1}$. Then $f \in \Pi$. Conversely if $f \in \Pi$ satisfies (1.58) with g non-increasing, then $g \in RV^{\infty}_{-1}$.
Moreover in this case tg(t) is an auxiliary function for f.
Similarly if

$$f(t) = c + \int_{t}^{\infty} g\,(s)\,ds \quad (1.59)$$

with $g \in RV^{0}_{-1}$, then $f \in \Pi^0$ (see def. 1.11). Conversely if $f \in \Pi^0$ satisfies (1.59) with g non-increasing, then $g \in RV^{0}_{-1}$.
Moreover in this case $t^{-1}g(t^{-1})$ is an auxiliary function for $f(t^{-1})$.
This property supplements a corresponding statement for functions in RV^{∞}_{α}, $\alpha \neq 0$ (cf. prop. 1.7.11).

6. If $f \in \Pi(a)$ there is a function f_1 with $(-1)^{n+1}f_1^{(n)} \in RV^{\infty}_{-n}$ for $n = 1,2,\ldots$ such that $f_1(t) - f(t) = o(a(t))$, $t\to\infty$.

Proofs

ad 1. The statement $f + g \in \Pi$ is a consequence of the representation (1.50) since the sum of two slowly varying functions is slowly varying (see proposition 1.7.2).
If $f \in \Pi(a)$ and $h \in RV^{\infty}_{\alpha}$, then for $x > 0$ we have

$$\lim_{t\to\infty} \frac{f\left(\frac{h(tx)}{h(t)}\,h(t)\right) - f(h(t))}{\alpha\, a(h(t))} = \log x$$ by the uniform convergence

theorem (thm. 1.14).

For the last statement we expand the function h:

$$\frac{h(f(tx)) - h(f(t))}{a(t)h'(f(t))} = \frac{f(tx) - f(t)}{a(t)} \cdot \frac{h'(f(t) + \theta\{f(tx) - f(t)\})}{h'(f(t))}$$

for some $0 < \theta = \theta(x, t) < 1$. Now the second factor on the right-hand side tends to 1 as $t\to\infty$ since $h' \in RV_\alpha^\infty$ and $f \in RV_0^\infty$ (see corollary 1.18) by the uniform convergence theorem (theorem 1.14).

ad 2. Define $\psi(t) := f(t) - t^{-1} \int_0^t f(s)\, ds$ for $t > 0$. If $f \in \Pi(a)$, we have by theorem 1.17

$$\lim_{t\to\infty} \frac{f(t) - f_1(t)}{a(t)} = \lim_{t\to\infty} \frac{\psi(t)}{a(t)} = 1.$$

As a consequence $f_1 \in \Pi(a)$ (see theorem 1.13).

Conversely suppose $f_1 \in \Pi(a)$. Then for $x > 0$ we have by definition $\int_0^t \psi(s)\, ds/s = f_1(t)$ and hence

$$\frac{f_1(tx) - f_1(t)}{a(t)} = \int_1^x \frac{\psi(ts)}{a(t)} \frac{ds}{s}.$$

Now fix $x > 1$. Since $f_1 \in \Pi(a)$ the above expression tends to $\log x$ as $t \to \infty$. Since f is non-decreasing, $t\psi(t)$ is non-decreasing. This implies

$$(1 - x^{-1}) \frac{\psi(t)}{a(t)} \le \int_1^x \frac{\psi(ts)}{a(t)} \frac{ds}{s} \text{ for } t > 0,$$

hence

$$\overline{\lim_{t\to\infty}} \frac{\psi(t)}{a(t)} \le \frac{\log x}{1-x^{-1}}$$ for $x > 1$. Similarly we find $$\underline{\lim_{t\to\infty}} \frac{\psi(t)}{a(t)} \ge \frac{-\log x}{x^{-1} - 1}$$

for $0 < x < 1$.

Finally let $x \to 1$ to obtain $\psi(t) \sim a(t)$ $(t\to\infty)$, which implies $\psi \in RV_0^\infty$. The proof is finished by application of theorem 1.17.

ad 3. We may assume without loss of generality that f is integrable on finite intervals of $\mathbb{R}^+$.

Define the functions f_i for $i = 1, 2, 3$ recursively by

$$f_i(t) := t^{-1}\int_0^t f_{i-1}(s)ds \text{ for } t > 0,$$

where $f_0 = f$.

Repeated application of theorem 1.13 and 1.17 gives

$$f(t) - f_3(t) = \sum_{i=0}^{2} \{f_i(t) - f_{i+1}(t)\} \sim 3a(t) \quad (t\to\infty).$$

Hence $f_3 \in \Pi(a)$ by theorem 1.13. Define $\bar{f}$ by $\bar{f}(t) := f_3(e^3 t)$, then $f(t) - \bar{f}(t) = o(a(t))$ $(t\to\infty)$. Furthermore $\bar{f}$ is twice differentiable and

$$t^2 f_3''(t) = (f_1(t) - f_2(t)) - 2(f_2(t) - f_3(t)) \sim - a(t) \quad (t\to\infty)$$

by theorem 1.17.

ad 4. From remark 2 following cor. 1.18 it follows that there exist functions a_0, b such that $a_0(t) \sim a(t)$, $b(t) = o\,(a(t))$ $(t\to\infty)$ and

$$f(t) = \int_{t'}^{t} \frac{a_0(s)}{s}\, ds + b(t) \text{ for } t > t'. \tag{1.60}$$

Then for all ε, δ_1, δ_3, $\delta_4 > 0$ there exists $t = t_0\,(\varepsilon, \delta_1, \delta_3, \delta_4)$ such that for all $t \geq t_0$, $x \geq 1$ we have

$$\begin{aligned} f(tx) - f(t) &= \int_1^x \frac{a_0(ts)}{s} ds + \frac{b(tx)}{a(tx)}\, a(tx) - b(t) \\ &\leq [(1 + \delta_3) \int_1^x s^{\delta_1 - 1}\, ds + \varepsilon(1 + \delta_4)\, x^{\delta_1} + \varepsilon]\, a(t) \\ &= \left\{[1 + \delta_3 + \varepsilon(1 + \delta_4)\delta_1]\, \frac{x^{\delta_1} - 1}{\delta_1} + \varepsilon(2 + \delta_4)\right\} a(t) \end{aligned}$$

using $a_0(t) \sim a(t)$, $b(t) = o(a(t))$ and prop. 1.7.5.

Hence f satisfies the stated upper inequality if we take ε, δ_3 and δ_4 such that $\max\{\delta_3 + \varepsilon(1 + \delta_4)\delta_1,\ \varepsilon(2 + \delta_4)\} = \delta_2$.

The proof of the lower inequality is similar.

ad 5. We give the proof of the first statement, the proof of the other statement is similar.

$$\frac{f(tx) - f(t)}{tg(t)} = \int_1^x \frac{g(ts)}{g(t)} ds. \tag{1.61}$$

If $g \in RV^{\infty}_{-1}$, then the right-hand side in (1.61) tends to log x $(t\to\infty)$ by the uniform convergence theorem for regularly varying functions (theorem 1.3). Next suppose $f \in \Pi(a)$. We have

$$\frac{f(tx) - f(t)}{a(t)} = \frac{tg(t)}{a(t)} \int_1^x \frac{g(ts)}{g(t)} ds,$$

and the integral is at most x-1 when x > 1. Hence for x > 1, since $f \in \Pi$, we get

$$\underline{\lim}_{t\to\infty} \frac{tg(t)}{a(t)} \geq \frac{\ln x}{x-1}.$$

Similarly we find $\overline{\lim}_{t\to\infty} \frac{tg(t)}{a(t)} \leq \frac{\ln x}{x-1}$ for $0 < x < 1$.

Let $x \to 1$ to obtain $tg(t) \sim a(t)$ $(t\to\infty)$ and the last function is slowly varying by theorem 1.12.

ad 6. See Corollary 2.16. ◊

Remark

A special case of the current subsection is obtained when the auxiliary function a satisfies $a(t) \to \rho > 0$ $(t\to\infty)$.

Note that the specialization of theorem 1.17 then gives the following statement:

Suppose $g: \mathbb{R}^+ \to \mathbb{R}^+$ is measurable.

Then $g \in RV^{\infty}_{\rho}$ if and only if log g is locally integrable on (t_0, ∞) for some $t_0 > 0$ and

$$\lim_{t\to\infty} \int_{t_0/t}^{1} \log\{\frac{g(ts)}{g(t)}\} ds = \int_0^1 \log s^{\rho} ds = -\rho.$$

This can be seen by applying theorem 1.17 for $f(t) = \log g(t)$.

Examples

The functions f defined by

$$f(t) = \log t + o(1) \quad (t\to\infty),$$

$$f(t) = (\log t)^{\alpha} (\log \log t)^{\beta} + o(\log t)^{\alpha-1} \ (t\to\infty),\ \alpha > 0,\ \beta \in \mathbb{R},$$

$$f(t) = \exp\{(\log t)^{\alpha}\} + o(\log t)^{\alpha-1} \exp\{(\log t)^{\alpha}\} \text{ for } 0 < \alpha < 1,\ t\to\infty,$$

$$f(t) = t^{-1}\log \Gamma(t) + o(1) \ (t\to\infty)$$

are in Π.
The functions f defined by

$$f(t) = [\log t]$$

$$f(t) = 2 \log t + \sin \log t$$

are in RV_0^{∞}, but not in Π.

The following result is a generalization of part of theorem 1.17 (the kernel function k below is constant in theorem 1.17).

Theorem 1.20

Suppose $f \in \Pi(a)$ is integrable on finite intervals of $\mathbb{R}^+$.

(i) If the measurable function $k : \mathbb{R}^+ \to \mathbb{R}$ is bounded on $(0,1)$, then

$$\int_0^1 k(s) \frac{f(ts) - f(t)}{a(t)} ds \to \int_0^1 k(s) \log s \, ds, \ t\to\infty . \tag{1.62}$$

(ii) If $t^{\varepsilon}k(t)$ is integrable on $(1,\infty)$ for some $\varepsilon > 0$, then

$$\int_1^{\infty} k(s) f(ts) \, ds < \infty \text{ for } t > 0$$

and

$$\int_1^{\infty} k(s) \frac{f(ts) - f(t)}{a(t)} ds \to \int_1^{\infty} k(s) \log s \, ds \ (t\to\infty).$$

Proof

(i) Note that for $0 < \varepsilon < 1$ the function $t^{-\varepsilon}k(t)$ is integrable on $(0,1)$. We proceed as in the first part of the proof of theorem 1.17. Applying corollary 1.15 we have

$$\int_{t_o/t}^{1} k(s)\,\frac{f(ts) - f(t)}{a(t)}\,ds \to \int_0^1 k(s)\log s\,ds$$

by Lebesgue's theorem on dominated convergence. Since k is bounded, $ta(t) \in RV_1^\infty$ and $f(t) = o(t^{1/2})$ $(t\to\infty)$, we have

$$\int_0^{t_o/t} k(s)\,\frac{f(ts) - f(t)}{a(t)}\,ds = \{\int_0^{t_o} k(s/t)f(s)ds - f(t)\int_0^{t_o} k(s/t)ds\}/ta($$

$\to 0,\ t\to\infty.$

(ii) The second statement is proved in a similar way. ◇

Definition 1.21

The functions f_1, f_2 : $\mathbb{R}^+ \to \mathbb{R}$ are inversely asymptotic (at infinity) if for every constant $c > 1$ there exists a $t_o = t_o(c)$ such that

$$f_1(t) \leq f_2(ct) \qquad t \geq t_o$$

and (1.63)

$$f_2(t) \leq f_1(ct) \qquad t \geq t_o.$$

Notation: $f_1 \overset{*}{\sim} f_2$ or $f_1(t) \overset{*}{\sim} f_2(t)$ $(t\to\infty)$. ◇

We use the notation $f_1(t) \overset{*}{\sim} f_2(t)$ $(t \to 0+)$ if $f_1(1/t) \overset{*}{\sim} f_2(1/t)$ $(t\to\infty)$.
It is easy to see that if f_1 and f_2 are increasing and unbounded, then $f_1 \overset{*}{\sim} f_2$ at infinity if and only if the inverse functions are asymptotically equal (i.e. $f_1^{\leftarrow}(t) \sim f_2^{\leftarrow}(t)$, $t\to\infty$).
The relevance of this definition for functions in RV_α and in the class Π follows from the next proposition.

Proposition 1.22

(i) Suppose $f_1 \in RV_\alpha^\infty$, $\alpha > 0$ and f_2 is measurable. Then $f_1 \overset{*}{\sim} f_2$ if and only if $f_1(t) \sim f_2(t)$ $(t\to\infty)$. It then follows that $f_2 \in RV_\alpha$.

(ij) Suppose $f_1 \in \Pi(a)$ and f_2 is measurable. Then $f_1 \overset{*}{\sim} f_2$ if and only if $f_1(t) - f_2(t) = o(a(t))$, $t\to\infty$. It then follows that $f_2 \in \Pi(a)$.

Proof

(i) Since $f_1 \in RV_\alpha$ the inequalities (1.63) imply that for every $c > 1$

$$c^{-\alpha} \le \varliminf_{t\to\infty} \frac{f_2(t)}{f_1(t)} \le \varlimsup_{t\to\infty} \frac{f_2(t)}{f_1(t)} \le c^{\alpha},$$

which implies $f_1(t) \sim f_2(t)$ $(t\to\infty)$.
Conversely, if $f_1(t) \sim f_2(t)$ and $f_1 \in RV_\alpha^\infty$ $(\alpha > 0)$, then for $t \ge t_0$
$f_1(ct) \ge c^{\alpha/2} f_1(t) \ge f_2(t)$.
The second inequality in (1.63) is obtained likewise.

(ij) The second statement follows similarly since $f_1 \in \Pi(a)$ implies
$f_1(ct) = f_1(t) + a(t) \log c + o(a(t))$ $(t\to\infty)$. ◇

As a consequence: if $f_1 \overset{*}{\sim} f_2$, $f_1 \in \Pi$, then there exist functions $a\colon \mathbb{R}^+ \to \mathbb{R}^+$, $b\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$\frac{f_i(tx) - b(t)}{a(t)} \to \log x \ (t\to\infty) \text{ for } i = 1,2,\ x > 0. \tag{1.64}$$

Note that every pair of admissible functions $a > 0$ and b gives rise to an equivalence class of functions $f \in \Pi$ satisfying (1.64). The next lemma shows that every equivalence class contains a smooth function.

Lemma 1.23

a) Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive. If $f \in RV_\alpha^\infty$ $(0 < \alpha < 1)$ or $f \in \Pi$, then there exists a positive decreasing continuous function s with $s(t) \to 0$ $(t\to\infty)$ such that

$$f(t) \overset{*}{\sim} \int_0^t s(x)dx \ (t \to \infty). \tag{1.65}$$

b) Suppose f satisfies (1.65) with s positive, eventually decreasing and $s(\infty) = 0$.
   (i) If $s \in RV_{\alpha-1}^\infty$, $\alpha > 0$, then $f \in RV_\alpha^\infty$ and if $s \in RV_{-1}^\infty$, then $f \in \Pi(a)$ with $a(t) \sim ts(t)$ $(t\to\infty)$.
   (ij) If $f \in RV_\alpha^\infty$ $(0 < \alpha < 1)$, then $s \in RV_{\alpha-1}^\infty$.
   If $f \in \Pi(a)$, then $s \in RV_{-1}^\infty$ and $a(t) \sim ts(t)$ $(t\to\infty)$.

Proof

a) If $f \in RV_\alpha^\infty$ $(\alpha > 0)$ the statement is an immediate consequence of prop. 1.7.3 and prop. 1.22. Next suppose $f \in \Pi$. By proposition 1.19.3 there exists

$\bar{f} \overset{*}{\sim} f$ which is twice differentiable and (by iteration) we may suppose its derivative $\bar{s}(t)$ to be convex and decreasing for $t \geq t_0$. Hence

$$\bar{f}(t) = \bar{f}(t_o) + \int_{t_o}^{t} \bar{s}(x)\, dx \text{ for all } t > t_o.$$

The right-hand side it not yet exactly of the required form.
Note that since $\bar{f}$ is eventually positive $\underline{\lim}_{t\to\infty} \bar{f}(t) - t\bar{s}(t) > 0$ by remark 1 following cor. 1.18.
Take $t_1 \geq t_0$ such that $\bar{f}(t) > t\bar{s}(t)$ for $t \geq t_1$. The function f_0 defined by $f_0(t) := \int_0^t s(x)dx$ with

$$s(x) = \begin{cases} \bar{f}(t_1)/t_1 & \text{for } 0 < x < t_1 \\ \bar{s}(x) & \text{for } x \geq t_1 \end{cases}$$

satisfies $f_0(t) \overset{*}{\sim} f(t)$ $(t\to\infty)$.

The final step is to redefine the function s on the interval $(0, t_1 + 1)$ without changing $\int_0^{t_1+1} s(x)dx$ in such a way that s is decreasing and continuous.

b) The implication $s \in RV^{\infty}_{\alpha-1}$, $\alpha > 0 \to f \in RV^{\infty}_{\alpha}$ is an immediate consequence of the propositions 1.7.4 and 1.22 (i). The converse implication is a consequence of the propositions 1.7.11 and 1.22 (i). The proof of the corresponding statements for the class Π are similar.

◇

Remark
A similar result holds for functions $f \in RV^{\infty}_{\alpha}$ with $\alpha > 1$ or $\alpha < 0$. We leave the formulation to the reader. The statement of lemma 1.23 will be used in chapter 2.

I.3 The class Γ

For RV functions propositions 1.7.9 and 1.7.10 show that (generalized) inversion gives again an RV function. For non-decreasing unbounded functions in the class Π we obtain the following class by inversion (cf. theorem 1.27 below).

Definition 1.24

A non-decreasing function $f\colon \mathbb{R} \to \mathbb{R}$ which is eventually positive, is said to belong to the class Γ if there exists a function $b\colon \mathbb{R} \to \mathbb{R}^+$ such that for all $x \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{f(t + x\, b(t))}{f(t)} = e^x. \tag{1.66}$$

Notation: $f \in \Gamma$ or $f \in \Gamma(b)$.

The function b is called an auxiliary function for f. ◇

Remarks

1. Note that (1.66) implies $f(\infty) = \infty$.

2. From lemma 1.25 below it follows that relation (1.66) holds uniformly on each bounded interval.
   Hence any positive function b_1 is an auxiliary function for f if and only if $b_1(t) \sim b(t)$ $(t\to\infty)$; the "only if" part of this statement follows by contradiction.

Lemma 1.25

Suppose the functions $f, f_n : \mathbb{R} \to \mathbb{R}$ are non-decreasing for $n=1,2,\ldots$, f is continuous and $f_n(x) \to f(x)$ $(n\to\infty)$ for $x \in \mathbb{R}$.
Then convergence is uniform on bounded intervals of $\mathbb{R}$.

Proof

By contradiction. Suppose there exists a sequence $x_1^1, x_2^1, \ldots \in [a,b]$ such that

$$f_n(x_n^1) - f(x_n^1) \geq c > 0, \tag{1.67}$$

say, for all n.
Let $x_1, x_2, \ldots$ be a subsequence of $x_1^1, x_2^1, \ldots$ with $x_n \to x_o \in [a,b]$ $(n\to\infty)$.
Choose n_0 such that $f(x_n) > f(x_o) - c/3$ for $n \geq n_0$.
Choose $\varepsilon > 0$ such that $f(x_o + \varepsilon) < f(x_o) + c/3$.
Choose n_1 such that $f_n(x_o + \varepsilon) < f(x_o + \varepsilon) + c/3$ for $n \geq n_1$.
Choose n_2 such that $x_n \leq x_o + \varepsilon$ for $n \geq n_2$.
Combination of the above four inequalities contradicts (1.67). ◇

In order to show that the class Γ consists of the fuctions which are inverse to a non-decreasing Π-function we need the following lemma. Recall (def 1.6) that if $f\colon (t_0, \infty) \to \mathbb{R}$ is bounded on intervals of the form (t_0, a) with $a < \infty$ and $\lim_{t\to\infty} f(t) = \infty$, then the generalized inverse function f^+ is defined by $f^+(x) = \inf\{y;\ f(y) \geq x\}$ for x sufficiently large.

Lemma 1.26

Suppose the functions $f_n\colon \mathbb{R}^+ \to \mathbb{R}$ are nondecreasing, $\lim_{t\to\infty} f_n(t) = \infty$ for $n = 1,2,\ldots$ and $f_n(x) \to f(x)$ $(n\to\infty)$ for every continuity point of f. Suppose also $\lim_{t\to\infty} f(t) = \infty$.
Then $f_n^{+}(y) \to f^+(y)$ $(n\to\infty)$ for every continuity point of f^+.

Proof

Let y be a continuity point of f^+. Fix $\varepsilon > 0$. We have to prove that for $n \geq n_o$

$$f_n^{+}(y) - \varepsilon \leq f^+(y) \leq f_n^{+}(y) + \varepsilon.$$

We are going to prove the right inequality, the proof of the left-hand inequality is similar.
Choose $0 < \varepsilon_1 < \varepsilon$ such that $f^+(y) - \varepsilon_1$ is a continuity point of f. This is possible since the continuity points of f form a dense set. Since f^+ is continuous in y, $f^+(y)$ is a point of increase for f, hence $f(f^+(y) - \varepsilon_1) < y$. Choose $\delta < y - f\,(f^+(y) - \varepsilon_1)$. Since $f^+(y) - \varepsilon_1$ is a continuity point of f, there exists n_o such that $f_n(f^+(y) - \varepsilon_1) < f(f^+(y) - \varepsilon_1) + \delta < y$ for $n \geq n_0$. The definition of the function f_n^{+} then implies $f^+(y) - \varepsilon_1 < f^{+}_{n}(y)$.

◇

Theorem 1.27

(i) Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}$.
If $f \in \Pi(a)$ and $f(\infty) = \infty$, then $f^+ \in \Gamma(b)$ with $b(t) \sim a(f^+(t))$ $(t\to\infty)$.

(ij) Conversely, if $g \in \Gamma(b)$, then $g^+ \in \Pi(a)$ with $a(t) \sim b(g^+(t))$ $(t\to\infty)$.

As in proposition 1.7.9 the domain of definition $f^+(g^+)$ can be extended to $\mathbb{R}$ ($\mathbb{R}^+$ respectively) by defining the function to be zero on a neighbourhood of $-\infty$ (0 respectively).

Proof

(i) Suppose $f \in \Pi(a)$. Note that f is locally bounded on intervals of the form (t_0, a) for t_0 sufficiently large, hence f^+ is well-defined. Using the definition of f^+ we have $f((1-\varepsilon) f^+(s)) \leq s \leq f((1+\varepsilon) f^+(s))$ for any $\varepsilon > 0$. As a consequence we have for $x > 0$

$$\frac{f(xf^+(s)) - s}{a(f^+(s))} \geq \frac{f(xf^+(s)) - f(f^+(s))}{a(f^+(s))} - \frac{f((1+\varepsilon) f^+(s)) - f(f^+(s))}{a(f^+(s))} .$$

The right-hand side in the above inequality tends to $\log (x/(1+\varepsilon))$ since $f \in \Pi(a)$ and $\lim_{s\to\infty} f^+(s) = \infty$.

Using a similar upper inequality we find

$\lim_{s\to\infty} \frac{f(xf^+(s)) - s}{a(f^+(s))} = \log x$ since $\varepsilon > 0$ is arbitrary. Application of lemma 1.26 then shows

$$\frac{f^+(s + xa(f^+(s)))}{f^+(s)} = \inf\{y;\ \frac{f(yf^+(s)) - s}{a(f^+(s))} \geq x\} \to e^x \ (s\to\infty).$$

(ij) Conversely suppose $g \in \Gamma(b)$. By the definition of g^+ we have

$$g(g^+(t) - 0) \leq t \leq g(g^+(t) + 0). \tag{1.68}$$

Hence for any $\varepsilon > 0$ we have $g(g^+(t) - \varepsilon\, b(g^+(t))) \leq t \leq g(g^+(t) + \varepsilon\, b(g^+(t)))$ since the function b is positive.

Division by $g(g^+(t))$ throughout and application of (1.66) shows that $t \sim g(g^+(t))$ $(t\to\infty)$.

We thus have by (1.66)

$$\lim_{t\to\infty} \frac{g(g^+(t) + x\, b(g^+(t)))}{t} = e^x \text{ for } x \in \mathbb{R}.$$

By lemma 1.26 we find

$$\frac{g^+(tx) - g^+(t)}{b(g^+(t))} = \frac{\inf\{v;\ g(v) \geq tx\} - g^+(t)}{b(g^+(t))} =$$

$$= \inf\{y;\ g(g^+(t) + y\, b(g^+(t))) \geq tx\} \to \ln x \ (t\to\infty)$$

for $x > 0$. ◇

Remarks

1. Note that if $f: \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing and $f(\infty) = \infty$, then for any continuity point of f we have $f(t) = \inf\{y;\ f^+(y) \geq t\}$, hence the generalized inverse of the generalized inverse gives us the original function.
2. If $g \in \Pi(b)$, then the composition $b \circ g^+ \in RV_0^\infty$.

Next we prove a representation theorem for the class Γ.

Theorem 1.28

Suppose $f: \mathbb{R} \to \mathbb{R}^+$ is non-decreasing.
The following statements are equivalent:

(i) $\quad f \in \Gamma.$ (1.69)

(ij) There exists a differentiable function $\beta: \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta'(x) \to 0$ $(x\to\infty)$ such that

$$f(t) \sim \exp\{\int_1^t \frac{ds}{\beta(s)}\}\ (t\to\infty). \tag{1.70}$$

(iij)
$$\lim_{t\to\infty} \frac{f(t).\int_0^t\int_0^x f(s)\,ds\,dx}{(\int_0^t f(s)\,ds)^2} = 1\ . \tag{1.71}$$

Proof

(i) → (ij)

Theorem 1.27 implies that $f^+ \in \Pi$. Proposition 1.19.3 shows that there exists a function g, twice differentiable with $-\ g'' \in RV_{-2}^\infty$ and $g \overset{*}{\sim} f^+$. The latter relation implies $g^+(t) \sim f(t)$, $t\to\infty$, by definition 1.21. (Note that $f \in \Gamma$ implies $f(t+) \sim f(t-)$, $t\to\infty$).
Since $-g'' \in RV_{-2}^\infty$ we have $\frac{tg''(t)}{g'(t)} \to -1$ $(t\to\infty)$ by theorem 1.4. Replacing t by $g^+(t)$ gives

$$\frac{-g^+(t)(g^+)''(t)}{\{(g^+)'(t)\}^2} = \frac{g^+(t)g''(g^+(t))}{g'(g^+(t))} \to -1\ (t\to\infty).$$

Hence $\{\frac{1}{(\ln g^{+})'(t)}\}' = 1- \frac{g^{+}(t)(g^{+})''(t)}{\{(g^{+})'(t)\}^2} \to 0 \ (t\to\infty)$.

Define the function β by $\beta(t) = 1/(\ln g^{+})'(t)$. Then β satisfies the requirements of the theorem and for some constant c

$$\log g^{+}(t) = \int_1^t \frac{ds}{\beta(s)} + c. \tag{1.72}$$

Then $f \sim g^{+}$ satisfies (1.70) since we can modify β on the interval (1,2) in such a way that (1.72) holds with $c = 0$.

<u>(ij) → (i)</u>

First we note that $\beta'(t) \to 0$ $(t\to\infty)$ implies

$$\frac{\beta(t + x\beta(t))}{\beta(t)} \to 1 \ (t\to\infty) \tag{1.73}$$

uniformly on finite intervals of $\mathbb{R}$, since

$$-1 + \beta(t + x\beta(t))/\beta(t) = x\beta'(t + \theta\, x\, \beta(t)) \text{ with } \theta = \theta(t,x) \text{ and } 0 < \theta < 1.$$

The right-hand side is easily seen to tend to zero uniformly as $t\to\infty$.
Now by (1.70)

$$\frac{f(t + x\beta(t))}{f(t)} \sim \exp\{\int_t^{t+x\beta(t)} \frac{ds}{\beta(s)}\} = \exp\{\int_0^x \frac{\beta(t)}{\beta(t+v\beta(t))}\, dv\}$$

and the integral on the right-hand side tends to x as $t\to\infty$.

<u>(i), (ij) → (iij)</u>

First we prove that $f \in \Gamma(b)$ implies $\int_0^t f(s)\, ds \in \Gamma(b)$. By the previous proof we may assume that $f \in \Gamma(\beta)$ with $\beta(t) \sim b(t)$ $(t\to\infty)$, β as in thm. 1.28 (ij) and such that (1.70) holds.
Define the function g by $g(t) = \exp\{\int_0^t \frac{ds}{\beta(s)}\}$.

Since $\beta' \to 0$ we have $(\beta g)' = \beta' g + g \sim g$, hence $\beta(t)g(t) \sim \int_0^t g(s)ds$ $(t\to\infty)$.

Since $f(t) \sim g(t)$ $(t\to\infty)$, this implies

$$\beta(t) \sim \frac{\int_0^t g(s)\, ds}{g(t)} \sim \frac{\int_0^t f(s)\, ds}{f(t)} \ (t\to\infty). \tag{1.74}$$

It follows that (cf. (1.73))

$$\int_0^{t+x\beta(t)} f(s)\, ds / \int_0^t f(s)\, ds \sim \frac{\beta(t + x\beta(t))}{\beta(t)} \frac{f(t + x\beta(t))}{f(t)} \to e^x.$$

This implies $\int_0^t f(s)ds \in \Gamma(\beta)$. The proof also shows that if a function h satisfies $h \in \Gamma(\beta)$, then $\beta(t) \sim \int_0^t h(s)\, ds/h(t)$ $(t\to\infty)$. Applying this for $h(t) := \int_0^t f(s)ds$ entails

$$\beta(t) \sim \int_0^t \int_0^x f(s)\, ds\, dx \Big/ \int_0^t f(s)\, ds \quad (t\to\infty). \tag{1.75}$$

The statement (1.71) is implied by (1.74) and (1.75).

<u>(iij) → (i)</u>

Define the function ε by $\varepsilon(t) = 1 - f(t) \left(\int_0^t \int_0^x f(s)dsdx\right)/\left(\int_0^t f(s)ds\right)^2$ and the function h by $h(t) = \int_0^t \int_0^x f(s)\, ds\, dx \Big/ \int_0^t f(s)\, ds$, $t > 0$.

Then $\varepsilon(t) \to 0$ $(t\to\infty)$ by (1.71) and $h(t) = h(1) + \int_1^t \varepsilon(s)ds$. It follows (as in the part (ij) → (i) of this proof) that

$$\lim_{t\to\infty} h(t + xh(t))/h(t) = 1 \tag{1.76}$$

for all x uniformly on finite intervals and hence

$$\int_0^t \int_0^x f(s)\, ds\, dx = c \exp\left\{\int_1^t \frac{ds}{h(s)}\right\} \in \Gamma(h). \tag{1.77}$$

Note that $c = \int_0^1 \int_0^x f(s)\, ds\, dx$.

By (1.76) and (1.77) we then have

$$\int_0^t f(s)\, ds = \frac{1}{h(t)} \int_0^t \int_0^x f(s)\, ds\, dx \in \Gamma(h) \tag{1.78}$$

and hence $f(t) \sim \left(\int_0^t f(s)\, ds\right)^2 \Big/ \int_0^t \int_0^x f(s)\, ds\, dx$ $(t\to\infty)$. Combination with (1.77) and (1.78) gives $f \in \Gamma(h)$. ◇

Remark

Note that for the implication (1.70) → (1.66) or (1.71) → (1.66) the monotonicity of f has not been used. The question of defining a function class like Γ without monotonicity remains unsettled.

Corollary 1.29

1. If $f \in \Gamma(b)$, then

$$b(t) \sim \frac{\int_0^t f(s)\,ds}{f(t)} \quad (t\to\infty).$$

Hence b can always be taken measurable.
Moreover the function β in the above theorem satisfies $\beta(t) \sim b(t)$ $(t\to\infty)$.
Hence if $f \in \Gamma(b)$, then $b(t + xb(t)) \sim b(t)$ $(t\to\infty)$ uniformly on finite intervals of $\mathbb{R}$.

2. $f \in \Gamma(b)$ implies $\int_0^t f(s)\,ds \in \Gamma(b)$.

3. We may replace (1.70) by $f(t) \sim \exp\{\int_1^t \frac{c(s)}{\beta(s)}\,ds\}$, where $c(s) \to c > 0$ $(s\to\infty)$.

4. $f \in \Gamma(b)$ implies $b(t)/t \to 0$, $t\to\infty$ (since the same holds for the function β.)

◇

The next theorem provides another characterization of the class Γ.

Theorem 1.30

If $f \in \Gamma$, then for all positive α

$$\lim_{t\to\infty} \frac{\int_0^t \{f(s)\}^{\alpha} ds}{\{f(t)\}^{\alpha-1} \int_0^t f(s)\,ds} = \frac{1}{\alpha}\,. \tag{1.79}$$

Conversely, if a positive non-decreasing function f satisfies (1.79) for some positive $\alpha \neq 1$, then $f \in \Gamma$.

Proof

Suppose $f \in \Gamma(b)$. Then $\lim\limits_{t\to\infty} \frac{\{f(t + xb(t)/\alpha)\}^{\alpha}}{f(t)^{\alpha}} = e^x$,

hence $f^{\alpha} \in \Gamma(b/\alpha)$. Applying corollary 1.29.1 twice, we get

$$\frac{\int_0^t \{f(s)\}^\alpha \, ds}{\{f(t)\}^\alpha} \sim \frac{b(t)}{\alpha} \sim \frac{\int_0^t f(s)\, ds}{\alpha\, f(t)} \quad (t\to\infty),$$

which is equivalent to (1.79).

For the proof of the converse statement we define the function ρ by

$$\rho(t) = \frac{1}{\alpha-1}\frac{\{f(t)\}^\alpha}{\int_0^t \{f(s)\}^\alpha ds}\left(1 - \frac{\int_0^t \{f(s)\}^\alpha\, ds}{(f(t))^{\alpha-1}\int_0^t f(s)\, ds}\right) g(t), \text{ where}$$

$$g(t) = \left(\alpha \int_0^t \{f(s)\}^\alpha\, ds \Big/ \int_0^t f(s) ds\right)^{1/(\alpha-1)}, \; t > 0. \qquad (1.80)$$

Then $g(t) = c + \int_0^t \rho(s)ds$, hence using (1.79) twice we find

$$\frac{\rho(t)}{g(t)} \sim \frac{1}{\alpha}\frac{\{f(t)\}^\alpha}{\int_0^t \{f(s)\}^\alpha ds} \sim \frac{f(t)}{\int_0^t f(s)ds} \quad (t\to\infty). \qquad (1.81)$$

Note that (1.79) and (1.70) imply $g(t) \sim f(t)$ $(t\to\infty)$.

Hence $\int_0^t g(s)ds \sim \int_0^t f(s)\, ds$ $(t\to\infty)$ and combination with (1.81) gives

$\lim_{t\to\infty} \rho(t) \int_0^t g(s)ds \,/\, \{g(t)\}^2 = 1$. By the proof of theorem 1.28 (cf. the remark following the theorem) we have $\lim_{t\to\infty} \rho(t + xb(t))/\rho(t) = e^x$ for all $x \in \mathbb{R}$ uniformly on finite intervals with $b(t) = g(t)/\rho(t)$.

Hence for $x \in \mathbb{R}$ since $g(t) \sim f(t)$ $(t\to\infty)$ and $f(t) \to \infty$ we have

$$\frac{g(t + xb(t))}{g(t)} - 1 = \frac{\int_t^{t+xb(t)} \rho(s)ds}{c + \int_0^t \rho(s)ds} \sim b(t) \int_0^x \frac{\rho(t + ub(t))}{\rho(t)} du \Big/ \left(\int_0^t \rho(s)ds/\rho(t)\right)$$

$$\sim e^x - 1 \quad (t\to\infty).$$

Hence $g \in \Gamma$. Since $f(t) \sim g(t)$ $(t\to\infty)$, we find $f \in \Gamma$. ◇

Next we list some <u>properties of the class Γ</u>

Proposition 1.31

1. If $f \in \Gamma$, then $\log f(t)/\log t \to \infty$ $(t\to\infty)$.

   Moreover $\lim_{t\to\infty} \frac{f(tx)}{f(t)} = \begin{cases} 0 \text{ if } 0 < x < 1 \\ \infty \text{ if } x > 1. \end{cases}$

2. The class Γ is closed under multiplication:
   if $f_1 \in \Gamma(b_1)$, $f_2 \in \Gamma(b_2)$, then $f_1 f_2 \in \Gamma(b)$
   with $b(t) := b_1(t)\, b_2(t)/\{b_1(t) + b_2(t)\}$.

3. If $f \in \Gamma$, $h \in RV_\alpha^\infty$ with $\alpha > 0$, then $h \circ f \in \Gamma$, where $h \circ f$ denotes the composition of the two functions. If $f \in \Gamma$ and h is differentiable with $h' \in RV_\alpha^\infty$, $\alpha > -1$, then $f \circ h \in \Gamma$.

4. If $f \in \Gamma(b)$ then $\int_0^t f(s)ds \in \Gamma(b)$. Conversely if $\int_0^t f(s)ds \in \Gamma(b)$ and f is non-decreasing, then $f \in \Gamma(b)$.

5. If $f \in \Gamma(b)$, there exists $f_0 \in C^\infty$ with $f_0^{(n)} \in \Gamma(b)$ for $n = 1, 2, \ldots$ such that $f_0(t) \sim f(t)$ $(t\to\infty)$.

6. Suppose $f \in \Gamma(b)$. If $\delta_1, \delta_2 > 0$ are arbitrary, there exists $t_0 = t_0(\delta_1, \delta_2)$ such that for $t \geq t_0$, $x \geq 0$

$$(1 - \delta_1)\left(\frac{f(t + xb(t))}{f(t)}\right)^{\delta_2} \leq \frac{b(t + xb(t))}{b(t)} \leq (1 + \delta_1)\left(\frac{f(t + xb(t))}{f(t)}\right)^{\delta_2}.$$

7. Suppose $f \in \Gamma(b)$. Define the function g by $g(t) = 1/\{\int_1^t ds/f(s)\}$. Then $g \in \Gamma(b)$.

Proof

ad 1. The proof is an immediate consequence of theorem 1.28, cor. 1.29.4 and de l'Hôpital's rule.

ad 2. By theorem 1.28 it is sufficient to prove that

$$\lim_{t\to\infty} \frac{d}{dt} \{\beta_1(t)^{-1} + \beta_2(t)^{-1}\} = 0 \text{ with } \beta_1 \text{ and } \beta_2 \text{ as defined there.}$$

This follows immediately since $\beta_1, \beta_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $\beta_i'(t) \to 0$ $(t\to\infty)$ for $i = 1, 2$.

ad 3. If $f \in \Gamma(b)$ and $h \in RV_\alpha^\infty$ with $\alpha > 0$, by the uniform convergence theorem for regularly varying functions we have $h(f(t + xb(t)/\alpha))/h(f(t)) \to e^x$ $(t\to\infty)$. If $f \in \Gamma(b)$ and h is differentiable with $h' \in RV_\alpha$, $\alpha > -1$, by the uniform convergence theorem for regularly varying functions (thm. 1.3) and lemma 1.25 we have for some $\theta = \theta(x, t) \in (0, 1)$

$$\frac{f(h(t + \frac{xb(h(t))}{h'(t)}))}{f(h(t))} = \frac{f(h(t) + xb(h(t))\,\frac{h'(t + \theta xb(h(t))/h'(t))}{h'(t)})}{f(h(t))} \to e^x$$

as $t\to\infty$ (note that $b(h(t))/th'(t) = \frac{b(h(t))}{h(t)} \cdot \frac{h(t)}{th'(t)} \to 0$ $(t\to\infty)$ by cor. 1.29.4 since $h' \in RV_\alpha$ with $\alpha > -1$).

ad 4. Since $\int_0^t f(s)ds \in \Gamma(b)$, for $x > 0$ the right-hand side of the inequality

$$\frac{f(t)\ b(t)}{\int_0^t f(s)ds} \le \frac{\int_t^{t+xb(t)} f(s)ds}{x\int_0^t f(s)ds}$$

tends to $(e^x - 1)/x$ as $t\to\infty$. Let now $x \to 0+$ to obtain $\overline{\lim}_{t\to\infty} f(t)\ b(t) \int_0^t f(s)ds \le 1$. Similarly, for $x < 0$ we have

$$-\frac{\int_{t+xb(t)}^t f(s)ds}{x\int_0^t f(s)ds} \le \frac{f(t)\ b(t)}{\int_0^t f(s)ds},$$

which implies $\underline{\lim}_{t\to\infty} f(t)\ b(t)/\int_0^t f(s)ds \ge 1$.

Hence $f(t)\ b(t) \sim \int_0^t f(s)ds$. Since $\int_0^t f(s)ds \in \Gamma(b)$ we have $b(t + xb(t)) \sim b(t)$ (cor. 1.29.1). Combining these results, we find $f \in \Gamma(b)$.

ad 5. From theorem 1.28 it follows that without loss of generality we may suppose f to be strictly increasing and continuous. Application of theorem 1.27 (ij) shows that the inverse function $f^{\leftarrow} \in \Pi(a)$ with $a(t) \sim b(f^{\leftarrow}(t))$ (or $a(f(t)) \sim b(t)$ $(t\to\infty)$).

Since $f^{\leftarrow} \in \Pi(a)$, there exists a function $g_0 \in \Pi(a)$ satisfying $g_0 \overset{*}{\sim} f^{\leftarrow}$ and $(-1)^{n+1} g_0^{(n)} \in RV^{\infty}_{-n}$ for $n = 1, 2, \ldots$ (see prop. 1.19.6 and prop. 1.22 (ij)).
By the definition of the relation $\overset{*}{\sim}$ we have $g_0^{\leftarrow}(t) \sim f(t)$ $(t\to\infty)$, hence $a(g_0^{\leftarrow}(t)) \sim a(f(t)) \sim b(t)$ (by the uniform convergence theorem for RV functions), which is equivalent to

$$a(t) \sim b(g_0(t)) \quad (t\to\infty). \tag{1.82}$$

On the other hand we find

$$a(t) \sim tg_0'(t) \quad (t\to\infty) \tag{1.83}$$

by application of lemma 1.23b.
We claim that the function f_0 defined by $f_0(t) := g_0^{\leftarrow}(t)$ for all suffiently large t satisfies the assumptions.
First note that $f_0(t) \sim f(t)$ which implies $f_0 \in \Gamma(b)$ since $f \in \Gamma(b)$. We shall prove that for $n \in \mathbb{N}$ we have

$$f_0^{(n)}(t) \sim \frac{f_0(t)}{\{b(t)\}^n} \quad (t\to\infty), \tag{1.84}$$

which implies $f_0^{(n)} \in \Gamma(b)$ (since $f_0 \in \Gamma(b)$).

Substituting $g_0(t)$ for t shows that (1.84) is equivalent to

$$f_0^{(n)}(g_0(t)) \; \{b(g_0(t))\}^n / t \to 1 \quad (t\to\infty).$$

Combination of (1.82) and (1.83) shows that the last limit relation is equivalent to

$$f_0^{(n)}(g_0(t)) \; \{tg_0'(t)\}^n t^{-1} \to 1 \quad (t\to\infty) \tag{1.85}$$

We will prove (1.85) by induction using Faà di Bruno's formula (see e.g. Abramowitz and Stegun, p. 823):
Since $f_0(g_0(t)) = t$ for all t sufficiently large, we have for $n > 1$

$$0 = (\frac{d}{dt})^n f_0(g_0(t)) = \sum_{m=0}^{n} f_0^{(m)}(g_0(t)) \ \Sigma' n! \prod_{k=1}^{n} (\frac{g_0^{(k)}(t)}{k!})^{a_k} \frac{1}{a_k!}, \quad (1.86)$$

where Σ' denotes summation over all a_k's satisfying $a_1 + 2a_2 + \ldots + na_n = n$ and $a_1 + \ldots + a_n = m$.
Since $(-1)^{n+1} g_0^{(n)} \in RV_{-n}^{\infty}$, we have by repeated application of theorem 1.4

$$\frac{t^{k-1} g_0^{(k)}(t)}{g_0'(t)} \to (-1)^{k-1}(k-1)! \ (t\to\infty) \text{ for all } k \geq 2. \quad (1.87)$$

Hence

$$\prod_{k=1}^{n} (\frac{g_0^{(k)}(t)}{k!})^{a_k} \frac{1}{a_k!} \sim (-1)^{n-m}(g_0'(t))^m t^{m-n} \prod_{k=1}^{n} (k^{a_k} \cdot a_k!)^{-1} \ (t\to\infty).$$

Substitution gives

$$0 = t^{n-1}(\frac{d}{dt})^n f_0(g_0(t)) = (1+o(1)) \sum_{m=0}^{n} \{f_0^{(m)}(g_0(t))(tg_0'(t))^m t^{-1}\}$$
$$(-1)^{n-m} n! \ \Sigma' \prod_{k=1}^{n} (k^{a_k} \cdot a_k!)^{-1} \ (t\to\infty). \quad (1.88)$$

The proof of (1.85) for $n = 1$ is immediate from $f_0(g_0(t)) = t$ and (1.87).
Now suppose (1.85) holds for $1 \leq n \leq N-1$.
Then the existence of $\lim_{t\to\infty} f_0^{(N)}(g_0(t)) \ \{tg_0'(t)\}^N t^{-1}$ is a consequence of (1.88). Moreover this limit is 1 since $\sum_{m=0}^{n} (-1)^m \ \Sigma' \prod_{k=1}^{n} (k^{a_k} a_k!)^{-1} = 0$ for $n \in \mathbb{N}$ (this can be seen e.g. by taking $f_0(t) = \exp t$ in (1.86)).

ad 6. We only prove the second inequality. The proof of the first inequality is similar. Suppose $f \in \Gamma(b)$. Since there exists a strictly increasing $f_1 \in \Gamma(b)$ satisfying $f_1(t) \sim f(t)$ $(t\to\infty)$ (theorem 1.28), we may suppose without loss of generality that f is strictly increasing. We apply proposition 1.7.5 to the function $b \circ f^{\leftarrow}$, which is slowly varying by theorem 1.27 (ij) and theorem 1.12 to obtain

$$\frac{b(f^{\leftarrow}(sy))}{b(f^{\leftarrow}(s))} \leq (1 + \delta_1)y^{\delta_2} \text{ for } s \geq s_0,\ y \geq 1.$$

Now take $s = f(t)$ and $y = f(t + xb(t))/f(t)$ in the resulting inequality.

ad 7. By theorem 1.28 there exists a function $\beta(t) \sim b(t)$ $(t\to\infty)$ such that $\beta'(t) \to 0$ and $1/f(t) \sim \exp\{-\int_1^t ds/\beta(s)\}$ $(t\to\infty)$.

Hence $g(t) = 1/\{\int_1^t ds/f(s)\} \sim 1/\int_1^t \exp(-\int_1^s du/\beta(u))ds$ $(t\to\infty)$.

Since $f \in \Gamma(\beta)$ we have

$$\frac{g(t + x\beta(t))}{g(t)} \sim \frac{\exp(-\int_1^t du/\beta(u))}{(1 + x\beta'(t)) \exp(-\int_1^{t+x\beta(t)} du/\beta(u))} \sim e^x \ (t\to\infty)$$

by de l'Hôpital's rule.

Hence $g \in \Gamma(\beta)$, which implies $g \in \Gamma(b)$ by remark 2 following def. 1.24.

◇

In theorem 1.17 it is shown that for $f \in \Pi(a)$ it is possible to construct a representation in terms of the function a. Our last result for the class Γ gives a similar statement for functions $f \in \Gamma(b)$.

Proposition 1.32

If $f \in \Gamma(b)$ with b such that $1/b$ is locally integrable on $\mathbb{R}^+$ (this can always be achieved since any auxiliary function is asymptotic to a positive continuous one), then there exists a non-decreasing function $f_1 \in RV_1$ such that

$$f(t) = f_1(h(t)), \tag{1.89}$$

where $h(t) := \exp\left(\int_0^t \frac{1}{b(s)}\, ds\right)$.

Conversely if f satisfies (1.89) with h as above, then $f \in \Gamma(b)$.

Proof

Suppose $f \in \Gamma(b)$. Define $f_1(t) := f(h^{\leftarrow}(t))$.

Note that $h^{\leftarrow} \in \Pi(a)$, where $a(t) \sim b(h^{\leftarrow}(t))$ $(t\to\infty)$ by theorem 1.27.

By lemma 1.25 we have for $x > 0$

$$\lim_{t\to\infty} \frac{f_1(tx)}{f_1(t)} = \lim_{t\to\infty} \frac{f(h^{\leftarrow}(t) + \{(h^{\leftarrow}(tx) - h^{\leftarrow}(t))/b(h^{\leftarrow}(t))\}\ b(h^{\leftarrow}(t)))}{f(h^{\leftarrow}(t))}$$

$$= \exp\left(\lim_{t\to\infty} \frac{h^{\leftarrow}(tx) - h^{\leftarrow}(t)}{b(h^{\leftarrow}(t))}\right) = x.$$

Conversely, if f satisfies (1.89), where $f_1 \in RV_1$, then for $x > 0$

$$\lim_{t\to\infty} \frac{f(t + x\, b(t))}{f(t)} = \lim_{t\to\infty} \frac{f_1(\{h(t + x\, b(t))/h(t)\}\ h(t))}{f_1(h(t))} =$$

$$= \lim_{t\to\infty} h(t + x\, b(t))/h(t) = e^x. \qquad \diamond$$

I.4 Beurling slowly varying functions

The class of auxiliary functions b for functions in the class Γ (cf. cor. 1.29.1) is an interesting class in its own right since it can be used in other contexts as well. We now give some results for this class of functions.

Definition 1.33

A measurable function $b: \mathbb{R} \to \mathbb{R}$ which is eventually positive is Beurling slowly varying (at infinity) if

$$\lim_{t\to\infty} \frac{b(t + xb(t))}{b(t)} = 1 \text{ for all } x \in \mathbb{R}. \qquad (1.90)$$

Notation: $b \in BSV$. $\diamond$

Remark

This class of functions was used by A. Beurling in connection with a generalization of Wiener's Tauberian theorem (unpublished, cf. Bloom 1976).

Before discussing further properties of the class Γ, we give two results concerning the class BSV.

Theorem 1.34

If $b \in BSV$ is continuous, relation (1.90) holds uniformly for $x \in [a, b]$ with $-\infty < a < b < \infty$.

Proof

We prove the result for $a = 0$, $b = 1$, the argument for an arbitrary interval being similar.

Suppose (1.90) does not hold locally uniformly. Then there exist $\varepsilon \in (0,1)$ and sequences $\{x_n\} \subset (0,1)$ and $t_n \to \infty$ $(n\to\infty)$ such that

$$|b(t_n + x_n b(t_n))/b(t_n) - 1| \geq \varepsilon \text{ for } n = 1, 2, \ldots$$

The function $f_n(t) := b(t_n + tb(t_n))/b(t_n) - 1$ is continuous and $\lim_{n\to\infty} f_n(t) = 0$ for fixed t.

Hence there is an integer N and a sequence $\alpha_n \in (0,1)$ $(n = 1, 2, \ldots)$ such that

$$|b(y_n)/b(t_n) - 1| = \varepsilon \text{ for } n \geq N, \tag{1.91}$$

where $y_n = t_n + \alpha_n b(t_n)$.

We introduce three sequences of sets:

$$V_n := \{\alpha \in (0,\ 2+\varepsilon);\ \left|\frac{b(t_n + \alpha b(t_n))}{b(t_n)} - 1\right| < \frac{\varepsilon}{2}\},$$

$$W_n := \{\mu \in (0,1);\ \left|\frac{b(y_n + \mu b(y_n))}{b(y_n)} - 1\right| < \frac{\varepsilon}{2(1+\varepsilon)}\},$$

$$W_n' = \{\alpha \in (0,\ 2+\varepsilon);\ \left|\frac{b(t_n + \alpha b(t_n))}{b(y_n)} - 1\right| < \frac{\varepsilon}{2(1+\varepsilon)}\} =$$

$$= \{\alpha \in (0,\ 2+\varepsilon);\ \alpha = \alpha_n + \mu\frac{b(y_n)}{b(t_n)} \text{ and } \mu \in W_n\}.$$

Since $b \in BSV$ we have $\lim_{n\to\infty} \lambda(V_n) = 2+\varepsilon$ and $\lim_{n\to\infty} \lambda(W_n) = 1$ (λ denotes Lebesgue measure).

Hence $\alpha_n \in (0,1)$ and (1.91) imply $\underline{\lim}_{n\to\infty} \lambda(W_n') \geq 1-\varepsilon$.

For $\alpha \in W_n'$ we have

$$\left|\frac{b(t_n + \alpha b(t_n))}{b(t_n)} - 1\right| = \left|\frac{b(y_n)}{b(t_n)} - 1 + \frac{b(y_n)}{b(t_n)}\left\{\frac{b(t_n + \alpha b(t_n))}{b(y_n)} - 1\right\}\right|$$

$$\geq \left|\frac{b(y_n)}{b(t_n)} - 1\right| - \frac{b(y_n)}{b(t_n)} \cdot \frac{\varepsilon}{2(1+\varepsilon)} \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

hence $V_n \cap W'_n = \phi$. Since $V_n, W'_n \subset (0, 2+\varepsilon)$, this implies

$$2+\varepsilon \geq \underline{\lim}_{n\to\infty} \lambda(V_n \cup W'_n) \geq \underline{\lim} (\lambda(V_n) + \lambda(W'_n)) \geq 2+\varepsilon + 1-\varepsilon = 3$$

which gives a contradiction. ◇

Next we prove a representation for BSV functions which satisfy (1.90) locally uniformly.

Theorem 1.35

If $b \in$ BSV and (1.60) holds uniformly on finite intervals, then there exists a integrable function ε such that $\lim_{t\to\infty} \varepsilon(t) = 0$ and

$$b(t) \sim \int_0^t \varepsilon(s)ds \quad (t\to\infty). \tag{1.92}$$

Conversely, if $b: \mathbb{R} \to \mathbb{R}$ is measurable, eventually positive and satisfies (1.92) with $\varepsilon(t) \to 0$ $(t\to\infty)$, then (1.90) holds uniformly on bounded intervals of $\mathbb{R}$.

Proof

Suppose (1.90) holds uniformly on finite intervals. Then there exists $t_0 > 0$ such that for $t \geq t_0$ and all $x \in [-1, 1]$ we have $b(t + xb(t)) \geq \frac{1}{2} b(t)$. Define the sequence $\{t_n\}$ by $t_{n+1} = t_n + b(t_n)$, $n = 0, 1, 2, \ldots$

Then the sequence $\{t_n\}$ is increasing and we claim that $t_n \to \infty$ $(n\to\infty)$.

If not then $\lim_{n\to\infty} t_n = p < \infty$. Now $t_n = t_0 + \sum_{k=1}^{n} b(t_{k-1})$. Then the series $\sum_{k=1}^{\infty} b(t_k)$ converges and, in particular, $\lim_{n\to\infty} b(t_n) = 0$. Since $p \geq t_0$,

$$b(y) \geq \tfrac{1}{2} b(p) \text{ for all } y \in [p - b(p), p + b(p)].$$

Note that $b(t)$ is positive for $t \geq t_0$ and this is contradicted by

$$0 = \lim_{n\to\infty} b(t_n) \geq \underline{\lim}_{y\to p} b(y) \geq \tfrac{1}{2} b(p).$$

This proves $t_n \to \infty$ $(n\to\infty)$.

Define the function b^* by $b^*(t) = 0$ on $(0, t_0)$, $b^*(t) = b(t)$ for $t = t_n$ $(n = 1, 2, \ldots)$ and linear on the interval $[t_n, t_{n+1}]$ $(n = 0, 1, 2, \ldots)$. Then, since convergence in (1.90) is uniform, we have $b(t) \sim b^*(t)$ $(t\to\infty)$.

Moreover b^* satisfies the representation

$$b^*(t) = \int_0^t \varepsilon(s)ds, \quad t > 0$$

with $\varepsilon(s) = 0$ on $(0, t_0)$ and

$$\varepsilon(s) = \frac{b^*(t_{n+1}) - b^*(t_n)}{t_{n+1} - t_n} = \frac{b(t_{n+1}) - b(t_n)}{b(t_n)}$$

for $t_n \leq s < t_{n+1}$, $n = 0, 1, 2, \ldots$ Since $\varepsilon(s) \to 0$ $(s \to \infty)$ b satisfies the required representation.

For the converse part we may assume that $b(t) = \int_0^t \varepsilon(s)ds$.

We prove that (1.90) holds uniformly for $x \in [a, b]$.

By (1.92) there exists t_0 such that

$$0 < \frac{b(t)}{t} < \frac{1}{2|a|} \text{ for } t \geq t_0.$$

For any $\varepsilon_0 > 0$ there exists $t_1 \geq t_0$ such that $|\varepsilon(t)| < \varepsilon_0$ for $t > t_1$. Consequently for $t > 2t_1$ we have $t(1 + v\frac{b(t)}{t}) > t_1$ and hence $|\varepsilon(t + vb(t)| < \varepsilon_0$ for all $v \geq \min(0, a)$. It follows that

$$\left|\frac{b(t + xb(t)) - b(t)}{b(t)}\right| \leq \left| \int_0^x |\varepsilon(t + vb(t))|dv \right| < \varepsilon_0|x| \leq \varepsilon_0 \max(|a|, b)$$

for $t > 2t_1$ and all $x \in [a, b]$. ◇

Remark

From the proof it follows that it is possible to take ε continuous in (1.92).

We close this section with an application of the Beurling slowly varying functions and the class Γ.

Theorem 1.36

Suppose y is a positive solution of the second order differential equation $y'' = fy$ satisfying $y(x) \to \infty$ $(x \to \infty)$, where f is continuous and $1/\sqrt{f} \in BSV$.

Then $y \in \Gamma(1/\sqrt{f})$.

Proof

a. First we suppose that f is differentiable and $(1/\sqrt{f(x)})' = -f'(x)/2f^{3/2}(x) \to 0$ $(x\to\infty)$. Define the function w by

$$w(x) := \frac{y'(x)}{y(x)\sqrt{f(x)}} \tag{1.93}$$

Then $w(x) > 0$ for all x sufficiently large (if not then $y'(x_k) = 0$, hence $y''(u_k) = y(u_k)\ f(u_k) = 0$ for some sequences x_k, $u_k \to \infty$ $(k\to\infty)$ which gives a contradiction).

Note that

$$w' = -\sqrt{f}\ \left(w + \frac{f'}{4f^{3/2}} + \sqrt{1 + \frac{(f')^2}{16f^3}}\right)\left(w + \frac{f'}{4f^{3/2}} - \sqrt{1 + \frac{(f')^2}{16f^3}}\right) \tag{1.94}$$

We shall prove $w(x) \to 1$ $(x\to\infty)$ and consider the following three cases:

a.1. $w'(x) > 0$ for all $x > x_0$.

Then w is increasing and $\lim\limits_{x\to\infty} w(x) =: A \in [0, \infty]$ exists. If $A = \infty$, then (1.94) implies $w'(x)/\sqrt{f(x)} \to -\infty$ $(x\to\infty)$ which contradicts $w' > 0$.

If $A < \infty$, from (1.94) it follows that $\lim\limits_{x\to\infty} \frac{w'(x)}{\sqrt{f(x)}} = 1 - A^2$.

If $A \neq 1$ this implies

$$w(x) \sim (1 - A^2) \int_0^x \sqrt{f(s)}ds. \tag{1.95}$$

Since $w(x) \leq A + 1$ for all sufficiently large x we have $\sqrt{f(x)} = \frac{y'(x)}{y(x)\ w(x)} \geq \frac{y'(x)}{(A+1)\ y(x)}$ and $y(x) \to \infty$ entails $\int_0^x \sqrt{f(s)}ds \to \infty$ $(x\to\infty)$.

Combination with (1.95) then gives $w(x) \to \infty$ which gives a contradiction as above. Hence $w(x) \to 1$ $(x\to\infty)$.

a.2. As in part a.1 we find $\lim\limits_{x\to\infty} w(x) = 1$ in case $w'(x) < 0$ for all $x > x_0$.

a.3. If $w'(x) = 0$ infinitely often we have

$$w(x) = -\frac{f'(x)}{4f(x)^{3/2}} + \sqrt{1 + \frac{\{f'(x)\}^2}{16f^3(x)}}$$

for every x where $w'(x) = 0$.

Since w is monotone between consecutive zeros of w' we find
$\lim_{x\to\infty} w(x) = 1$.

The proof can be completed as follows.
Since $f = y''/y$ and $\lim_{x\to\infty} w(x) = 1$ we have $(y'(x))^2/y(x)y''(x) \to 1$ $(x\to\infty)$ which is equivalent to $y'' \in \Gamma(b)$ by theorem 1.28. Moreover since $w(x) \to 1$ we have

$$b(x) \sim \frac{y'(x)}{y''(x)} \sim \frac{y(x)}{y'(x)} \sim \frac{1}{\sqrt{f(x)}} \quad (x\to\infty).$$

Application of cor. 1.29.2 then finishes the proof.

b. If f is not differentiable then, by theorem 1.35 there exists a differentiable function g such that $g(x) \sim f(x)$ $(x\to\infty)$ and $\left(\frac{1}{\sqrt{g(x)}}\right)' \to 0$ $(x\to\infty)$.
Then for any $\varepsilon > 0$ there exists $x_1 = x_1(\varepsilon)$ such that
$(1-\varepsilon)g(x) \leq f(x) \leq (1+\varepsilon)g(x)$ for $x \geq x_1$.
Now consider the positive solutions of the differential equations
$u'' = (1-\varepsilon)gu$ and $v'' = (1+\varepsilon)gv$ which tend to infinity.
Note that for $x \geq x_1$ we have

$$\frac{d}{dx}\{y'(x)u(x) - u'(x)y(x)\} = y(x)u(x)\{f(x) - (1-\varepsilon)g(x)\} \geq 0.$$

Hence for $x \geq x_1$ we have

$$y'(x)u(x) - u'(x)y(x) \geq c, \text{ where c is a constant.}$$

This implies

$$\frac{y'(x)}{y(x)\sqrt{f(x)}} \geq \frac{c}{u(x)y(x)\sqrt{f(x)}} + \frac{u'(x)}{u(x)\sqrt{f(x)}} \text{ for } x \geq x_1. \tag{1.96}$$

By part a of the proof we have $u'(x) \sim u(x)\sqrt{(1-\varepsilon)g(x)}$ $(x\to\infty)$, which implies

$$\frac{u'(x)}{u(x)\sqrt{f(x)}} \to \sqrt{1-\varepsilon} \tag{1.97}$$

hence $u(x)y(x)\sqrt{f(x)} \sim \sqrt{1-\varepsilon}^{\,-1}u'(x)y(x) \to \infty$ since $y(x) \to \infty$ $(x\to\infty)$ and $u'(x) \to \infty$ (note that $u' \in \Gamma$).

Combination of (1.96) and (1.97) gives $\underline{\lim}_{x\to\infty} \frac{y'(x)}{y(x)\ \sqrt{f(x)}} \geq \sqrt{1-\varepsilon}$ and similarly we find $\overline{\lim}\ y'(x)/\{y(x)\ \sqrt{f(x)}\} \leq \sqrt{1+\varepsilon}$.
Since $\varepsilon > 0$ is arbitrary this implies $y'(x) \sim y(x)\ \sqrt{f(x)}$ and the proof can be completed as in a.1. ◇

I. 5 Sequential versions of regular variation.

In this paragraph we consider representation and embedding theorems for RV-sequences and Π-sequences. We start with a formal definition.

Definition 1.37

A sequence of positive numbers $\{c_n\ ;\ n = 0, 1, 2, \ldots\}$ is said to be regularly varying (RV) if

$$\lim_{n\to\infty} c_{[xn]}/c_n = x^{\alpha} \quad (x > 0) \text{ for some } \alpha \in \mathbb{R}. \tag{1.98}$$

Notation: $\{c_n\}$ is a RV_α - sequence. ◇

It is clear that if the function $c_{[x]}$ is regularly varying with index α, then $\{c_n\}$ is a RV_α-sequence.
The next theorem gives a converse result, which enables us to use earlier results about RV-functions by "embedding" the sequence in an RV-function.

Theorem 1.38

If the sequence $\{c_n\}$ of positive numbers satisfies $\lim_{n\to\infty} c_{[xn]}/c_n = \psi(x)$ for $x > 0$, where $0 < \psi(x) < \infty$ for $x > 0$, then $f(x) := c_{[x]}$ is regularly varying. In particular $\{c_n\}$ is a RV_α-sequence.

Proof

We first prove $c_{n-1}/c_n \to 1$ $(n\to\infty)$.

Since $\pi^{-1}[n\pi] = \max\{\frac{k}{\pi};\ \frac{k}{\pi} < n\}$, we have $[\pi^{-1}[n\pi]] = n-1$ for all $n \in \mathbb{N}$.
Hence

$$\frac{c_{n-1}}{c_n} = \frac{c_{[\pi^{-1}[n\pi]]}}{c_n} = \frac{c_{[\pi^{-1}[n\pi]]}}{c_{[n\pi]}} \cdot \frac{c_{[n\pi]}}{c_n} \to \psi(\pi^{-1}) \cdot \psi(\pi). \tag{1.99}$$

Hence $\lim_{n\to\infty} c_{n-1}/c_n$ exists.

Since $\frac{c_{[[n/2]2]}}{c_n} = \begin{cases} 1 & \text{if } n \text{ is even} \\ c_{n-1}/c_n & \text{if } n \text{ is odd} \end{cases}$ and moreover $\lim_{n\to\infty} c_{[[n/2]2]}/c_n =$ $\psi(2)\ \psi(\frac{1}{2})$, we find $\psi(2)\ \psi(\frac{1}{2}) = \psi(\pi)\ \psi(\pi^{-1}) = 1$.
Combination with (1.99) then gives $\lim_{n\to\infty} c_{n-1}/c_n = 1$.

Then also $c_{n+k}/c_n \to 1$ $(n\to\infty)$ for any fixed $k \in \mathbb{Z}$.

Since $0 \leq tx - [t]\ x \leq [x] + 1$ we have for any fixed $x > 0$

$$\frac{f(tx)}{f(t)} = \frac{c_{[x[t]]}}{c_{[t]}} \cdot \frac{c_{[tx]}}{c_{[[t]x]}} \to \psi(x)\ (t\to\infty),$$

hence $f \in RV_\alpha$ by theorem 1.2. ◇

Corollary 1.39

$\{c_n\}$ is a RV_α-sequence with $\alpha > -1$ if and only if

$$\lim_{n\to\infty} \frac{1}{n\ c_n} \sum_{k=1}^{n} c_k = \frac{1}{\alpha+1} \text{ with } \alpha > -1.$$

Proof:

Use theorem 1.38, 1.2 and 1.4. ◇

Next we prove a similar statement concerning the class Π. Here however, we have to require beforehand that the auxiliary function is in RV_0.

Definition 1.40

A sequence of positive numbers $\{c_n,\ n = 0, 1, 2, \ldots\}$ is said to be a $\Pi(a)$-sequence if there exists a RV_0-sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} \frac{c_{[nx]} - c_n}{a_n} = \log x \text{ for all } x > 0. \tag{1.100}$$

◇

The next result shows that it is possible to use earlier results obtained in this chapter for the class Π.

Theorem 1.41

If the sequence $\{c_n\}$ is a $\Pi(a)$-sequence, then the function f defined by $f(t) := c_{[t]}$ belongs to the class Π.

Proof

Since $\{a_n\}$ is a RV_0 sequence, by definition 1.40 we have

$$\lim_{n\to\infty} \frac{c_{[[nx]z]} - c_n}{a_n} = \lim_{n\to\infty} \frac{c_{[[nx]z]} - c_{[nx]}}{a_{[nx]}} \cdot \frac{a_{[nx]}}{a_n}$$

$$+ \lim_{n\to\infty} \frac{c_{[nx]} - c_n}{a_n} = \ln z + \ln x \text{ for all } x,\ z > 0.$$

This implies (take $x = \pi$, $z = \pi^{-1}$) that

$$\lim_{n\to\infty} \frac{c_{n-1} - c_n}{a_n} = 0, \text{ which implies } \lim_{n\to\infty} \frac{c_{n+k} - c_n}{a_n} =$$

$$- \lim_{n\to\infty} \sum_{j=1}^{k} \frac{c_{n+j-1} - c_{n+j}}{a_{n+j}} \frac{a_{n+j}}{a_n} = 0 \text{ for } k \in \mathbb{Z} \text{ fixed,}$$

since $\{a_n\}$ is a RV_0-sequence.

Hence for all $x > 0$ we have

$$\lim_{t\to\infty} \frac{c_{[tx]} - c_{[t]}}{a_{[t]}} = \lim_{t\to\infty} \frac{c_{[tx]} - c_{[[t]x]}}{a_{[tx]}} \frac{a_{[tx]}}{a_{[t]}} +$$

$$+ \lim_{t\to\infty} \frac{c_{[[t]x]} - c_{[t]}}{a_{[t]}} = \ln x$$

(use the fact that $[tx] - [[t]x]$ is bounded). ◇

The final theorem is not concerned with RV sequences proper but it provides a criterion for regular variation when one only has convergence through a certain sequence of reals tending to infinity.

Theorem 1.42

Let $f: \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and let the positive sequence $\{a_n\}$ satisfy $a_n \to \infty$ and $a_{n+1}/a_n \to 1$ $(n\to\infty)$.

Suppose $\lim_{n\to\infty} \frac{f(a_n t)}{b_n} = \psi(t)$ exists for all t in an open interval V of $\mathbb{R}^+$, where b_n and $\psi(t)$ are finite and positive for $n \geq 1$ and $t \in V$.

Then $f \in RV_\alpha^\infty$ for some $\alpha \in \mathbb{R}$.

Proof

Note that $V \cap u^{-1} V \neq \emptyset$ for all u in a non-empty interval K.

If t, ut $\in$ V we have $\frac{f(a_n ut)}{f(a_n t)} \to \frac{\psi(ut)}{\psi(t)}$ $(n \to \infty)$.

Hence if we write $f_u(t) = f(ut)/f(t)$ we have

$$f_u(a_n t) \to \frac{\psi(ut)}{\psi(t)} \; (n \to \infty)$$

for all $t \in V \cap (u^{-1}V)$.

Now write $f^*(t) := f_u(e^t)$, $a_n^* := \log a_n$.
Then $f^*(t + a_n^*)$ converges as $n \to \infty$ for all t in a non-empty open interval J.
Let $\varepsilon > 0$ be arbitrary.
Define for $k \in \mathbb{Z}$, $m \in \mathbb{N}$

$$C_{k,m} := \bigcap_{n \geq m} \{t \in \mathbb{R};\; f^*(t + a_n^*) \in [k\varepsilon - \varepsilon, k\varepsilon + \varepsilon]\}.$$

Hence, since the set $C_{k,m}$ is closed for all k, m, J is non-empty and open and $J \subset \bigcup_{k,m} C_{k,m}$, we can apply Baire's category theorem (see Hewitt and Stromberg p. 68). It follows that one of the sets $C_{k,m}$ contains an open interval I, which means that

$$k\varepsilon - \varepsilon \leq f^*(t + a_n^*) \leq k\varepsilon + \varepsilon \text{ for } n \geq m,\; t \in I.$$

Since $a_n^* \to \infty$, $a_{n+1}^* - a_n^* \to 0$, it follows that $\bigcup_{n \geq m} a_n^* + I$ contains an interval of the form $[t_0, \infty)$, hence

$$k\varepsilon - \varepsilon \leq f^*(t) \leq k\varepsilon + \varepsilon \text{ for all } t \geq t_0.$$

Hence $\lim_{t \to \infty} f_u(t) = \lim_{t \to \infty} f(ut)/f(t)$ exists and is finite and positive for all $u \in K$. The proof is finished by an application of theorem 1.2. ◇

I.6 Discussion

We do not attempt to give a full bibliographic account of the material of this chapter. Instead, we give at least one key reference for each of the main topics.

Most of the results from section 1.1 are already present in J. Karamata's papers (1930) and (1933) in some form. The present form of the uniform convergence theorem (th. 1.3) and of the representation theorem (th. 1.5) is due to Van Aardenne-Ehrenfest et al. (1949) and de Bruijn (1959). Our method of proof for theorem 1.2 stems from Cziszar and Erdös (1964) and has been used also by Bingham and Goldie (1982). Theorem 1.8 (a general-kernel Abelian theorem) is due to Karamata (1962).
Properties 1 up to 4 from prop. 1.7 originate from Karamata's original papers. A reference for the inequalities in properties 5 up to 7 is Pitman (1968). A reference for the statements on inverses of RV functions is de Haan (1970).
Property 11 has been taken from Feller (1971) and property 12 from de Haan (1977).
Some of the results of section 1.2 (class Π) appear in Bojanic and Karamata (1963), many of them have been taken from de Haan (1970) after a recursion to monotone functions via the uniform convergence theorem (th. 1.14). Theorem 1.20 is a version of a theorem due to Bingham and Teugels (1980); the present form is believed to be new. The notion of inversely asymptotic functions and its applications have been taken from Balkema, de Haan and Geluk (1979).
The material of section 1.3 (class Γ) has been taken mainly from de Haan (1970). Some of the properties of prop. 1.31 are new. The problem how to extend the theory of the class Γ to functions which are not monotone is still open. The results on Beurling slowly varying functions (section 1.4) are due (with different proofs) to Bloom (1976).
The application to differential equations (th. 1.36) is due to Omey (1981).
The section on regularly varying sequences is a compilation (except for the material on $\Pi(a)$ sequences) of the articles by Bojanic and Seneta (1973), Galambos and Seneta (1973) and Weissman (1974), the latter with improved proof.
Theorem 1.42, due to Kendall, has been presented here with a new short proof due to Balkema.
We end with some remarks about generalizations.
A theory of regular variation for functions f: $\mathbb{R} \to \mathbb{C}$ has been developed by Vuilleumier (1976).
A reference for the notion of regular variation and Π-variation for functions f: $\mathbb{R}_n^+ \to \mathbb{R}^+$ is de Haan and Omey (1983).
Generalizations of the class Π can be found in Geluk: Π-regular variation (1981) and Omey and Willekens: Π-variation with remainder (1986). The latter title alludes to the notion of "slow variation with remainder", see Aljancić, Bojanić and Tomić (1974), Goldie and Smith (1987). A somewhat different generalization will be discussed in chapter 3.

II. Transforms of regularly varying functions

In this chapter all functions we consider are assumed to be measurable unless otherwise stated.

In Chapter 1 we have seen that regular variation is preserved under certain transformations. Under suitable regularity conditions we have for example:

$$\text{if } f \in RV_\alpha \ (\alpha > 0), \text{ then } \int_0^t f(s)ds \in RV_{\alpha+1}, \ \sup_{0<t\leq x} f(t) \in RV_\alpha$$

and the generalised inverse $f^{+} \in RV_{\alpha^{-1}}$. Under somewhat more restrictive conditions the converse statements also hold. In this chapter we study two other transforms that preserve regular variation: the complementary function (see definition 2.1) and the Laplace transform (see definition 2.11). In fact, when discussing the Laplace transform we will need the results about the complementary function.

II 1. The complementary function

Definition 2.1

a. Suppose $f\colon \mathbb{R}^{+} \to \mathbb{R}$ is bounded on finite intervals of $\mathbb{R}^{+}$, $f(\infty) = \infty$ and $f(t) = o(t)$ $(t\to\infty)$. Then the complementary function f^{c} is defined by

$$f^{c}(y) = \sup \{f(x) - xy;\ x > 0\},\ y > 0.$$

b. Suppose $f\colon \mathbb{R}^{+} \to \mathbb{R}$ is bounded in every interval (a,∞) for $a > 0$ and $f(0+) = \infty$. Then the inverse complementary function f_c is defined by

$$f_c(x) = \inf \{f(y) + xy;\ y > 0\},\ x > 0. \qquad \diamond$$

We shall concentrate on results for the complementary function, which plays a role in Tauberian theorems for Laplace transforms. Similar results hold for inverse complementary functions.

In case

$$f(x) = \int_0^x s(t)\, dt < \infty \quad \text{for } x > 0 \qquad (2.1)$$

with s: $(0, \infty) \to (0, \infty)$ continuous and strictly decreasing, then the transform f^c takes a particularly simple form. We find the supremum by differentiation:

$$f^c(y) = f(s^{\leftarrow}(y)) - ys^{\leftarrow}(y) = \int_0^{s^{\leftarrow}(y)} s(u)du - ys^{\leftarrow}(y) = \int_y^{\infty} s^{\leftarrow}(u)du, \qquad (2.2)$$

where $s^{\leftarrow}$ is the inverse function of s.

Note that any complementary function f^c is convex, since the concave upper hull of f has the same complementary function as f itself. Compare (2.1) and (2.2).

Now a regularly varying function with index between 0 and 1 is close in a certain sense to a concave function (see lemma 1.23). In order to derive relations similar to (2.1) and (2.2) for functions in RV or Π, we use the concept of inversely asymptotic functions (see definition 1.21). The following lemma is an immediate consequence of definition 2.1 and enables us to derive the asymptotic behaviour of f^c from the behaviour of f.

<u>Lemma 2.2</u>

Suppose f_1, f_2: $\mathbb{R}^+ \to \mathbb{R}$ are bounded on finite intervals of $\mathbb{R}^+$, tend to ∞ and $f_i(t) = o(t)$ for $t\to\infty$, i = 1, 2.

(i) If $f_1 \leq f_2$ then $f_1^c \leq f_2^c$.

(ii) If $f_1 = f_2$ on a neighbourhood of ∞, then $f_1^c = f_2^c$ on a right-neighbourhood of 0.

(iii) If $f_2(t) = f_1(at)$ with $a > 0$, then $f_2^c(s) = f_1^c(s/a)$.

◇

<u>Theorem 2.3</u>

Suppose f satisfies the assumptions of definition 2.1 and let s: $\mathbb{R}^+ \to \mathbb{R}^+$ be decreasing and continuous, $s(t) \to 0$ $(t\to\infty)$ and

$$\int_0^1 s(x)dx < \infty.$$

Then

$$f(t) \overset{*}{\sim} \int_0^t s(x)dx \quad (t\to\infty) \qquad (2.3)$$

implies

$$f^c(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+), \tag{2.4}$$

where $s^{\leftarrow}$ is the inverse function of s.
Conversely if f is non-decreasing and $s \in RV_{-\gamma}^{\infty}$ with $0 < \gamma \leq 1$ (hence $s^{\leftarrow} \in RV^0_{-\gamma^{-1}}$), then (2.4) implies (2.3). ◊

Before giving the proof of the theorem we state the following corollary which is immediate by lemma 1.23.

Corollary 2.4
Suppose f: $\mathbb{R}^+ \to \mathbb{R}$ satisfies the assumptions of definition 2.1a.

(i) Let α,β be related by $\alpha^{-1} + \beta^{-1} = 1$.
Then

$$f \in RV_\alpha^\infty \text{ with } 0 < \alpha < 1 \tag{2.5}$$

implies

$$f^c \in RV_\beta^0 \text{ with } \beta < 0 . \tag{2.6}$$

(ij) Also

$$f \in \Pi \tag{2.7}$$

implies

$$f^c \in \Pi^0. \tag{2.8}$$

Conversely if f is non-decreasing (2.6) implies (2.5) and (2.8) implies (2.7). ◊

Proof of theorem 2.3
First we prove the Abelian part (the implication (2.3) → (2.4)).
Recall that (definition 1.21) the relation (2.3) means: For every a > 1 there exists a constant $t_0 = t_0(a)$ such that

$$\int_0^{t/a} s(x)dx \leq f(t) \leq \int_0^{ta} s(x)dx \text{ for } t \geq t_0.$$

The three implications in lemma 2.2 give for some $u_0 > 0$

$$\int_{ua}^\infty s^{\leftarrow}(x)dx \leq f^c(u) \leq \int_{u/a}^\infty s^{\leftarrow}(x)dx \text{ for } 0 < u \leq u_0,$$

which means that (2.4) holds.

Conversely, suppose (2.4) holds and f is non-decreasing.
Note that the function f^c satisfies the assumptions of definition 2.1b. Hence $(f^c)_c$ exists and is in fact the concave upper hull of f. Application of the analogue of the Abelian part ((2.3) → (2.4)) of theorem 2.3 for the inverse complementary function of f^c shows that

$$f_1(t) := (f^c)_c(t) \overset{*}{\sim} \int_0^t s(x)dx \ (t \to \infty).$$

Application of lemma 1.23 gives $f_1 \in \Pi$ or $RV^{\infty}_{1-\gamma}$ with $0 < \gamma < 1$.
By the definition of the classes Π and RV this implies for $\gamma \in (0,1]$ and $0 < \varepsilon < 1$

$$\lim_{t\to\infty} \frac{f_1(t(1+\varepsilon)) - f_1(t)}{f_1(t) - f_1(t(1-\varepsilon))} = \int_1^{1+\varepsilon} s^{-\gamma}ds / \int_{1-\varepsilon}^{1} s^{-\gamma}ds < 1, \tag{2.9}$$

where the case $\gamma = 1$ corresponds to $f_1 \in \Pi$.
As a consequence of this asymptotic concavity, for fixed $a > 1$ any interval (t,at) for t sufficiently large will contain a point x with $f(x) = f_1(x)$ (apply (2.9) with $(1 + \varepsilon)/(1 - \varepsilon) = a$). Hence since f is non-decreasing $f_1(t) \le f_1(x) = f(x) \le f(at)$. Since obviously $f \le f_1$, we find $f \overset{*}{\sim} f_1$ and hence f satisfies (2.3). ◊

It follows from the above discussion that theorem 2.3 and corollary 2.4 above give results in case $f \in RV_\alpha$ with $0 < \alpha < 1$ and in case $f \in \Pi$, which can be seen as an extension to $\alpha = 0$. It is also possible to prove an extension for $\alpha = 1$. In order to see which order of magnitude for f is appropriate for such an extension, we recall that the existence of f^c requires $f(s) = o(s)$, $s\to\infty$ (definition 2.1a). It turns out that, as in the case $\alpha = 0$, the appropriate function class is again closely related to the class Π.
In order to formulate the results we define two classes of functions, related with the classes Π and Γ and the relation $\overset{\cdot\cdot}{\sim}$ (def. 2.6), which is the analogue of the relation $\overset{*}{\sim}$ (see definition 1.21) appropriate for this context.

<u>Definition 2.5</u>
A measurable function $f: \mathbb{R}^+ \to \mathbb{R}$ is said to belog to the class Π^- if there exists a positive function a such that for all $x > 0$

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = - \log x.$$

Notation: $f \in \Pi^-$ or $f \in \Pi^-(a)$.
If the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing and if there exists a positive function b such that

$$\lim_{u \to 0+} \frac{f(u+x\, b(u))}{f(u)} = e^{-x} \text{ for } x > 0, \tag{2.10}$$

then f belongs to the class $\Gamma^{(0)}$.
(compare definition 1.24). ◇
Notation: $f \in \Gamma^{(0)}$.

Note that $f \in \Pi^-$ if and only if $-f \in \Pi$. Also it can be proved to see that $f(t) \in \Gamma^{(0)}$ if and only if $f(1/t) \in \Gamma$.
From theorem 1.28 it follows by a change of variable that if $f \in \Gamma^{(0)}$, then there exists a differentiable function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta'(u) \to 0$, $\beta(u) \to 0$ $(u \to 0+)$ and

$$f(u) \sim \exp \left\{ \int_u^\infty \frac{dx}{\beta(x)} \right\} \quad (u \to 0+). \tag{2.11}$$

Conversely, if

$$f(u) \sim \exp \left\{ \int_u^\infty \frac{c(x)}{\beta(x)}\, dx \right\} \quad (u \to 0+), \tag{2.12}$$

where $c(x) \to c > 0$ $(x \to 0+)$ and β as above, then $f \in \Gamma^{(0)}$.

Definition 2.6
Suppose $f_1, f_2\colon \mathbb{R}^+ \to \mathbb{R}$. We say

$$f_1(t) \overset{*}{\sim} f_2(t),\ t\to\infty \text{ (or } f_1 \overset{*}{\sim} f_2) \tag{2.13}$$

if for every constant $a > 1$ there exists a $t_0 = t_0(a)$ such that for all $t \geq t_0$

$$f_1(ta) \leq a\, f_2(t)$$
and
$$f_2(ta) \leq a\, f_1(t). \tag{2.14}$$
◇

Note that $f_1(t) \overset{*}{\sim} f_2(t)$ $(t\to\infty)$ if and only if $-\frac{f_1(t)}{t} \overset{*}{\sim} -\frac{f_2(t)}{t}$.

Before formulating an extension of theorem 2.3 above to the case $\gamma = 0$ (or $\alpha = 1$ in cor. 2.4) we give a lemma that is helpful for understanding the role of the classes Π^- and $\Gamma^{(0)}$.

Lemma 2.7

(i) Suppose $f: \mathbb{R}^+ \to \mathbb{R}^+$. Then $f(t)/t \in \Pi^-$ if and only if there is an eventually decreasing continuous function $s \in \Pi^-$ such that

$$f(t) \overset{*}{\sim} \int_0^t s(x)dx \qquad (t\to\infty). \tag{2.15}$$

(ij) Suppose $f: \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing. Then $f \in \Gamma^{(0)}$ if and only if there is a decreasing $t \in \Gamma^{(0)}$ such that

$$f(u) \sim \int_u^\infty t(x)dx \qquad (u \to 0+). \tag{2.16}$$

Proof

(i) Suppose $f(t)/t \in \Pi^-$. By Lemma 1.23a there exists a decreasing continuous function ψ with $-\psi \in \Pi(a)$ such that $- f(t)/t \overset{*}{\sim} -\psi(t)$ $(t\to\infty)$.
Application of theorem 1.17 gives

$$-\psi(t) + (te)^{-1} \int_0^{te} \psi(x)dx = o(a(t)).$$

Hence $-\psi(t) \overset{*}{\sim} - t^{-1} \int_0^t \psi(xe)dx$ by proposition 1.22 (ij) and (2.15) is satisfied with $s(x) = \psi(xe)$.

Conversely, if $s \in \Pi^-$ then $t^{-1} \int_0^t s(x)dx \in \Pi^-$ (see theorem 1.17).
From (2.15) and prop. 1.22 (ij) it then follows that $f(t)/t \in \Pi^-$.

(ij) If $f \in \Gamma^{(0)}$ we have the representation (2.11). The derivative of the right-hand side of (2.11) is

$$t(u) = - c \exp\left\{\int_u^1 \frac{1 + \beta'(x)}{\beta(x)}dx\right\} \quad (c \in \mathbb{R}) ,$$

which is in $\Gamma^{(0)}$ since it satisfies the representation (2.12).
The converse part is a consequence of the analogue for $\Gamma^{(0)}$ of corollary 1.29.2 and proposition 1.31.7.

◊

Theorem 2.8

Suppose f satisfies the assumptions of definition 2.1a and let s: $\mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, $s(\infty) = 0$ and $\int_0^1 s(x)dx < \infty$.
Then

$$f(t) \stackrel{\bullet}{\sim} \int_0^t s(x)dx \quad (t\to\infty) \tag{2.17}$$

implies

$$f^c(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \ (u \to 0+). \tag{2.18}$$

Conversely if $f(t)/t$ is non-increasing, $s \in \Pi^-$ is decreasing, then (2.18) implies (2.17). ◇

Before we prove theorem 2.8 we state the following corollary which is an immediate consequence of theorem 2.8 and lemma 2.7.

Corollary 2.9

Suppose f: $\mathbb{R}^+ \to \mathbb{R}$ satisfies the assumptions of definition 2.1a.
Then $f(t)/t \in \Pi^-$ implies $f^c \in \Gamma^{(0)}$.
Conversely if $f(t)/t$ is non-increasing $f^c \in \Gamma^{(0)}$ implies $f(t)/t \in \Pi^-$. ◇

Proof of theorem 2.8

Suppose f satisfies (2.17). By definition 2.6 this means : for every $a > 1$ there exists a constant $t_o(a)$ such that for all $t \geq t_o$

$$\frac{1}{a}\int_0^{ta} s(x)\ dx \leq f(t) \leq a \int_0^{t/a} s(x)\ dx.$$

Application of lemma 2.2 then gives for some $u_o > 0$

$$\frac{1}{a}\int_u^\infty s^{\leftarrow}(x)dx \leq f^c(u) \leq a \int_u^\infty s^{\leftarrow}(x)dx \text{ for } 0 < u \leq u_o,$$

which means that (2.18) holds.
Conversely, suppose (2.18) holds and f is non-decreasing.
Note that the function f^c satisfies the assumptions of definition 2.1b.
Hence $(f^c)_c$ exists and is in fact the concave upper hull of f.

Application of the analogue of the Abelian part ((2.17) → (2.18)) of theorem 2.8 for the inverse complementary function of f^c shows that

$$f_1(t) := (f^c)_c(t) \overset{\cdot}{\sim} \int_0^t s(x)dx \quad (x\to\infty). \tag{2.19}$$

Application of lemma 1.21 gives $f_1(t)/t \in \Pi^-$.
Suppose a is the auxiliary function of $s \in \Pi^-$. Then (2.19) implies

$$\lim_{t\to\infty} \frac{f_1(t(1+\varepsilon)) - 2f_1(t) + f_1(t(1-\varepsilon))}{ta(t)} =$$

$$\lim_{t\to\infty}[(1+\varepsilon)\{\frac{f_1(t(1+\varepsilon))}{t(1+\varepsilon)} - \frac{f_1(t)}{t}\} + (1-\varepsilon)\ \{\frac{f_1(t(1-\varepsilon))}{t(1-\varepsilon)} - \frac{f_1(t)}{t}\}]/a(t)$$

$$= -(1+\varepsilon)\ \ln(1+\varepsilon) - (1-\varepsilon)\ \ln(1-\varepsilon) < 0$$

for $0 < \varepsilon < 1$. Since f_1 is the concave upper hull of f it follows that for fixed $a > 1$ any interval (t, at) contains a point x such that $f_1(x) = f(x)$ provided t is sufficiently large. Since f_1 is concave, $f_1(t)/t$ is non-increasing. Hence

$$\frac{f_1(at)}{at} \le \frac{f_1(x)}{x} = \frac{f(x)}{x} \le \frac{f(t)}{t}\ .$$

for all t sufficiently large. On the other hand we find since $f_1 \ge f$

$$\frac{f_1(t)}{t} \ge \frac{f_1(at)}{at} \ge \frac{f(at)}{at}.$$

This proves $f \overset{\cdot}{\sim} f_1$, hence (2.17). ◇

Results similar to theorem 2.8 with a suitable definition of the complementary function can be given in case $\alpha > 1$ and $\alpha < 0$. This possibility is mentioned in the paper of Bingham and Teugels (1975).

II 2. The Laplace transform

J. Karamata introduced the concept of regular variation in 1930 for use as a suitable condition for Abelian and Tauberian theorems for Laplace transforms. His Tauberian theorem generalized an earlier result of Hardy and Littlewood

(1930) for functions f(x) asymptotic to x^{α} ($\alpha \geq 0$) as $x \to \infty$. We start here with Karamata's result (theorem 2.11). Next we treat a similar generalization of the case $f(x) = c \log x + o(1)$ (this involves the class Π; see theorems 2.14 and 2.16). We proceed with a generalization of the case $\log f(x) \sim x^{\alpha}$ ($0 < \alpha < 1$), due to Kohlbecker (corollary 2.20a) and end with the borderline cases $\alpha = 0$ (which corresponds in some sense to the case $\alpha = \infty$ in Karamata's Tauberian result; see cor. 2.20b) and $\alpha = 1$ (see theorem 2.26). That way the whole spectrum from functions like log x to functions like exp(x/log x) is covered. Note that for $\alpha > 1$ the Laplace transform does not converge (see definition 2.10).

Definition 2.10

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is measurable and $\int_0^\infty e^{-tx}|f(x)|dx < \infty$ for all $t > 0$. The Laplace transform $\hat{f}$ of the function f is for $t > 0$ defined by

$$\hat{f}(t) = t \int_0^\infty e^{-tx} f(x)\, dx. \tag{2.20}$$

◇

If f is non-decreasing and $f(0+) = 0$ we can write $\hat{f}(t) = \int_0^\infty e^{-tx}\, df(x)$.

Theorem 2.11 (Karamata, 1931)

Suppose $\alpha \geq 0$ and f satisfies the assumptions of definition 2.10. If

$$f \in RV_{\alpha}^{\infty} \tag{2.21}$$

then

$$\hat{f} \in RV_{-\alpha}^{0} \tag{2.22}$$

and

$$\hat{f}(1/t) \sim \Gamma(1 + \alpha) f(t) \quad (t \to \infty). \tag{2.23}$$

Conversely if $x^{\beta} f(x)$ is positive and non-decreasing for some $\beta \in [0,1)$ and all $x > 0$, then (2.22) implies (2.21). ◇

Proof

The implication (2.21) $\to$ (2.23) is a special case of theorem 1.8. Now (2.22) follows.

Next suppose $x^{\beta}f(x)$ is non-decreasing for some $0 \leq \beta < 1$ and (2.22) holds. For $a,v > 0$

$$\hat{f}(v) \geq v \int_a^{\infty} e^{-vx} f(x)dx \geq f(a)/q(av)$$

with $q(v) := \{v^{\beta} \int_v^{\infty} u^{-\beta}e^{-u}du\}^{-1}$.

Hence for $t,x,p > 0$

$$x^{\beta}f(xt)/\hat{f}(p/t) \leq x^{\beta}q(px). \tag{2.24}$$

Since $x^{\beta}f(xt)/\hat{f}(t^{-1})$ is non-decreasing in x for all $t > 0$ and bounded by $x^{\beta}q(x)$, we can apply the selection principle (Widder (1941), p. 26): if $t_n \to \infty$, there exists a subsequence $t_{n'} \to \infty$ and a function ϕ such that

$$\lim_{n' \to \infty} f(xt_{n'})/\hat{f}(t_{n'}^{-1}) = \phi(x) \tag{2.25}$$

for each continuity point x of ϕ. It is now sufficient to prove that each such function is of the form $\phi(x) = x^{\alpha}/\Gamma(1+\alpha)$. Note that (2.22) and (2.25) imply

$$\lim_{n' \to \infty} f(xt_{n'})/\hat{f}(p/t_{n'}) = p^{\alpha}\, \phi(x) \tag{2.26}$$

for each continuity point x of ϕ.

By Lebesgue's theorem on dominated convergence (note that $q(v) \sim e^{v}$, $v \to \infty$, and $q(v) \sim cv^{-\beta}$, $v \to 0+$), we get for $s > p$ (using (2.24) and (2.26))

$$\lim_{n' \to \infty} s \int_0^{\infty} e^{-xs}\, f(xt_{n'})/\hat{f}(p/t_{n'})\, dx = s \int_0^{\infty} e^{-xs}\, p^{\alpha}\phi(x)\, dx < \infty.$$

But then also, since $\hat{f}(p/t_{n'}) \sim \hat{f}(t_{n'}^{-1})p^{-\alpha}$ $(n' \to \infty)$,

$$\lim_{n' \to \infty} s \int_0^{\infty} e^{-xs}\, f(xt_{n'})/\hat{f}(t_{n'}^{-1})dx = s \int_0^{\infty} e^{-xs}\, \phi(x)dx$$

which is now true for all $s > 0$. On the other hand we know

$$\lim_{n'\to\infty} s\int_0^\infty e^{-xs}f(xt_{n'})/\hat{f}(t_{n'}^{-1})dx = \lim_{n'\to\infty} \hat{f}(s/t_{n'})/\hat{f}(t_{n'}^{-1}) = s^{-\alpha}$$

The uniqueness property of the Laplace transform (Widder (1941) p, 80) now gives $\phi(x) = x^\alpha/\Gamma(1+\alpha)$ for $x > 0$. ◇

Remarks

1. For non-decreasing f it is also possible to prove the implication (2.23) → $f \in RV$. For details the reader is referred to Drasin's paper (1968).

2. Note that if $f : \mathbb{R}^+ \to \mathbb{R}$ is locally bounded and measurable and if

$$\log f(x) = o(x) \quad (x \to \infty),$$

then $\hat{f}(t) < \infty$ for $t > 0$.
In particular this is true if $\log f(x) = o(x)$ is replaced by $f \in RV$.

Corollary 2.12 (= proposition 1.7.12)

Any $f \in RV_\alpha^\infty$ with $\alpha + 1 \in \mathbb{N}$ is asymptotic to a function f_1 with the property that the absolute values of all its derivatives are regularly varying.

Proof

If $\alpha > 0$, there is an increasing function $f_o(t) \sim f(t)$ $(t\to\infty)$ by proposition 1.7.3.
Define $f_1(t) = \hat{f}_o(1/t)/\Gamma\,(1 + \alpha)$. For $\alpha < 0$ a similar proof can be given. ◇

Our next result contains an o-version of the above theorem.

Theorem 2.13

Suppose f satisfies the assumptions of definition 2.10 and let $g \in RV_\alpha^\infty$ with $\alpha \geq 0$.
If

$$f(t) = o(g(t)) \quad (t\to\infty) \tag{2.27}$$

then

$$\hat{f}(1/t) = o(g(t)) \quad (t\to\infty). \tag{2.28}$$

Conversely if f is non-decreasing and (2.28) holds, then (2.27) is true.

Proof

Suppose (2.27) with $g \in RV_\alpha$, $\alpha \geq 0$. Without loss of generality we may suppose that g satisfies the assumptions of definition 2.10.
For $\varepsilon > 0$ arbitrary, there exists t_o such that $f(t) \leq \varepsilon\, g(t)$ for $t \geq t_o$. Hence

$$t^{-1} \int_{t_o}^{\infty} e^{-s/t} f(s)ds \leq \varepsilon\, t^{-1} \int_{t_o}^{\infty} e^{-s/t}\, g(s)ds \leq 2\, \varepsilon\, \Gamma(1+\alpha)\, g(t).$$

for t sufficiently large by theorem 2.11.
Since $g \in RV_\alpha$ with $\alpha \geq 0$ we have

$$|t^{-1} \int_0^{t_o} e^{-s/t} f(s)ds| \leq t^{-1} \int_0^{t_o} |\, f(s)|ds = o(g(t))\ \ (t \to \infty).$$

Combination of the above inequalities then gives (2.28) since $\varepsilon > 0$ is arbitrary. The converse implication for non-decreasing f follows immediately since $f(t) \leq e\, \hat{f}(1/t)$ by (2.24). ◊

For positive functions $f \in \Pi$ it is possible to improve the result for $\alpha = 0$ in theorem 2.11.

Theorem 2.14

Suppose f satisfies the assumptions of definition 2.10. Then

$$f \in \Pi \tag{2.29}$$

implies

$$\hat{f} \in \Pi^0. \tag{2.30}$$

Conversely if f is non-decreasing then (2.30) implies (2.29). Moreover $f \in \Pi(a)$ implies

$$\lim_{t \to \infty} \frac{f(t) - \hat{f}(t^{-1})}{a(t)} = \gamma\ , \tag{2.31}$$

where γ is Euler's constant. ◊

Proof

The implication (2.29) $\rightarrow$ (2.31) is a special case of theorem 1.18.

Note that $f \in \Pi(a)$ and (2.31) imply $\hat{f} \in \Pi^0$ (see theorem 1.13).

In case $f(t) \neq 0$ on $(0, t_o)$, note that $\left| t^{-1} \int_0^{t_o} e^{-s/t} f(s)\, ds \right| \leq t^{-1} \int_0^{t_o} |f(s)|ds$ and the right-hand side is $o(a(t))$ $(t\rightarrow\infty)$, which shows that (2.31) is also satisfied in this case. Conversely suppose $\hat{f} \in \Pi^0$ and f is non-decreasing. Without loss of generality we may suppose that $f(0+) = 0$. Then the Laplace transform of the non-decreasing function g defined by

$g(t) = \int_0^t s df(s)$ satisfies

$$\hat{f}(t) = \int_t^\infty \hat{g}(s)ds. \tag{2.32}$$

Hence $\hat{g} \in RV^0_{-1}$ by proposition 1.19.5. This in turn implies $g \in RV^0_1$ by theorem 2.11. Application of theorem 1.17 finally gives $f \in \Pi$. ◇

Corollary 2.15

If $f(tx) - f(t) \rightarrow \log x$ $(t\rightarrow\infty)$, then $\hat{f}(t^{-1}x^{-1}) - \hat{f}(t^{-1}) \rightarrow \log x$ $(t\rightarrow\infty)$.

The converse holds under the assumption f is non-decreasing. Moreover then $f(t) - \hat{f}(t^{-1}) \rightarrow \gamma$, $t\rightarrow\infty$.

It is possible to give a Mercerian result here of a restricted type:

if $(f(t) - \hat{f}(t^{-1}))/t^{-1}\int_0^t s df(s) \rightarrow \gamma$ $(t\rightarrow\infty)$ with f non-decreasing and $f(0+) = 0$, then $f \in \Pi$. See Embrechts (1978). ◇

Example

If $f(t) = c\log t + o(1)$ $(c > 0)$ then $\hat{f}(t^{-1}) = c \log t - c\gamma + o(1)$, $t\rightarrow\infty$.

The converse holds under the assumption f is non-decreasing.

Note that the statement "$f(t) - \hat{f}(t^{-1}) \rightarrow \gamma$ implies $f(t) = \log t + o(1)$" is not correct: take for example $f(t) = t + \log (t+1)$.

Corollary 2.16 = Proposition 1.19.6

Any $f \in \Pi(a)$ has a companion function f_1 such that $(-1)^{n+1} f_1^{(n)} \in RV^\infty_{-n}$ for $n = 1,2,\ldots$ and $f_1(t) - f(t) = o(a(t))$, $t\rightarrow\infty$ (define f_1 by $f_1(t) = \hat{f}(t^{-1}e^{-\gamma})$).

Theorem 2.17

Suppose that $f: \mathbb{R}^+ \to \mathbb{R}$ is integrable on finite intervals of $\mathbb{R}^+$ and that $L \in RV_0^\infty$. Then

$$\lim_{t\to\infty} \frac{f(tx) - f(t)}{L(t)} = 0 \text{ for every } x > 0 \tag{2.33}$$

implies

$$\lim_{s\to 0+} \frac{\hat{f}(sx) - \hat{f}(s)}{L(1/s)} = 0 \text{ for every } x > 0. \tag{2.34}$$

and

$$\lim_{t\to\infty} \frac{f(t) - \hat{f}(t^{-1})}{L(t)} = 0. \tag{2.35}$$

Conversely if f is non-decreasing, then (2.34) implies (2.33).

Proof

Define the function g by $g(t) = tf(t) - \int_0^t f(s)ds$.

Note that $t^{-1}\, g(t)$ is locally bounded on $t > 0$ and that conversely $f(t) = \frac{g(t)}{t} + \int_0^t \frac{g(s)}{s^2}\, ds$. We then have

$$\frac{f(t) - \hat{f}(1/t)}{L(t)} = \frac{g(t)}{tL(t)} - \int_0^\infty e^{-s} \frac{g(ts)}{tsL(t)}\, ds + \int_0^1 \frac{1-e^{-s}}{s} \frac{g(ts)}{tsL(t)}\, ds +$$

$$- \int_1^\infty \frac{e^{-s}}{s} \frac{g(ts)}{tsL(t)}\, ds.$$

If (2.33) holds the first term on the right-hand side tends to zero as $t\to\infty$ by remark 3 following corollary 1.18, the second term tends to zero by theorem 2.13 and the last two terms tend to zero by similar arguments as in the proof of theorem 2.14. This proves (2.35). Now (2.34) follows from (2.33) and (2.35) since $L \in RV_0^\infty$. Conversely suppose (2.34) holds. Then with the function g as defined above we have

$$\frac{\hat{f}(2^{-1}t^{-1}) - \hat{f}(t^{-1})}{L(t)} = \int_{\frac{1}{2}}^1 \frac{\hat{g}\,(st^{-1})}{tL(t)}\, ds \geq \frac{2^{-1}\,\hat{g}(t^{-1})}{tL(t)}\ .$$

Hence $\hat{g}(t^{-1}) = o\,(tL(t))\ (t\to\infty)$. Application of theorem 2.13 and remark 3 following corollary 1.18 then gives (2.33). This finishes the proof. ◊

Next we turn to Tauberian theorems for functions that grow faster than polynomials.

Roughly speaking we shall prove theorems connecting regular variation of log f at infinity with regular variation of log $\hat{f}$ at zero. It is now convenient to switch notation: instead of log f we will write f. This has the consequence that log $\hat{f}$ has to be considered as a function of log f, which is done in the next definition:

Definition 2.18

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is such that the Laplace transform of exp f is finite. We define the function $\tilde{f}$ by the relation

$$\tilde{f}(s) = \log s \int_0^\infty \exp\{f(t) - st\}\, dt,\ s > 0\ . \tag{2.36}$$

◊

In the proof of theorem 2.19 we use the concept of an inversely asymptotic function (see definition 1.21) in order to treat the cases Π and RV_α with $0 < \alpha < 1$ simultaneously. It turns out that the transform $\tilde{f}$ defined above and the complementary function f^c whose properties were described in the first part of this chapter, are the same up to $\overset{*}{\sim}$ equivalence.

Theorem 2.19

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is such that $\tilde{f}(s)$ is finite for $s > 0$ and let

$$s: \mathbb{R}^+ \to \mathbb{R}^+ \text{ be decreasing, continuous, } \int_0^1 s(x)dx < \infty,\ t\, s(t) \to \infty\ (t\to\infty) \tag{2.37}$$

and

$$s \in RV_\alpha^\infty \text{ with } -1 \le \alpha < 0. \tag{2.38}$$

Then

$$f(t) \overset{*}{\sim} \int_0^t s(x)\, dx,\ t\to\infty \tag{2.39}$$

implies

$$\tilde{f}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)\, dx, \quad u \to 0+ , \tag{2.40}$$

where $s^{\leftarrow}$ is the inverse function of s.
Conversely if f is non-decreasing and if there exists a function s satisfying (2.37) and (2.38), then (2.40) implies (2.39). ◇

<u>Corollary 2.20</u>
Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is such that $\tilde{f}(s)$ is finite for all $s > 0$.

a. Let p, q be related by $p^{-1} + q^{-1} = 1$.
Then $f \in RV_p^\infty$ with $0 < p < 1$ (i.e. $\alpha = p-1 > -1$) implies $\tilde{f} \in RV_q^0$ with $q < 0$.
If $f \in RV_p^\infty$, then (2.39) is equivalent to (cf. prop. 1.22)

$$f(t) \sim \int_0^t s(x)dx \sim p^{-1}\, t\, s(t) \quad (t\to\infty)$$

and (2.40) is equivalent to

$$\tilde{f}(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \sim -q^{-1}\, u\, s^{\leftarrow}(u) \quad (u \to 0+).$$

b. The case $p = 0$ translates into the following: $f \in \Pi$ (i.e. $\alpha = -1$) with auxiliary function $ts(t)\to\infty(t\to\infty)$ (s decreasing) implies $\tilde{f} \in \Pi^0$ with auxiliary function $b(u) \sim u\, s^{\leftarrow}(u) \to \infty\ (u \to 0+)$. If $f \in \Pi$, then (2.39) is equivalent to (cf. prop. 1.22)

$$f(t) = \int_0^t s(x)dx + o(t\, s(t)) \quad (t\to\infty)$$

and (2.40) is equivalent to

$$\tilde{f}(u) = \int_u^\infty s^{\leftarrow}(x)dx + o(u\, s^{\leftarrow}(u)) \quad (u \to 0+).$$

Converse statements are true under the assumption that f is non-decreasing. ◇

We prove the two statements in theorem 2.19 separately. For the proof of the Abelian part we need three lemmas.

Lemma 2.21

Suppose $s : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing, $s(\infty) = 0$, $\int_0^1 s(x)\,dx < \infty$, $ts(t) \to \infty$ $(t\to\infty)$ and let $\exp f(t)$ be locally integrable.
Define the function f_o by

$$f_o(t) := \int_0^t s(x)dx. \tag{2.41}$$

Then

$$f(t) \overset{*}{\sim} f_o(t) \qquad (t\to\infty) \tag{2.42}$$

implies

$$\tilde{f}(u) \overset{*}{\sim} \tilde{f}_o(u) \qquad (u \to 0+). \tag{2.43}$$

◇

Proof

Fix $c > 1$. We claim that

$$\exp \tilde{f}_o(u) - \exp \tilde{f}_o(cu) \to \infty \text{ for } u \to 0+. \tag{2.44}$$

If we define t_o such that $e^{-t_o} = ce^{-ct_o}$, then $t_o < 1$ and

$$\exp \tilde{f}_o(u) - \exp \tilde{f}_o(cu) = \int_0^\infty e^{f_o(t)}(ue^{-ut} - cue^{-cut})dt$$

$$= \int_0^\infty e^{f_o(t/u)}(e^{-t} - ce^{-ct})\,dt =$$

$$= \int_0^\infty (e^{f_o(t/u)} - e^{f_o(t_o/u)})(e^{-t} - ce^{-ct})dt.$$

Since f_o is non-decreasing the integrand in the last expression is non-negative, hence the right-hand side is at least

$$(e^{f_o(2/u)} - e^{f_o(t_o/u)}) \int_2^\infty (e^{-t} - ce^{-ct})dt. \tag{2.45}$$

Note that $ts(t) \to \infty$ $(t\to\infty)$ implies $\exp(f_o(2/u) - f_o(t_o/u)) =$

$$\exp \left\{\int_{t_o}^2 s(x/u)/u\,dx\right\} \to \infty,\ u \to 0+.$$

Hence the expression (2.45) tends to infinity which proves (2.44).

Now by (2.42) there exists $t_1 = t_1(c)$ such that $f(ct) \geq f_o(t)$ for $t \geq t_1$. Define the function f_1 by $f_1(t) := \min(f_o(t), f(ct))$.
Then $e^{\tilde{f}_o(u)} - e^{\tilde{f}_1(u)} = u \int_0^{t_1} (e^{f_o(t)} - e^{f_1(t)})\, e^{-ut}\, dt = o(1)$ $(u \to 0+)$. Together with (2.45) this gives $\tilde{f}_o(cu) \leq \tilde{f}_1(u)$ for all u sufficiently small. The right-hand side is at most $\tilde{f}(u/c)$ since $f_1(t) \leq f(ct)$.
Hence $\tilde{f}_o(cu) \leq \tilde{f}\,(u/c)$ for $u \leq u_o$.
Similarly we find $\tilde{f}(cu) \leq \tilde{f}_o\,(u/c)$ for $u \leq u_1$. This finishes the proof since $c > 1$ is arbitrary. ◇

Lemma 2.22

If f is non-decreasing and f^c, $\tilde{f}$ are well-defined, then

$$\tilde{f}(s) \geq f^c(s) \text{ for } s > 0. \tag{2.46}$$

Proof

For u, s > 0 we have

$$\exp \tilde{f}(s) = s \int_0^\infty e^{-xs+f(x)}dx \geq s \int_u^\infty e^{-xs+f(x)}dx \geq e^{f(u)}\ s \int_u^\infty e^{-xs}dx$$
$$= \exp\{f(u) - s\,u\}.$$

The proof is finished by taking the supremum over u > 0 on the right-hand side. ◇

Lemma 2.23

If $f(t) = \int_0^t s(x)dx$, $t > 0$, where the function s is continuous, decreasing and $s \in RV_\alpha$ with $-1 \leq \alpha < 0$, then for all $t > 0$

$$\tilde{f}(s(t)) = f^c(s(t)) + \log\{s(t) \int_{-t}^\infty e^{-\Delta(u)}du\}, \tag{2.47}$$

where the function Δ is defined by

$$\Delta(u) := u\, s(t) - f(t+u) + f(t). \tag{2.48}$$

Moreover the function Δ is convex, positive for $u \neq 0$, $u > -t$, $\Delta(0) = 0$ and satisfies the inequality

$$\Delta(t+u) \geq (1-2^{\alpha'})\ u\ s(t) \text{ for } u > 0 \tag{2.49}$$

and all t sufficiently large, where $\alpha < \alpha' < 0$.

Proof

By the definition of the complementary function f^c we have

$$\tilde{f}(s(t)) = f(t) - t\ s(t) + \log\{s(t) \int_{-t}^{\infty} e^{-\Delta(u)} du\}$$

$$= f^c(s(t)) + \log\{s(t) \int_{-t}^{\infty} e^{-\Delta(u)} du\}.$$

Since $s \in RV_\alpha$ we have $s(2t) \leq 2^{\alpha'} s(t)$ for $t > t_0$, where $\alpha < \alpha' < 0$. Now fix $t > t_0$. Then since s is decreasing we have

$$\Delta'(u) = s(t) - s(t+u) \geq s(t) - s(2t) \geq (1 - 2^{\alpha'})s(t)$$

for $u > t$. Hence we have

$$\Delta(t+u) = \int_0^{t+u} \Delta'(x)\ dx \geq \int_t^{t+u} \Delta'(x)dx \geq (1 - 2^{\alpha'})\ us(t)$$

for $u > 0$. ◇

Proof of theorem 2.19 (Abelian part)

In view of lemma 2.21 we may assume that $f(t) = \int_0^t s(x)dx$ with $s \in RV_\alpha$ $(-1 \leq \alpha < 0)$ continuous, decreasing and $ts(t) \to \infty$ $(t\to\infty)$.

Application of lemma 2.23 gives

$$\tilde{f}(s(t)) = f^c(s(t)) + \log\{s(t) \int_{-t}^{\infty} e^{-\Delta(u)} du\}$$

$$\leq f^c(s(t)) + \log\ s(t)\ \{\int_{-t}^{t} 1.du + \int_0^{\infty} e^{-\Delta(t+u)} du\} \leq$$

$$\leq f^c(s(t)/c) - (f^c(s(t)/c) - f^c(s(t))) + \log\{2ts(t) + (1-2^{\alpha'})^{-1}\}.$$

Now we have for any $c > 1$

$$f^c(s(t)/c) - f^c(s(t)) = \int_{s(t)/c}^{s(t)} s^{\leftarrow}(x)dx \geq (1-c^{-1})ts(t),$$

hence

$$\tilde{f}(s(t)) \leq f^c(s(t)/c) - (1-c^{-1})ts(t) + \log\{2ts(t) + (1-2^{\alpha'})^{-1}\}.$$

Now let $t\to\infty$. Then $s(t) \to 0$ and $ts(t) \to \infty$ by assumption. The last inequality then gives $\tilde{f}(s) \leq f^c(s/c)$ for sufficiently small s.
Combination with lemma 2.22 now gives $\tilde{f}(s) \overset{*}{\sim} f^c(s)$, $s \to 0+$.
In view of theorem 2.3 this finishes the proof. ◇

Before giving a proof of the Tauberian part of theorem 2.19 we discuss its main line. We have seen that under the main assumptions (2.38) and (2.39) the complementary transform and the $\sim$-transform have the same behaviour up to $\overset{*}{\sim}$ equivalence.
If $\tilde{f}$ satisfies (2.40), by the analogue of theorem 2.6 for the transform f_c its inverse complementary function $(\tilde{f})_c$ satisfies

$$h(t) := (\tilde{f})_c\ (t) \overset{*}{\sim} \int_0^t s(x)\, dx\ (t\to\infty).$$

It is then sufficient to prove $h \overset{*}{\sim} f$. The proof is by contradiction. We show that if the relation $f \overset{*}{\sim} h$ is not true, then $\tilde{f} \overset{*}{\sim} \tilde{h}$ cannot be true.

Since on the other hand $\tilde{h}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)dx\ (u \to 0+)$ by theorem 2.3 and

$$\tilde{f}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)\, dx\ (u \to 0+) \text{ by assumption,}$$

this gives the required contradiction. In order to evaluate $\tilde{h}$ we separate the domain of integration $(0,\infty)$ into two parts: an interval I and its complement I^c.

Lemma 2.24 below shows that the contribution of I^c is small, i.e.

$$\log s(t) \int_{I^c} e^{h(t) - ts(t)}dt \leq \tilde{h}(cs(t))$$

for t sufficiently large.

Lemma 2.24

Suppose $f(t) = \int_0^t s(x)\,dx$ $(t > 0)$ with s continuous, decreasing and $s \in RV_\alpha^\infty$ with $-1 \leq \alpha < 0$. Suppose moreover $ts(t) \to \infty$ $(t \to \infty)$. Then for every $0 < \beta < 1$ there exist constants $c > 1$ and t_o such that for $t \geq t_o$

$$\log s(t) \int_{I^c} e^{f(u)-us(t)}du \leq \tilde{f}\,(cs(t)), \tag{2.50}$$

where $I = (t - \beta t,\ t + \beta t)$.

Proof

Fix $t > t_o$ and define the function Δ as in (2.48). Application of lemma 2.23 gives $\Delta(\beta t) = \Delta(\frac{\beta}{2}t + \frac{\beta}{2}t) \geq (1 - 2^{\alpha'})\ \frac{\beta}{2}ts(\frac{\beta}{2}t) \geq \gamma_1 \beta ts(t)$ for some $\gamma_1 > 0$ not depending on β and $t \geq t_1$.

Similarly $\Delta(-\beta t) \geq \gamma_2\ \beta ts(t)$ for $t \geq t_2$. Since Δ is convex and $\Delta(0) = 0$,

$$s(t) \int_{t+\beta t}^{\infty} e^{f(u)-us(t)}du = e^{f(t) - ts(t)} s(t) \int_{\beta t}^{\infty} e^{-\Delta(u)}du \leq$$

$$e^{f(t)-ts(t)} s(t) \int_{\beta t}^{\infty} e^{-\Delta(\beta t)-\Delta(u-\beta t)}du$$

$$\leq e^{f(t)-ts(t)} s(t) e^{-\gamma_1 \beta ts(t)} \int_0^{\infty} e^{-\Delta(u)}du = e^{-\gamma_1 \beta ts(t)} s(t) \int_t^{\infty} e^{f(u) - us(}$$

This together with the corresponding inequality for the integral over $(0,t-\beta t)$ gives

$$\log s(t) \int_{I^c} e^{f(u)-us(t)}du \leq \tilde{f}(s(t)) - \gamma ts(t), \tag{2.51}$$

where $\gamma := \beta \min (\gamma_1, \gamma_2)$.

Application of theorem 2.19 (Abelian part) gives

$$\tilde{f}(u) \overset{*}{\sim} \int_u^{\infty} s^{\leftarrow}(x)dx\ ,\ u \to 0+. \tag{2.52}$$

Hence for any $\varepsilon > 0$ there exists u_o such that for $u \leq u_o$

$$\tilde{f}(u) \leq \int_{(1-\varepsilon)u}^{\infty} s^{\leftarrow}(x)dx \leq \int_{(1+2\varepsilon)u}^{\infty} s^{\leftarrow}(x)dx + 3\ \varepsilon\ u\ s^{\leftarrow}((1-\varepsilon)u) \leq$$

$$\tilde{f}((1+\varepsilon)u) + 3\ \varepsilon\ u\ s^{\leftarrow}((1-\varepsilon)u) \tag{2.53}$$

since s^{+} is decreasing. Since $s^{+} \in RV^{0}_{-1/\alpha}$ (see proposition 1.7.9) we have $s^{+}((1-\varepsilon)u) \leq c_o\, s^{+}(u)$ for u sufficiently small, where $c_o = c_o(\varepsilon) > (1-\varepsilon)^{-1/\alpha}$ is a constant. Substitution in (2.53) gives for $u \leq u_o$

$$\tilde{f}(u) \leq \tilde{f}((1+\varepsilon)u) + 3\,\varepsilon\, c_o u\, s^{+}(u).$$

The proof is completed by application of the inequality (2.51) if we take $c = 1 + \varepsilon > 1$ so that $3\,\varepsilon\, c_o < \gamma$ and $u = s(t)$. ◇

<u>Proof of theorem 2.19 (Tauberian part)</u>

Define the function h by $h(t) := (\tilde{f})_c(t)$. From $\tilde{f} \geq f^c$ (lemma 2.22) it follows that

$$h(t) \geq (f^c)_c(t) \geq f(t). \tag{2.54}$$

The latter inequality follows since $(f^c)_c$ is the concave upper hull of f.

From $\tilde{f}(u) \overset{*}{\sim} \int_u^\infty s^{+}(x)\,dx$ $(u \to 0+)$ it follows by theorem 2.3 that

$$h(t) \overset{*}{\sim} \int_0^t s(x)dx,\ t \to \infty \text{ with } s \in RV^{\infty}_{\alpha}\ (-1 \leq \alpha < 0). \tag{2.55}$$

It remains to prove that $f(t) \overset{*}{\sim} h(t)$ $(t \to \infty)$. The proof is by contradiction. If $f(t) \overset{*}{\sim} h(t)$ is not true, then since f and h satisfy (2.54), there exists a sequence $\tau_n \to \infty$ $(n \to \infty)$ and a constant $c > 1$ such that $h(\tau_n/c) \geq f(\tau_n c)$. This implies since h and f are non-decreasing, that $h(t/c) \geq f(t)$ for $\tau_n < t < c\tau_n$ or (with $t_n = \tau_n \sqrt{c}$, $\beta = 1 - c^{-\frac{1}{2}}$)

$$h(t/c) \geq f(t) \quad \text{for} \quad t \in I_n := (t_n - \beta t_n, t_n + \beta t_n). \tag{2.56}$$

Together with (2.54) this gives for $s > 0$

$$\tilde{f}(s) = \log s \int_0^\infty e^{f(u)-us}du \leq \log \Big(s \int_{I_n} e^{h(u/c)-us}du + s \int_{I_n^c} e^{h(u)-us}du\Big). \tag{2.57}$$

Note that (2.55) implies by lemma 1.23 that $h(t) = \int_0^t s_1(x)\,dx$ where s_1 non-increasing and $s_1(t) \sim s(t)$ $(t \to \infty)$. So if we take $f = h$ and $t = t_n$ in lemma

2.24 we can estimate the second integral at the right-hand side in (2.57). As a consequence there exists $c' > 1$ such that (with $s_n := s_1(t_n)$)

$$\tilde{f}(s_n) \leq \log\left(e^{\tilde{h}(cs_n)} + e^{\tilde{h}(c's_n)}\right) \leq \tilde{h}(c''s_n) + 1, \qquad (2.58)$$

$$c'' = \min(c,c') > 1.$$

Now application of the direct statement of theorem 2.19 shows that (2.55) implies $\tilde{h}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)dx$, $u \to 0+$. Since also $xs^{\leftarrow}(x) \to \infty$ $(x \to 0+)$, we have $\tilde{h}(s_n\sqrt{c''}) - \tilde{h}(s_n c'') \to \infty$ $(n\to\infty)$.

This takes (2.58) into the form $\tilde{f}(s_n) \leq \tilde{h}(s_n\sqrt{c''})$ for sufficiently large n, hence $\tilde{f}$ and $\tilde{h}$ are not inversely asymptotic.
On the other hand by assumption $\tilde{f}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)dx$ and we already found $\tilde{h}(u) \overset{*}{\sim} \int_u^\infty s^{\leftarrow}(x)dx$, $u \to 0+$, hence a contradiction is obtained. ◇

Now that the proof of theorem 2.19 has been completed, let us pause to consider its place with respect to the previous results. In theorem 2.11 we considered functions $f \in RV_\alpha^\infty$ $(0 \leq \alpha < \infty)$. Theorem 2.19 concerns functions f such that $\log f \in RV_{\alpha'}^\infty$, $(0 < \alpha' < 1)$ or $\log f \in \Pi$ (the case $\alpha' = 0$). We argue that the case $\alpha' = 0$ of theorem 2.19 can also be considered as the borderline case $\alpha = \infty$ of theorem 2.11. To this purpose note that $f \in RV_\alpha^\infty$ is equivalent $(\alpha > 0)$ to

$$\lim_{t\to\infty} \left(\frac{f(tx)}{f(t)}\right)^{1/a(t)} = x \qquad (2.59)$$

for all $x > 0$ provided that $\lim_{t\to\infty} a(t) = \alpha$. Now (2.59) with $\lim_{t\to\infty} a(t) = \infty$ is equivalent to $\log f \in \Pi(a)$ which is the condition for theorem 2.19 with $\alpha' = 0$.
Incidentally, also the condition for theorem 2.14 (the refinement of theorem 2.11 for $\alpha = 0$, which is $f \in \Pi$, is of the form (2.59) namely with the condition $\lim_{t\to\infty} a(t) = 0$:

$$\log x \sim \log\left(\frac{f(tx)}{f(t)}\right)^{1/a(t)} = \left\{\log \frac{f(tx)}{f(t)}\right\}/a(t) \sim \left\{\frac{f(tx)}{f(t)} - 1\right\}/a(t) \quad (t\to\infty$$

We mention that an alternative result for the case $\alpha' = 0$ of theorem 2.19 has been proved by Parameswaran (1961). Without proof we mention the result here for completeness.

Theorem 2.25

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is such that $\tilde{f}(s)$ is finite for $s > 0$.
Then $f(t) \sim \{L(t)\}^{-1}$ $(t\to\infty)$, $L \in RV_0^\infty$ and L^* is non-decreasing imply $\tilde{f}(u) \sim L^*(1/u)$ $(u \to 0+)$.
Conversely if f is non-decreasing, $\tilde{f}(u) \sim L^*(1/u)$ $(u \to 0+)$ with $L \in RV_0^\infty$, then $f(t) \sim \{L(t)\}^{-1}$ $(t\to\infty)$. ◇

In the above theorem the funtion L^* is the conjugate slowly varying function as defined in chapter 1 (see the remark following theorem 1.8).

The final theorem of this chapter gives a result for functions growing even faster than the functions from theorem 2.19. This result can be considered as the borderline case $\alpha = 1$ of theorem 2.19. Note that $\alpha > 1$ is impossible, since then the Laplace transform does not exist any more.

Theorem 2.26

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is such that $\tilde{f}(s)$ is finite for $s > 0$. Let (see definition 2.5)

$$s: \mathbb{R}^+ \to \mathbb{R}^+ \text{ be decreasing, continuous, } s(\infty) = 0, \int_0^1 s(x)dx < \infty \text{ and } s \in \Pi^-(a). \tag{2.60}$$

Then

$$f(t) \stackrel{\cdot}{\sim} \int_0^t s(x)dx, \quad t\to\infty \tag{2.61}$$

implies

$$\tilde{f}(u) \sim \int_u^\infty s^{\leftarrow}(x)dx, \quad u \to 0+. \tag{2.62}$$

Conversely suppose f is non-decreasing, $f(t)/t$ is non-increasing and s satisfies (2.60). Then (2.62) implies (2.61). ◇

Remark

Theorem 2.26 does not hold without the condition f(t)/t non-increasing. For a counterexample the reader is referred to Geluk, de Haan, Stadtmüller [1986]. It is also possible to relax the assumptions on s in the theorem.

Corollary 2.27

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is such that $\tilde{f}(s)$ is finite for $s > 0$.
If $f(t)/t \in \Pi^-$ (see definition 2.5) and $\lim_{t\to\infty} f(t)/t = 0$, then $\tilde{f} \in \Gamma^{(0)}$ (def. 2.5).
Conversely, if f(t) is non-decreasing, f(t)/t non-increasing,

$$\lim_{t\to\infty} f(t)/t = 0 \text{ and } \tilde{f} \in \Gamma^{(0)}, \text{ then } f(t)/t \in \Pi^-.$$

Moreover (2.61) with s as in (2.60) is equivalent to

$$f(t) = ts(t/e) + o(ta(t)),\ t\to\infty. \tag{2.63}$$

◇

Proof of cor. 2.27

For the first part (the implications $f(t)/t \in \Pi \to \tilde{f} \in \Gamma^{(0)}$) we use theorem 2.26 and lemma 2.7. In order to prove (2.63) notice that $-s \in \Pi(a)$ and

$$-\frac{f(t)}{t} \overset{*}{\sim} -\frac{1}{t}\int_0^t s(x)\,dx\ (t\to\infty),$$

hence by proposition 1.22 and theorem 1.17

$$\frac{f(t)}{t} = \frac{1}{t}\int_0^t s(x)dx + o(a(t)) =$$

$$= s(t) + \left\{-s(t) + t^{-1}\int_0^t s(x)\,dx\right\} + o(a(t))$$

$$= s(t) + a(t) + o(a(t)) = s(t/e) + o(a(t))\ (t\to\infty).$$

The last equality follows directly from the definition of Π^- (a). ◇

We prove the two statements of theorem 2.26 separately.
For the proof of the Abelian part we need two lemmas.

Lemma 2.28

If s satisfies (2.60) and $f(t) = \int_0^t s(x)dx$, then (2.62) holds.

Proof

Since $s \in \Pi^-(a)$ we have $s(t) - s(2t) \geq \frac{1}{2}a(t) \log 2$ for $t > t_0$. Fix $t > t_0$ and define the function Δ as in (2.48). Recall that $\Delta(u)$ is convex and non-negative for $u \geq -t$. Moreover $\Delta'(t) = s(t) - s(2t) \geq \frac{1}{2} a(t) \log 2$, whence for $u > 0$

$$\Delta(u+2t) \geq \Delta(t) + (u+t)\Delta'(t) \geq u\Delta'(t) \geq \tfrac{1}{2}u\, a(t) \log 2. \tag{2.64}$$

We have (lemma 2.22 and 2.23)

$$f^c(s(t)) \leq \tilde{f}(s(t)) = f^c(s(t)) + \log s(t) \int_{-t}^{\infty} e^{-\Delta(u)}du . \tag{2.65}$$

The integral on the right-hand side can be estimated using (2.63):

$$s(t) \int_{-t}^{\infty} e^{-\Delta(u)}du \leq s(t) \int_{-t}^{2t} 1 \,.\, du + s(t) \int_0^{\infty} e^{-\Delta(u+2t)}du \leq$$

$$\leq 3ts(t) + 2s(t)/(a(t) \log 2). \tag{2.66}$$

Consider this expression for $t\to\infty$. Now $s \in \Pi^-$, $a \in RV_0^{\infty}$ and hence $ta(t) \to \infty$, $\log s(t) = o(\log t)$, $t\to\infty$ (prop. 1.7.1). So

$$\log s(t) \int_{-t}^{\infty} e^{-\Delta(u)}du \sim \log t \;(t\to\infty). \tag{2.67}$$

By theorem 2.8 $f^c(u) \sim \int_u^{\infty} s^{\leftarrow}(x)dx$, $u \to 0+$.

In view of (2.65) and (2.66) the proof is finished if we show that

$$\frac{\log t}{\int_{s(t)}^{\infty} s^{\leftarrow}(x)dx} = \frac{\log t}{\int_0^t s(x)dx - ts(t)} \to 0 \text{ as } t\to\infty. \tag{2.68}$$

An application of theorem 1.17 shows that $\int_0^t s(x)dx - ts(t)$ is in RV_1^{∞}.

It follows that the left-hand side of (2.68) is in RV_{-1}^{∞}, hence (2.62) follows.

Lemma 2.29

If the assumptions of lemma 2.28 are satisfied, for any $\delta > 1$

$$(f/\delta)^{\sim}(u) - \tilde{f}(\delta u)/\delta^2 \to \infty, \quad u \to 0+ \tag{2.69}$$

$$(\delta f)^{\sim}(u) - \delta^2\tilde{f}(u/\delta) \to -\infty, \quad u \to 0+. \tag{2.70}$$

By $(af)^{\sim}$ we mean $\tilde{g}$ with $g(t) := af(t)$ for $a > 0$.

Proof

Fix $\delta > 1$. Since $f^c \leq \tilde{f}$ (lemma 2.22) and $\tilde{f}(u) \sim f^c(u)$ (thm. 2.3 and lemma 2.27)

$$(f/\delta)^{\sim}(u) \geq (f/\delta)^c(u) = f^c(\delta u)/\delta \sim \tilde{f}(\delta u)/\delta, \quad u \to 0+.$$

This proves (2.69) since $\tilde{f}(u) \to \infty$ as $u \to 0+$. In order to prove (2.70) note that

$$\delta\, \tilde{f}(u/\delta) \geq \delta\, f^c(u/\delta) = (\delta f)^c(u) \sim (\delta f)^{\sim}(u), \quad u \to 0+. \qquad \diamond$$

Proof of theorem 2.26 (Abelian part)

Define the function f_o by $f_o(t) = \int_0^t s(x)dx$.

Fix $\delta > 1$ and define the function f_1 by $f_1(t) = \min\{f_o(t)/\delta, f(t/\delta)\}$. Since $f \doteq f_o$ it follows that $f_1(t) = f_o(t)/\delta$ for $t > t_o$. Hence

$$\exp(f_o/\delta)^{\sim}(u) - \exp(\tilde{f}_1(u)) = \tag{2.71}$$

$$u\int_0^{t_o} \{\exp(f_o(t)/\delta) - \exp(f_1(t))\}e^{-ut}dt = o(1), \quad u \to 0+.$$

Now (2.69), (2.71) and $f_1(t) \leq f(t/\delta)$ imply $\tilde{f}(\delta u) \geq \tilde{f}_1(u) > \tilde{f}_0(\delta u)/\delta^2$ for u sufficiently small. Similarly we find $\delta^2\tilde{f}_o(u/\delta) \geq \tilde{f}(u/\delta)$ for u sufficiently small by introducing the function $f_2(t) = \max(\delta f_o(t), f(t\delta))$.

This proves $\tilde{f}(u) \sim \tilde{f}_o(u), u \to 0+$, and the latter is asymptotic to $\int_u^\infty s^+(x)dx$ by lemma 2.28. $\diamond$

In order to prove the Tauberian part of theorem 2.26 we need an analogue of lemma 2.24.

<u>Lemma 2.30</u>

Suppose $f(t) = \int_0^t s(x)dx$ with $s \in \Pi^-(a)$, s non-increasing, continuous and $s(\infty) = 0$.

For every $0 < \beta < 1$ there exist $c > 1$ and $t_o > 0$ such that for $t \geq t_o$

$$\log s(t) \int_{I^c} e^{f(u)-us(t)}du \leq \tilde{f}(s(t))/c, \tag{2.72}$$

where $I = (t-\beta t,\ t+\beta t)$.

<u>Proof</u>

For $t > 0$ fixed and $u > -t$ define $\Delta(u) := us(t) - f(t+u) + f(t)$ as in (2.49). Then as before

$$\Delta'(u) \geq s(t) - s(t+u)$$

and, using $s \in \Pi^-(a)$, for $t \geq t_o$, $u \geq \beta t/2$

$$\Delta'(u) \geq \tfrac{1}{2}\ a(t)\ \log\ (1 + \tfrac{\beta}{2}) =: 2\ c_o\ a(t).$$

This implies

$$\Delta(\beta t) = \Delta(\beta t/2) + \int_{\beta t/2}^{\beta t} \Delta'(u)\ du \geq c_o\ \beta t\ a(t)$$

and since Δ is convex and $\Delta(0) = 0$, for $t \geq t_o$

$$s(t) \int_{t+\beta t}^{\infty} e^{f(u)-us(t)}\ du = e^{f(t)\ -\ ts(t)} s(t) \int_{\beta t}^{\infty} e^{-\Delta(u)}du$$

$$\leq e^{f(t)-ts(t)}\ s(t) \int_{\beta t}^{\infty} e^{-\Delta(\beta t)\ -\ \Delta(u\ -\ \beta t)}du$$

$$\leq e^{f(t)\ -\ ts(t)\ -\ c_o\ \beta ta(t)} s(t) \int_0^{\infty} e^{-\Delta(u)}du =$$

$$e^{-c_o\beta ta(t)} s(t) \int_t^{\infty} e^{f(u)-us(t)}du.$$

Similarly for some $c_1 > 0$ and t sufficiently large

$$s(t) \int_0^{t-\beta t} e^{f(u) - us(t)}du \leq e^{-c_1\beta ta(t)} s(t) \int_0^t e^{f(u) - us(t)}du.$$

Combination of the two inequalities gives for some $c_2 > 0$ and t sufficiently large

$$\log s(t) \int_{I^c} e^{f(t) - us(t)}du \leq \tilde{f}\,(s(t)) - c_2\beta ta(t).$$

It is now sufficient to prove that $ta(t) \sim \tilde{f}(s(t))$ $(t\to\infty)$.
This follows from the direct statement of theorem 2.26:

$$\tilde{f}(s(t)) \sim \int_{s(t)}^{\infty} s^{\leftarrow}(u)du = t\,\{t^{-1}\int_0^t s(u)du - s(t)\} \sim ta(t)\ (t\to\infty),$$

the last asymptotic equality being a consequence of $s \in \Pi^-(a)$ and theorem 1.17. ◇

Proof of theorem 2.26 (Tauberian part)

Suppose (2.62) holds. The function h defined by $h(t) := (\tilde{f})_c(t)$ is concave by the definition of the inverse complementary function (definition 2.1). Moreover h is eventually positive. Hence $h(t)/t$ is eventually non-increasing. By lemma 2.22 we have $h(t) \geq (f^c)_c(t)$. Also $(f^c)_c$ is (by definition) the convex upper hull of f, hence $(f^c)_c \geq f$.
As a consequence, for $c > 1$ we have for sufficiently large t

$$h(t) \geq (f^c)_c(t) \geq f(t).$$

From (2.62) it follows by the analogue of theorem 2.8 for the inverse complementary function that

$$h(t) \stackrel{\bullet}{\sim} \int_0^t s(x)dx\ (t\to\infty). \tag{2.73}$$

It remains to prove that $f(t) \stackrel{\bullet}{\sim} h(t)$. The proof is by contradiction. Suppose that $f(t) \stackrel{\bullet}{\sim} h(t)$ is not true, then, since $f(t) \leq h(t)$, there exists a sequence $\tau_n \to \infty$ and a constant $c > 1$ such that

$$\frac{f(\tau_n)}{\tau_n} < \frac{h(c\tau_n)}{c\tau_n}.$$

Now $f(t)/t$ and $h(t)/t$ are non-increasing, so for $\tau_n < t < \tau_n \sqrt{c}$ we have $f(t)/t \leq f(\tau_n)/\tau_n < h(\tau_n c)/\tau_n c \leq h(t\sqrt{c})/t\sqrt{c}$. Hence for $t_n = \tau_n c^{1/4}$ and $\beta = 1 - c^{-1/4}$ we get

$$\frac{f(t)}{t} < \frac{h(tc)}{tc} \text{ on } I_n := (t_n - \beta t_n, t_n + \beta t_n). \tag{2.74}$$

We want to apply lemma 2.30 with $f = h$. In order to do so, we have to show that there exists a non-increasing function s_1 such that $h(t) = \int_0^t s_1(x)dx$ and $s_1 \in \Pi^-$. Since h is concave, there exists a non-increasing function s_1 such that $h(t) = \int_0^t s_1(x)dx$. Since $s \in \Pi^-(a)$ we have, by lemma 2.7 (i), in view of (2.73) $h(t)/t \in \Pi^-(a)$. This, together with proposition 1.19.2, shows that $s_1 \in \Pi^-(a)$.

Now we can apply lemma 2.30 with $f = h$. For n sufficiently large with $s_n = s_1(t_n)$ by (2.74)

$$\tilde{f}(s_n) \leq \log\{s_n \int_{I_n} \exp(h(tc)/c - ts_n)dt + s_n \int_{I_n^c} \exp(h(t) - s_n t)dt\}$$

$$\leq \log\{\exp(h/c)^{\sim}(s_n/c) + \exp(\tilde{h}(s_n)/c)\}.$$

Since $(h/c)^{\sim}(s_n/c) \sim (h/c)^c\,(s_n/c) = h^c(s_n)/c \sim \tilde{h}(s_n)/c$ $(n\to\infty)$ by the Abelian parts of the theorems 2.8 and 2.26, we find for all $\varepsilon > 0$ and sufficiently large n

$$\tilde{f}\,(s_n) \leq \log\,(e^{\tilde{h}(s_n)(1+\varepsilon)/c} + e^{\,\tilde{h}(s_n)/c}) \leq \tilde{h}(s_n)(1+\varepsilon)/c + 1$$

and hence $\tilde{f}(s_n) \leq \tilde{h}(s_n)/\sqrt{c}$ for $n \geq n_0$, which means that $\tilde{f}(s) \sim \tilde{h}(s)$ $(s \to 0+)$ cannot be true. But on the other hand since $h(t) \overset{\cdot}{\sim} \int_0^t s(x)dx$ $(t\to\infty)$ implies that $\tilde{h}(s) \sim \int_s^\infty s^{\leftarrow}(x)dx \sim \tilde{f}(s)$ $(s \to 0+)$ by the Abelian parts of the theorems 2.8 and 2.26 and we have obtained a contradiction. ◇

We give some examples, showing the scope of applicability of the above results.

<u>Example 1</u>

Suppose

$$f(t) = (\log t)^{\alpha} (\log \log t)^{\beta} + o((\log t)^{\alpha-1}(\log\log t)^{\beta}) \quad (t\to\infty) \qquad (2.75)$$

$\alpha > 0$, $\beta \in \mathbb{R}$.
By theorem 2.14

$$\hat{f}(u) = |\log u|^{\alpha} (\log|\log u|)^{\beta} - (\gamma + o(1)) \; |\log u|^{\alpha-1}(\log|\log u|)^{\beta} \quad (u \to 0+). \qquad (2.76)$$

Conversely, if f is non-decreasing, (2.76) implies (2.75).

<u>Example 2</u>
Suppose

$$f(t) \sim t^{\alpha}(\log t)^{\beta} \; (t \to \infty), \; \alpha > 0, \; \beta \in \mathbb{R}. \qquad (2.77)$$

By theorem 2.11

$$\hat{f}(t) \sim \Gamma(1+\alpha) \; u^{-\alpha}|\log u|^{\beta} \; (u \to 0+). \qquad (2.78)$$

Conversely if f is non-decreasing (2.78) implies (2.77).

<u>Example 3</u>
Suppose $f(t) \sim t^{\alpha}(\log t)^{\beta}$, $t\to\infty$, $\beta \in \mathbb{R}$, $0 < \alpha < 1$. In order to derive the asymptotic behaviour of $\tilde{f}$ we can apply theorem 2.19 : relation (2.39) is equivalent to $f(t) \sim \int_0^t s(x)dx$ $(t\to\infty)$ (see proposition 1.22).

The function s satisfies $s(t) \sim \alpha \, t^{\alpha-1} \log^{\beta} t$, $t\to\infty$.
We define the function ϕ by $\phi(t) := \frac{\alpha}{s(t)} \sim t^{1-\alpha}\log^{-\beta} t$, $t\to\infty$.

As in the remark following theorem 1.8 we find

$$\phi^{+}(t) \sim (1 - \alpha)^{-\beta/(1-\alpha)} t^{1/(1-\alpha)} (\log t)^{\beta/(1-\alpha)}, \; t\to\infty.$$

As a consequence the inverse function of s satisfies

$$s^{\leftarrow}(u) \sim \phi^{+}(\tfrac{\alpha}{u}) \sim (1-\alpha)^{-\beta/(1-\alpha)} \; (\tfrac{\alpha}{u})^{1/(1-\alpha)} (-\log u)^{\beta/(1-\alpha)}, \; u\to 0+.$$

Application of theorem 2.19 then gives

$\tilde{f}(u) \sim \int_u^\infty s^+(x)dx$ and the last expression is asymptotic to $\frac{1-\alpha}{\alpha} u\, s^+(u)$ by theorem 1.4. Hence we find

$$\tilde{f}(u) \sim (1-\alpha)^{(1-\alpha-\beta)/(1-\alpha)}\ \alpha^{\alpha/(1-\alpha)} u^{-\alpha/(1-\alpha)} (-\log u)^{\beta/(1-\alpha)}, \quad u \to 0+$$

and a converse statement holds under the assumption that f is non-decreasing.

Example 4

Consider the function

$$f(t) = t/(\log t)^\beta + o(t/(\log t)^{\beta+1}, \quad t\to\infty, \quad \beta > 0. \tag{2.79}$$

We want to derive the asymptotic behaviour of the transform $\tilde{f}$ defined by

$$\tilde{f}(s) = \log s \int_0^\infty \exp\{f(u) - su\}\, du$$

as $s \to 0+$ (see definition 2.18).

Note that the function $f(t)/t$ is in $\Pi^-(a)$ with $a(t) \sim \beta(\log t)^{-(\beta+1)}$, $t\to\infty$ (see definition 2.5). So we may apply theorem 2.26 and we have to find a functon $s \in \Pi^-(a)$ satisfying (2.60) and such that

$$f(t) \overset{*}{\sim} \int_0^t s(x)dx, \quad t\to\infty. \tag{2.80}$$

Our first try is $\frac{d}{dx}(x/(\log x)^\beta) = (\log x)^{-\beta} - \beta(\log x)^{-\beta-1} = ((\log(ex))^{-\beta} + o((\log x)^{-\beta-1})\ (x\to\infty)$.

Now $(\log ex))^{-\beta}$ is positive and decreasing for $x > 1$ and $f(t) = \int_1^t (\log ex)^{-\beta}dx + o(t(\log t)^{-\beta-1})$, $t\to\infty$.

We define $s(x) := (\log ex)^{-\beta}$ for $x > 1$ and decreasing and integrable on (0, 1).

Note that $-s \in \Pi(a)$, hence $-\frac{1}{t}\int_0^t s(x)dx \in \Pi(a)$.

Since $f(t)/t = t^{-1}\int_0^t s(x)dx + O(a(t))$, we have (see proposition 1.22)

$-f(t)/t \overset{*}{\sim} -t^{-1}\int_0^t s(x)dx\ (t\to\infty)$, hence (2.80). Application of theorem 2.26 then gives

$$\tilde{f}(u) \sim \int_u^\infty s^+(x)dx \quad (u \to 0+).$$

It remains to evaluate the right-hand side. Now $s^{+}(x) = \exp(-1 + x^{-1/\beta})$ and hence

$$\tilde{f}(u) \sim \alpha\, u^{1+1/\alpha} \exp(-1 + u^{-1/\alpha}), \quad (u \to 0+) \tag{2.81}$$

by de l'Hopital's rule. Note that by theorem 2.8 the complementary function f^c has the same asymptotic behaviour.
Conversely if $\tilde{f}$ satisfies (2.81), f is non-decreasing and f(t)/t non-increasing, then f satisfies (2.79).

<u>Example 5</u>
If $f(t) = t/(\log t)^{\beta} + \beta(1+\beta)t\,(\log\log t)/(\log t)^{1+\beta} + \beta\,(1 - \log\beta)$
$t/(\log t)^{1+\beta} + o(t/(\log t)^{1+\beta}),\ t\to\infty$ (2.82)
for some $\beta > 0$, then $\tilde{f}(s) \sim \exp(s^{-1/\beta})$, $s \to 0+$ and the converse statement holds under the assumptions f non-decreasing and f(t)/t non-increasing.

<u>Proof</u>
Suppose $\tilde{f}$ satisfies $\tilde{f}(s) \sim \exp(s^{-1/\beta})$, $s \to 0+$, f is non-decreasing and f(t)/t is non-increasing. We derive the asymptotic behaviour of f using theorem 2.26. Note that $\tilde{f} \in \Gamma^{(0)}$.
Since $\tilde{f}(u) \sim \frac{1}{\beta}\int_u^{\infty} x^{-1-1/\beta}\exp(x^{-1/\beta})dx$ $(u \to 0+)$
we have $s^{+}(x) \sim \beta^{-1}x^{-1-1/\beta}\exp(x^{-1/\beta})$ $(x \to 0+)$
and $s^{+}((\log y)^{-\beta}) \sim \beta^{-1}y(\log y)^{1+\beta}$ $(y\to\infty)$.
As in the remark following theorem 1.8 we find by inversion
$s(x) = \{\log\beta x - (1+\beta)\log\log\beta x + o(1)\}^{-\beta}$.
Hence by lemma 2.7, theorem 2.26 and corollary 2.27 we find

$$f(t) \stackrel{\sim}{\cdot} \int_0^t \frac{dx}{\{\log\beta x - (1+\beta)\log\log\beta x\}^{\beta}} \stackrel{\sim}{\cdot} \beta^{-1}\int_0^{\beta t} \frac{dx}{\{\log x - (1+\beta)\log\log x\}^{\beta}}$$

and

$$f(te/\beta) = \frac{te/\beta}{\{\log t - (1+\beta)\log\log t\}^{\beta}} + o(t/(\log t)^{1+\beta}).$$

Since

$$\left\{\frac{\log t}{\log t - (1+\beta)\log\log t}\right\}^{\beta} =$$

$$= \left\{1 + (1+\beta)\,\frac{\log\log t}{\log t} + (1+o(1))(1+\beta)^2\left(\frac{\log\log t}{\log t}\right)^2\right\}^{\beta}$$

$$= 1 + \beta(1+\beta)\,\frac{\log\log t}{\log t} + \beta(1+\beta)^2(1+o(1))\left(\frac{\log\log t}{\log t}\right)^2$$

we find

$$f(te/\beta) = \frac{te/\beta}{(\log t)^{\beta}} + e(1+\beta)\,\frac{t\log\log t}{(\log t)^{1+\beta}} + o\left(\frac{t}{(\log t)^{1+\beta}}\right),$$

which is (2.82).

II.3. General kernel transforms

Two important subjects in the preceding section of this chapter were Abelian and Tauberian theorems for the Laplace transform of functions belonging to the classes RV and Π. In this section we replace the Laplace transform by a more general kernel and derive Tauberian results. We restrict our attention to positive kernels and use the following notation.

Definition 2.31

Suppose $k, f : \mathbb{R}^+ \to \mathbb{R}$ are measurable. In this section the transform $\hat{f}$ is defined for $t > 0$ by

$$\hat{f}(t) = \int_0^\infty k(s)\, f(ts)ds \tag{2.83}$$

and is supposed to be finite for $t > 0$. ◇

In ch. 1 it was observed (thm. 1.8) that if $f \in RV_\alpha^\infty$ and $t^\alpha k(t)\max(t^{-\varepsilon}, t^{+\varepsilon})$ is integrable on $(0, \infty)$ for some $\varepsilon > 0$, then

$$\hat{f}(t)/f(t) \to \int_0^\infty s^\alpha k(s)\, ds \quad (t\to\infty).$$

As a consequence, if the last integral is positive, the function $\hat{f} \in RV_\alpha$.

We prove a converse statement, thereby using Wiener's Tauberian theorem.

Definition 2.32

The function $g : \mathbb{R}^+ \to \mathbb{R}$ is slowly decreasing if

$$\lim_{\mu\to 1+} \underline{\lim}_{t\to\infty} \inf_{u\in[1,\mu]} \{g(tu) - g(t)\} \geq 0. \tag{2.84}$$

Without proof we quote the following result (see e.g. Hardy (1948)).

Lemma 2.33 (Wiener-Pitt)

Suppose the kernel $k_o \in L^1(0, \infty)$ satisfies the condition

$$\int_0^\infty k_o(s)\, s^{-ix}\, ds \neq 0 \text{ for all } x \in \mathbb{R} \tag{2.85}$$

and the function $g : \mathbb{R}^+ \to \mathbb{R}$ is bounded and slowly decreasing.
Then

$$\frac{1}{t}\int_0^\infty k_o\left(\frac{s}{t}\right) g(s)\, ds \to c \int_0^\infty k_o(s)\, ds \quad (t\to\infty) \tag{2.86}$$

implies

$$g(t) \to c \ (t\to\infty). \tag{2.87}$$

◇

Two Tauberian theorems are proved, the first one for functions in RV, the second one for functions in Π. The corresponding Abelian statements were derived in theorems 1.8 and 1.20 respectively. ◇

Theorem 2.34

Suppose $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies: $t^\beta f(t)$ is non-decreasing for some $\beta \geq 0$ and $\hat{f} \in RV_\alpha^\infty$ with $\alpha \geq 0$. If the kernel k is non-negative, $t^\alpha k(t) \max(t^\varepsilon, t^{-\varepsilon})$ is integrable on $(0,\infty)$ and

$$\int_0^\infty k(s)\, s^{\alpha-ix} ds \neq 0 \text{ for all } x \in \mathbb{R},$$

then

$$f(t) \sim \left(\int_0^\infty s^\alpha k(s)ds\right)^{-1} \cdot \hat{f}(t) \quad (t \to \infty), \text{ hence } f \in RV_\alpha^\infty .$$

Proof

Without loss of generality we may assume $\int_1^\infty k(s)\, ds/s^\beta > 0$ (if not reformulate the theorem for $k_1(t) := k(ct)$ for a suitable constant $c > 0$). For $t > 0$ we have the inequality

$$\hat{f}(t) = \int_0^\infty k(s)\, f(ts)\, ds \geq f(t) \int_1^\infty k(s)\, ds/s^\beta > 0,$$

hence the function θ defined by

$$\theta(t) = f(t)/\hat{f}(t) \text{ is bounded for } t > 0 \text{ and positive.}$$

Note that for $1 < u \leq \mu$, $t > 0$

$$\theta(tu) - \theta(t) \geq \frac{f(t)}{\hat{f}(t)} \left\{\frac{\hat{f}(t)}{u^\beta \hat{f}(tu)} - 1\right\} \geq \frac{f(t)}{\hat{f}(t)} \left\{\frac{\hat{f}(t)}{\mu^\beta \sup\limits_{u\in[1,\mu]} \hat{f}(tu)} - 1\right\}.$$

Since $\hat{f} \in RV_\alpha$, by the uniform convergence theorem (theorem 1.3.3), we have

$$\varliminf_{t\to\infty} \inf_{u\in[1,\mu]} \{\theta(tu) - \theta(t)\} \geq \varliminf_{t\to\infty} \frac{f(t)}{\hat{f}(t)} -(1-\mu^{-\alpha-\beta}) = -(1-\mu^{-\alpha-\beta}) \varlimsup_{t\to\infty} \theta(t),$$

which implies that θ is slowly decreasing.

We proceed as in the proof of theorem 1.8, applying Lebesgue's theorem while using the inequalities from prop. 1.7.5 and the fact that θ is bounded. It follows that

$$\int_0^\infty k(s)\, \theta(ts) \left(\frac{\hat{f}(ts)}{\hat{f}(t)} - s^\alpha\right) ds \to 0 \quad (t\to\infty) .$$

Since

$$\int_0^\infty k(s)\, \theta(ts) \frac{\hat{f}(ts)}{\hat{f}(t)}\, ds = \int_0^\infty \frac{k(s)\, f(ts) ds}{\hat{f}(t)} = 1 ,$$

this implies

$$\int_0^\infty k(s)\, \theta(ts)\, s^\alpha\, ds \to 1 \quad (t\to\infty).$$

Application of the Wiener-Pitt theorem (lemma 2.33) above with $k_o(t) = k(t)t^\alpha$ and $f(t) = \Theta(t)$ shows that

$$\Theta(t) = f(t)/\hat{f}(t) \to \left(\int_0^\infty k(s)\ s^\alpha ds\right)^{-1} \ (t\to\infty). \qquad \diamond$$

Note that the Tauberian part of theorem 2.11 (Karamata's theorem) is a special case of theorem 2.34.

Remark

Under certain additional assumptions it is possible to prove that $\hat{f}(t)/f(t) \to a\ (t\to\infty)$ implies $f \in RV$ (Jordan 1974).

Theorem 2.35

Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and $\hat{f} \in \Pi(a)$. Suppose the kernel k is non-negative, $k(t)\ \max(t^\varepsilon, t^{-\varepsilon})$ is integrable on $(0, \infty)$ and k satisfies the Wiener condition

$$\int_0^\infty k(s)s^{-ix}dx \neq 0 \text{ for all } x \in \mathbb{R}.$$

Then $f \in \Pi(a_o)$ with

$$a_o(t) \sim \left(\int_0^\infty k(s)\ ds\right)^{-1} a(t)\ (t\to\infty) \text{ and}$$

$$\frac{\hat{f}(t) - f(t)\int_0^\infty k(s)\ ds}{a(t)} \to \int_0^\infty k(s)\ \log s\ ds\ (t\to\infty).$$

Proof

As in the proof of theorem 1.17 we define the function ψ by

$$\psi(t) = f(t) - t^{-1}\int_0^t f(s)\ ds\ ,\ t > 0.$$

Now observe that $\hat{\psi}(t) = \hat{f}(t) - t^{-1}\int_0^t \hat{f}(s)\ ds$. Application of theorem 1.17 ($a \to c$) gives, since $\hat{f} \in \Pi(a)$, $\hat{\psi}(t) \sim a(t)\ (t\to\infty)$ and $\hat{\psi} \in RV_0^\infty$.

Since f is non-decreasing, the function $t\ \psi(t)$ is non-decreasing and we can apply theorem 2.34 to obtain $\psi \in RV_0$ and $\psi(t) \sim a(t)\ \left(\int_0^\infty k(s)\ ds\right)^{-1}\ (t\to\infty)$.

A second application of theorem 1.17 (c → a) now gives $f \in \Pi(a_o)$

with $$a_o(t) \sim \left(\int_0^\infty k(s)\, ds\right)^{-1} a(t).$$

The last limit relation is a consequence of theorem 1.20 (the Abelian counterpart of the present theorem). ◇

Note that the Tauberian part of theorem 2.14 is a special case of theorem 2.35.

II.4. Discussion

The connection between an RV function and its complementary function has been noted first by Matuszewska (1962). See also Bingham and Teugels (1975). The present exposition, both for RV and Π/Γ (th. 2.3 and th. 2.8) has been adapted from Balkema, Geluk and de Haan (1979).
The main theorem for the Laplace transform, theorem 2.11 (for RV functions) is of course due to Karamata (1931). No exposition is given here of the Mercerian implication: $\hat{f}(1(t) \sim \Gamma(1+\alpha)\, f(t)$ $(t\to\infty)$ implies $f \in RV$. This has been proved by Drasin (1968).
Theorem 2.14 (class Π) has been adapted from de Haan (1976). The o-results of theorems 2.13 and 2.16 stem from Geluk and de Haan (1981). Theorem 2.19 (concerning functions like e.g. $\exp\{(\log x)^\beta x^\alpha\}$, $0 \le \alpha < 1$, $\beta > 0$) is a combination of results from Kohlbecker (1958) and Balkema, Geluk and de Haan (1979) for the cases $0 < \alpha < 1$ and $\alpha = 0$ respectively.
Finally theorem 2.25 (concerning functions like e.g. $\exp\{x/\log x\}$) has been adapted from Geluk, de Haan and Stadtmüller (1986). Wagner (1968) contains a somewhat similar result.
General kernel results like th. 2.34 and 2.35 can be found in Bingham and Teugels (1979) and Bingham and Teugels (1980) respectively.

III. O-Regular variation and O-versions of the class Π.

In this chapter we investigate what can be said if we only assume

$$\overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} < \infty \text{ for } x > 0, \tag{3.1}$$

instead of the existence of the limit (i.e. $f \in RV$) as in chapter 1. We also consider a similar extension of the class Π.
For this wider class of functions it is possible to derive results which are analogous to those of chapter 1. In fact straightforward generalizations of many of the characterizations from that chapter are possible. Part of the material of this chapter has been treated in greater generality in two articles by Bingham and Goldie (1982).

III. 1. O-regular variation

The following notation is useful in this section:

Definition 3.1
The functions f and g are of the same order at infinity, notation $f(x) \asymp g(x)$ $(x\to\infty)$ if f and g are both positive and if there exist $0 < c_1 < c_2 < \infty$ and x_0 such that $c_1 \leq f(x)/g(x) \leq c_2$ for $x \geq x_0$. ◊

Theorem 3.2 below offers results analogous to the results of theorems 1.4, 1.5 and prop. 1.7 for regularly varying functions. Recall from theorem 1.2 that if $\lim_{t\to\infty} f(tx)/f(t)$ exists for all $x > 0$, then the limit has the form x^α for some index $\alpha \in \mathbb{R}$. Theorem 3.2 also offers an analogue of this result (part iij) for functions satisfying (3.1).

Theorem 3.2
Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive. The following statements are equivalent:

(i) $$\overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} < \infty \text{ for all } x > 0. \tag{3.2}$$

(ij) There exist $\alpha, \beta \in \mathbb{R}$, t_0 and $c > 1$ such that

$$c^{-1}x^\beta \leq \frac{f(tx)}{f(t)} \leq c\, x^\alpha \text{ for all } x \geq 1,\ t \geq t_0. \tag{3.3}$$

(iij) $$A := \lim_{x\to\infty} \frac{\log \overline{\lim}_{t\to\infty} f(tx)/f(t)}{\log x} \text{ exists and } A < +\infty \tag{3.4}$$

and

$$B := \lim_{x\to\infty} \frac{\log \underline{\lim}_{t\to\infty} f(tx)/f(t)}{\log x} \text{ exists and } B > -\infty. \tag{3.5}$$

(iv) There exist $t_0 \geq 0$ and $\sigma \in \mathbb{R}$ such that

$$\int_{t_0}^{t} s^{+\sigma-1} f(s)ds \asymp t^{\sigma} f(t) \quad (t\to\infty). \tag{3.6}$$

(v) There exists $\tau \in \mathbb{R}$ such that

$$\int_{t}^{\infty} s^{\tau-1} f(s)ds \asymp t^{\tau} f(t) \quad (t\to\infty). \tag{3.7}$$

(vi) There exist $t_1 \geq 0$ and measurable functions a and c with $c(t)$ 1 and a bounded such that for $t > t_1$

$$f(t) = c(t) \exp \left\{\int_{t_1}^{t} a(s)ds/s\right\}. \tag{3.8}$$

(vii) There exist $\alpha, \beta \in \mathbb{R}$, $t_2 > 0$, $x_1 > 1$ such that

$$x^{\beta} \leq \frac{f(tx)}{f(t)} \leq x^{\alpha} \text{ for } t \geq t_2,\ x \geq x_1.$$

Proof

(i) → (ij)

Define the function F by $F(t) := \ln f(e^t)$. First we prove that if $I \subset \mathbb{R}$ is an arbitrary finite interval, then

$$\overline{\lim_{t\to\infty}} \sup_{u\in I} \{F(t+u) - F(t)\} < \infty. \tag{3.9}$$

Suppose the contrary holds. Then there exist sequences $t_n \to \infty$, $x_n \in I$ $(n = 1, 2, \ldots)$ such that

$$F(t_n + x_n) - F(t_n) > n.$$

For an arbitrary finite interval $J \subset \mathbb{R}$ we consider the sets

$$Y_{1,n} = \{y \in J;\ F(t_n + y) - F(t_n) > \tfrac{n}{2}\} \text{ and}$$

$$Y_{2,n} = \{y \in J;\ F(t_n + x_n) - F(t_n + y) > \tfrac{n}{2}\}.$$

The above sets are measurable for each n and $Y_{1,n} \cup Y_{2,n} = J$, hence either $\lambda(Y_{1,n}) \geq \frac{1}{2}\lambda(J)$ or $\lambda(Y_{2,n}) \geq \frac{1}{2}\lambda(J)$ (or both) where λ denotes the Lebesgue measure.

Now define

$$Z_n = \{z; F(t_n + x_n) - F(t_n + x_n - z) > \tfrac{n}{2},\ x_n - z \in J\} = \{z;\ x_n - z \in Y_{2,n}\}.$$

Then $\lambda(Z_n) = \lambda(Y_{2,n})$ and thus we have either

$$\lambda(Y_{1,n}) \geq \tfrac{1}{2}\ \lambda(J) \text{ or } \lambda(Z_n) \geq \tfrac{1}{2}\lambda(J)$$

for infinitely many $n \in \mathbb{N}$ (or both), where all the $Y_{1,n}$'s and Z_n's are subsets of a fixed finite interval.

Hence we have $\lambda(\limsup_{n\to\infty} Y_{1,n}) = \lim_{k\to\infty} \lambda(\bigcup_{n=k}^{\infty} Y_{1,n}) \geq \frac{1}{2}\lambda(J)$ or a similar expression for the Z_n's (or both). This implies the existence of a real number x_0 contained in infinitely many $Y_{1,n}$ or in infinitely many Z_n. This contradicts the assumption $\overline{\lim_{t\to\infty}}\ F(t + x_0) - F(t) < \infty$. Hence (3.9) is proved.

Next we apply (3.9) with $I = [0,1]$. There exists a constant c_0 such that $F(t+u) - F(t) \leq c_0$ for all $0 \leq u \leq 1$ and $t \geq t_0$. Then for $t \geq t_0$ and $y > 0$

$$F(t+y) - F(t) = F(t+y) - F(t+[y]) + \sum_{k=0}^{[y]-1} \{F(t+k+1) - F(t+k)\}$$

$$\leq ([y]+1)\ c_0 \leq c_0 y + c_0.$$

This finishes the proof of the right-hand inequality in (3.3). The proof of the left-hand inequality can be given if we replace f by 1/f in the above proof.

<u>(ij) → (iij)</u> Trivial.

(iij) → (i)

From (iij) it follows that for some α, $\beta \in \mathbb{R}$, $x_0 > 1$ we have

$$x^{\beta} \le \phi(x) \le \Phi(x) \le x^{\alpha}$$

for all $x \ge x_0 > 1$ with

$$\Phi(x) := \overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} \tag{3.10}$$

and

$$\phi(x) := \underline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)}. \tag{3.11}$$

This gives (3.2) for $x \ge x_0$. For $x \in (1, x_0)$ we have the inequality

$$\Phi(x) = \overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} \le \overline{\lim_{t\to\infty}} \frac{f(txx_0)}{f(t)} \Big/ \underline{\lim_{t\to\infty}} \frac{f(txx_0)}{f(tx)} \le \frac{(xx_0)^{\alpha}}{x_0^{\beta}} < \infty.$$

Similarly one proves $\phi(x) > 0$ for all $x > 1$. These two inequalities imply $\Phi(x) < \infty$ for all $x > 0$.

(ij) → (iv) and (ij) → (v)

The function

$$\gamma(t) := \int_{t_0}^{t} s^{\sigma-1} f(s)ds/t^{\sigma}f(t) = \int_{t_0/t}^{1} s^{\sigma-1} \frac{f(st)}{f(t)}ds \tag{3.12}$$

is bounded away from zero and infinity by (3.3) if we choose t_0 as in (3.3) and $\sigma > -\beta$. The proof of (ij) → (v) is similar.

(iv) → (vi) and (v) → (vi)

With γ defined as in (3.12) we have

$$\int_{t_0}^{t} \frac{ds}{s\,\gamma(s)} = \log \int_{t_o}^{t} s^{\sigma-1}f(s)ds + c_0$$

for $t > t_0$ and some $c_0 \in \mathbb{R}$ (since both sides have the same derivatives a.e.). The last relation implies

$$\exp\{\int_{t_0}^{t} \frac{ds}{s\,\gamma(s)}\} \asymp \int_{t_0}^{t} s^{\sigma-1} f(s)ds = t^{\sigma} f(t)\ \gamma(t) \asymp t^{\sigma} f(t).$$

Hence f has the required representation with $a(s) := \gamma(s)^{-1} - \sigma$.

The proof of (v) → (vi) is similar.

<u>(vi) → (i) and (ii) → (vij) → (iij)</u>
Trivial.
This finishes the proof of the theorem. ◇

<u>Definition 3.3</u>
A function f is <u>O-regularly varying</u> (at infinity) if f satisfies the conditions of theorem 3.2.
Notation: $f \in RO$.
The limits at the left-hand sides of (3.4) and (3.5) are called the <u>upper</u> and <u>lower</u> index of f respectively.
Notation: $\overline{\text{index}}$ f and $\underline{\text{index}}$ f. ◇

<u>Remark</u>
Note that if $f \in RO$, g measurable and $f(t) \asymp g(t)$ $(t\to\infty)$, then $g \in RO$.

It is obvious that if $k = 1/f$, then $\overline{\text{index}}\ k = -\ \underline{\text{index}}\ f$ and $\underline{\text{index}}\ k = -\ \overline{\text{index}}\ f$.

<u>Examples</u>
1. $f(x) = \exp[\ln x]$. Then $\underline{\text{index}}\ f = \overline{\text{index}}\ f = 1$, but $f \notin RV_1^{\infty}$.

2. Let $f(x) = 0 \quad x < e$

$\quad = \exp\{\alpha \log x + \beta(\log x)\ (\sin \log \log x)\},\ x \geq e.$

Then for every sequence $\{t_k\}$ with $t_k \to \infty$ we have

$$\lim_{k\to\infty} f(t_k x)/f(t_k) = \phi(x)$$

if and only if

$$\lim_{k\to\infty}\{g(s_k + y) - g(s_k)\} = \log \phi(e^y)$$

with $g(x) := \log f(e^x)$, $y = \log x$ and $s_k = \log t_k$.
Because $s_k\{\sin(\log s_k + \log(1 + y/s_k) - \sin \log s_k\} - y \cos \log s_k \to 0$ as $k\to\infty$ we have

$$\lim_{k\to\infty}\{g(s_k + y) - g(s_k)\} = \alpha\, y + \beta\, y \lim_{k\to\infty} (\sin \log(s_k + y) + \cos \log s_k).$$

The limit points of $f(tx)/f(t)$ are thus given by

$$\phi(x) = x^c \text{ with } c \in [\alpha - |\beta|\sqrt{2},\ \alpha + |\beta|\sqrt{2}].$$

Hence $\overline{\text{index}}\ f = \alpha + |\beta|\sqrt{2}$ and $\underline{\text{index}}\ f = \alpha - |\beta|\sqrt{2}$.
Note that, if $\beta = 1$ and $1 < \alpha < \sqrt{2}$, $\lim_{t\to\infty} f(0) = \infty$ but $\underline{\text{index}}\ f < 0$.

3. In example 2 the limit functions $\phi(x) := \lim_{k\to\infty} f(t_k x)/f(t_k)$ have the form $\phi(x) = x^c$. It is not necessarily true that the limit function is of this form however.
Example: if $f(t) = t^\beta(2 + \sin(\log t))$, $t > 0$, $\beta \in \mathbb{R}$, then $f \in RO$ and
$\phi(x) = x^\beta \cdot \dfrac{2 + \sin(\alpha + \log x)}{2 + \sin\alpha}$, $\alpha \in \mathbb{R}$.
An example of a monotone RO function of this type is
$f(t) = \exp\left(\int_1^t \{2 + \sin(\log s)\}ds/s\right)$.
In that case we have $\phi(x) = x^2 \exp(-\cos(\alpha + \log x) + \cos\alpha)$.

In the above theorem the two-sided bounds can not be replaced by one-sided bounds. For example: the right-hand inequality in (3.3) is not equivalent to (3.4). The following is a counterexample: take $F(x) = \ln f(e^x)$ and let F be continuous, piecewise linear with $F(3n) = F(3n + 2) = -(n-1)^2$ and $F(3n + 1) = -n^2$. Then (3.4) is satisfied, but not the right-hand inequality in (3.3).

Corollary 3.4

(i) The constants α, β and c in (ij) and (vij) are not uniquely determined. If $f \in RO$ we can take any $\beta < \underline{\text{index}}\ f$ and $\alpha > \overline{\text{index}}\ f$. The constant $c > 1$ in (3.3) however cannot be taken arbitrarily small for given f as the following example shows:
$f(t) = 2^n \chi_{[2^n,\, 2^{n+1})}(t)$, $n \in \mathbb{N}$, where χ is the indicator-function.

(ij) Note that (3.6) holds for any $\sigma > -\underline{\text{index}}\ f$ and (3.7) holds for any $\tau < -\overline{\text{index}}\ f$.

(iij) If $f \in RO$, there exists $f_0(t) \asymp f(t)$ with f_0 continuous. It is even possible to obtain $f_0 \in C^n$ by a construction similar to the one in remark 2 following theorem 1.5. ♢

<u>Remark</u>

If f is non-decreasing, we can omit the lower inequalities in theorem 3.2. Also, instead of $\overline{\lim}_{t\to\infty} f(tx)/f(t) < \infty$ for all $x > 1$, it then is sufficient to require

$$\overline{\lim_{t\to\infty}}\, f(tx_0)/f(t) < \infty \text{ for some } x_0 > 1. \tag{3.13}$$

<u>Proof</u>

Suppose (3.13). Then $f(tx_0)/f(t) \leq c$ for $t \geq t_0$ and some $x_0 > 1$. With $\rho = \ln c/\ln x_0$ we find

$$\frac{f(tx_0^{\,n})}{f(t)} = \frac{f(tx_0^{\,n})}{f(tx_0^{\,n-1})}\ \frac{f(tx_0^{\,n-1})}{f(tx_0^{\,n-2})} \cdots \frac{f(tx_0)}{f(t)} \leq x_0^{\,n\rho}.$$

Hence if $x > 1$ is arbitrary, there exists $n \in \mathbb{N}$ such that $x_0^{\,n-1} \leq x < x_0^{\,n}$ and $f(tx)/f(t) \leq x_0^{\,n\rho} \leq x_0^{\,\rho}x^{\rho}$.

This shows that the right-hand inequality in (3.3) is satisfied for all $x > 1$. The left-hand inequality follows immediately from the monotonicity of f. ♢

In view of the use of this class of functions for Tauberian theorems, we are especially interested in monotone RO-functions. We do not restrict ourselves to the class of functions described in the previous remark however, but consider next a class of RO functions for which there is a positive lower bound on the growth of the function:

Note that if $f \in RO$ and <u>index</u> $f > \varepsilon > 0$, then f is at least of the same order of magnitude as a monotone function that increases as a power function:

$$f(t) \asymp t^{\varepsilon} \int_{t_0}^{t} f(s)s^{-1-\varepsilon}ds,$$

$t \to \infty$ ((3.6), cf. cor. 3.4).

The next theorem characterizes this class of functions.

Theorem 3.5

Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is measurable. The following statements are equivalent

(i) $$\Phi(x) = \overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} < \infty \text{ for all } x \geq 1 \tag{3.14}$$

and there exists $x_o > 1$ such that

$$\phi(x) = \underline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} > 1 \text{ for all } x \geq x_o. \tag{3.15}$$

(ij) There exist $\alpha, \beta > 0$, $t_o \geq 0$ and $c > 1$ such that

$$c^{-1} x^{\beta} \leq \frac{f(tx)}{f(t)} \leq cx^{\alpha} \text{ for all } x \geq 1,\ t \geq t_o. \tag{3.16}$$

(iij) $$\lim_{x\to\infty} \frac{\ln \overline{\lim_{t\to\infty}}\ f(tx)/f(t)}{\ln x} < \infty$$

and

$$\lim_{x\to\infty} \frac{\ln \underline{\lim_{t\to\infty}}\ f(tx)/f(t)}{\ln x} > 0 .$$

(iv) There exist $t_o \geq 0$ and $\sigma \geq 0$ such that

$$\int_{t_o}^{t} s^{-\sigma-1} f(s)\, ds \asymp t^{-\sigma} f(t) \quad (t\to\infty) .$$

(v) There exists $\tau \geq 0$ such that

$$\int_{t}^{\infty} s^{-\tau-1} f(s)\, ds \asymp t^{-\tau} f(t) \quad (t\to\infty) .$$

(vi) There exist $t_o \geq 0$ and measurable functions a and c with

$c(t) \asymp 1$ and $a(t) \asymp 1$ $(t\to\infty)$ such that

$$f(t) = c(t)\exp\left\{\int_{t_o}^{t} a(s)\, ds/s\right\} . \tag{3.17}$$

(vij) $\overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} < \infty$ for all $x > 0$ and there exists $x_1 > 1$ such that

$$\underline{\lim_{t\to\infty}} \frac{f(tx_1)}{f(t)} > 1.$$

(viij) There exist $0 < \beta < \alpha < \infty$, $t_0 > 0$, $x_1 > 1$ such that

$$x^\beta \leq \frac{f(tx)}{f(t)} \leq x^\alpha \text{ for } t \geq t_0,\ x \geq x_1.$$

Proof

(i) → (vij)

We have to prove that $\Phi(x) < \infty$ for all $x > 0$, where the function Φ is defined by $\Phi(x) := \overline{\lim_{t\to\infty}}\, f(tx)/f(t)$.

Since $\phi(x) > 1$ for $x \geq x_o$ we have $\Phi(x) < 1$ for $x < x_o^{-1}$.

By assumption $\Phi(y) < \infty$ for $y > 1$.

Now the inequality $\Phi(xy) \leq \Phi(x)\,\Phi(y)$ and the last two statements show that $\Phi(x) < \infty$ for $x \in (x_o^{-1}, 1)$, which finishes the proof.

(vij) → (i)

Since $f \in RO$, by (3.3) we have $\phi(\tau) \geq c^{-1}\tau^\beta \geq c_o$ for $\tau \in [1, x_1]$, where $c_o := \min(c^{-1}, c^{-1} x_1^\beta) > 0$.

Define $n_o = \min\{n;\ c_o\,\phi(x_1)^n > 1\}$. Then for $x \geq x_1^{n_o}$ there exists $m \geq n_o$ such that $x_1^m \leq x < x_1^{m+1}$ and $\phi(x) \geq \phi(x/x_1^m)\cdot\phi(x_1^m) \geq c_o\,\phi(x_1)^m > 1$.

(i), (vij) → (ij)

Since $f \in RO$ the second inequality in (3.16) follows and we only have to prove that the second inequality holds for sufficiently large x. Take $x_1 > x_o^2$, define $X_i = \log x_i$, $i = 0,1$ and define the function F as in the proof of theorem 3.2. We shall prove that for an arbitrary finite interval $I \subset [X_1, \infty)$

$$\underline{\lim}_{t\to\infty} \inf_{u\in I} \{F(t+u) - F(t)\} > 0. \tag{3.18}$$

First we shall prove this for $I = [X_1, 2X_1]$.

Suppose the contrary holds. Then there exist sequences $t_n \to \infty$, $x_n \in I$ $(n = 1,2,\ldots)$ such that $F(t_n+x_n) - F(t_n) < 1/n$. Define

$$J = [X_o, X_1/2],$$

$$Y_{1,n} = \{y;\ F(t_n + y) - F(t_n) < 1/2n,\ y \in J\},$$

$$Y_{2,n} = \{y;\ F(t_n + x_n) - F(t_n+y) < 1/2n,\ y \in J\}.$$

and

$$Z_n = \{z;\ F(t_n+x_n) - F(t_n+x_n-z) < 1/2n,\ x_n - z \in J\}$$
$$= \{z;\ x_n - z \in Y_{2,n}\} \subset [\tfrac{1}{2}X_1,\ 2X_1 - X_0].$$

Proceeding exactly as in the proof of theorem 3.2 one obtains (3.18) for $I = [X_1, 2X_1]$, i.e. there exist constants $c_o > 0$ and t_o such that $F(t+u) - F(t) \geq c_o$ for all $t \geq t_o$ and $u \in [X_1, 2X_1]$. Then for $t \geq t_o$ and $y > X_1$ we have

$$F(t+y) - F(t) = F(t+y) - F(t+\{[y/X_1]-1\}X_1) +$$

$$+ \sum_{k=0}^{[y/X_1]-2} \{F(t+(k+1)X_1) - F(t+kX_1)\} \geq [y/X_1]c_o \geq$$

$$(-1 + y/X_1)c_o.$$

This proves (3.18) and the second inequality in (3.16). We omit the rest of the proof, which is similar to the proof of theorem 3.2. ◇

Definition 3.6.

Suppose $f:\mathbb{R}^+ \to \mathbb{R}$ is measurable and eventually positive. The function f is of bounded increase ($f \in BI$) if f satisfies (3.14).

The function f is of positive increase ($f \in PI$) if f satisfies (3.15). As a consequence, if f satisfies the assumptions of theorem 3.5 above, then $f \in BI \cap PI$. ◇

Corollary 3.7

a. If $f \in BI \cap PI$ and $g: \mathbb{R}^+ \to \mathbb{R}$ is measurable, $g(t) \asymp f(t)$ $(t\to\infty)$, then $g \in BI \cap PI$.

b. $BI \cap PI \subset RO$.

c. If $f \in RO$, then there exists $\beta \geq 0$ such that $t^\beta f(t) \in BI \cap PI$.

d. $f \in BI \cap PI$ if and only if $f \in RO$ and index $f > 0$.

e. If $f \in BI \cap PI$, then there exists a strictly increasing function f_o such that $f(t) \asymp f_o(t)$, $t\to\infty$. It follows that if $f \in BI \cap PI$ is locally bounded, then $\sup_{0<x\leq t} f(x) \asymp \inf_{x\geq t} f(x) \asymp f(t)$, $t\to\infty$.

f. If $f_o(t) = \exp\{\int_{t_o}^{t} a(s)ds/s\}$ with $a(s) \asymp 1$ $(s\to\infty)$, then the inverse function $f_o^{\leftarrow}$ is in BI ∩ PI. (Proof: similar to the proof of proposition 1.7.8).

g. If $f \in$ BI ∩ PI is bounded on finite intervals of $\mathbb{R}^+$, the generalized inverse function f^+ is as in definition 1.6 and f_o is as in e above, then $f^+(t) \asymp f_o^{\leftarrow}(t)$ $(t\to\infty)$, hence $f^+ \in$ BI ∩ PI.
(Proof : by theorem 3.5 there exist $c > 1$ and $t_o = t_o(c)$ such that
$c^{-1} f_o(t) \leq f(t) \leq c\, f_o(t)$ for $t > t_o$.
Hence $f_o^{\leftarrow}(t/c) \leq f^+(t) \leq f_o^{\leftarrow}(ct)$.
Also $f_o^{\leftarrow}(ct) \asymp f_o^{\leftarrow}(t) \asymp f_o^{\leftarrow}(t/c)$ by property e above.) ◊

h. If $f \in$ BI ∩ PI and $f(t) = f(t_0) + \int_{t_0}^{t} \psi(s)ds$ for $t \geq t_0$ with ψ monotone, then $t\,\psi(t) \asymp f(t)$ $(t\to\infty)$. (Proof: similar to the proof of prop. 1.7.11).

In the sequel we need the following lemma, which can be obtained from cor. 3.7 in a way similar to the proof of proposition 1.7.6 and 1.7.7.

Lemma 3.8

a. Suppose $f \in$ BI ∩ PI is bounded on finite intervals of $\mathbb{R}^+$.
For arbitrary $\xi > 0$, there exist $c > 0$ and t_o such that $f(tx)/f(t) \leq c$ for $t \geq t_o$ and $0 < x \leq \xi$.

b. Suppose $f \in$ RO is bounded on finite intervals of $\mathbb{R}^+$. For arbitrary $\xi > 0$ and $\alpha < \underline{\text{index}}\, f$, there exist $c > 0$ and t_o such that $f(tx)/f(t) \leq cx^{\alpha}$ for $t \geq t_o$ and $0 < x \leq \xi$. ◊

The reader is invited to prove the equivalence of the following statements for non-decreasing $f \in$ PI.

Exercise

Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing.
Then the following statements are equivalent:

a. $\underline{\lim}_{t\to\infty} \dfrac{f(tx)}{f(t)} > 1$ for some $x > 1$.

b. There exist β, t_0 and $c > 0$ such that

$$\frac{f(tx)}{f(t)} \geq cx^{\beta} \text{ for all } x \geq 1,\ t \geq t_0.$$

c. $1/\int_t^{\infty} \frac{ds}{sf(s)} \in PI$.

d. $\overline{\lim\limits_{t\to\infty}} \frac{\int_0^t f(s)\,ds}{tf(t)} < 1$.

e. There exists $\varepsilon > 0$ such that $t^{-1-\varepsilon} \int_0^t f(s)ds$ is increasing. ◇

III. 2. <u>O-versions of the class Π: asymptotically balanced functions</u>

<u>Definition 3.9</u>

Suppose $f:\mathbb{R}^+ \to \mathbb{R}$ is measurable. The function f is <u>asymptotically balanced</u> if there exists a function $a : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) $$\Psi(x) := \overline{\lim\limits_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} < \infty \text{ for all } x > 1. \qquad (3.19)$$

(ii) $$\psi(x) := \underline{\lim\limits_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} > -\infty \text{ for all } x > 0. \qquad (3.20)$$

(iii) There exists $x_0 > 1$ such that

$$\psi(x) = \underline{\lim\limits_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} > 0 \text{ for all } x \geq x_0. \qquad (3.21)$$

Notation: $f \in AB$ or $f \in AB(a)$. ◇

<u>Examples</u>

$f(t) = \log t + O(1)$ $(t\to\infty)$ is in $AB(1)$.

$f(t) = c - t^{-\alpha}$, $c \in \mathbb{R}$, $\alpha > 0$ is in $AB(t^{-\alpha})$.

The function $\exp(f(t)) \in BI \cap PI$ if and only if $f \in AB(1)$.

<u>Lemma 3.10</u>

If $f \in AB(a)$, then $\overline{\lim\limits_{t\to\infty}}\, a(tx)/a(t) < \infty$ for all $x > 0$. Moreover we may take a measurable in definition 3.9 and hence in RO.

Proof

Fix $x > 0$ and define $y := 1 + \max(x_0, x^{-1})$ with x_0 as in (3.21).
Since

$$\frac{a(tx)}{a(t)} = \left\{\frac{f(txy) - f(t)}{a(t)} - \frac{f(tx) - f(t)}{a(t)}\right\} / \frac{f(txy) - f(tx)}{a(tx)},$$

we have

$$\overline{\lim_{t\to\infty}} \frac{a(tx)}{a(t)} \leq \frac{\Psi(xy) - \psi(x)}{\psi(y)} < \infty. \tag{3.22}$$

This finishes the first part of the proof, since $x > 0$ is arbitrary. The proof is finished by observing that we may take $a(t) := f(tx_0) - f(t)$ where x_0 is as in (3.21). ◊

We are going to prove a characterization theorem for functions in the class AB. To this end we need two lemmas.

Lemma 3.11

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is measurable and $a \in RO$.

(i) Suppose there exists $x_0 \geq 1$ such that

$$\Psi(x) := \overline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} < \infty \text{ for all } x > x_0.$$

Then for any $x_1 > x_0^2$ there exist t_0 and $\sigma \in \mathbb{R}$ such that

$$\frac{f(tx) - f(t)}{a(t)} \leq x^\sigma \text{ for } x \geq x_1,\ t \geq t_0. \tag{3.23}$$

In (3.23) we may take any $\sigma > \overline{\text{index}}\ a$.

(ij) Suppose there exists $x_0 \geq 1$ such that

$$\psi(x) := \underline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} > 0 \text{ for all } x > x_0.$$

Then for any $x_1 > x_0^2$ there exist t_0 and $c > 0$ such that

$$\frac{f(tx) - f(t)}{a(t)} \geq c \text{ for } x \geq x_1,\ t \geq t_0. \tag{3.24}$$

In (3.24) we may replace c by x^τ for any $\tau <$ index a.

Proof

(i) Similar to the proof of theorem 3.2 (i) → (ij).
We pass to additive arguments and write $F(x) := f(e^x)$, $A(x) := a(e^x)$ and $X_i := \log x_i$ for $i = 0,1$. First we shall prove that for an arbitrary finite interval $I \subset [X_1, \infty)$

$$\overline{\lim_{t\to\infty}} \sup_{u\in I} \{F(t+u) - F(t)\} / A(t) < \infty. \tag{3.25}$$

We first prove this for $I = [X_1, 2X_1]$. Suppose the contrary holds, then there exist sequences $t_n \to \infty$, $x_n \in I$ $(n = 1,2,\dots)$ such that

$$\{F(t_n + x_n) - F(t_n)\}/A(t_n) > n.$$

Define $J := [X_0, X_1/2]$ and

$$Y_{1,n} = \{y;\ (F(t_n + y) - F(t_n))/A(t_n) > n/2,\ y \in J\},$$

$$Y_{2,n} = \{y;\ (F(t_n + x_n) - F(t_n + y))/A(t_n) > n/2,\ y \in J\},$$

$$Z_{1,n} = \{z;\ (F(t_n + x_n) - F(t_n + x_n - z))/A(t_n) > n/2,\ x_n - z \in J\}.$$

Since $a \in RO$ we have $c > 0$, n_0 such that $A(t_n) \geq c\, A(t_n + x_n - z)$ for $n \geq n_0$ and $z \in Z_{1,n}$ by theorem 3.2. As a consequence $Z_{1,n} \subset Z_{2,n}$ for $n \geq n_0$, where $Z_{2,n}$ is defined by
$Z_{2,n} = \{z;\ (F(t_n + x_n) - F(t_n + x_n - z))/A(t_n + x_n - z) > cn/2,\ x_n - z \in J\} \subset [X_0, 2X_1 - X_0]$.

As before we find $\lambda(\limsup_{n\to\infty} Z_{2,n}) \geq \lambda(\limsup_{n\to\infty} Z_{1,n}) \geq \frac{1}{2}\lambda(J)$ or $\lambda(\limsup_{n\to\infty} Y_{1,n}) \geq \frac{1}{2}\lambda(J)$, which contradicts our assumption.

As a consequence we find that for some c_1, t_1
$\{f(tx) - f(t)\}/a(t) \leq c_1$ for $x_1 \leq x \leq x_1^2$, $t \geq t_1$. We shall choose $c_1 > 0$.
Finally choose $x \geq x_1$, then $x_1^m \leq x < x_1^{m+1}$ for some $m \geq 1$.
Since $a \in RO$, there exist α, $c_2 > 0$ such that $a(tx)/a(t) \leq c_2\, x^\alpha$ for $x \geq 1$ and $t \geq t_2$.
Hence for $t \geq \max(t_1, t_2)$

$$\frac{f(tx)-f(t)}{a(t)} = \sum_{k=1}^{m-1} \frac{f(x_1^{k}t)-f(x_1^{k-1}t)}{a(x_1^{k-1}t)} \frac{a(x_1^{k-1}t)}{a(t)} +$$

$$+ \frac{f(xt)-f(x_1^{m-1}t)}{a(x_1^{m-1}t)} \frac{a(x_1^{m-1}t)}{a(t)} \leq$$

$$\leq \sum_{k=1}^{m} c_1 \, a(x_1^{k-1}t)/a(t) \leq \sum_{k=1}^{m} c_1 c_2 x_1^{(k-1)\alpha}$$

$$= c_1c_2(x_1^{m\alpha}-1)/(x_1^{\alpha}-1) \leq c_1c_2(x_1^{\alpha}-1)^{-1}x^{\alpha}, \text{ where } c_1 \in \mathbb{R}.$$

(ij) We omit the proof of the second part, which is similar. ◊

Lemma 3.12

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be measurable and $a \in RO$. If ψ and Ψ are defined as in lemma 3.11 and $-\infty < \psi(x) \leq \Psi(x) < \infty$ for all $x > 1$, then there exist constants t_0 and σ, $c > 0$ such that

$$|\{f(tx)-f(t)\}/a(t)| \leq c\, x^{\sigma} \text{ for } x \geq 1,\ t \geq t_0. \tag{3.26}$$

Moreover for any $\sigma > \overline{\text{index}}\ a$ there exist $c > 0$ and t_0 such that (3.26) holds.

Proof

By lemma 3.11 we have $\{f(ty)-f(t)\}/a(t) \leq y^{\sigma}$ for $t \geq t_0$ and $y \geq x_1$, where σ is a positive constant.
Then for $x \in [1,2]$ and $t \geq t_0$.

$$\frac{f(tx)-f(t)}{a(t)} = \frac{f(2x_1t)-f(t)}{a(t)} - \frac{f(2x_1\,t)-f(xt)}{a(xt)} \frac{a(tx)}{a(t)}$$
$$\geq \frac{f(2x_1t)-f(t)}{a(t)} - \frac{(2x_1)^{\sigma}\, a(tx)}{a(t)} \geq \frac{f(2x_1t)-f(t)}{a(t)} - c_0x^{\alpha}$$

for some $\alpha \in \mathbb{R}$ and $c_0 > 0$ since $a \in RO$. Hence

$$\varliminf_{t\to\infty} \inf_{x\in[1,2]} \frac{f(tx)-f(t)}{a(t)} \geq \psi(2x_1) - c_0 \max(1,2^{\alpha}) > -\infty. \tag{3.27}$$

Replacing f by -f we find a similar upper inequality.

An iteration procedure as in the proof of lemma 3.11 then gives (3.26). ◇

We now proceed to give a characterization for functions of the class AB. First we derive a representation for monotone functions of the class AB. We then show that any function of the class is close to a monotone function in a certain sense.

Theorem 3.13

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing. Then $f \in AB(a)$ if and only if there exists $r > 0$ such that the function g defined by

$$g(t) := \int_0^t s^r \, df(s) \tag{3.28}$$

is in BI ∩ PI. In that case we have $g(t) \asymp t^r a(t)$ $(t\to\infty)$.

Proof

Assume that $f \in AB(a)$.

Since we may take $a \in RO$ by lemma 3.10, we have $t^r a(t) \in BI \cap PI$ (see cor. 3.7) for arbitrary r>-index a. It is thus sufficient to prove $t^r a(t) \asymp g(t)$ $(t\to\infty)$.

Application of Fatou's lemma gives

$$\varliminf_{t\to\infty} \frac{g(t)}{t^r a(t)} = \varliminf_{t\to\infty} \quad r \int_0^1 \frac{f(t) - f(tv)}{a(t)} v^{r-1} dv \geq$$

$$\geq r \int_0^1 \varliminf_{t\to\infty} \frac{f(t) - f(tv)}{a(tv)} \frac{a(tv)}{a(t)} v^{r-1} dv > 0$$

since $f \in AB(a)$ and $a \in RO$. Next we prove $\varlimsup_{t\to\infty} g(t)/t^r a(t) < \infty$. By lemma 3.11 (i) and theorem 3.2, there exist $c, \alpha, \sigma, t_0, x_1 > 1$ such that

$$\frac{f(t) - f(tv)}{a(tv)} \frac{a(tv)}{a(t)} \leq v^{-\sigma} \cdot cv^{\alpha} \tag{3.29}$$

for $tv \geq t_0$ and $v^{-1} > x_1$. Write

$$\frac{g(t)}{t^r a(t)} = r\{\int_0^{t_0/t} + \int_{t_0/t}^{1/x_1} + \int_{1/x_1}^{1}\} \frac{f(t) - f(tv)}{a(t)} v^{r-1} dv.$$

By (3.29) we have

$$\overline{\lim_{t\to\infty}}\ r \int_{t_0/t}^{1/x_1} \frac{f(t) - f(tv)}{a(t)}\ v^{r-1}dv \le r\ c \int_0^{1/x_1} v^{\alpha-\sigma+r-1}dv$$

and the last integral is finite if we take $r > \sigma - \alpha$.
Moreover we have

$$\overline{\lim_{t\to\infty}}\ |r \int_0^{t_0/t} \frac{f(t) - f(tv)}{a(t)}\ v^{r-1}dv| \le \overline{\lim_{t\to\infty}}\ \frac{|f(t)|}{a(t)} \left(\frac{t_0}{t}\right)^r +$$

$$f(t_0)t_0^r\ \overline{\lim_{t\to\infty}}\ t^{-r}/a(t).$$

The last expression is finite if we choose r sufficiently large, since $|f(t)| \le t^{\alpha_0}$ and $a(t) \le t^{\beta_0}$ for $t \ge t_0$ and some $\alpha_0, \beta_0 \in \mathbb{R}$ by lemma 3.12 and theorem 3.2 respectively.
Finally, since f is non-decreasing and $f \in AB(a)$, we have

$$\overline{\lim_{t\to\infty}}\ r \int_{1/x_1}^{1} \frac{f(t) - f(tv)}{a(t)}\ v^{r-1}dv \le \overline{\lim_{t\to\infty}}\ \frac{f(t) - f(t/x_1)}{a(t)} (1 - x_1^{-r}) < \infty.$$

Combination of the above results gives $g\ (t) \asymp t^r\ a(t)$.

Conversely, assume that $g \in BI \cap PI$. Using (3.28) we obtain

$$f(t) = f(0)+ \int_0^t s^{-r}\ d\ g(s) \tag{3.30}$$

and hence

$$\frac{f(tx) - f(t)}{t^{-r}g(t)} = r \int_1^x \frac{g(tu)}{g(t)}\ u^{-r-1}du + \frac{(xt)^{-r}g(tx)}{t^{-r}g(t)} - 1. \tag{3.31}$$

Since g is monotone, for $x > 1$

$$\overline{\lim_{t\to\infty}}\ \frac{f(tx) - f(t)}{t^{-r}g(t)} \le (1 - x^{-r})\ \overline{\lim_{t\to\infty}}\ \frac{g(tx)}{g(t)} + x^{-r}\ \overline{\lim_{t\to\infty}}\ \frac{g(tx)}{g(t)} - 1 < \infty .$$

Also, with the function h defined by $h(x): = \underline{\lim}_{t\to\infty}\ g(tx)/g(t)$, by (3.31),

$$\underline{\lim}_{t\to\infty}\ \frac{f(tx) - f(t)}{t^{-r}g(t)} \ge r \int_1^x u^{-r-1}h(u)du + x^{-r}h(x)-1 = : k\ (x).$$

Since $h(t) \ge 1$ for $t \ge 1$, but $h(t) \ne 1$ on $(1,\infty)$, we find that for x sufficiently large

$$k(x) > r \int_1^x u^{-r-1}\,du + x^{-r} - 1 = 0.$$

Hence $f \in AB(a)$ with $a(t) \asymp t^{-r}\, g(t),\ t \to \infty$. ◇

Remark

It follows from the proof that in the above result we may take any $r > -$ index a.

Corollary 3.14

Suppose f is non-decreasing. Then $f \in AB$ if and only if there exists a non-decreasing function $g \in BI \cap PI$ and constants $r > 0$ and c such that

$$f(t) = c + g(t)\, t^{-r} + r \int_0^t s^{-r-1}\, g(s)\, ds.$$

Proof

This is (3.30). ◇

In the sequel we will need the following variant of this result.

Lemma 3.15

Suppose $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is measurable and the function f defined by

$$f(t) := \int_0^t g(s)ds/s^2 \text{ is finite for all } t > 0.$$

Then $g \in BI \cap PI$ implies $f \in AB(a)$ with index $a > -1$. The converse statement is true if g is non-decreasing.

Proof

Suppose $g \in BI \cap PI$. Since

$$\frac{f(tx) - f(t)}{t^{-1} g(t)} = \int_1^x \frac{g(ts)}{g(t)} \frac{ds}{s^2},$$

$f \in AB(a)$ with $a(t) = g(t)/t$ and index $a > -1$.

Conversely if $f \in AB(a)$ with index $a > -1$ we obtain for $x > 1$ (use the monotonicity of g)

$$\frac{f(tx) - f(t)}{a(t)} = \int_1^x \frac{g(ts)}{ta(t)} \frac{ds}{s^2} \geq \frac{g(t)}{ta(t)} (1 - \frac{1}{x})$$

and for $0 < x < 1$

$$\frac{f(t) - f(tx)}{a(t)} \leq \frac{g(t)}{ta(t)} (\frac{1}{x} - 1),$$

hence $g(t) \asymp ta(t)$. Then $g \in BI \cap PI$ follows. ◇

For regularly varying functions and functions in the class Π, the notion of inversely asymptotic functions (see definition 1.21) proved useful. Lemma 1.23 a shows that for any function $f \in \Pi$ or $f \in RV$ it is possible to find a smooth function f_0 which is inversely asymptotic to f, i.e. for all $a > 1$ there exists $t_0(a)$ such that $f_0(t/a) \leq f(t) \leq f_0(at)$ for $t \geq t_0$ (this is relation $\overset{*}{\sim}$, see definition 1.21). We show that for any function $f \in AB$, there exists a smooth (namely non-decreasing) function f_0 such that the above inequalities hold for <u>some</u> $a > 1$ and all t sufficiently large.
We start with a formal definition.

<u>Definition 3.16</u>
The functions $f, f_0 : \mathbb{R}^+ \to \mathbb{R}$ are <u>O-inversely asymptotic</u> if there exist constants $a > 1$ and $t_0 = t_0(a)$ such that

$$f(t) \leq f_0(at) \qquad t \geq t_0$$

and (3.32)

$$f_0(t) \leq f(at) \qquad t \geq t_0.$$

Notation : $f \overset{O}{\sim} f_0$ or $f(t) \overset{O}{\sim} f_0(t)$, $t\to\infty$. ◇

The reader should compare this with definition 1.21 (relation $\overset{*}{\sim}$). It is easy to see that if f and f_0 are increasing and unbounded, then $f \overset{O}{\sim} f_0$ if and only if the inverse functions satisfy $f^{\leftarrow} = O(f_0^{\leftarrow})$ and $f_0^{\leftarrow} = O(f^{\leftarrow})$, in other words, if $f^{\leftarrow} \asymp f_0^{\leftarrow}$.
The relevancy of this definition for functions f in $BI \cap PI$ follows from the following lemmas.

<u>Lemma 3.17</u>
Suppose $f, f_0 \in BI \cap PI$.

Then $f(t) \overset{0}{\sim} f_0(t)$ $(t\to\infty)$ if and only if $f(t) \asymp f_0(t)$, $t\to\infty$.

Proof

Suppose $f \overset{0}{\sim} f_0$. We then have $f(t) \leq f_0(at) \leq ca^{1 + \overline{\text{index}}\, f_0} f_0(t)$ by (3.32) and theorem 3.5. A lower inequality is obtained similarly.

Conversely, suppose $f(t) \leq b\, f_0(t)$ for $t \geq t_0$ and $b > 0$.

By theorem 3.5 we have

$$f_0(at) \geq c^{-1}\, a^{\frac{1}{2}\, \underline{\text{index}}\, f_0}\, f_0(t) \text{ for } t \geq t_1,\ a \geq 1 \text{ and some } c > 1.$$

Hence $f(t) \leq f_0(at)$ for $t \geq \max(t_0, t_1)$ if we choose $a > 1$ such that $c^{-1} a^{\frac{1}{2}\underline{\text{index}}\, f_0} \geq b$.

The proof of a converse inequality is similar. ◇

Lemma 3.18

Suppose $f_0 \in BI \cap PI$, $f : \mathbb{R}^+ \to \mathbb{R}$ measurable and $f \overset{0}{\sim} f_0$.

Then $f \in BI \cap PI$.

Proof

Directly from lemma 3.17 and cor. 3.7a. ◇

For asymptotically balanced functions a statement analogous to that of lemma 3.18 is correct, although the proof is somewhat different, since the analogue of lemma 3.17 is no longer true.

Lemma 3.19

Suppose $f_0 \in AB(a)$, $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable and $f \overset{0}{\sim} f_0$. Then $f \in AB(a)$ and

$$f(t) - f_0(t) = O(a(t)),\ t\to\infty. \tag{3.33}$$

Proof

Fix $x > 1$.

For t sufficiently large, by definition 3.16, there exists $c > 1$ such that

$$\frac{f_0(tcx) - f_0(t/c)}{a(t)} \geq \frac{f(tx) - f(t)}{a(t)} \geq \frac{f_0(tx/c) - f_0(tc)}{a(t)}. \tag{3.34}$$

For x sufficiently large, the right-hand side in (3.34) has a positive limes inferior as $t\to\infty$, since $f_0 \in AB(a)$.
The rest of the proof is easy. ◇

Remark

There is a statement like that of lemma 3.19 for the class Π: if $f \overset{*}{\sim} f_0$ and $f_0 \in \Pi(a)$, then $f(t) - f_0(t) = o(a(t))$. See proposition 1.22 (ij). The latter relation has a converse : if $f_0 \in \Pi$, $f(t) - f_0(t) = o(a(t))$, then $f \overset{*}{\sim} f_0$ and hence $f \in \Pi$ (see theorem 1.13 and prop. 1.22 (ij)). The corresponding converse of relation (3.33) is not correct as the following example shows. Note that this remark reduces the value of corollary 3.21 below.

Example

Take $f_0(t) = t$, $f(t) = t + (-1)^{[\log t]}t$.
Then $f_0 \in AB(a)$ with $a(t) = t$ and $f(t) - f_0(t) = O(a(t))$, $t\to\infty$, but for $x = e^{2n+1}$ and $e^{2m} \le t < e^{2m+1}$ $(m,n \in \mathbb{N})$ we have $\{f(tx) - f(t)\}/t = -2$, hence $\varliminf_{t\to\infty} \{f(tx) - f(t)\}/t < 0$ for $x = e^3, e^5, e^7, \ldots$, i.e. f is not in AB(a).

Note that the relation $\overset{0}{\sim}$ is an equivalence relation for functions of the class AB. The next theorem shows that every equivalence class contains a smooth function, namely a non-decreasing function and for such functions a representation is available.

Theorem 3.20

Suppose $f\colon \mathbb{R}^+ \to \mathbb{R}$ is measurable. Then the following statements are equivalent

(i) $f \in AB(a)$.

(ij) There exists a non-decreasing function $f_0 \in AB(a)$ such that $f(t) \overset{0}{\sim} f_0(t)$ $(t\to\infty)$.

Proof

(i) → (ij)

Suppose $f \in AB$. Then, by lemma 3.11 (ij) there exist t_0 and x_1 such that $f(tx) \ge f(t)$ for $t \ge t_0$, $x \ge x_1$, Now define the function f_0 by $f_0(t_0x_1^n) = f(t_0x_1^n)$ for $n = 0, 1, 2, \ldots$ and linear in between. Note that f_0 is non-decreasing. Further for $s > 2$ we have

$$f_0(t_0x_1^{s-2}) \leq f_0(t_0x_1^{[s]-1}) = f(t_0x_1^{[s]-1}) \leq f(t_0x_1^{s})$$

$$\leq f(t_0x_1^{[s]+2}) = f_0(t_0x_1^{[s]+2}) \leq f_0(t_0x_1^{s+2}).$$

Hence we obtain $f_0(t/x_1^2) \leq f(t) < f_0(tx_1^2)$ for $t > t_0x_1^2$. Note that $f_0 \in AB(a)$ by lemma 3.19.

(ij) → (i)

This is an immediate consequence of lemma 3.19. ◊

Corollary 3.21

If $f \in AB(a)$, then there exists a non-decreasing $f_0 \in AB(a)$ such that $f(t) = f_0(t) + O(a(t))$, $t\to\infty$.

Proof

Use lemma 3.19 and theorem 3.20. ◊

We use the above corollary to derive a more specific result.

Theorem 3.22

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is measurable.

Then the following statements are equivalent.

(i) $f \in AB(a)$

(ij) There exists a non-decreasing function $g \in BI \cap PI$ and constants $r > 0$ and c such that $g(t) \asymp t^r\, a(t)$ and

$$f(t) \overset{0}{\sim} c + r \int_0^t s^{-r-1}\, g(s)ds. \tag{3.35}$$

Proof

Suppose $f \in AB(a)$. Application of theorem 3.20 and corollary 3.14 shows that we have

$$f(t) \overset{0}{\sim} f_0(t) := c + g(t)t^{-r} + r \int_0^t s^{-r-1}\, g(s)ds,$$

where $g \in BI \cap PI$ is non-decreasing, $r > 0$ and $g(t)$ $t^r a(t)$.

We prove that $f_0(t) \overset{0}{\sim} f_1(t)$ $(t\to\infty)$, where $f_1(t)$ denotes the right-hand side in (3.35).

By theorem 3.5 (viij) and the monotonicity of g for some $x_1 > 1$, $t_0 > 0$, $0 < \beta < r$ we have for $t \geq t_0$ and $x > y > x_1$

$$\frac{f_1(tx) - f_0(t)}{t^{-r}g(t)} = r \int_1^x \frac{g(ts)}{g(t)} s^{-r-1} ds - 1 \geq r \int_1^y s^{-r-1} ds + \tag{3.36}$$

$$r \int_y^x s^{\beta-r-1} ds - 1 = y^{-r}\left(\frac{r}{r-\beta} y^{\beta} - 1\right) - \frac{r}{r-\beta} x^{\beta-r}.$$

We take $y = y_0 > x_1$ such that $\delta := \frac{r}{r-\beta} y_0^{\beta} - 1 > 0$. Then the right-hand side in (3.36) is positive for all x satisfying $x^{-\beta+r} > \frac{r\, y_0^r}{(r-\beta)\ \delta}$. Hence $f_1(tx) \geq f_0(t)$ for some $x > 1$ and $t > t_0$.

The reverse inequality follows since $f_1(t) \leq f_0(t) \leq f_0(tx)$. Hence we find $f_0(t) \overset{0}{\sim} f_1(t)$ $(t \to \infty)$, which implies (3.35).

Conversely, if f satisfies (ij), we have by Fatou's lemma

$$\underline{\lim}_{t\to\infty} \frac{f_1(tx) - f_1(t)}{t^{-r}g(t)} \geq r \int_1^x \underline{\lim}_{t\to\infty} \frac{g(ts)}{g(t)} s^{-r-1} ds > 0$$

for x sufficiently large, since $g \in BI \cap PI$ is non-decreasing. ◇

Corollary 3.23

If $f \in AB(a)$, then there exists a non-decreasing function $g \in BI \cap PI$ and constants $r > 0$ and c such that $g(t) \asymp t^r a(t)$ and

$$f(t) = c + r \int_0^t s^{-r-1} g(s) ds + O(t^{-r} g(t)), \quad t \to \infty. \tag{3.37}$$

◇

Remark

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable and satisfies (3.37) with $r > 0$ and $r < \underline{\text{index}}\ g \leq \overline{\text{index}}\ g < \infty$.

Then $f \in AB(a)$ with $a(t) \asymp t^{-r}\ g(t)$. This is a partial converse of corollary 3.21. ◇

III.3. Discussion

A reference for O-regularly varying functions is Aljancić and Arandelović (1977).

A reference for the classes BI, PI and AB is de Haan and Resnick (1984).

There are many other possible generalizations of the classes RV and Π; see the two papers by Bingham and Goldie (1982). We have chosen the present ones since they seem to be useful and since the results and proofs for those classes follow quite closely the theory of RV and Π. The results presented after def. 3.16 are new and partly due to Balkema.

IV. Tauberian theorems for O-varying functions.

In this chapter Tauberian theorems are proved for the classes of functions RO and AB (O-regularly varying functions and asymptotically balanced functions). Note that the results are straightforward generalizations of the corresponding statements for the classes of functions RV and Π respectively (Karamata's theorem - theorem 2.11 - and theorem 2.14).

IV. 1. The Laplace transform

Theorem 4.1

Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ is measurable and has a finite Laplace transform $\hat{f}(t)$ for $t > 0$.

If

$$f \in RO \text{ with } \underline{\text{index}}\ f > -1 \qquad (4.1)$$

(see definition 3.2), then

$$\hat{f}(1/t) \in RO \text{ with } \underline{\text{index}}\ \hat{f}(1/t) > -1 \qquad (4.2)$$

and

$$f(t) \asymp \hat{f}(1/t). \qquad (4.3)$$

Conversely if $t^\alpha f(t)$ is non-decreasing for some $\alpha \in [0,1)$, then (4.2) or (4.3) implies (4.1).

Proof

First suppose (4.1) holds.

Since $tf(t) \in BI \cap PI$ we may apply theorem 3.5.

By (3.16) there exist $c > 1$, α, $\beta > 0$ such that

$$c^{-1}x^{\alpha-1} \leq f(tx)/f(t) \leq cx^{\beta-1} \text{ for } xt \geq t_0,\ 0 < x \leq 1.$$

Now write

$$\frac{\hat{f}(t^{-1})}{f(t)} = \int_0^{t_0/t} e^{-x}\frac{f(tx)}{f(t)}\,dx + \int_{t_0/t}^{1} e^{-x}\frac{f(tx)}{f(t)}\,dx + \int_1^\infty e^{-x}\frac{f(tx)}{f(t)}\,dx$$

$$=: I_1 + I_2 + I_3.$$

Then $0 < \underline{\lim}_{t\to\infty} I_i \leq \overline{\lim}_{t\to\infty} I_i < \infty$ for $i = 2, 3$. Next we consider I_1.
Since <u>index</u> $f > -1$ we have $tf(t) \to \infty$ $(t\to\infty)$. Hence

$$|I_1| \leq \int_0^{t_0} e^{-x/t}|f(x)|dx/tf(t) \leq \int_0^{t_0} |f(x)|dx/tf(t) \to 0 \quad (t\to\infty).$$

This proves $\hat{f}(t^{-1})$ $f(t)$ $(t\to\infty)$. Since $f \in RO$, it follows that $\hat{f}(t^{-1}) \in RO$.
Hence (4.1) implies (4.2) and (4.3).
Conversely suppose $t^{\alpha}f(t)$ is non-decreasing and (4.2) holds. Then

$$\hat{f}(s) = \int_0^{\infty} e^{-t} f(t/s)\, dt \geq a^{\alpha} f(a/s) \int_a^{\infty} t^{-\alpha} e^{-t}\, dt = : c(a)f(a/s) \quad (4.4)$$

for all $s, a > 0$.
Hence for $\beta > 1$ and sufficiently small s

$$\hat{f}(s) = \int_0^{\infty} e^{-t} f(t/s)\, dt \leq \beta^{\alpha}f(\beta/s) \int_0^{\beta} t^{-\alpha}e^{-t}dt + c(1)^{-1} \int_{\beta}^{\infty} e^{-t} \hat{f}(s/t)dt$$

$$\leq \beta^{\alpha} f(\beta/s) \int_0^{\beta} t^{-\alpha}e^{-t}dt + c(1)^{-1}\hat{f}(s) \int_{\beta}^{\infty} e^{-t}ct^{\gamma}dt,$$

for some $c > 0$, $\gamma \in \mathbb{R}$, the last inequality being a consequence of theorem 3.2 (applied to the function $\hat{f}(1/x)$).
Now choose $\beta = \beta_o > 1$ in such a way that $c(1)^{-1} \int_{\beta_o}^{\infty} e^{-t}ct^{\gamma}dt \leq \frac{1}{2}$.
Then we find

$$\hat{f}(s) \leq 2\, \beta_o^{\alpha}\, f(\beta_o/s) \int_0^{\beta_o} t^{-\alpha}e^{-t}dt = : c_2\, f(\beta_o/s) \text{ for all } s \leq s_o. \quad (4.5)$$

Combination of (4.4) and (4.5) gives

$$\overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} \leq c_2c(1)^{-1} \overline{\lim_{t\to\infty}} \frac{\hat{f}(1/tx)}{\hat{f}(\beta_0/t)} < \infty$$

for $x > 1$. Note that <u>index</u> $f \geq -\alpha > -1$ since $t^{\alpha}f(t)$ is non-decreasing. Hence $f \in RO$.
Finally suppose $t^{\alpha} f(t)$ is non-decreasing (for some $\alpha \in [0,1)$) and (4.3) holds.
We use the inequality (4.4) again and find for $x > 0$ fixed

$$\overline{\lim_{t\to\infty}} \frac{f(tx)}{f(t)} \leq c(x)^{-1} \overline{\lim_{t\to\infty}} \frac{\hat{f}(1/t)}{f(t)} < \infty.$$

Hence $f \in RO$. ◇

Theorem 4.2

Suppose $f : \mathbb{R}^+ \to \mathbb{R}$ is measurable and has a finite Laplace transform $\hat{f}(t)$ for $t > 0$.

If

$$f \in AB\ (a_f) \text{ with } \underline{\text{index}}\ a_f > -1, \tag{4.6}$$

then

$$\hat{f}(1/t) \in AB\ (a_{\hat{f}}) \text{ with } \underline{\text{index}}\ a_{\hat{f}} > -1 \tag{4.7}$$

and

$$f(t) \overset{0}{\sim} \hat{f}(1/t)\ (t \to \infty). \tag{4.8}$$

(see def. 3.16).

Conversely if f is non-decreasing, then (4.7) implies (4.6).

Proof

Since we may replace $f(t)$ by $f(t) + c$ without affecting f or $\hat{f} \in AB$, we may suppose without loss of generality $f(0+) = 0$.

a. We first prove the equivalence of (4.6) and (4.7) under the assumption that f is non-decreasing. Suppose (4.6) holds. Since $\underline{\text{index}}\ a_f > -1$, we may apply theorem 3.13 with $r = 1$. Hence the function g, defined by

$$g(t) := tf(t) - \int_0^t f(s)ds$$

is monotone, in $BI \cap PI$ and $g(t) \asymp ta(t)$ $(t \to \infty)$. Theorem 4.1 then gives $\hat{g}(1/t) \in BI \cap PI$.

Now observe that

$$\frac{d}{ds} \hat{f}\ (1/s) = -\frac{1}{s} \hat{f}\ (\frac{1}{s}) + \frac{1}{s^3} \int_0^\infty e^{-t/s}\ t\ f(t)dt = \hat{g}\ (1/s)/s^2$$

and hence (note that $\hat{f}(\infty) = f(0+) = 0$)

$$\hat{f}(1/s) = \int_0^s \hat{g}\ (1/t)\ dt/t^2.$$

Application of lemma 3.15 then gives $\hat{f}(1/s) \in AB\ (a_{\hat{f}})$ and $a_{\hat{f}}(t) \asymp \hat{g}(1/t)/t$, hence $\underline{\text{index}}\ a_{\hat{f}} > -1$. It is clear that this reasoning can be followed in reversed order. Hence we have proved the equivalence of (4.6) and (4.7) in case f is non-decreasing.

b. Next we prove (4.6) → (4.7) without the assumption of monotonicity for f. For arbitrary f ∈ AB, by theorem 3.20, there exists a non-decreasing function f_o and constants t_o, $x_o > 1$ such that for $t \geq t_o$

$$f_o(t/x_o) \leq f(t) \leq f_o(tx_o). \tag{4.9}$$

Define the function f_1 by

$$f_1(t) = \max(f(t), f_o(tx_o)).$$

Then $f_1(t) = f_o(tx_o))$ for $t \geq t_o$, hence

$$\hat{f}_1(1/t) - \hat{f}_o(1/tx_o) = \int_0^{t_o/t} e^{-s}\{f_1(ts) - f_o(tx_os)\}ds \leq$$

$$\leq t^{-1}\int_0^{t_o} e^{-s/t}|f_1(s) - f_o(x_os)|ds \leq c/t \tag{4.10}$$

for all t sufficiently large, where $c := \int_0^{t_o} |f_1(s) - f_o(x_os)|ds$.
Since $f(t) \leq f_1(t)$, we have $\hat{f}(1/t) \leq \hat{f}_1(1/t)$. Combination with (4.10) gives

$$\hat{f}(1/t) \leq \hat{f}_o(1/tx_o) + c/t \leq \hat{f}_o(1/tx_1)$$

for some $x_1 > x_o$ and all t sufficiently large, since $\hat{f}_o(1/t) \in AB(a_{\hat{f}_o})$ with index $a_{\hat{f}_o} > -1$ by part a of the proof.
Introducing the function $f_2(t) = \min\{f(t), f_o(t/x_o)\}$ one finds similarly

$$\hat{f}(1/t) \geq \hat{f}_o(x_2/t) \text{ for some } x_2 > x_o \text{ and t sufficiently large.}$$

Hence $\hat{f}(1/t) \overset{0}{\sim} \hat{f}_o(1/t)$, $t \to \infty$.

By part a we have $\hat{f}_o(1/t) \in AB(a_{\hat{f}_o})$. Application of lemma 3.19 then finishes the proof of (4.7).

c. Finally we prove the implication (4.6) → (4.8).
By theorem 3.22

$$f(t) \overset{0}{\sim} c + r\int_0^t s^{-r-1}g(s)ds =: f_2(t) \tag{4.11}$$

where $r > 0$ and c are constants and $g \in BI \cap PI$ is non-decreasing.

Since we have proved in part b that if $f \overset{0}{\sim} f_o$ and f_o non-decreasing then $\hat{f}(1/t) \overset{0}{\sim} \hat{f}_o(1/t)$, it is sufficient to prove $\hat{f}_2(1/t) \overset{0}{\sim} f_2(t)$ $(t\to\infty)$.

Since $f \in AB(a)$ with <u>index</u> $a > -1$, we may take r=1 in (4.11). (see the remark following theorem 3.13.)
Now it follows that

$$\varliminf_{t\to\infty} \frac{f_2(tx) - \hat{f}_2(1/t)}{t^{-1}g(t\ )} \geq \int_0^x \frac{1-e^{-s}}{s^2} \varliminf_{t\to\infty} \frac{g(st)}{g(t)}\, ds - \int_x^\infty \frac{e^{-s}}{s^2} \varlimsup_{t\to\infty} \frac{g(ts)}{g(t)}\, ds \tag{4.12}$$

by Fatou's lemma, theorem 3.2 and the dominated convergence theorem.
Since the first integral is positive and the second integral is finite, the right-hand side in (4.12) is positive for x sufficiently large. This proves $f_2(tx_0) \geq \hat{f}_2(1/t)$ for $t \geq t_0$ and some $x_0 > 1$. The proof of the converse inequality needed for (4.8) is similar. Hence $f_2(t) \overset{0}{\sim} \hat{f}_2(1/t)$ $(t\to\infty)$. ◇

<u>Corollary 4.3</u>
Under the assumptions of theorem 4.3 we have $f(t) - \hat{f}(1/t) = O(a(t))$, $t\to\infty$.

<u>Examples:</u>

1. Suppose $f(t) = (\log\ t)^\alpha + O((\log\ t)^{\alpha-1})$ $(\alpha > 0)$ and $\hat{f}(u) < \infty$ for $u > 0$. Since $f \in AB\ (a)$ with $a(t) = (\log\ t)^{\alpha-1}$, we have $\hat{f}(1/t) = (\log\ t)^\alpha + O((\log\ t)^{\alpha-1})$ and the converse implication is true if f is non-decreasing.

2. A condition like <u>index</u> $a > -1$ is necessary for the theorem. This is shown by the following example:
   Let $f(t) = 0$ on $[0, 1]$ and $f(t) = 1 - t^{-\alpha}$ for $t > 1$, where $\alpha \geq 1$ is a constant.
   Then $f(tx) - f(t) \asymp t^{-\alpha}$ as $t \to \infty$, whereas

$$\hat{f}(1/tx) - \hat{f}(1/t) \asymp \begin{cases} t^{-1}\ln t & \text{if } \alpha = 1 \\ 1/t & \text{if } \alpha > 1. \end{cases}$$

<u>Remark</u>
Without proof we mention the following variant of theorem 4.2 and corollary 4.3. See de Haan, Stadtmüller (1985).

Suppose $a \in RO$ with $\underline{\text{index}}\ a > -1$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, $f(0+) = 0$ and $\hat{f}(t) < \infty$ for $t > 0$. Then the statements

$$\text{(i)} \qquad \overline{\lim_{t\to\infty}} \frac{f(tx) - f(t)}{a(t)} < \infty \text{ for all } x > 1 \tag{4.13}$$

$$\text{(ii)} \qquad \overline{\lim_{t\to\infty}} \frac{\hat{f}(1/tx) - \hat{f}(1/t)}{a(t)} < \infty \text{ for all } x > 1, \tag{4.14}$$

are equivalent and they imply

$$f(t) - \hat{f}(1/t) = O\,(a(t)),\ t\to\infty. \tag{4.15}$$

◇

IV.2. <u>General kernel transforms</u>

Next we prove a generalization of theorem 4.1 for more general kernels. We restrict our attention to positive kernels as we did in the corresponding theorem on RV functions. Moreover the monotonicity assumption for f is weakened (see condition (4.17) below).

From now on we use the notation

$$\hat{f}(t) = \int_0^\infty k(s)\ f(ts)\ ds \text{ (see definition 2.31).} \tag{4.16}$$

<u>Theorem 4.4</u>

a. Let $f \in RO$ with $\underline{\text{index}}\ f > -1$ and suppose f is (Lebesgue) integrable on finite intervals of $\mathbb{R}^+$. If the function $k:\mathbb{R}^+ \to \mathbb{R}^+$ is bounded on (0,1) and

$$0 < \int_0^1 s^\alpha k(s)ds < \infty$$

and

$$0 < \int_1^\infty s^\beta k(s)ds < \infty$$

for some $\alpha < \underline{\text{index}}\ f$ and $\beta > \overline{\text{index}}\ f$, then

$$f(t) \quad \hat{f}(t) \quad (t\to\infty), \text{ hence } \hat{f} \in RO.$$

b. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is measurable, $\lim_{t\to\infty} f(t) = \infty$ and there exist $\lambda > 1$, $c > 0$ such that

$$\inf_{t\le t'\le \lambda t} \{f(t') - f(t)\} > -\ c \text{ for all } t > 0. \tag{4.17}$$

Suppose $\hat{f}(t)$ is finite for $t > 0$ and $\hat{f} \in RO$.

Suppose the kernel $k \in L^1(0, \infty)$ is non-negative and satisfies the assumptions

$$\int_0^1 k(s)\, ds > 0, \int_1^\infty k(s)\, ds > 0, \int_1^\infty s^\beta k(s)\, ds < \infty$$

for some $\beta > \overline{\text{index}}\, \hat{f}$,

$$\sum_{j=0}^\infty j\lambda^{-j} k(\lambda^{-j} s) \text{ and } \sum_{j=0}^\infty j\lambda^j k(\lambda^j s)$$

are bounded on finite intervals of $\mathbb{R}^+$. (4.18)

Then $f \in RO$.

Proof

a. Since there exist $c > 1$ such that $f(tx)/f(t) \leq cx^\alpha$ for $tx \geq t_0$, $0 < x \leq 1$ by theorem 3.2 and cor. 3.4, we have

$$\overline{\lim_{t\to\infty}} \int_{t_0/t}^1 k(s) \frac{f(ts)}{f(t)}\, ds \leq c \int_0^1 k(s)\, s^\alpha ds < \infty.$$

Similarly we find

$$\underline{\lim}_{t\to\infty} \int_{t_0/t}^1 k(s) \frac{f(ts)}{f(t)}\, ds > 0.$$

Since k is bounded on $(0, 1)$ and $\underline{\text{index}}\, f > -1$ we have

$$\left| \int_0^{t_0/t} k(s) \frac{f(ts)}{f(t)}\, ds \right| \leq (tf(t))^{-1} \int_0^{t_0} k(s/t)\, |f(s)|\, ds = o(1)\ (t\to\infty).$$

Similarly we find that $\int_1^\infty k(s) \frac{f(ts)}{f(t)}\, ds$ is bounded away from zero and infinity. This completes the proof of part a.

b. We write

$$\hat{f}(t) = \int_0^\gamma k(s)\, f(ts)\, ds + \int_\gamma^\infty k(s)\, f(ts)\, ds, \tag{4.19}$$

where $\gamma > 0$ is to be determined later and start by estimating the first term at the right-hand side. There exists $t_o \in [\, t, \lambda t]$ such that

$$\inf_{t \le t' \le \lambda t} \{f(\lambda t) - f(t')\} \ge f(\lambda t) - f(t_o) - 1 \ge$$

$$\ge \inf_{t_o \le t' \le \lambda t_o} \{f(t') - f(t_o)\} - 1 \ge - c - 1 \text{ by (4.17).}$$

Since k is non-negative this implies

$$\int_0^\gamma k(s)\, f\,(ts)\, ds \le \sum_{j=0}^{\infty} \{ \sup_{\gamma\lambda^{-j-1} \le \xi \le \gamma\lambda^{-j}} f\,(\xi t)\} \int_{\gamma\lambda^{-j-1}}^{\gamma\lambda^{-j}} k(s)ds \le$$

$$\le \sum_{j=0}^{\infty} \{f(\gamma\lambda^{-j}t) + c + 1\} \int_{\gamma\lambda^{-j-1}}^{\gamma\lambda^{-j}} k(s)ds.$$

Repeated application of (4.17) gives $f(\gamma\, \lambda^{-j}t) \le f(\gamma t) + jc$.
Hence

$$\int_0^\gamma k(s)\, f\,(ts)\, ds \le c_1\, f(\gamma t) + c_2, \tag{4.20}$$

where $c_1 = \int_0^\gamma k(s)ds > 0$ and $c_2 = c \sum_{j=0}^{\infty} (j+1) \int_{\gamma\lambda^{-j-1}}^{\gamma\lambda^{-j}} k(s)ds + c_1 < \infty$

by assumption (4.18).
We are now going to estimate the integral over (γ, ∞) in (4.19).

Write $c_3 = \int_1^\infty k(s)\, ds$. Then by (4.17) and (4.18) for $t > 0$

$$\hat{f}(t) = c_3\, f(t) + \int_0^1 k(s)\, f(ts)ds + \sum_{j=0}^{\infty} \int_{\lambda^j}^{\lambda^{j+1}} k(s)\, \{f(ts) - f(t)\}ds$$

$$\ge c_3\, f(t) + \sum_{j=0}^{\infty} \int_{\lambda^j}^{\lambda^{j+1}} k(s)\, \{f(ts) - f(t\lambda^j) - cj\}ds \tag{4.21}$$

$$\ge c_3\, f(t) - c \int_1^\infty k(s)ds - c \int_1^\lambda \sum_{j=0}^{\infty} j\, \lambda^j\, k(\lambda^j s)ds =: c_3(f(t) - c_4).$$

Hence

$$\int_\gamma^\infty k(s)\, f(ts)ds \le \int_\gamma^\infty k(s)\, \{c_4 + \hat{f}(ts)/c_3\}ds.$$

By theorem 3.2, since $\hat{f} \in RO$, there exist t_o, $c_5 > 0$ such that $\hat{f}(ts) \leq c_5 \hat{f}(t)\, s^{\beta}$ for $t > t_o$, $s > 1$ where $\beta > \overline{\text{index}}\ \hat{f}$.

Hence

$$\int_{\gamma}^{\infty} k(s)\ f(ts)ds \leq \frac{c_5}{c_3}\ \hat{f}(t) \int_{\gamma}^{\infty} s^{\beta}k(s)\ ds + c_4 \int_{\gamma}^{\infty} k(s)ds. \tag{4.22}$$

Now choose γ such that $c_6 := \frac{c_5}{c_3} \int_{\gamma}^{\infty} s^{\beta}k(s)\ ds \leq 1/2$.

Combination of (4.20) and (4.22) then gives

$$\hat{f}(t) \leq \tfrac{1}{2}\ \hat{f}(t) + c_1\ f(\gamma t) + c_2 + c_4 \int_{\gamma}^{\infty} k(s)ds,$$

hence

$$\hat{f}(t) \leq 2\ c_1\ f(\gamma t) + 2\ \{c_2 + c_4 \int_{\gamma}^{\infty} k(s)\ ds\}.$$

The last inequality, together with (4.21), $\lim_{t\to\infty} f(t) = \infty$ and $\hat{f} \in RO$ imply

$$f(t) \quad \hat{f}(t)\ (t\to\infty).$$

Hence $f \in RO$. ◇

IV.3. Discussion

The results of theorems 4.1 and 4.2 (Tauberian theorems for RO and AB) have been adapted from de Haan and Stadtmüller (1985). Theorem 4.4 (a general kernel Tauberian theorem) has been adapted from Geluk (1985).

References.

T. v. Aardenne-Ehrenfest, N.G. de Bruijn, J. Korevaar, A note on slowly oscillating functions, Nw. Arch. Wisk, 23, (1949), 77-86.

M. Abramowitz, I.A. Stegun, Handbook of mathematical functions, Dover, (1965).

S. Aljancić, R. Bojanić, M. Tomić, Slowly varying functions with remainder term and their applications in analysis, Serbian Acad. Sci. and Arts Monographs, vol. CDLXII. Section Nat. Math. Sci., 41, (1974), Beograd.

S. Aljancić, D. Arandelović, O-regularly varying functions, Publ. l'Inst. Math. 22 (36), (1977), 5-22.

A.A. Balkema, Monotone transformations and limit laws, Mathematical Centre tracts 45, Amsterdam (1973).

A.A. Balkema, J.L. Geluk, L. de Haan, An extension of Karamata's Tauberian theorem and its connection with complementary convex functions, Quart. J. Math. Oxford, (2), 30, (1979), 385-416.

N.H. Bingham, C.M. Goldie, Extensions of regular variation I, II, Proc. Lond. Math. Soc., (3), 44, (1982), 473-496, 497-532.

N.H. Bingham, J.L. Teugels, Duality for regularly varying functions, Quart. J. Math. Oxford, (3), 26, (1975), 333-353.

N.H. Bingham, J.L. Teugels, Tauberian theorems and regular variation, Nieuw Arch. v. Wisk. (27), 2, (1979), 153-186.

N.H. Bingham, J.L. Teugels, Mercerian and Tauberian theorems for differences, Math. Z., 170, (1980), 247-262.

N.H. Bingham, C.M. Goldie, On one-sided Tauberian conditions, Analysis, 3, (1983), 159-188.

S. Bloom, A characterization of B-slowly varying functions, Proc. Amer. Math. Soc., 54, (1976), 243-250.

R. Bojanic, J. Karamata, On a class of functions of regular asymptotic behavior, Math. Res. Centre, U.S. Army, Madison, Tech. Summ. Rep. No. 436, (1963).

R. Bojanic, E. Seneta, A unified theory of regularly varying sequences, Math. Z., 134, (1973), 91-106.

N.G. de Bruijn, Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform., Nw. Arch. Wisk., 7, (1959), 20-26.

I. Csiszar, P. Erdös, On the function $g(t) = \lim\sup_{x\to\infty} \{f(x+t) - f(x)\}$, Magyar Tud. Akad., Mat. Kut. Int. Közl., Ser A, 9(1964), 603-606.

D. Drasin, Tauberian theorems and slowly varying functions, Trans. Amer. Math. Soc. 133, (1968), 333-356.

P. Embrechts, A second-order theorem for Laplace transforms, J.Lond. Math. Soc, 17, (1978), 102-106.

W. Feller, An introduction to probability theory and its applications, vol II, Wiley, (1971).

J. Galambos, E. Seneta, Regularly varying sequences, Proc. Amer. Math. Soc., 41, (1973), 110-116.

J.L. Geluk, Abelian and Tauberian theorems for O-regularly varying functions, Proc. Am. Math. Soc, 93, (1985), 235-241.

J.L. Geluk, Π-Regular variation, Proc. Amer. Math. Soc.,82, (1981), 565-571.

J.L. Geluk, L. de Haan, On functions with small differences, Proc. Kon. Ned. Akad. v. Wet., Indag. Math., 84, (1981), 187-194.

J.L. Geluk, L. de Haan, U. Stadtmüller, A Tauberian theorem of exponential type, Can. J. Math., 38, (1986), 697-718.

C.M. Goldie, R.L. Smith, Slow variation with remainder: a survey of the theory and its applications, Quart. J. Math. Oxford 2, 38, (1987), 45-71.

L. de Haan, On regular variation and its application to the weak convergence of sample extremes, Math. Centre Tracts 32, Amsterdam (1970).

L. de Haan, An Abel-Tauber theorem for Laplace transforms, J. London Math. Soc., 13, (1976), 537-542.

L. de Haan, On functions derived from regularly varying functions, J. Austral. Math. Soc. Ser A, 23, (1977), 431-438.

L. de Haan, E. Omey, Integrals and derivatives of regularly varying functions in R^d and domains of attraction of stable distributions, Stoch. Proc. Appl. 16, (1983), 157-170.

L. de Haan, S. Resnick, Asymptotically balanced functions and stochastic compactness of sample extremes, Ann. Prob., 12, (1984), 588-608.

L. de Haan, G. Ridder, Stochastic compactness of sample extremes, Ann. Prob. 7, (1979), 290-303.

L. de Haan, U. Stadtmüller, Dominated variation and related concepts and Tauberian theorems for Laplace transforms, J.Math. Anal. Appl., 108, (1985), 344-365.

P.R. Halmos, Measure theory, Springer (1950).

G.H. Hardy, Divergent series, Oxford (1948).

E. Hewitt, K. Stromberg, Real and abstract analysis, Springer (1969).

G.S. Jordan, Regularly varying functions and convolutions with real kernels, Trans. Amer. Math. Soc., 194, (1974), 177-194.

J. Karamata, Sur un mode de croissance régulière des fonctions, Mathematica (Cluj), 4, (1930), 38-53.

J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamentaux. Bull. Soc. Math. France, 61, (1933), 55-62.

J. Karamata, Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen, J. für reine u. ang. Math. 164, (1931), 27-39.

J. Karamata, Some theorems concerning slowly varying functions, Math. Research Center, Univ. of Wisconsin, Techn. Rep., (1962).

D.G. Kendall, Delphic Semi-groups, infinitely divisible regenerative phenomena and the arithmetic of p-functions, Zeitschr. f. Wahrsch., 9, (1967/1968), 163-195.

E.E. Kohlbecker, Weak asymptotic properties of partitions, Trans. Amer. Math. Soc., 88, (1958), 346-365

W. Matuszewska, Regularly increasing functions in connection with the theory of $L^{*\phi}$-spaces, Studia. Math., 21, (1962), 317-344.

W. Matuszewska, A remark on my paper "Regularly increasing functions in connection with the theory of $L^{*\phi}$-spaces", Studia Math., 25, (1965), 265-269.

E. Omey, Regular variation and its applications to second order linear differential equations, Bull. Soc. Math. Belg. Ser. B, 33, (1981), 207-229.

E. Omey, E. Willekens, Π-variation with remainder, Technical Report, University of Leuven, 1986.

S. Parameswaran, Partition functions whose logarithms are slowly oscillating, Trans. Amer. Math. Soc., 100, (1961), 217-240.

E.J.G. Pitman, On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin, J. Austr. Math. Soc. 8, (1968), 423-443.

E. Seneta, Regularly varying functions, Springer (1976).

G. Simons, W. Stout, A weak invariance principle with applications to domains of attraction, Ann. Prob. 6, (1978), 294-315.

U. Stadtmüller, R. Trautner, Tauberian theorems for Laplace transforms, J. Reine Ang. Math., 311-312, (1979), 283-290).

M. Vuilleumier, Slowly varying functions in the complex plane, Trans. Am. Math. Soc., 218, (1976), 343-348.

E. Wagner, Ein reeller Tauberscher Satz für die Laplace-Transformation, Math. Nachr. 36 (1968), 323-331.

I. Weissman, A note on Bojanic-Seneta theory of regularly varying sequences, Math. Z., 151, (1976), 29-30.

D.V. Widder, The Laplace transform, Princeton (1941).

List of symbols

	page
RV_α^∞, RV_α^0	3
index, regularly varying, slowly varying	3
λ Lebesgue measure	4
f^+ generalized inverse function	9
$f_1 \circ f_2$ composition of f_1 and f_2	9
$f^\leftarrow$ inverse function	11
L^* conjugate slowly varying function	15
Π, $\Pi(a)$	19
Π^0	20
$f_1 \overset{*}{\sim} f_2$ inversely asymptotic	32
Γ, $\Gamma(b)$	35
BSV, Beurling slowly varying functions	48
RV_α-sequence	54
$\Pi(a)$-sequence	55
f^c, f_c complementary (inverse complementary) function	59
Π^-, $\Pi^-(a)$	62
$\Gamma^{(0)}$	63
$f_1 \overset{\cdot\cdot}{\sim} f_2$	63
$\hat{f}$ Laplace transform	67
$\tilde{f}$	73
$\hat{f}$ (general kernel)	92
slowly decreasing	93
$f(x) \asymp g(x)$	97
RO, O-regularly varying	101
$\overline{\text{index}}$, $\underline{\text{index}}$	101
BI, Bounded increase	106
PI, Positive increase	106
AB, Asymptotically balanced	108
$f_1 \underset{\sim}{O} f_2$ O-inversely asymptotic	115

MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

2 A.R. Bloemena. *Sampling from a graph.* 1964.

3 G. de Leve. *Generalized Markovian decision processes, part I: model and method.* 1964.

4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

6 M.A. Maurice. *Compact ordered spaces.* 1964.

7 W.R. van Zwet. *Convex transformations of random variables.* 1964.

8 J.A. Zonneveld. *Automatic numerical integration.* 1964.

9 P.C. Baayen. *Universal morphisms.* 1964.

10 E.M. de Jager. *Applications of distributions in mathematical physics.* 1964.

11 A.B. Paalman-de Miranda. *Topological semigroups.* 1964.

12 J.A.Th.M. van Berckel, H. Brandt Corstius, R.J. Mokken, A. van Wijngaarden. *Formal properties of newspaper Dutch.* 1965.

13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

18 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 2.* 1968.

19 J. van der Slot. *Some properties related to compactness.* 1968.

20 P.J. van der Houwen. *Finite difference methods for solving partial differential equations.* 1968.

21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

22 T.J. Dekker. *ALGOL 60 procedures in numerical algebra, part 1.* 1968.

23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

24 J.W. de Bakker. *Recursive procedures.* 1971.

25 E.R. Paërl. *Representations of the Lorentz group and projective geometry.* 1969.

26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

28 J. Oosterhoff. *Combination of one-sided statistical tests.* 1969.

29 J. Verhoeff. *Error detecting decimal codes.* 1969.

30 H. Brandt Corstius. *Exercises in computational linguistics.* 1970.

31 W. Molenaar. *Approximations to the Poisson, binomial and hypergeometric distribution functions.* 1970.

32 L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* 1970.

33 F.W. Steutel. *Preservation of infinite divisibility under mixing and related topics.* 1970.

34 I. Juhász, A. Verbeek, N.S. Kroonenberg. *Cardinal functions in topology.* 1971.

35 M.H. van Emden. *An analysis of complexity.* 1971.

36 J. Grasman. *On the birth of boundary layers.* 1971.

37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

41 A. Verbeek. *Superextensions of topological spaces.* 1972.

42 W. Vervaat. *Success epochs in Bernoulli trials (with applications in number theory).* 1972.

43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

44 H. Bart. *Meromorphic operator valued functions.* 1973.

45 A.A. Balkema. *Monotone transformations and limit laws.* 1973.

46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

58 W. Albers. *Asymptotic expansions and the deficiency concept in statistics.* 1975.

59 J.L. Mijnheer. *Sample path properties of stable processes.* 1975.

60 F. Göbel. *Queueing models involving buffers.* 1975.

63 J.W. de Bakker (ed.). *Foundations of computer science.* 1975.

64 W.J. de Schipper. *Symmetric closed categories.* 1975.

65 J. de Vries. *Topological transformation groups, 1: a categorical approach.* 1975.

66 H.G.J. Pijls. *Logically convex algebras in spectral theory and eigenfunction expansions.* 1976.

68 P.P.N. de Groen. *Singularly perturbed differential operators of second order.* 1976.

69 J.K. Lenstra. *Sequencing by enumerative methods.* 1977.

70 W.P. de Roever, Jr. *Recursive program schemes: semantics and proof theory.* 1976.

71 J.A.E.E. van Nunen. *Contracting Markov decision processes.* 1976.

72 J.K.M. Jansen. *Simple periodic and non-periodic Lamé functions and their applications in the theory of conical waveguides.* 1977.

73 D.M.R. Leivant. *Absoluteness of intuitionistic logic.* 1979.

74 H.J.J. te Riele. *A theoretical and computational study of generalized aliquot sequences.* 1976.

75 A.E. Brouwer. *Treelike spaces and related connected topological spaces.* 1977.

76 M. Rem. *Associons and the closure statement.* 1976.

77 W.C.M. Kallenberg. *Asymptotic optimality of likelihood ratio tests in exponential families.* 1978.

78 E. de Jonge, A.C.M. van Rooij. *Introduction to Riesz spaces.* 1977.

79 M.C.A. van Zuijlen. *Emperical distributions and rank statistics.* 1977.

80 P.W. Hemker. *A numerical study of stiff two-point boundary problems.* 1977.

81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2.* 1976.

83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

88 A. Schrijver. *Matroids and linking systems.* 1977.

89 J.W. de Roever. *Complex Fourier transformation and analytic functionals with unbounded carriers.* 1978.

90 L.P.J. Groenewegen. *Characterization of optimal strategies in dynamic games.* 1981.

91 J.M. Geysel. *Transcendence in fields of positive characteristic.* 1979.

92 P.J. Weeda. *Finite generalized Markov programming.* 1979.

93 H.C. Tijms, J. Wessels (eds.). *Markov decision theory.* 1977.

94 A. Bijlsma. *Simultaneous approximations in transcendental number theory.* 1978.

95 K.M. van Hee. *Bayesian control of Markov chains.* 1978.

96 P.M.B. Vitányi. *Lindenmayer systems: structure, languages, and growth functions.* 1980.

97 A. Federgruen. *Markovian control problems; functional equations and algorithms.* 1984.

98 R. Geel. *Singular perturbations of hyperbolic type.* 1978.

99 J.K. Lenstra, A.H.G. Rinnooy Kan, P. van Emde Boas (eds.). *Interfaces between computer science and operations research.* 1978.

100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

101 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 2.* 1979.

102 D. van Dulst. *Reflexive and superreflexive Banach spaces.* 1978.

103 K. van Harn. *Classifying infinitely divisible distributions by functional equations.* 1978.

104 J.M. van Wouwe. *Go-spaces and generalizations of metrizability.* 1979.

105 R. Helmers. *Edgeworth expansions for linear combinations of order statistics.* 1982.

106 A. Schrijver (ed.). *Packing and covering in combinatorics.* 1979.

107 C. den Heijer. *The numerical solution of nonlinear operator equations by imbedding methods.* 1979.

108 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 1.* 1979.

109 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 2.* 1979.

110 J.C. van Vliet. *ALGOL 68 transput, part I: historical review and discussion of the implementation model.* 1979.

111 J.C. van Vliet. *ALGOL 68 transput, part II: an implementation model.* 1979.

112 H.C.P. Berbee. *Random walks with stationary increments and renewal theory.* 1979.

113 T.A.B. Snijders. *Asymptotic optimality theory for testing problems with restricted alternatives.* 1979.

114 A.J.E.M. Janssen. *Application of the Wigner distribution to harmonic analysis of generalized stochastic processes.* 1979.

115 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 1.* 1979.

116 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 2.* 1979.

117 P.J.M. Kallenberg. *Branching processes with continuous state space.* 1979.

118 P. Groeneboom. *Large deviations and asymptotic efficiencies.* 1980.

119 F.J. Peters. *Sparse matrices and substructures, with a novel implementation of finite element algorithms.* 1980.

120 W.P.M. de Ruyter. *On the asymptotic analysis of large-scale ocean circulation.* 1980.

121 W.H. Haemers. *Eigenvalue techniques in design and graph theory.* 1980.

122 J.C.P. Bus. *Numerical solution of systems of nonlinear equations.* 1980.

123 I. Yuhász. *Cardinal functions in topology - ten years later.* 1980.

124 R.D. Gill. *Censoring and stochastic integrals.* 1980.

125 R. Eising. *2-D systems, an algebraic approach.* 1980.

126 G. van der Hoek. *Reduction methods in nonlinear programming.* 1980.

127 J.W. Klop. *Combinatory reduction systems.* 1980.

128 A.J.J. Talman. *Variable dimension fixed point algorithms and triangulations.* 1980.

129 G. van der Laan. *Simplicial fixed point algorithms.* 1980.

130 P.J.W. ten Hagen, T. Hagen, P. Klint, H. Noot, H.J. Sint, A.H. Veen. *ILP: intermediate language for pictures.* 1980.

131 R.J.R. Back. *Correctness preserving program refinements: proof theory and applications.* 1980.

132 H.M. Mulder. *The interval function of a graph.* 1980.

133 C.A.J. Klaassen. *Statistical performance of location estimators.* 1981.

134 J.C. van Vliet, H. Wupper (eds.). *Proceedings international conference on ALGOL 68.* 1981.

135 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part I.* 1981.

136 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part II.* 1981.

137 J. Telgen. *Redundancy and linear programs.* 1981.

138 H.A. Lauwerier. *Mathematical models of epidemics.* 1981.

139 J. van der Wal. *Stochastic dynamic programming, successive approximations and nearly optimal strategies for Markov decision processes and Markov games.* 1981.

140 J.H. van Geldrop. *A mathematical theory of pure exchange economies without the no-critical-point hypothesis.* 1981.

141 G.E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds.* 1981.

142 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 1.* 1981.

143 J.M. Schumacher. *Dynamic feedback in finite- and infinite-dimensional linear systems.* 1981.

144 P. Eijgenraam. *The solution of initial value problems using interval arithmetic; formulation and analysis of an algorithm.* 1981.

145 A.J. Brentjes. *Multi-dimensional continued fraction algorithms.* 1981.

146 C.V.M. van der Mee. *Semigroup and factorization methods in transport theory.* 1981.

147 H.H. Tigelaar. *Identification and informative sample size.* 1982.

148 L.C.M. Kallenberg. *Linear programming and finite Markovian control problems.* 1983.

149 C.B. Huijsmans, M.A. Kaashoek, W.A.J. Luxemburg, W.K. Vietsch (eds.). *From A to Z, proceedings of a symposium in honour of A.C. Zaanen.* 1982.

150 M. Veldhorst. *An analysis of sparse matrix storage schemes.* 1982.

151 R.J.M.M. Does. *Higher order asymptotics for simple linear rank statistics.* 1982.

152 G.F. van der Hoeven. *Projections of lawless sequences.* 1982.

153 J.P.C. Blanc. *Application of the theory of boundary value problems in the analysis of a queueing model with paired services.* 1982.

154 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part I.* 1982.

155 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part II.* 1982.

156 P.M.G. Apers. *Query processing and data allocation in distributed database systems.* 1983.

157 H.A.W.M. Kneppers. *The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic.* 1983.

158 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 1.* 1983.

159 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 2.* 1983.

160 A. Rezus. *Abstract AUTOMATH.* 1983.

161 G.F. Helminck. *Eisenstein series on the metaplectic group, an algebraic approach.* 1983.

162 J.J. Dik. *Tests for preference.* 1983.

163 H. Schippers. *Multiple grid methods for equations of the second kind with applications in fluid mechanics.* 1983.

164 F.A. van der Duyn Schouten. *Markov decision processes with continuous time parameter.* 1983.

165 P.C.T. van der Hoeven. *On point processes.* 1983.

166 H.B.M. Jonkers. *Abstraction, specification and implementation techniques, with an application to garbage collection.* 1983.

167 W.H.M. Zijm. *Nonnegative matrices in dynamic programming.* 1983.

168 J.H. Evertse. *Upper bounds for the numbers of solutions of diophantine equations.* 1983.

169 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 2.* 1983.

CWI TRACTS

1 D.H.J. Epema. *Surfaces with canonical hyperplane sections.* 1984.

2 J.J. Dijkstra. *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility.* 1984.

3 A.J. van der Schaft. *System theoretic descriptions of physical systems.* 1984.

4 J. Koene. *Minimal cost flow in processing networks, a primal approach.* 1984.

5 B. Hoogenboom. *Intertwining functions on compact Lie groups.* 1984.

6 A.P.W. Böhm. *Dataflow computation.* 1984.

7 A. Blokhuis. *Few-distance sets.* 1984.

8 M.H. van Hoorn. *Algorithms and approximations for queueing systems.* 1984.

9 C.P.J. Koymans. *Models of the lambda calculus.* 1984.

10 C.G. van der Laan, N.M. Temme. *Calculation of special functions: the gamma function, the exponential integrals and error-like functions.* 1984.

11 N.M. van Dijk. *Controlled Markov processes; time-discretization.* 1984.

12 W.H. Hundsdorfer. *The numerical solution of nonlinear stiff initial value problems: an analysis of one step methods.* 1985.

13 D. Grune. *On the design of ALEPH.* 1985.

14 J.G.F. Thiemann. *Analytic spaces and dynamic programming: a measure theoretic approach.* 1985.

15 F.J. van der Linden. *Euclidean rings with two infinite primes.* 1985.

16 R.J.P. Groothuizen. *Mixed elliptic-hyperbolic partial differential operators: a case-study in Fourier integral operators.* 1985.

17 H.M.M. ten Eikelder. *Symmetries for dynamical and Hamiltonian systems.* 1985.

18 A.D.M. Kester. *Some large deviation results in statistics.* 1985.

19 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 1: Philosophy, framework, computer science.* 1986.

20 B.F. Schriever. *Order dependence.* 1986.

21 D.P. van der Vecht. *Inequalities for stopped Brownian motion.* 1986.

22 J.C.S.P. van der Woude. *Topological dynamix.* 1986.

23 A.F. Monna. *Methods, concepts and ideas in mathematics: aspects of an evolution.* 1986.

24 J.C.M. Baeten. *Filters and ultrafilters over definable subsets of admissible ordinals.* 1986.

25 A.W.J. Kolen. *Tree network and planar rectilinear location theory.* 1986.

26 A.H. Veen. *The misconstrued semicolon: Reconciling imperative languages and dataflow machines.* 1986.

27 A.J.M. van Engelen. *Homogeneous zero-dimensional absolute Borel sets.* 1986.

28 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 2: Applications to natural language.* 1986.

29 H.L. Trentelman. *Almost invariant subspaces and high gain feedback.* 1986.

30 A.G. de Kok. *Production-inventory control models: approximations and algorithms.* 1987.

31 E.E.M. van Berkum. *Optimal paired comparison designs for factorial experiments.* 1987.

32 J.H.J. Einmahl. *Multivariate empirical processes.* 1987.

33 O.J. Vrieze. *Stochastic games with finite state and action spaces.* 1987.

34 P.H.M. Kersten. *Infinitesimal symmetries: a computational approach.* 1987.

35 M.L. Eaton. *Lectures on topics in probability inequalities.* 1987.

36 A.H.P. van der Burgh, R.M.M. Mattheij (eds.). *Proceedings of the first international conference on industrial and applied mathematics (ICIAM 87).* 1987.

37 L. Stougie. *Design and analysis of algorithms for stochastic integer programming.* 1987.

38 J.B.G. Frenk. *On Banach algebras, renewal measures and regenerative processes.* 1987.

39 H.J.M. Peters, O.J. Vrieze (eds.). *Surveys in game theory and related topics.* 1987.

40 J.L. Geluk, L. de Haan. Regular variation, extensions and Tauberian theorems. 1987.