REGULARITY AND SINGULARITY ESTIMATES ON HYPERSURFACES MINIMIZING PARAMETRIC ELLIPTIC VARIATIONAL INTEGRALS

BY

Part I. R. SCHOEN and L. SIMON

Stanford University, Stanford, USA(1)

Part II. F. J. ALMGREN, Jr(2)

Princeton University, Princeton, USA

Introduction

In this paper we study the structure of n dimensional rectifiable currents in $\mathbf{R}^{n+1}$ which minimize the integrals of parametric elliptic integrands. The existence of such minimizing surfaces is well known [7, 5.1.6] as is their regularity almost everywhere [7, 5.3.19]. In Part I of the present paper we give a new geometric construction from which regularity estimates can be obtained for minimizing hypersurfaces. In this construction we replace the parametric problem for n dimensional surfaces in $\mathbf{R}^{n+1}$ by a nonparametric problem for which the minimizing hypersurface is a graph in $\mathbf{R}^{n+2}$ with horizontal slices closely approximating in a certain sense the hypersurface(s) minimizing the original problem. Analysis of the associated Euler-Lagrange partial differential equation carried out in § 2 of Part I yields an upper bound for the integral of the square of the second fundamental form over the approximating graphs, hence over the regular parts of the original surface. Since a neighbourhood of a singular point must contribute substantially to this integral (see Theorem 1.3 and the remark following it), we can thus conclude by an argument similar to that given by Miranda [13] that the Hausdorff $(n-2)$-dimensional measure of the interior singular set is locally finite (Theorem 3.1).

In Part II of this work we show that the singular sets in question must have Hausdorff

(1) Present addresses: R. Schoen, U. C. Berkeley; L. Simon, University of Minnesota.

(2) This research was supported in part by grants from the National Science Foundation. Part of the work of the second author was carried out at the Courant Institute of Mathematical Sciences and was supported by a grant from the Alfred P. Sloan Foundation. Part of the work of the third author was supported by a grant from the John Simon Guggenheim Foundation.

$(n-2)$-dimensional measure zero (actually the $(n-2)$-dimensional upper Minkowski content must locally vanish). We also show that for constant coefficient integrands the maximum Hausdorff dimension of interior singular sets of minimizing surfaces is upper semicontinuous as a function of integrands in the class 2 topology. We conclude, in particular, that for integrands close to the n dimensional area integrand the maximum Hausdorff dimension of singular sets can be not much more than $n-7$.

It is perhaps worth mentioning explicitly that the results described above imply in particular that there are *no* interior singularities for 2-dimensional hypersurfaces minimizing parametric elliptic integrals.

This paper represents a composite of results discovered independently by the various authors. The combined results are stronger than those obtained independently and their joint presentation permits the elimination of substantial duplication.

PART I

I.1. Preliminaries

Except in explicitly indicated instances, we will use the standard notation of Federer [7]. $\mathbf{U}^n(x_0, \varrho)$, $\mathbf{B}^n(x_0, \varrho)$ denote respectively the open and closed balls in $\mathbf{R}^n$ with radius ϱ and centre x_0. $\mathcal{L}^n$ denotes Lebesgue measure in $\mathbf{R}^n$.

We will be concerned mainly with locally rectifiable n-dimensional currents in $\mathbf{R}^{n+1}$; that is, with currents $T \in \mathcal{R}_n^{\text{loc}}(\mathbf{R}^{n+1})$, $n > 1$. Given such a current T, $\|T\|$ denotes the associated variation measure and $\vec{T}(x) \in \Lambda_n(\mathbf{R}^{n+1})$ denotes the "unit tangent direction" of T ([7], 4.1.7); thus for each smooth n-form ω with compact support in $\mathbf{R}^{n+1}$ we have

$$T(\omega) = \int_{\mathbf{R}^{n+1}} \langle \vec{T}(x), \omega(x) \rangle \, d\|T\|(x). \tag{1}$$

$\nu^T = (\nu_1^T, \ldots, \nu_{n+1}^T) \in \mathbf{S}^n$ ($\mathbf{S}^n = \partial \mathbf{B}^{n+1}(0, 1)$) will denote the unit normal of T, defined by

$$\nu^T(x) = * \vec{T}(x), \tag{2}$$

where

$$*: \Lambda_n(\mathbf{R}^{n+1}) \to \mathbf{R}^{n+1}$$

is the linear isometry characterized by

$$* e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_{n+1} = (-1)^{n+1-i} e_i, \ i = 1, \ldots, n+1.$$

Here $e_1, \ldots, e_{n+1}$ is the usual orthonormal basis for $\mathbf{R}^{n+1}$.

Note that if ω is expressed in the form

$$\omega = \sum_{i=1}^{n+1} (-1)^{n+1-i} \omega_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_{n+1},$$

where ω_i are smooth functions with compact support in $\mathbf{R}^{n+1}$, then

$$\langle \vec{T}(x), (x) \rangle = \sum_{i=1}^{n+1} \nu_i^T(x)\, \omega_i(x),$$

and hence (1) can be written

$$T(\omega) = \sum_{i=1}^{n+1} \int_{\mathbf{R}^{n+1}} \nu_i^T(x)\, \omega_i(x)\, d\|T\|. \tag{3}$$

Of special importance will be the case when T can be represented in the form

$$T = (\partial \mathbf{E}^{n+1} \llcorner V) \llcorner A, \tag{4}$$

where A, V are Lebesgue measurable subsets of $\mathbf{R}^{n+1}$ and

$$\mathbf{E}^{n+1} = \mathcal{L}^{n+1} \wedge e_1 \wedge \ldots \wedge e_{n+1}.$$

It will be convenient to use the abbreviation $[\![V]\!]$ for $\mathbf{E}^{n+1} \llcorner V$; hence (4) becomes

$$T = \partial [\![V]\!] \llcorner A.$$

Also, if M is an oriented m-dimensional C^2 submanifold of $\mathbf{R}^{n+1}$ and B is a Borel subset of M, then we let $[\![B]\!]_M$ denote the current defined by

$$[\![B]\!]_M(\omega) = \int_B \omega, \tag{5}$$

The expression on the right denoting integration of the m-form ω over $B \subset M$ in the usual sense of differential geometry. (To be strictly precise we should write $\int_B i^{\#}\omega$ on the right of (5), where i denotes the inclusion map of M into $\mathbf{R}^{n+1}$.) When no confusion is likely to arise, we will write $[\![B]\!]$ instead of $[\![B]\!]_M$.

Now suppose we have a mapping

$$F\colon \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$$

such that F has locally Lipschitz second order partial derivatives on $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \sim \{0\}$. $\mathbf{F}$ will denote the corresponding functional, defined for $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ by

$$\mathbf{F}(T) = \int_{\mathbf{R}^{n+1}} F(x, \nu^T(x))\, d\|T\|(x).$$

It will always be assumed that $\mathbf{F}$ is a *parametric functional* in the sense that

$$F(x, \mu p) = \mu F(x, p),\ \mu > 0,\ x \in \mathbf{R}^{n+1},\ p \in \mathbf{R}^{n+1}, \tag{6}$$

and F is assumed to be both *positive* and *elliptic* in the sense that

$$F(x,p)\geqslant |p|,\ x\in\mathbf{R}^{n+1},\ p\in\mathbf{R}^{n+1} \tag{7}$$

$$\sum_{i,j=1}^{n+1} F_{p_ip_j}(x,p)\,\xi_i\xi_j \geqslant |p|^{-1}|\xi'|^2,\ \xi'=\xi-\left(\xi\cdot\frac{p}{|p|}\right)\frac{p}{|p|},\ x,\xi\in\mathbf{R}^{n+1},\ p\in\mathbf{R}^{n+1}\sim\{0\}. \tag{8}$$

Note that, up to a scalar factor, (8) is the strongest convexity condition possible in view of (6).

It can be shown that (6), (7), (8) are precisely the conditions for $\Phi(x,\alpha)\equiv F(x,\ast\alpha)$, $x\in\mathbf{R}^{n+1}$, $\alpha\in\Lambda_n(\mathbf{R}^{n+1})$ ($\ast$ as in (2)) to be a positive elliptic parametric integrand in the sense [7], 5.1.1, 5.1.2.

We will let $\mathfrak{F}(\lambda,\varrho_0)$ denote the class of F satisfying (6)–(8) together with the following bounds:

$$F(x,\nu)+|F_p(x,\nu)|+\sum_{i,j=1}^{n+1}|F_{p_ip_j}(x,\nu)|+\sum_{i,j,k=1}^{n+1}|F_{p_ip_jp_k}(x,\nu)|+\varrho_0\sum_{i,j,k=1}^{n+1}|F_{x_ip_jp_k}(x,\nu)|$$
$$+\varrho_0\left|\sum_{i,j=1}^{n+1}F_{p_ix_j}(x\ \nu)\right|+\varrho_0^2\sum_{i,j,k=1}^{n+1}|F_{x_ix_jp_k}(x,\nu)|\leqslant\lambda,\quad x\in\mathbf{R}^{n+1},\ \nu\in\mathbf{S}^n. \tag{9}$$

Here $\lambda\geqslant 1$ and ϱ_0 are constants; much of the subsequent work in this paper will be carried out in the ball $\mathbf{U}^{n+1}(0,\varrho_0)$, and the presence of the factors ϱ_0, ϱ_0^2 in the left side of (9) is then appropriate if one wishes to obtain estimates and conclusions which can be stated independent of ϱ_0.

We note the important special case $F(p)\equiv|p|$; for this case we have

$$\mathbf{F}(T)=\mathbf{M}(T),$$

where $\mathbf{M}(T)$ denotes the mass of T, defined by

$$\mathbf{M}(T)=\|T\|(\mathbf{R}^{n+1})=\sup_{\|\omega\|=1}T(\omega). \tag{10}$$

Here $\|\omega\|$ denotes the comass of ω, ω an arbitrary smooth n-form with compact support in $\mathbf{R}^{n+1}$.

For later reference we note that (6) implies

$$p\cdot F_p(x,p)=F(x,p) \tag{11}$$

$$\sum_{i=1}^{n+1}p_iF_{p_ip_j}(x,p)=0,\quad j=1,\dots,n+1, \tag{12}$$

for all $(x, p) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \sim \{0\}$ and all $F \in \mathfrak{F}(\lambda, \varrho_0)$. One consequence of (12) is that

$$F_{p_i p_j}(x, p)\, \xi_i = F_{p_i p_j}(x, p)\, \xi_i', \quad \xi' = \xi - \left(\xi \cdot \frac{p}{|p|}\right) \frac{p}{|p|}, \tag{13}$$

so that, in particular, we can deduce from (9) that

$$\sum_{i,j=1}^{n+1} F_{p_i p_j}(x, p)\, \xi_i \xi_j \leqslant \lambda |\xi'|^2 \tag{14}$$

for all $x, \xi \in \mathbf{R}^{n+1}$, $p \in \mathbf{R}^{n+1} \sim \{0\}$.

Also, by using the extended mean value theorem

$$h(1) = h(0) + h'(0) + \int_0^1 (1-t)\, h''(t)\, dt$$

with $h(t) \equiv F(x, \nu + t(\eta - \nu))$, where $\eta, \nu \in \mathbf{S}^n$, we obtain the identity

$$F(x, \eta) = F(x, \nu) + (\eta - \nu) \cdot F_p(x, \nu) + \sum_{i,j=1}^{n+1} (\eta_i - \nu_i)(\eta_j - \nu_j) \int_0^1 (1-t)\, F_{p_i p_j}(x, \nu + t(\mu - \nu))\, dt,$$

and by (11) and (8) we then have

$$\begin{aligned} F(x, \eta) &\geqslant \eta \cdot F_p(x, \nu) + \int_0^1 (1-t) \frac{d^2}{dt^2} |\nu + t(\eta - \nu)|\, dt \\ &\equiv \eta \cdot F_p(\nu) + (1 - \eta \cdot \nu), \quad \eta, \nu \in \mathbf{S}^n. \end{aligned}$$

Thus, since $1 - \eta \cdot \nu = \frac{1}{2}|\eta - \nu|^2$, we obtain

$$F(x, \eta) \geqslant \eta \cdot F_p(x, \nu) + \tfrac{1}{2}|\eta - \nu|^2, \ \eta, \nu \in \mathbf{S}^n, \ x \in \mathbf{R}^{n+1}. \tag{15}$$

We now wish to use (15) to obtain an inequality (inequality (20) below) which will play a key role in the non-parametric approximation arguments to be given later. We let Ω be a bounded C^2 domain in $\mathbf{R}^n$, let $u \in C^2(\bar{\Omega})$, let G denote the graph of u, and let ν denote the upward unit normal function defined on $\bar{\Omega} \times \mathbf{R}$ by

$$\begin{aligned} \nu(x) \equiv \nu(x') = (-Du(x'), 1)/(1 + |Du(x')|^2)^{1/2}, \ x = (x_1, \ldots, x_{n+1}) \in \bar{\Omega} \times \mathbf{R}, \\ x' = (x_1, \ldots, x_n). \end{aligned} \tag{16}$$

We suppose that $F \in \mathfrak{F}(\lambda, \varrho_0)$ satisfies $F_{x_{n+1}}(x, p) \equiv 0$ (i.e. $F(x, p)$ is independent of x_{n+1}) and that

$$\operatorname{div} F_p(x, \nu) \equiv 0 \quad \text{on} \quad \Omega \times \mathbf{R}. \tag{17}$$

Note that by (16) and (6) we can write $F_p(x, \nu) = F_p(x, -Du(x'), 1)$ and hence equation (17) is equivalent to the requirement that u satisfy

$$\sum_{i=1}^{n} \frac{d}{dx_i} F_{p_i}(x', u(x'), -Du(x'), 1) = 0 \tag{18}$$

for $x' \in \Omega$. By virtue of the fact that $F_{x_{n+1}}(x, p) \equiv 0$, this is precisely the Euler-Lagrange equation for the non-parametric functional

$$\Phi(u) = \int_\Omega F(x, u(x), -Du(x), 1)\, d\mathcal{L}^n(x) \left(= \int_G F(x, \nu(x))\, d\mathcal{H}^n(x) \right).$$

Now define an n-form ω on $\overline{\Omega} \times \mathbf{R}$ by

$$\omega(x) = \sum_{i=1}^{n+1} (-1)^{n+1-i} F_{p_i}(x, \nu(x))\, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_{n+1}. \tag{19}$$

Then one easily checks that

$$d\omega = \operatorname{div} F_p(x, \nu)\, dx_1 \wedge \ldots \wedge dx_{n+1} \equiv 0 \quad \text{on } \overline{\Omega} \times \mathbf{R}$$

by (17).

Next take any current $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with

$$\partial T = [\![\partial G]\!],$$

and spt $T \subset \overline{\Omega} \times \mathbf{R}$. Since $\mathbf{H}_n(\overline{\Omega} \times \mathbf{R}) \cong \mathbf{H}_n(\overline{\Omega}) = 0$ ($\mathbf{H}_n$ denoting the n^{th} homology group with integer coefficients: [7], 4.4.1, 4.4.5) we then have $R \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with spt $R \subset \overline{\Omega} \times \mathbf{R}$ and

$$\partial R = T - [\![G]\!].$$

Then

$$T(\omega) - [\![G]\!](\omega) = \partial R(\omega) = R(d\omega) = 0;$$

that is

$$T(\omega) = [\![G]\!](\omega).$$

This is easily seen to imply, by (3), that

$$\int_{\mathbf{R}^{n+1}} \nu^T \cdot F_p(x, \nu)\, d\|T\| - \int_G \nu \cdot F_p(x, \nu)\, d\mathcal{H}^n = 0,$$

and hence, using (11) and (15), we obtain

$$\frac{1}{2} \int_{\mathbf{R}^{n+1}} |\nu - \nu^T|^2 d\|T\| \leqslant \mathbf{F}(T) - \mathbf{F}([\![G]\!]). \tag{20}$$

$T \in \mathcal{R}_n^{\mathrm{loc}}(\mathbf{R}^{n+1})$ is said to be (absolutely) $\mathbf{F}$ minimizing in A (A any subset of $\mathbf{R}^{n+1}$ and $F \in \mathcal{J}(\lambda, \varrho_0)$) if

$$\mathbf{F}(T \llcorner K) \leqslant \mathbf{F}(S) \tag{21}$$

for each compact $K \subset A$ and each $S \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with $\partial S = \partial(T \llcorner K)$ and spt $S \subset A$. Notice that if $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ and spt $T \subset A$, then T is $\mathbf{F}$ minimizing in A if and only if

$$\mathbf{F}(T) \leqslant \mathbf{F}(S) \tag{22}$$

for each $S \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with $\partial S = \partial T$ and spt $S \subset A$.

Henceforth for $F \in \mathcal{J}(\lambda, \varrho_0)$

$$\mathcal{M}(F, \varrho_0)$$

will denote the collection of $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ which are $\mathbf{F}$-minimizing in $\mathbf{B}(0, \varrho_0)$ and which can be expressed in the form

$$T = \partial[\![V]\!] \llcorner \mathbf{U}(0, \varrho_0) \tag{23}$$

for some Lebesgue measurable subset V of $\mathbf{U}(0, \varrho_0)$. Also, given $T \in \mathcal{M}(F, \varrho_0)$ we will let V_T denote a Lebesgue measurable subset of $\mathbf{U}(0, \varrho_0)$ such that (23) holds with $V = V_T$. We can always assume that V_T is open and

$$\partial V_T \cap \mathbf{U}^{n+1}(0, \varrho_0) = \operatorname{spt} T \cap \mathbf{U}^{n+1}(0, \varrho_0). \tag{24}$$

(In (24) ∂V_T denotes the ordinary topological boundary of V_T.) We can arrange this by first taking any Lebesgue measurable subset V of $\mathbf{U}^{n+1}(0, \varrho_0)$ such that (23) holds, and then defining V_T to be the union of those components W of $\mathbf{U}^{n+1}(0, \varrho_0) \sim \operatorname{spt} T$ such that $\mathcal{L}^{n+1}(W \sim V) = 0$. This procedure works because $\mathcal{L}^{n+1}(\operatorname{spt} T \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0$. In fact $\mathcal{H}^n(\operatorname{spt} T \cap \mathbf{U}^{n+1}(0, \varrho_0)) < \infty$; this follows directly from [7, 4.1.28(4)] together with I.1(28), (33) below.

The notation

$$\mathcal{M}(\lambda, \varrho_0) = \bigcup_{F \in \mathcal{J}(\lambda, \varrho_0)} \mathcal{M}(F, \varrho_0)$$

will also be used subsequently.

If $F \in \mathcal{J}(\lambda, \varrho_0)$ and if T is $\mathbf{F}$-minimizing in A, where $A \subset \mathbf{R}^{n+1}$ is such that there is a Lipschitz retraction h of an open set $\mathbf{U} \supset A$ onto A, with

$$\operatorname{dist}(A, \partial U) = \theta > 0, \ \sup_U |Dh| \leqslant \beta,$$

then we have the isoperimetric inequality

$$(\mathbf{M}(T \llcorner K))^{(n-1)/n} \leqslant c_1 \mathbf{M}(\partial(T \llcorner K)), \tag{25}$$

where K is a compact subset of $\mathbf{R}^{n+1}$ such that $\partial(T\llcorner K)\in\mathcal{R}_{n-1}(\mathbf{R}^{n+1})$ and where c_1 is a constant depending only on n, λ, θ and β.

To prove this we first notice that by [7], 4.4.2(2), p. 466, we can find $S\in\mathcal{R}_n(\mathbf{R}^{n+1})$ with $\partial S=\partial(T\llcorner K)$, spt $S\subset A$ and

$$(\mathbf{M}(S))^{(n-1)/n}\leqslant c_2\mathbf{M}(\partial S),$$

where c_2 depends only on n, θ and β. Hence (1.25), with $c_1=\lambda^{(n-1)/n}c_2$, follows from this because (9) implies $\mathbf{M}(T\llcorner K)\leqslant\lambda\mathbf{M}(S)$.

We remark also that we have, for any T as in (25), the Sobolev-type inequality

$$\left\{\int_{\mathbf{R}^{n+1}}h^{n/(n-1)}d\|T\|\right\}^{(n-1)/n}\leqslant c_1\int_{\mathbf{R}^{n+1}}|\delta^T h|d\|T\|, \tag{26}$$

where c_1 is as in (25) and h is any C^1 function on $\mathbf{R}^{n+1}$ such that spt h is compact and spt $h\cap$ spt $\partial T=\varnothing$. In (26) δ^T is the tangential gradient operator relative to T, defined $\|T\|$ – almost everywhere by

$$\delta^T h = Dh-(\nu^T\cdot Dh)\nu^T. \tag{27}$$

Inequality (26) follows directly from (25) by the argument of [5], Lemma 1.

(25) can also be used (as in [7] 5.1.6 pp. 522–3) to prove the lower bound

$$\mathbf{M}(T\llcorner\mathbf{U}^{n+1}(x_0,\varrho))\geqslant c_2\varrho^n, \tag{28}$$

where $x_0\in$ spt T and ϱ is such that $\mathbf{U}^{n+1}(x_0,\varrho)\cap$ spt $(\partial T)=\varnothing$, and where c_2 is a positive constant depending only on n, λ, θ and β.

If $T\in\mathcal{M}(F,\varrho_0)$, we can get an upper bound for $\mathbf{M}(T\llcorner\mathbf{U}^{n+1}(x_0,\varrho))$ as follows. First note that for almost every $\varrho\in(0,\varrho_0-|x_0|)$ we have

$$\partial[\![V_T\cap\mathbf{U}^{n+1}(x_0,\varrho)]\!]=T\llcorner\mathbf{U}^{n+1}(x_0,\varrho)+(\partial[\![\mathbf{U}^{n+1}(x_0\,\varrho)]\!])\llcorner V_T. \tag{29}$$

This holds whenever $\mathcal{H}^n(\text{spt } T\cap\partial\mathbf{U}^{n+1}(x_0,\varrho))=0$. For such ϱ

$$-\partial(T\llcorner\mathbf{U}^{n+1}(x_0,\varrho))=\partial((\partial[\![\mathbf{U}^{n+1}(x_0,\varrho)]\!])\llcorner V_T), \tag{30}$$

and hence, since $T\in\mathcal{M}(F,\varrho_0)$,

$$\mathbf{F}(T\llcorner\mathbf{U}^{n+1}(x_0,\varrho))\leqslant\mathbf{F}(-(\partial[\![\mathbf{U}^{n+1}(x_0,\varrho)]\!])\llcorner V_T). \tag{31}$$

Similarly, since $T=-\partial[\![\sim V_T]\!]\llcorner\mathbf{U}^{n+1}(x_0,\varrho)$, we have, again for almost all $\varrho\in(0,\varrho_0-|x_0|)$,

$$\mathbf{F}(T\llcorner\mathbf{U}^{n+1}(x_0,\varrho))\leqslant\mathbf{F}((\partial[\![\mathbf{U}^{n+1}(x_0,\varrho)]\!])\llcorner(\sim V_T)). \tag{32}$$

Using (8), (9), we then deduce from (31), (32) that

$$\mathbf{M}(T \llcorner \mathbf{U}^{n+1}(x_0, \varrho)) \leqslant \tfrac{1}{2}\lambda \mathcal{H}^n(\partial \mathbf{U}^{n+1}(x_0, \varrho)) = \tfrac{1}{2}(n+1)\boldsymbol{\alpha}(n+1)\lambda\varrho^n, \tag{33}$$

$\boldsymbol{\alpha}(n+1)$ = volume of unit ball in $\mathbf{R}^{n+1}$, for all $\varrho \in (0, \varrho_0 - |x_0|)$.

We can also show that there is a lower bound for $\mathcal{L}^{n+1}(V_T \cap \mathbf{U}^{n+1}(x_0, \varrho))$ in terms of ϱ as follows. First, by the isoperimetric inequality for currents in $\mathbf{I}_{n+1}(\mathbf{R}^{n+1})$ and by (29), (31), we have for almost all $\varrho \in (0, \varrho_0 - |x_0|)$

$$\begin{aligned}\{\mathcal{L}^{n+1}(V_T \cap \mathbf{U}^{n+1}(x_0, \varrho))\}^{n/(n+1)} &\leqslant \beta(n)\mathbf{M}(\partial [\![V_T \cap \mathbf{U}^{n+1}(x_0, \varrho)]\!])\\ &\leqslant \beta(n)\{\mathbf{M}(T \llcorner \mathbf{U}^{n+1}(x_0, \varrho)) + \mathcal{H}_n(\partial \mathbf{U}^{n+1}(x_0, \varrho) \cap V_T)\\ &\leqslant (1+\lambda)\beta(n)\mathcal{H}^n(\partial \mathbf{U}^{n+1}(x_0, \varrho) \cap V_T) = (1+\lambda)\beta(n)\frac{d}{d\varrho}\mathcal{L}^{n+1}(\mathbf{U}^{n+1}(x_0, \varrho) \cap V_T).\end{aligned} \tag{34}$$

Here $\beta(n) = \{(n+1)(\boldsymbol{\alpha}(n+1))^{1/(n+1)}\}^{-1}$ is the isoperimetric constant. Integration with respect to ϱ in (3.4) now gives

$$(n+1)\{\mathcal{L}^{n+1}(V_T \cap \mathbf{U}^{n+1}(x_0\, \varrho))\}^{1/(n+1)} \geqslant \frac{\varrho}{(1+\lambda)\beta(n)};$$

that is

$$\mathcal{L}^{n+1}(V_T \cap \mathbf{U}^{n+1}(x_0, \varrho)) \geqslant (1+\lambda)^{-(n+1)}\boldsymbol{\alpha}(n+1)\varrho^{n+1}. \tag{35}$$

The following theorem contains some basic compactness and semi-continuity results.

THEOREM 1.1. *Let f be a non-negative Lipschitz function with compact support in $\mathbf{R}^{n+1}$, and let $\varrho_0 = \sup f$, $A_\varrho = \{x: f(x) > \varrho\}$, $\varrho \in [0, \varrho_0)$. Further, let $S_r = \partial [\![U_n]\!] \llcorner A_0 \in \mathcal{R}_n(\mathbf{R}^{n+1})$, $r = 1, 2, \ldots$, be such that*

$$\limsup_{r\to\infty} \mathbf{M}(S_r \llcorner A_0) < \infty.$$

Then there is a subsequence $\{S_k\}$ of $\{S_r\}$ and a current $S = \partial [\![U]\!] \llcorner A_0 \in \mathcal{R}_n(\mathbf{R}^{n+1})$ such that

(i) $\mathcal{L}^{n+1}((U_k \Delta U) \cap A_0) \to 0$ *as* $k \to \infty$,

(ii) $\mathbf{M}(S \llcorner A_\varrho) \leqslant \liminf_{k\to\infty} \mathbf{M}(S_k \llcorner A_\varrho)$, $\varrho \in (0, \varrho_0)$.

Furthermore, if $R_k^{(\varrho)}$ is defined by

$$R_k^{(\varrho)} = \partial [\![A_\varrho]\!] \llcorner (U_k \sim U) - \partial [\![A\varrho]\!] \llcorner (U \sim U_k),$$

then for almost all $\varrho \in [0, \varrho_0)$ we have $R_k^{(\varrho)} \in \mathcal{R}_n(\mathbf{R}^{n+1})$ and

(iii) $(S_k - S) \llcorner A_\varrho = \partial\{([\![U_k]\!] - [\![U]\!]) \llcorner A\varrho\} + R_k^{(\varrho)}$,

(iv) $\mathbf{M}(R_k^{(\varrho)}) \to 0$ *as* $k \to \infty$.

If $F^r \in \mathcal{F}(\lambda, \varrho_0)$ *and* $F^r \to F$ *uniformly on* $A_0 \times S^n$, *then*

(v) $\mathbf{F}(S\llcorner A_\varrho) \leqslant \liminf_{k\to\infty} \mathbf{F}^k(S_k\llcorner A_\varrho)$

for all $\varrho \in (0, \varrho_0)$. *If it is also true that each* S_r *is* $\mathbf{F}^r$*-minimizing in* A_0, *then*

(vi) S *is* $\mathbf{F}$*-minimizing in* A_0

and

(vii) $\mathbf{F}(S\llcorner A_\varrho) \geqslant \limsup_{k\to\infty} \mathbf{F}^k(S_k\llcorner A_\varrho)$

for almost all $\varrho \in (0, \varrho_0)$.

Proof.

(i) is a well-known result (see [7], 4.2.17 for a more general result).

(ii) follows from the definition (10) of $\mathbf{M}(T)$ together with the fact that, for fixed ω, $S_k(\omega) \to S(\omega)$ by (i).

(iii) and (iv) follow from the theory of [7], 4.2.1, 4.3.6, together with (i).

Because of (iii) and (iv), (v) follows from a slight modification of [7], 5.1.5. ([7] treats the case $F^r \equiv F$, $r = 1, 2, \dots$.)

To prove (vi) and (vii) we first take ϱ such that (iii) and (iv) hold, and let $R \in \mathcal{R}_n(\mathbf{R}^{n+1})$ be such that spt $(R) \subset A_0$ and $\partial R = \partial(S\llcorner A_\varrho)$. Then by (iii)

$$\partial(R + R_k^{(\varrho)}) = \partial(S^k\llcorner A_\varrho),$$

and hence since S_κ is absolutely $\mathbf{F}_k$-minimizing in A_0, we have

$$\begin{aligned}\mathbf{F}^k(S_k\llcorner A_\varrho) &\leqslant \mathbf{F}^k(R + R_k^{(\varrho)}) \leqslant \mathbf{F}^k(R) + \mathbf{F}^k(R_k^{(\varrho)})\\ &\leqslant \mathbf{F}^k(R) + \lambda\mathbf{M}(R_k^{(\varrho)}).\end{aligned}$$

Hence

$$\limsup_{k\to\infty} \mathbf{F}^k(S_k\llcorner A_\varrho) \leqslant \mathbf{F}(R) \tag{36}$$

by (iv). Combining (v) and (36) we then have

$$\mathbf{F}(S\llcorner A_\varrho) \leqslant \mathbf{F}(R);$$

that is, $S\llcorner A_\varrho$ is absolutely $\mathbf{F}$-minimizing in A_0. (vi) now clearly follows.

Finally, to prove (vii) we replace R in (36) by $S\llcorner A_\varrho$.

The following regularity theorem will be of basic importance in what follows. In stating this theorem we let sing T denote the singular set of a current T of the form (4); i.e,

$$\text{sing } T = \text{spt } T \sim \{x\colon \text{spt } T \cap \mathbf{U}(x, \varrho) \text{ is a } C^2 \text{ hypersurface for some } \varrho > 0\}.$$

Note that by definition sing T is closed. $X \in \operatorname{spt} T$ will be called a *singular point* if $x \in \operatorname{sing} T$. We will say x is a *regular point* of spt T if $x \in \operatorname{reg} T$, where

$$\operatorname{reg} T = \operatorname{spt} T \sim \operatorname{sing} T.$$

A theorem like the regularity theorem below was first proved by De Giorgi [6] in the case $F(x, p) \equiv |p|$ (i.e. in the area case) and by Almgren [1] in the case of arbitrary $F \in \mathfrak{J}(\lambda, \varrho_0)$. Almgren's results also apply to appropriate **F**-minimizing currents and varifolds in the case of codimension > 1, and the condition that the current be absolutely **F**-minimizing can be relaxed. Allard has obtained a regularity theorem for *stationary* varifolds in [4].

THEOREM 1.2. *There are constants $\varepsilon > 0$, $\beta \in (0, 1)$, depending only on n and λ, such that if $T \in \mathcal{M}(\lambda, \varrho_0)$, if $x_0 \in \operatorname{spt} T$, if $\varrho \in (0, \varrho_0 - |x_0|)$ and if*

$$\operatorname{spt} T \cap \mathbf{U}^{n+1}(x_0, \varrho) \subset \{x\colon \operatorname{dist}(x, H) < \varepsilon\varrho\} \tag{37}$$

for some hyperplane H containing x_0, then spt $T \cap \mathbf{U}^{n+1}(x_0, \beta\varrho)$ *is a connected C^2 hypersurface M with $\bar{M} \sim M \subset \partial \mathbf{U}^{n+1}(x_0, \beta\varrho)$ and with unit normal $\nu = \nu^T$ satisfying*

$$|\nu(x) - \nu(\tilde{x})| \leqslant c\,\frac{|x - \tilde{x}|}{\varrho}, \quad x, \tilde{x} \in M. \tag{38}$$

Here c is a constant depending only on n and λ.

A new proof of this theorem, based on an approximation by solutions of the non-parametric Euler-Lagrange equation, is given in [18].

Remarks. 1. There is a constant $\eta > 0$, depending only on ε, n and λ, such that if $2\varrho \in (0, \varrho_0 - |x_0|)$, if H is a hyperplane intersecting $\mathbf{U}^{n+1}(x_0, \varrho)$, if H_+ is a halfspace with $\partial H_+ = H$, and if

$$\mathcal{L}^{n+1}((H_+ \Delta V_T) \cap \mathbf{U}^{n+1}(x_0, 2\varrho)) < \eta\varrho^{n+1}, \tag{39}$$

then (37) holds. This assertion is easily checked by using the volume estimate (35).

Since $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$, it follows from this (see [7], 4.3.17) that for $\mathcal{H}^n$-almost all $x_0 \in \operatorname{spt} T \cap \mathbf{U}^{n+1}(0, \varrho_0)$, there is a $\varrho \in (0, \varrho_0 - |x_0|)$ such that (37) holds. That is, we deduce that

$$\mathcal{H}^n(\operatorname{sing} T \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0,$$

because in a neighbourhood of a point x_0 where an inequality of the form (37) holds, we can apply standard regularity theory for elliptic equations (see Lemma 2.3 below) to deduce that spt T is a C^2 hypersurface near x_0.

2. We also remark that there is an $\eta > 0$, again depending only on ε, n and λ, such that if $2\varrho \in (0, \varrho_0 - |x_0|)$ and if

$$\int_{\mathbf{U}^{n+1}(x_0, 2\varrho)} |\nu^T - \nu^0|^2 d\|T\| \leq \eta \varrho^n \tag{40}$$

for some $\nu^0 \in \mathbf{S}^n$, then (37) holds if we take H to be the hyperplane normal to ν_0 and containing x_0. This assertion is established for example in [18].

3. If $S_r = \partial[\![V_r]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0)$, $r = 1, 2, \ldots$ and $S = \partial[\![V]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0)$ are in $\mathcal{M}(F, \varrho_0)$, if

$$\mathcal{L}^{n+1}((V_r \Delta V) \cap \mathbf{U}^{n+1}(0, \varrho_0)) \to 0$$

as $r \to \infty$, and if $x_0 \in \operatorname{reg} S$, then by taking r_0 sufficiently large and letting H be the tangent hyperplane to reg S at x_0, we clearly have that there exists $\varrho > 0$ such that (39) holds with V_r in place of V_T, $r > r_0$. If we assume for convenience that $\nu^S(x_0) = e_{n+1} = (0, \ldots, 0, 1)$ and that $x_0 = 0$, then by remark 1 it follows from the theorem that there are open subsets W_r, $W \subset \mathbf{R}^n$ and a $\varrho > 0$ such that

$$\mathbf{U}^n(0, \varrho/2) \subset \Big(\bigcap_{r > r_0} W_r\Big) \cap W$$

and such that spt $S_r \cap \mathbf{U}^{n+1}(0, \varrho)$, $r > r_0$, and spt $S \cap \mathbf{U}^{n+1}(0, \varrho)$ can be represented in the non-parametric form

$$x_{n+1} = u_r(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in W_r, \; r > r_0,$$
$$x_{n+1} = u(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in W,$$

where u_r, u are C^2 solutions of (18) with $|Du_r| < 1$ and $u_r \to u$ (uniformly) on $\mathbf{U}^n(0, \varrho/2)$. Furthermore from (38) we deduce a uniform Lipschitz estimate for Du_r, $r > r_0$, and hence (by the Schauder estimates for linear elliptic equations) we have

$$Du_r \to Du, \quad D^2 u_r \to D^2 u,$$

where the convergence is uniform on $\mathbf{U}^n(0, \sigma)$, $\sigma < \varrho/2$.

4. Finally we remark that (38) implies that the unit normal ν^T of T satisfies

$$\sup_M |\delta \nu|^2 \leq c/\varrho^2, \quad (M = \operatorname{spt} T \cap \mathbf{U}^{n+1}(x_0, \varrho/2), \; \nu = \nu^T), \tag{41}$$

where c depends only on n and λ. In (41), and in what follows, $\delta = \delta^T$ denotes the tangential gradient operator associated with T as described in (27); if h is a C^1 function on reg T, then

$$\delta h = D\bar{h} - (\nu^T \cdot D\bar{h}) \nu^T$$

on reg T, where $\tilde{h}$ is any C^1 extension of h to a neighbourhood of reg T, and $D=(D_1, .., D_{n+1})$ is the usual gradient operator in $\mathbf{R}^{n+1}$. (Of course, δ so defined, is independent of the particular C^1 extension of h that one chooses to use.)

The quantity $|\delta\nu|^2$ appearing in (41) is geometrically just the sum of squares of principal curvatures of the hypersurface $M=\text{reg } T$. That is, if $\varkappa_1, ..., \varkappa_n$ are the principal curvatures of M at x_0, then

$$\sum_{i=1}^{n} \varkappa_i^2 = \sum_{i,j=1}^{n+1} (\delta_i \nu_j(x_0))^2 = |\delta\nu(x_0)|^2. \tag{42}$$

The following theorem asserts that a sufficiently small L_1 bound on the principal curvatures of a minimizing current is enough to guarantee the hypothesis (37) of the regularity theorem.

THEOREM 1.3. *For each $\varepsilon>0$, there is an $\eta>0$, depending only on ε, n and λ, such that if $T\in\mathcal{M}(\lambda, \varrho_0)$, if $x_0\in \text{spt } T$, if $\varrho\in(0, \varrho_0-|x_0|)$ and if*

$$\int_{\mathbf{U}^{n+1}(x_0,\varrho)\cap \text{reg } T} |\delta^T\nu^T|\, d\mathcal{H}^n \leqslant \eta\varrho^{n-1}, \tag{43}$$

then there is a hyperplane H containing x_0 such that

$$\text{spt } T \cap \mathbf{U}^{n+1}(x_0, \theta\varrho) \subset \{x: \text{dist}(x, H) < \varepsilon\theta\varrho\}.$$

Here $\theta\in(0,1)$ depends only on n and λ.

Remarks. 1. A consequence of the theorem is that if $T\in\mathcal{M}(\lambda, \varrho_0)$, and if $x_0\in \text{sing } T\cap \mathbf{U}^{n+1}(0, \varrho_0-\varrho)$, then

$$\int_{\mathbf{U}^{n+1}(x_0,\varrho)\cap \text{reg } T} |\delta^T\nu^T|\, d\mathcal{H}^n \geqslant \eta\varrho^{n-1}, \tag{44}$$

where η is a positive constant depending only on n, λ.

2. We will first prove the lemma subject to the assumption that sing $T=\emptyset$. Actually for the purposes of Part I we only need the above lemma in this case. Thus to treat the case sing $T\neq\emptyset$, we can (and will) use the conclusions of the main theorem in I.3 in order to appropriately modify the argument given below for the case sing $T=\emptyset$.

Proof. By introducing the transformation of x variables given by $\hat{x}=\varrho^{-1}(x-x_0)+x_0$, one easily checks that $F\in\mathcal{F}(\lambda, \varrho_0)$ is transformed to $\hat{F}\in\mathcal{F}(\lambda, 1)$. Hence it suffices to prove the theorem in the case $\varrho=\varrho_0=1$ and $x_0=0$.

Then if the theorem is false, we have λ and $\varepsilon>0$, and a sequence $\{T^r\}$ with $T^r = \partial[\![U_r]\!] \llcorner \mathbf{U}^{n+1}(0,1) \in \mathcal{M}(F^r, 1)$, $F^r \in \mathcal{J}(\lambda, 1)$, $r=1, 2, \ldots$, such that

$$\int_{\mathbf{U}^{n+1}(0,1)} |\delta^r \nu^r| \, d\|T^r\| \to 0 \quad \text{as} \quad r\to\infty, \tag{45}$$

and such that for each hyperplane H containing 0

$$\operatorname{spt}(T^r) \cap \mathbf{U}^{n+1}(0, 1/r) \not\subset \{x: \operatorname{dist}(x, H) < \varepsilon/r\}. \tag{46}$$

Here δ^r, ν^r denote respectively the gradient and unit normal associated with T^r. Using Theorem 1.1 we then have $F \in \mathcal{J}(\lambda, 1)$ and

$$T = \partial[\![U]\!] \llcorner \mathbf{U}^{n+1}(0,1) \in \mathcal{M}(F, 1) \tag{47}$$

such that $\mathcal{L}^n((U_r \Delta U) \cap \mathbf{U}^{n+1}(0,1)) \to 0$ as $r\to\infty$. Also, by (46) and remark 3 following Theorem 1.2, we have

$$0 \in \operatorname{sing} T, \tag{48}$$

and (by (45)) each component of reg T is contained in a hyperplane. If we let $h^r = (\sum_{i=1}^{n+1} \delta_i^r \nu_i^r)\nu^r$ be the mean curvature vector of T^r, then the first variation formula for T^r ([7, 5.1.8]) gives

$$\int_{\mathbf{U}^{n+1}(0,1)} (\delta^r \varphi - \varphi h^r) \, d\|T^r\| = 0, \; \varphi \in C_0^1(\mathbf{U}^{n+1}(0,1)). \tag{49}$$

But by virtue of (45) and remark 3 following Theorem 1.2, this implies that T is stationary; that is,

$$\int_{\mathbf{U}^{n+1}(0,1)} \delta^T \varphi \, d\|T\| = 0, \; \varphi \in C_0^1(\mathbf{U}^{n+1}(0,1)). \tag{50}$$

We now want to use the dimension reducing argument of Federer [8, p. 769]. The relevant part of [8] deals with absolutely area minimizing currents; however the argument on p. 769 of [8], and the necessary preliminaries in [8] and [7], apply if the absolutely area minimizing hypothesis is replaced by (47) and (50). It follows that

$$\mathcal{H}^{n-1}(\operatorname{sing} T \cap \mathbf{U}^{n+1}(0,1)) = 0. \tag{51}$$

(Otherwise the dimension reducing argument of [8] implies that there exists a 1-dimensional oriented cone in $\mathcal{R}_1^{\mathrm{loc}}(\mathbf{R}^2)$ which has a singularity at the origin and which minimizes a parametric elliptic integrand in $\mathbf{R}^2$, and this is clearly impossible.)

Combining (51) with the fact that each component of reg T is contained in a hyperplane as noted above, we can then deduce that $T = \sum_{i=1}^{R} [\![H_i]\!] \llcorner \mathbf{U}^{n+1}(0, 1)$, where $H_1, ..., H_R$ are hyperplanes with $H_i \cap H_j \cap \mathbf{U}^{n+1}(0, 1) = \emptyset$, $i \neq j$. But this contradicts (48).

Thus the proof of the theorem for the class of currents $T \in \mathcal{M}(\lambda, 1)$ with sing $T = \emptyset$ is complete.

We now turn to the general situation when sing $T \neq \emptyset$; we still work with $\varrho_0 = \varrho = 1$ and $x_0 = 0$ as above. As explained in the remark prior to the beginning of the proof, we can use the results of the main theorem in I.3 (Theorem 3.1). In particular we can use the fact that $\mathcal{H}^{n-1}(\text{sing } T \cap \mathbf{U}^{n+1}(0, 1)) = 0$. Thus for each $\gamma > 0$ and each $\varrho \in (0, 1)$ we can find balls $\mathbf{U}^{n+1}(x^{(1)}, \varrho_1), ..., \mathbf{U}^{n+1}(x^{(N)}, \varrho_N)$ covering sing $T \cap \mathbf{U}^{n+1}(0, \varrho)$ and such that $\varrho_i < \gamma$, $i = 1, ..., N$, and

$$\sum_{i=1}^{N} \varrho_i^{n-1} < \gamma. \tag{52}$$

Thus if we let ξ_i be a non-negative smooth function with $\xi_i \in [0, 1]$ on $\mathbf{R}^{n+1}$, $\xi_i \equiv 0$ on $\mathbf{U}^{n+1}(x^{(i)}, \varrho_i)$, $\xi_i \equiv 1$ on $\mathbf{R}^{n+1} \sim \mathbf{U}^{n+1}(x^{(i)}, 2\varrho_i)$ and $\sup_{\mathbf{R}^{n+1}} |D\xi_i| \leq 3/\varrho_i$, then we have, by virtue of I.1 (33) and (52), that

$$\int_{\text{spt } T \cap \mathbf{U}^{n+1}(0,1)} \left(\sum_{i=1}^{N} |\delta \xi_i| \right) d\mathcal{H}^n \leq c\gamma, \tag{53}$$

where c depends only on n and λ. Thus using $(\prod_{i=1}^{N} \xi_i)\varphi$ in place of φ in (49), and then letting $\gamma \to 0$ and using (53), we can deduce that (50) in once again valid. The above argument is then concluded as before.

The following technical lemma will also be needed subsequently.

LEMMA 1.1. *Suppose M is a connected C^2 oriented hypersurface contained in $\mathbf{U}^{n+1}(0, \varrho_0)$, suppose that*

$$\mathcal{H}^{n-1}((\bar{M} - M) \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0 \tag{54}$$

and suppose there is a constant c such that

$$\mathcal{H}^n(M \cap \mathbf{U}^{n+1}(x_0, \varrho)) \leq c\varrho^n \tag{55}$$

whenever $x_0 \in \bar{M}$ and $\varrho \in (0, \varrho_0 - |x_0|)$.

Then

$$\partial [\![M]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) = 0, \tag{56}$$

and $\mathbf{U}^{n+1}(0, \varrho_0) \sim \bar{M}$ has exactly two components V_1, V_2 with

$$\partial V_1 \cap \mathbf{U}^{n+1}(0, \varrho_0) = \partial V_2 \cap \mathbf{U}^{n+1}(0, \varrho_0) = \bar{M} \cap \mathbf{U}^{n+1}(0, \varrho_0). \tag{57}$$

If $\mathcal{C}$ denotes any non-empty collection of connected oriented C^2 hypersurfaces M which satisfy (54) *and* (55), *and if $\mathcal{C}$ is such that $M \cap M' = \emptyset$ for each distinct pair $M, M' \in \mathcal{C}$, then for each $M_0 \in \mathcal{C}$ we have*

$$\mathcal{H}^{n-1}((\bigcup_{M \in \mathcal{C} \sim \{M_0\}} \bar{M}) \cap M_0) = 0. \tag{58}$$

Proof. We can suppose without loss of generality that $\varrho_0 = 1$. Let $\varrho \in (0, 1)$ be arbitrary, and let x_i, ϱ_i, ξ_i be as in the previous proof.

Then, if ω is any smooth $(n-1)$-form with support in $\mathbf{U}^{n+1}(0, \varrho)$, we have by Stokes' theorem that

$$\begin{aligned} 0 &= [\![M]\!]\left(d\left(\left(\prod_{i=1}^{N} \xi_i\right)\omega\right)\right) \\ &= [\![M]\!]\left(\left(\prod_{i=1}^{N} \xi_i\right) \wedge d\omega\right) + \sum_{j=1}^{N} [\![M]\!]\left(\left(\prod_{i \neq 1}^{N} \xi_i\right)(d\xi_j \wedge \omega)\right). \end{aligned}$$

By virtue of (55) we still have an inequality of the form (53), hence letting $\gamma \to 0$, we obtain $[\![M]\!](d\omega) = 0$. In view of the arbitraryness of ϱ, this gives (56) as required.

Next, by (56), [7, 4.5.17] and the connectedness of M, one can quite easily prove that there is a connected open set V with $\partial V \cap \mathbf{U}^{n+1}(0, 1) = \bar{M} \cap \mathbf{U}^{n+1}(0, 1)$. Then, setting $V_1 = V$ and $V_2 = \mathbf{U}^{n+1}(0, 1) \sim \bar{V}$, (51) holds as required.

It remains to prove (58). Let U^+, U^- be the two components of $\mathbf{U}^{n+1}(0, 1) \sim \bar{M}$. It is quite easy to check that for any $M \in \mathcal{C} \sim \{M_0\}$, precisely one of the components, say $V(M)$, of $\mathbf{U}^{n+1}(0, 1) \sim \bar{M}$ has the properties that

$$\bar{M} \cap M_0 = \overline{V(M)} \cap M_0, \quad \text{and either} \quad V(M) \subset U^+ \quad \text{or} \quad V(M) \subset U^-. \tag{59}$$

Notice that the first assertion here follows from the latter pair of alternatives. That at least one of the alternatives in (59) holds is clear; indeed otherwise we would have a component V of $\mathbf{U}^{n+1}(0, 1) \sim \bar{M}$ such that $V \cap \bar{M}_0 \neq \emptyset$ (and hence $V \cap M_0 \neq \emptyset$), and one can then show by the connectedness of M_0 and the Poincaré inequality [7, 4.5.3] that $\mathcal{H}^{n-1}(\bar{M} \cap \bar{M}_0) > 0$. But this implies $M \cap M_0 \neq \emptyset$, contrary to hypothesis. By a similar argument we can show that for any pair M, M'

$$\text{either } V(M) \subset V(M') \quad \text{or} \quad V(M') \subset V(M) \quad \text{or} \quad V(M') \cap V(M) = \emptyset. \tag{60}$$

We now introduce an equivalence relation $\approx$ on $\mathcal{C} \sim \{M_0\}$ by writing $M \approx M'$ if either $V(M) \subset V(M')$ or $V(M') \subset V(M)$. There is at most a countable collection $\mathcal{C}_1, \mathcal{C}_2, \ldots$ of equivalence classes (since otherwise we deduce by (60) that there is an uncountable collection of pairwise disjoint open subsets of $\mathbf{U}^{n+1}(0, 1)$). Further, within each equivalence class C_j we can find $M_1^j, M_2^j, \dots$ such that $\bigcup_{M\in C_j}\overline{V(M)}=\bigcup_{k=1}^{\infty}\overline{V(M_k^j)}$. Thus by (59) we have

$$\begin{aligned}\mathcal{H}^{n-1}\Big(\bigcup_{M\in C\sim\{M_0\}}\overline{M}\cap M_0\Big)&=\mathcal{H}^{n-1}\Big(\bigcup_{M\in C\sim\{M_0\}}\overline{V(M)}\cap M_0\Big)\\&=\mathcal{H}^{n-1}\Big(\bigcup_{j,k=1}^{\infty}\overline{V(M_k^j)}\cap M_0\Big)=\mathcal{H}^{n-1}\Big(\bigcup_{j,k=1}^{\infty}\overline{M}_k^j\cap M_0\Big)=0\end{aligned}$$

as required.

I.2. Some non-parametric results

In this section we wish to look at solutions of the non-parametric Euler-Lagrange equation corresponding to functionals $\mathbf{F}$, where $F\in\mathcal{J}(\lambda, \varrho_0)$; that is, we study equations of the form

$$\sum_{i=1}^{n}\frac{d}{dx_i}F_{p_i}(x, u(x), -Du(x), 1)=F_{x_{n+1}}(x, u(x), -Du(x), 1),\quad x\in\Omega, \tag{1}$$

where Ω is a domain in $\mathbf{R}^n$ and $F\in\mathcal{J}(\lambda, \varrho_0)$.

The results obtained here for solutions of equations of the form (1) will be applied in Theorems 2.1, 2.2 and in I.3 to give the central result of Part I; viz. that if $T\in\mathcal{M}(F, \varrho_0)$, then the cylinder $T\times\mathbf{R}$ can be approximated in a certain sense by $C^2(\mathbf{U}^{n+1}(0, \varrho_0))$ solutions of the equation

$$\sum_{i=1}^{n+1}\frac{d}{dx_i}\tilde{F}_{p_i}(x, -Du(x), 1)=0. \tag{2}$$

Here the notation is as follows: $F\in\mathcal{J}(\lambda, \varrho_0)$ and $\tilde{F}$ is defined on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+2}$ by

$$\tilde{F}(x,p)=\int_{\mathbf{R}^{n+1}}\psi(y)\,(F^2(x, p'+p_{n+2}\varphi(p_{n+2}^{-1}|p'|)y)+p_{n+2}^2)^{1/2}dy,\ p=(p', p_{n+2})\in\mathbf{R}^{n+2}\sim\{0\},$$

where $\psi\in C_0^\infty(\mathbf{U}^{n+1}(0,1))$ with $\psi\geqslant 0$ and $\int\psi(y)\,dy=1$, and where $\varphi\in C^3(\mathbf{R})$ with $\varphi(t)=0$ for $|t|>\frac{1}{2}$ and $0<\varphi(t)<1$ for $|t|<\frac{1}{2}$.

Thus $\tilde{F}(x,p)=(F^2(x,p')+p_{n+2}^2)^{1/2}$ for $|p'|\geqslant\frac{1}{2}|p_{n+2}|$, and $\tilde{F}(x,p)$ is obtained by applying a smoothing operator for $|p'|<\frac{1}{2}|p_{n+2}|$, $\tilde{F}$ is a $C^{2,1}$ function on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+2}\sim\{0\}$, and $\|\varphi\|_{C^3}$ small enough (which we always assume subsequently) a positive multiple of $\tilde{F}$ satisfies conditions like I.1 (6), (7), (8), (9). (The checking of I.1 (8) is partly facilitated by the uniform convexity in q of $(F^2(x,q)+1)^{1/2}$, $0<|q|\leqslant 1$.)

The associated functional $\tilde{\mathbf{F}}$ is defined by

$$\tilde{\mathbf{F}}(T)=\int_{\mathbf{R}^{n+2}}\tilde{F}(x, \nu^T(x,t))\,d\|T\|(x,t),\ T\in\mathcal{R}_{n+1}(\mathbf{R}^{n+1}), \tag{3}$$

so that

$$\tilde{\mathbf{F}}(S\times[\![(a,b)]\!])=(b-a)\mathbf{F}(S),\ \ S\in\mathcal{R}_n(\mathbf{R}^{n+1}),\ (a,b)\subset\mathbf{R}. \tag{4}$$

Notice that the equation (2) has the same general form as (1), except that there is *no explicit u dependence* in (2). For this reason we will be especially interested in equations of the form (1), where as in I.1 (17)–(20)

$$\frac{\partial}{\partial t} F(x, t, p) \equiv 0, \; x \in \mathbf{R}^n, \; t \in \mathbf{R}, \; p \in \mathbf{R}^{n+2} \sim \{0\}. \tag{5}$$

For F as in (5) we will often write $F(x, p)$ $(x \in \mathbf{R}^n)$ instead of $F(x, t, p)$. Using this convention, the equation (1) becomes

$$\sum_{i=1}^{n} \frac{d}{dx_i} F_{p_i}(x, -Du, 1) = 0, \quad x \in \Omega. \tag{6}$$

(Notice that the form of (6) is the same as (2) with n in place of $n+1$.)

For equations of the form (6) there is a particularly nice existence and regularity theory, some of which we will develop here. Some of the results given below are new, others involve slight modifications of known results.

We begin with two lemmas concerning solutions of the equation (6). In the statement of these lemmas we let G be the graph of a solution u of (6); that is

$$G = \{(x, u(x)) : x \in \Omega\}, \tag{7}$$

where u satisfies (6). $[\![G]\!]$ will be the n-dimensional current associated with G; it will always be supposed that $\nu^{[G]}$ is the *upward* unit normal of G.

LEMMA 2.1. *$[\![G]\!]$ is absolutely* $\mathbf{F}$*-minimizing in* $\Omega \times \mathbf{R}$.

Proof. Let K be an arbitrary compact subset of $\Omega \times \mathbf{R}$ and let T be any current in $\mathcal{R}_n(\mathbf{R}^{n+1})$ with spt $T \subset \Omega \times \mathbf{R}$ and $\partial T = \partial([\![G]\!] \llcorner K)$. Analogously to I.1 (19)–(20), we can then find R with $\partial R = T - [\![G]\!] \llcorner K$, spt $R \subset \Omega \times \mathbf{R}$, such that I.1 (20) holds with $[\![G]\!] \llcorner K$ in place of $[\![G]\!]$.

LEMMA 2.2. *If Ω is a bounded Lipschitz domain, if ψ is a given real-valued function on $\partial\Omega$ such that $A = \{(x, t) : x \in \partial\Omega, \; t < \psi(x)\}$ is a Borel set, if $K_1 \leqslant \psi \leqslant K_2$ (K_1, K_2 constants) and if*

$$\partial [\![G]\!] = B,$$

where $B = \partial [\![A]\!]$(1) $\in \mathcal{R}_{n-1}(\mathbf{R}^{n+1})$, *then*

$$\sup_{\Omega} u \leqslant K_1 + c, \quad \inf_{\Omega} u \geqslant K_2 - c,$$

(1) Here, and subsequently, $[\![A]\!]$ is such that $\nu^{[A]}$ is the *inward* unit normal to Ω.

where c depends only on n, λ and Ω. In case $\Omega = \mathbf{U}^n(0, \varrho_1)$, c has the form $c_1\varrho_1$ where c_1 depends only on n and λ.

Remark. The constant c above does not depend on ϱ_0; this is because (as will be clear from the proof) no bounds for the derivatives $F_{x_i x_j p_k}$, $F_{p_i p_j x_k}$ need be assumed.

Proof. By Lemma 2.1 we know that $[\![G]\!]$ is minimizing in $\Omega \times \mathbf{R}$; since Ω is Lipschitz it easily follows that $[\![G]\!]$ is minimizing in $\bar{\Omega} \times \mathbf{R}$. Also, since Ω is a bounded Lipschitz domain we can find a Lipschitz retraction of $\Omega \cup \{x: \operatorname{dist}(x, \partial\Omega) < \theta\}$ onto $\bar{\Omega}$ for some $\theta > 0$. Thus there is a Lipschitz retraction of $(\Omega \times \mathbf{R}) \cup \{x: \operatorname{dist}(x, \partial\Omega \times \mathbf{R}\}$ onto $\bar{\Omega} \times \mathbf{R}$, and we can apply I.1 (28) with $T = [\![G]\!]$ to give

$$\mathcal{H}^n(G \cap \mathbf{U}^{n+1}(x_0, \varrho)) \geqslant c_1 \varrho^n \tag{8}$$

whenever $(\bar{G} \sim G) \cap \mathbf{U}^{n+1}(x_0, \varrho) = \emptyset$, where c_1 is a constant depending only on n, λ and Ω. We now let

$$s = \sup_{\Omega} (u - K_2).$$

If $s > 0$ we can choose $x_0 = (y, u(y)) \in G$ such that $u(y) > K_2 + s/2$. Taking $\varrho = s/2$ in (8) then gives

$$\sup_{\Omega} (u - K_2) \leqslant c_2 (\mathcal{H}^n(G_+))^{1/n}, \tag{9}$$

where c_R depends on n, λ and Ω, and where

$$G_+ = \{(x, t) \in G: t > K_2\}.$$

But now since $[\![G]\!]$ is $\mathbf{F}$-minimizing in $\bar{\Omega} \times \mathbf{R}$ we have

$$\mathbf{F}([\![G]\!] \llcorner \bar{U}_\varepsilon) \leqslant \mathbf{F}(S_\varepsilon), \tag{10}$$

where

$$U_\varepsilon = \{(x, t) \in \Omega \times \mathbf{R}: K_2 + \varepsilon < t < u(x)\},\ S_\varepsilon = \partial [\![U_\varepsilon]\!] - [\![G]\!] \llcorner \bar{U}_\varepsilon.$$

Since $\operatorname{spt}(\partial [\![U_\varepsilon]\!] - [\![G]\!] \llcorner \bar{U}_\varepsilon) \subset \bar{\Omega} \times \{K_2 + \varepsilon\}$, it follows that

$$\mathbf{F}(S_\varepsilon) \leqslant \lambda \mathcal{L}^n(\Omega). \tag{11}$$

By combining (9), (10), (11) (after letting $\varepsilon \to 0^+$) we then have $\sup_\Omega u \leqslant K_2 + c_3$; c_3 depending only on n, λ and Ω. In case $\Omega = \mathbf{U}^{n+1}(0, \varrho_1)$, an examination of the proof shows that $c_3 = c_4 \varrho_1$ with c_4 depending only on n and λ.

The proof that $\inf_\Omega u \geqslant K_1 - c$ is similar.

The next lemma is a well-known regularity result from the general theory of quasilinear elliptic equations.

LEMMA 2.3. *Suppose u is a Lipschitz weak solution of* (1) *on Ω; that is, u is Lipschitz on Ω and*

$$\sum_{i=1}^{n}\int_{\Omega}F_{p_i}(x,u,-Du,1)\,\zeta_{x_i}\,dx=\int_{\Omega}F_{x_{n+1}}(x,u,-Du,1)\,\zeta\,dx \tag{12}$$

for every smooth ζ with compact support in Ω.

Then u has locally Hölder continuous second partial derivatives on Ω. In fact for each $\gamma\in(0,1)$ and for each ball $\mathbf{U}^n(x_0\,\varrho)\subset\Omega$ with $\varrho\leqslant\varrho_0$, we have a bound of the form

$$\varrho^{1+\gamma}\sum_{i,j=1}^{n}|D_iD_ju|_{(\gamma),\,\mathbf{U}^n(x_0,\varrho/2)}\leqslant c,$$

where c depends only on n,λ,γ and $\sup_\Omega|Du|$. Here $|D_iD_ju|_{(\gamma)}$ denotes the Hölder coefficient corresponding to exponent γ.

If $F\in C^{r+1}$, $r\geqslant2$, on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}\sim\{0\}$, then u is $C^{r+\gamma}$ on Ω for every $\gamma\in(0,1)$. If F is analytic on $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}\sim\{0\}$, then u is analytic on Ω.

For a discussion of such regularity results the reader should see for example [12].

The next lemma is a consequence of the De Giorgi-Nash-Moser theory for linear elliptic equations.

LEMMA 2.4. *Suppose u_1 and u_2 are solutions of* (12) *on a ball $\mathbf{U}^n(x_0\,\varrho)$, suppose $Du_1=Du_2$ at each point of the set*

$$C=\{x\in\mathbf{U}^n(x_0,\varrho)\colon u_1(x)=u_2(x)\}$$

and suppose $C\neq\emptyset$. Then $u_1\equiv u_2$ on $\mathbf{U}^n(x_0,\varrho)$.

Proof. We note that $\max(u_1,u_2)$ and $\min(u_1,u_2)$ are $C^{1,1}$ functions which satisfy the strong form of (12) almost everywhere on $\mathbf{U}^n(x_0,\varrho)$. Hence $\max(u_1,u_2)$ and $\min(u_1,u_2)$ are both weak solutions of (12). However, it is well known (and easily checked) that if we take the difference $v=v_1-v_2$ of any two solutions v_1, v_2 of (12), then v satisfies a linear elliptic equation of the form

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}v_{x_j})=\frac{\partial}{\partial x_i}(b_i v)$$

where the a_{ij}, b_i are bounded functions (determined by F, v_1, v_2) and (a_{ij}) is positive definite. Hence by the De Giorgi-Nash-Moser theory we have that if $v\geqslant0$ on $\mathbf{U}^n(x_0,\varrho)$ and if $v=0$ at some point of $\mathbf{U}^n(x_0,\varrho)$, then $v\equiv0$. This follows, for example, from the Harnack inequality. Applying this to the solution $v = v_1 - v_2$ with $v_1 = \max(u_1, u_2)$ and $v_2 = \min(u_1, u_2)$, we then have the required result.

The remaining results in this section concern solutions of (6). G (as in (7)) denotes the graph of a solution u of (6).

Preparatory to the first 3 results here, we wish to derive an important identity involving second derivatives of u. The derivation is essentially based on an idea of Bernstein, and the final identity ((17) below) is of a type that plays a key role in [5], [11] and [16].

We begin by writing (6) in the weak form

$$\sum_{i=1}^{n} \int_\Omega F_{p_i}(x, -Du, 1)\,\zeta_{x_i}\,dx = 0 \tag{6'}$$

for each smooth ζ with compact support in Ω. Replacing ζ by ζ_{x_l} and integrating by parts we then have

$$\sum_{i=1}^{n} \int_\Omega \frac{d}{dx_l}\{F_{p_i}(x, -Du, 1)\}\,\zeta_{x_i}\,du = 0.$$

If we use the chain rule and the homogeneity condition I.1 (6), this is easily seen to give

$$\sum_{i,j=1}^{n} \int_\Omega v^{-1} F_{p_i p_j}(x, \nu)\,u_{x_j x_l} - \delta_{ij} F_{p_j x_l}(x, \nu)\}\,\zeta_{x_i}\,dx = 0,$$

where ν is as in I.1 (16) and $v = \sqrt{1 + |Du|^2}$.

Replacing ζ by ζu_{x_l}, summing over l, and using the identity

$$\sum_{l=1}^{n} u_{x_l} u_{x_l x_j} = v v_{x_j}, \quad j = 1, \ldots, n,$$

we then have

$$\sum_{i,j=1}^{n} \int_\Omega \{v^{-1} F_{p_i p_j}(x, \nu)\,u_{x_i x_l} u_{x_j x_l}\zeta + F_{p_i p_j}(x, \nu)\,v_{x_j}\zeta_{x_i}\}\,dx = \sum_{i,l=1}^{n} \int_\Omega F_{p_i x_l}(x, \nu)\,(\zeta u_{x_l})_{x_i}\,dx$$

$$= -\sum_{i,l=1}^{n} \int_\Omega \frac{d}{dx_i}\{F_{p_i x_l}(x, \nu)\}\,\zeta u_{x_l}\,dx = -\sum_{i,l=1}^{n} \int_\Omega \left\{\sum_{j=1}^{n+1} u_{x_l} F_{p_i p_j x_l}(x, \nu)\,\nu_{j x_i} + u_{x_l} F_{p_i x_l x_i}(x, \nu)\right\}\zeta\,dx.$$

From now on we interpret all functions $\varphi = \varphi(x)$, defined for $x \in \Omega$, as functions which are defined on $\Omega \times \mathbf{R}$ but which happen to be independent of the $(n+1)^{\text{th}}$ variable; that is, we will henceforth not distinguish notationally between φ and the function φ^* defined on $\Omega \times \mathbf{R}$ by $\varphi^*(x, t) \equiv \varphi(x)$, $x \in \Omega$.

Then we have the identity

$$\sum_{i,j,l=1}^{n} v^{-2} F_{p_i p_j}(x, \nu)\,u_{x_i x_l} u_{x_j x_l} = \sum_{i,j,l=1}^{n+1} F_{p_i p_j}(x, \nu)\,\delta_i \nu_l \delta_j \nu_l + \sum_{i,j=1}^{n+1} F_{p_i p_j}(x, \nu)\,\delta_i w \delta_j w, \tag{13}$$

where δ denotes the tangential gradient operator on G (that is, $\delta = D - \nu(\nu \cdot D)$) and where $w = \log v$. (13) is easily checked by computing the quantities on the right and then using I.1 (12).

We also have the identities

$$\sum_{i,j=1}^{n} v^{-1} F_{p_i p_j}(x, \nu)\, v_{x_j} \zeta_{x_j} = \sum_{i,j=1}^{n+1} F_{p_i p_j}(x, \nu)\, \delta_j w \delta_j \zeta \tag{14}$$

and

$$\sum_{i=1}^{n} F_{p_i p_j x_i}(x, \nu)\, \nu_{j x_i} = \sum_{i=1}^{n+1} F_{p_i p_j x_i}(x, \nu)\, \delta_i \nu_j, \tag{15}$$

which easily follow from I.1 (12).

By using (14)–(16) in (13), and noting that $v dx$ is the volume form for G, we then have

$$\sum_{i,j=1}^{n+1} \int_G \left\{ \sum_{l=1}^{n+1} F_{p_i p_j}(x, \nu)\, \delta_i \nu_j \delta_j \nu_l \zeta + F_{p_i p_j}(x, \nu)\, \delta_i w \delta_j w \zeta + F_{p_i p_j}(x, \nu)\, \delta_j w \delta_i \zeta \right\} d\mathcal{H}^n$$
$$= \sum_{l=1}^{n} \int_G \nu_l \left\{ \sum_{i,j=1}^{n+1} F_{p_i p_j x_l}(x, \nu)\, \delta_i \nu_j + \sum_{i=1}^{n+1} F_{p_i x_i x_l}(x, \nu) \right\} \zeta \, d\mathcal{H}^n. \tag{17}$$

We remark that if we replace ζ by $\nu_{n+1}\zeta$ in (17), then, using the fact that $\nu_{n+1} = v^{-1}$, we obtain

$$\sum_{i,j=1}^{n+1} \int_G \left\{ \nu_{n+1} \sum_{l=1}^{n+1} F_{p_i p_j}(x, \nu)\, \delta_i \nu_l \delta_j \nu_l \zeta - F_{p_i p_j}(x, \nu)\, \delta_j \nu_{n+1} \delta_i \zeta \right\} d\mathcal{H}^n$$
$$= \sum_{l=1}^{n+1} \int_G \nu_{n+1} \nu_l \left\{ \sum_{i,j=1}^{n+1} F_{p_i p_j x_l}(x, \nu)\, \delta_i \nu_j + \sum_{i=1}^{n+1} F_{p_i x_i x_l}(x, \nu) \right\} \zeta \, d\mathcal{H}^n. \tag{18}$$

Writing $\zeta(x) = \varphi^2(x, u(x))$ in (17), where φ has compact support in $\Omega \times \mathbf{R}$, and using the inequalities I.1 (8), (9), and (14), cf. analogous arguments in [11] and [16], we then deduce

$$\int_G (|\delta \nu|^2 + |\delta w|^2) \varphi^2 d\mathcal{H}^n \leq c_3 \int_G (|\delta \varphi|^2 + \varrho_0^{-2} \varphi^2) d\mathcal{H}^n, \tag{19}$$

where c_3 depends only on n and λ.

Choosing φ such that $\operatorname{spt} \varphi \subset \mathrm{U}^{n+1}(x_0, \varrho)$, $\varphi \equiv 1$ on $\mathrm{U}^{n+1}(x_0, \varrho/2)$ and $\sup |D\varphi| \leq 3/\varrho$, we obtain the bound for $|\delta \nu|^2$ in the following lemma. This bound will be of central importance in what follows.

LEMMA 2.5. *Suppose u satisfies (6) on Ω. If $x_0 \in G$ and $\mathrm{U}^{n+1}(x_0, \varrho) \cap (\bar{G} \sim G) = \emptyset$, where $\varrho < \varrho_0$, then*

$$\int_{G \cap \mathrm{U}^{n+1}(x_0, \varrho/2)} |\delta \nu|^2 d\mathcal{H}^n \leq c \varrho^{n-2}, \tag{20}$$

where c is a constant depending only on n and λ.

Next we have an interior gradient bound for solutions of (6). Note that such a result is false in general for solutions of (1). Gradient bounds of the type obtained here were first obtained for arbitrary dimension n in [5]; the result was extended to equations of the general type (6) in [11] and [16].

LEMMA 2.6. *Suppose u satisfies (6) on Ω, suppose $\varrho \in (0, \varrho_0)$, and suppose $\mathbf{U}^n(x_0\, \varrho) \subset \Omega$. Then*

$$|Du(x_0)| \leqslant c_1 \exp\{c_2 m_\varrho^+/\varrho\} \tag{21}$$

$$|Du(x_0)| \leqslant c_1 \exp\{c_2 m_\varrho^-/\varrho\}, \tag{22}$$

where

$$m_\varrho^+ = \sup_{\mathbf{U}^n(x_0,\, \varrho)} (u - u(x_0)),\ m_\varrho^- = \sup_{\mathbf{U}^n(x_0,\, \varrho)} (u(x_0) - u),$$

and where c_1, c_2 are constants depending only on n and λ.

The next lemma shows that if the principal curvatures are pointwise bounded, then ν_{n+1} satisfies a Harnack inequality on G. In the minimal surface case a similar result has been proved in [17].

LEMMA 2.7. *Suppose u satisfies (6), suppose $\varrho < \varrho_0$, $\mathbf{U}^n(x_0, \varrho) \subset \Omega$ and*

$$\sup_{G \cap \mathbf{U}^{n+1}(y_0,\, \varrho)} |\delta\nu|^2 \leqslant K/\varrho^2 \tag{23}$$

where $y_0 = (x_0, u(x_0))$ and K is a constant. Then

$$\sup_{G \cap \mathbf{U}^{n+1}(y_0,\, \varrho/2)} \nu_{n+1} \leqslant c \inf_{G \cap \mathbf{U}^{n+1}(y_0,\, \varrho/2)} \nu_{n+1},$$

where c depends only on n, K and λ.

Proof. We first note that there is $\theta \in (0, 1)$, depending only on K, n and λ, such that $G \cap \mathbf{U}^{n+1}(y_0, \theta\varrho)$ is *connected* and

$$|\nu(x) - \nu(y_0)| \leqslant c\theta,\ x \in G \cap \mathbf{U}^{n+1}(y_0, \theta\varrho). \tag{24}$$

This is fairly easy to prove by elementary means, but it is convenient here to simply note that by (23) and I.1 (33)

$$\int_{G \cap \mathbf{U}^{n+1}(y_0,\, \theta\varrho)} |\delta\nu|\, d\mathcal{H}^n \leqslant c\, \frac{\sqrt{K}(\theta\varrho)^n}{\varrho} \leqslant c\sqrt{K}\theta(\theta\varrho)^{n-1},$$

where c depends only on n and λ; hence we can use Theorems 1.2 and 1.3 to yield (24) and the required connectedness.

We can now introduce new orthogonal coordinates in the tangent hyperplane of G at y_0; with respect to such coordinates the equation (18) gives a uniformly elliptic equation for γ_{n+1} (see [17] for a detailed argument in the minimal surface case). Hence by Harnack's inequality for uniformly elliptic equations we deduce for small enough θ

$$\sup_{U^{n+1}(y_0,\theta\varrho/2)} \nu_{n+1} \leqslant c_2 \inf_{U^{n+1}(y_0,\theta\varrho/2)} \nu_{n+1},$$

which is the required inequality with $\theta\varrho$ in place of ϱ. Since we can vary y_0, the lemma now follows.

The following lemma contains the information concerning the Dirichlet problem which will be needed later.

LEMMA 2.8. *Suppose Ω is a bounded C^2 domain such that the distance function d, defined by $d(x) = dist\,(x, \partial\Omega)$ for $x \in \Omega$ and $d(x) = -dist\,(x, \partial\Omega)$ for $x \in \mathbf{R}^n \sim \Omega$, satisfies*

$$\sum_{i=1}^{n} \frac{d}{dx_i}\{F_{p_i}(x, Dd(x), 0)\} \leqslant 0 \text{ and } \sum_{i=1}^{n} \frac{d}{dx_i}\{F_{p_i}(x, -Dd(x), 0)\} \leqslant 0 \tag{25}$$

at each point $x \in \partial\Omega$, and suppose ψ is an arbitrary bounded real-valued function on $\partial\Omega$.

Then there is a $C^2(\Omega)$ solution u of (6) satisfying the condition

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} u(x) = \psi(x_0) \tag{26}$$

at each point $x_0 \in \partial\Omega$ where ψ is continuous. Furthermore, if

$$W = \{(x, t) \in \partial\Omega \times \mathbf{R}: t < \psi(x)\}$$

is such that

$$B = \partial[\![W]\!] \in \mathcal{R}_{n-1}(\mathbf{R}^{n+1})$$

and if the set of discontinuities of ψ are contained in a closed set of $\mathcal{H}^{n-1}$-measure zero, then the boundary values ψ are attained globally in the sense that

$$\partial[\![G]\!] = B. \tag{27}$$

In this case, $[\![G]\!]$ is absolutely $\mathbf{F}$-minimizing in $\bar{\Omega} \times \mathbf{R}$; if $T \in \mathcal{R}_n(\mathbf{R})^{n+1}$, $\partial T = B$, spt *$T \subset \bar{\Omega} \times \mathbf{R}$, then*

$$\frac{1}{2}\int_{\Omega\times\mathbf{R}} |\nu - \nu^T|^2 d\|T\| \leqslant \mathbf{F}(T) - \mathbf{F}([\![G]\!]). \tag{28}$$

Remarks. 1. Note that (28) guarantees uniqueness of the u satisfying (6) and (27).

2. In the special case when $\Omega = \mathbf{U}^n(0, \varrho_0)$, we have $d(x) = \varrho_0 - |x|$; hence (25) requires

$$\sum_{i=1}^{n} \frac{d}{dx_i} F_{p_i}\left(x, -\frac{x}{|x|}, 0\right) \leq 0 \quad \text{and} \quad \sum_{i=1}^{n} -\frac{d}{dx_i} F_{p_i}\left(x, \frac{x}{|x|}, 0\right) \leq 0.$$

But, using I.1 (8), one easily checks that

$$\sum_{i=1}^{n} \pm \frac{d}{dx_i} F_{p_i}\left(x, \mp\frac{x}{|x|}, 0\right) = -\frac{1}{|x|}\sum_{i=1}^{n} F_{p_i p_i}\left(x, \mp\frac{x}{|x|}, 0\right) \pm \sum_{i=1}^{n} F_{p_i x_i}\left(x, \mp\frac{x}{|x|}, 0\right) \leq -\varrho_0^{-1} = \lambda_1$$

for $x \in \partial\mathbf{U}^n(0, \varrho_0)$, where $\lambda_1 = \sup_{\partial \mathbf{U}^n(0,\varrho_0)} |\sum_{i=1}^n F_{p_i x_i}(x, \pm(x/|x|), 0)| \leq \lambda$.

Hence (25) holds in this case for any $\varrho_0 \leq \lambda^{-1}$ (and strict inequality holds in (25) if $\varrho_0 < \lambda^{-1}$).

In the constant coefficient case, i.e. $F_{x_i}(x, p) \equiv 0$, $i = 1, ..., n$, $p \in \mathbf{S}^n$, we have $\lambda_1 = 0$, and hence (25) holds for *every* $\varrho_0 > 0$.

Proof. The condition (25) is sufficient for the existence of boundary barriers for equations of the form (6) (see the discussion in [16], §5). Then in view of the a-priori bounds of Lemmas 2.2, 2.6 it is a standard matter ([14], alternatively see [16], Theorem 4) to deduce that (6) has a C^2 solution satisfying (26).

To prove (27) it suffices to show that

$$\partial[\![G^-]\!] = [\![G]\!] - [\![W]\!], \tag{29}$$

where

$$G^- = \{(x, t) \in \Omega \times \mathbf{R}: t < u(x)\}.$$

((27) follows from this by applying ∂ and using $\partial^2 = 0$.) Since the set of discontinuities of ψ is contained in a closed set of $\mathcal{H}^{n-1}$-measure zero, (29) follows from (26) and the fact that $\mathcal{H}^n(G) < \infty$. (28) holds by I.1 (20). Thus the proof is complete.

We now replace n by $n+1$ and apply the above results to solutions of the equation (2). In particular, if we apply the last theorem above, then we can prove that if $\varrho_0 < \lambda_1^{-1}$(1), if A is an open subset of $\partial\mathbf{U}^{n+1}(0, \varrho_0)$ such that

$$B = \partial[\![A]\!] \in \mathcal{R}_{n-1}(\mathbf{R}^{n+1}) \tag{30}$$

$$\mathcal{H}^{n-1}(\operatorname{spt} B) < \infty, \tag{31}$$

then for each $r = 1, 2, ...$ we have a $C^2(\mathbf{U}^{n+1}(0, \varrho_0))$ solution u^r of (2) with

$$u_r \equiv r \quad \text{on } A, \quad u_r \equiv 0 \quad \text{on} \quad \partial\mathbf{U}^{n+1}(0, \varrho_0) \sim \bar{A} \tag{32}$$

(1) Here λ_1 is as in the remark 2 following Lemma 2.8.

and with graph G_r such that

$$\partial[\![G_r]\!] = B\times[\![(0, r)]\!]+[\![A \times\{r\}]\!]-[\![A \times\{0\}]\!]. \tag{33}$$

We now fix A, B as in (30), (31) and introduce the following further notation for $F\in\mathcal{F}(\lambda, \varrho_0)$:

$$\mathcal{M}_A(F, \varrho_0) = \{T = \partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0, \varrho_0)\in\mathcal{M}(F, \varrho_0)\colon \partial[\![V]\!]\llcorner\partial\mathbf{U}^{n+1}(0, \varrho_0) = [\![A]\!]\}.$$

(Note that than any $T=\partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0, \varrho_0)\in\mathcal{M}_A(F, \varrho_0)$ must satisfy

$$T-[\![A]\!] = \partial[\![V]\!], \tag{34}$$

and, in particular, $\partial T = B$.)

We always take $\varrho_0<\lambda_1^{-1}$, λ_1 as in remark 2 following Lemma 2.8.

$\mathcal{M}'_A(F, \varrho_0)$ will denote the collection of $T=\partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0, \varrho_0)\in\mathcal{M}_A(F, \varrho_0)$ such that there is a subsequence $\{u_k\}$ of $\{u_r\}$ (u_r as in (32), (33)) and a sequence $\{d_k\}$ of reals such that for each $\varrho>0$

$$\mathcal{L}^{n+1}((U_k\Delta U)\cap(\mathbf{U}^n(0, \varrho_0)\times(-\varrho, \varrho)))\to 0 \quad \text{as} \quad k\to\infty, \tag{35}$$

where

$$U = V\times\mathbf{R}$$

and

$$U_k = \{x\in\mathbf{U}(0, \varrho_0)\times\mathbf{R}\colon x_{n+1} < u_k(x_1, \ldots, x_n)-d_k\}.$$

We note that the sequence d_k must satisfy

$$d_k\to\infty,\ k-d_k\to\infty \quad \text{as} \quad k\to\infty,$$

otherwise $U = V\times\mathbf{R}$ would be impossible by (33).

We note also that $\mathcal{M}'_A(F, \varrho_0)$ is closed in the sense that if

$$T_r=\partial[\![V_r]\!]\llcorner\mathbf{U}^{n+1}(0, \varrho_0)\in\mathcal{M}'_A(F, \varrho_0) \quad \text{and if} \quad T = \partial[\![V]\!]\llcorner\mathbf{U}^{n+1}_{\iota}(0, \varrho_0),$$

then

$$\mathcal{L}^{n+1}(V_r\Delta V)\to 0 \quad \text{as} \quad r\to\infty \quad \text{implies} \quad T\in\mathcal{M}'_A(F, \varrho_0). \tag{36}$$

The following lemma concerning $\mathcal{M}(F, \varrho_0)$ is of central importance, and is a consequence of Lemma 2.5. In [13] Miranda considered arbitrary convergent sequences of solutions of the minimal surface equation (converging in the same current sense as here) and proved a result like (i); we here use a similar argument to prove (i).

THEOREM 2.1. *If $T\in\mathcal{M}'_A(F, \varrho_0)$, then*

(i) $$\mathcal{H}^{n-2}(\text{sing } T\cap\mathbf{U}^{n+1}(0, \varrho)) < \infty, \quad \forall\varrho<\varrho_0,$$

and

(ii) $$\int_{\mathbf{U}^{n+1}(0,\varrho)\cap \operatorname{reg} T} |\delta^T \nu^T|^2 d\mathcal{H}^n \leqslant c\varrho^{n-2}, \quad \forall \varrho < \varrho_0,$$

where c depends only on n, λ and ϱ/ϱ_0.

Furthermore, each component M of reg T *satisfies*

(iii) $$\partial [\![M]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) = 0,$$

and if M is appropriately oriented(1)

(iv) $$[\![M]\!] \in \mathcal{M}(F, \varrho_0).$$

Proof. By Lemma 2.5 we have for $\varrho < \varrho_0$

$$\int_{G_r \cap \mathbf{U}^{n+2}(0,\varrho)} |\delta^r \nu^r|^2 d\mathcal{H}^{n-1} \leqslant c\varrho^{n-1}, \tag{37}$$

where

$$\delta^r = \delta^{[G_r]}, \; \nu^r = \nu^{[G_r]}$$

and where c depends only on n, λ and ϱ/ϱ_0.

We now let $T \in \partial [\![V]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) \in \mathcal{M}'_A(F, \varrho_0)$ and let $\varepsilon > 0$.

Defining

$$S = T \times [\![\mathbf{R}]\!], \quad U = V \times \mathbf{R}$$

we have that (35) holds for some sequence $\{d_k\}$ of reals. We now let $\varrho < \varrho_0$ and define

$$(\operatorname{sing} S)_\varrho = \operatorname{sing} S \cap \mathbf{B}^{n+2}(0, \varrho).$$

Then $(\operatorname{sing} S)_\varrho$ is compact, and hence for sufficiently small $\delta \in (0, \frac{1}{2}(\varrho_0 - \varrho))$, we can find points $x^{(1)}, \ldots, x^{(N)} \in (\operatorname{sing} S)_\varrho$ such that

$$(\operatorname{sing} S)_\varrho \subset \bigcup_{j=1}^{N} \mathbf{U}^{n-2}(x^{(j)}, 2\delta) \tag{38}$$

$$\mathbf{U}^{n+2}(x^{(j)}, \delta) \cap \mathbf{U}^{n+2}(x^{(k)}, \delta) = \emptyset, \quad k \neq j, \tag{39}$$

and

$$\mathcal{H}^{n-1}((\operatorname{sing} S)_\varrho) \leqslant 2^{n-1} N \boldsymbol{\alpha}(n-1) \delta^{n-1} + \varepsilon. \tag{40}$$

(Note that here we have used the definition of Hausdorff measure.)

---

(1) Given a component M of reg T, $T \in \mathcal{M}'_A(F, \varrho_0)$, we always take $[\![M]\!]$ such that $\nu^{[M]} = \nu^T$ on M.

Now let $x^{(j,k)} \in \operatorname{spt} S_k (S_k = [\![G_k]\!])$ be such that $|x^{(j)} - x^{(j,k)}| = \operatorname{dist}\{x^{(j)}, \operatorname{spt} S_k\}$. Since $x^{(j,k)} \to x^{(j)}$ as $k \to \infty$, we have

$$\mathbf{U}^{n+2}(x^{(j,k)}, \delta/2) \subset \mathbf{U}^{n+2}(x^{(j)}, \delta),\ k \geq k_0, \quad j = 1, \dots, N. \tag{41}$$

We now claim that there is a constant $\eta > 0$, depending only on n and λ, such that for $j = 1, \dots, N$ and for $k \geq k_1 \geq k_0$

$$\int_{\operatorname{spt} S_k \cap \mathbf{U}^{n+2}(x^{(j,k)}, \delta/2)} |\delta^{S_k} \nu^{S_k}|^2 \, d\mathcal{H}^{n+1} \geq \eta \delta^{n-1}. \tag{42}$$

This must hold because otherwise, for sufficiently small $\eta > 0$ and some subsequence $\{k'\} \subset \{k\}$, we would have by Theorem 1.3 that the hypothesis I.1 (37) holds with $n+1$, $\theta\delta$, $x^{(j,k')}$ and $S_{k'}$ in place of n, ϱ, x_0 and T respectively. Thus we would have that for $k' \geq k_1$, $\operatorname{spt} S_{k'} \cap \mathbf{U}^{n+2}(x^{(j,k')}, \theta\delta/2)$ is a connected C^2 hypersurface with

$$|\nu^{S_{k'}}(x) - \nu^{S_{k'}}(y)| \leq c|x-y|,\ x, y \in \mathbf{U}^{n+2}(x^{(j,k')}, \theta\delta/2) \cap \operatorname{spt} S_{k'},$$

where c depends only on n and λ. Since $x^{(j,k')} \to x^{(j)}$ this would clearly imply that $x^{(j)} \in \operatorname{reg} S$, and this contradicts the choice of $x^{(j)}$.

Summing over $j = 1, \dots, N$ in (42) and using (39)–(41) we have that for sufficiently large k

$$\eta \mathcal{H}^{n-1}((\operatorname{sing} S)_\varrho) - \eta\varepsilon \leq \int_{\operatorname{spt} S_k \cap \mathbf{U}^{n+2}(0, (\varrho_0 + \varrho)/2)} |\delta^{S_k} \nu^{S_k}|^2 \, d\mathcal{H}^{n+1},$$

and by (37) this gives (since $\varepsilon > 0$ was arbitrary)

$$\mathcal{H}^{n-1}((\operatorname{sing} S)_\varrho) \leq c\varrho^{n-1},$$

where c depends only on n, λ and ϱ/ϱ_0. Then since $S = T \times [\![R]\!]$, this clearly implies (i).

To prove (ii) we notice that if

$$(\operatorname{reg} S)_\sigma = \operatorname{reg} S \sim \{x : \operatorname{dist}(x, \operatorname{sing} S) < \sigma\},$$

then for $\varrho < \varrho_0$

$$\begin{aligned}\int_{(\operatorname{reg} S)_\sigma \cap \mathbf{U}^{n+2}(0, \varrho)} |\delta^S \nu^S|^2 \, d\mathcal{H}^{n+1} &= \lim_{k \to \infty} \int_{(\operatorname{spt} S^k \sim \{x : \operatorname{dist}(x, \operatorname{sing} S) < \sigma\}) \cap \mathbf{U}^{n+2}(0, \varrho)} |\delta^{S_k} \nu^{S_k}|^2 \, d\mathcal{H}^{n+1} \\ &\leq \lim_{k \to \infty} \int_{\operatorname{spt} S^k \cap \mathbf{U}^{n+2}(0, \varrho)} |\delta^{S_k} \nu^{S_k}|^2 \, d\mathcal{H}^{n+1} \leq c\varrho^{n-1}\end{aligned} \tag{43}$$

by (20). (43) holds because of the convergence described in remark 3 following Theorem 1.2. Since σ was arbitrary, (ii) easily follows from (43).

The remaining conclusions of the lemma are a direct consequence of Lemma 1.1.

In view of the definition of $\mathcal{M}'_A(F, \varrho_0)$ it is natural to ask whether or not, for every choice of constants d_r satisfying $d_r \to \infty$ and $r - d_r \to \infty$ as $r \to \infty$, there is a subsequence $\{u_k - d_k\}$ of $\{u_r - d_r\}$ such that (35) holds for some $U = V \times \mathbf{R}$. The following theorem answers this question. In this theorem, and in what follows, we continue to assume $\varrho_0 < \lambda_1^{-1}$, λ_1 as in remark 2 following Lemma 2.8. Here and subsequently we take sup φ (φ as in the definition of $\tilde{F}$) small enough to ensure that $T \times [\![\mathbf{R}]\!]$ minimizes $\tilde{\mathbf{F}}$ if and only if T minimizes $\mathbf{F}$, $T \in \mathbf{I}_n(\mathbf{R}^{n+2})$. That this can be done follows from (4) together with the fact that, by [7, 3.2.22, 4.1.28], for small enough sup φ we have $\tilde{\mathbf{F}}(R) \geqslant \int_{\mathbf{R}} \mathbf{F}(R_t)\, dt$, $R \in \mathbf{I}_{n+1}(\mathbf{R}^{n+2})$, where R_t denotes the slice by $x_{n+1} = t$ ([7, 4.3]).

THEOREM 2.2. *Let $\{d_r\}$ be any sequence of reals with*

(i) $$r - d_r \to \infty,\ d_r \to \infty \quad \text{as} \quad r \to \infty,$$

and let

$$U_r = \{(x, t) \in \mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}: t < u_r(x) - d_r\}.$$

Then there is a Lebesgue measurable $U \subset \mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}$ and a subsequence $\{k\} = \{r_s\}_{s=1,2,\ldots}$ of $\{r\}$ such that for each $\varrho > 0$

$$\mathcal{L}^{n+2}[(U_k \Delta U) \cap \mathbf{U}^{n+1}(0, \varrho_0) \times (-\varrho, \varrho))] \to 0$$

as $k \to \infty$. U is such that either

(ii) $$[\![U]\!] = [\![V \times \mathbf{R}]\!]$$

for some subset $V \subset \mathbf{U}^n(0, \varrho_0)$ with

(iii) $$T = \partial [\![V]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) \in \mathcal{M}'_A(F, \varrho_0),$$

or

(ii)′ $$[\![U]\!] = [\![V \times \mathbf{R}]\!] + [\![G^-]\!],$$

where V is as in (ii), (iii) *and where G^- has the form*

(iii)′ $$G^- = \{(x, t): x \in W,\ t < u(x)\},$$

with W an open subset of $\mathbf{U}^{n+1}(0, \varrho_0)$ and u a $C^2(W)$ solution of (2).

Remark. It *can* happen that the case (ii)′ occurs: consider for example the case $n = 1$, $\varrho_0 = 1$, $F(x, p) \equiv |p|$ and $A = \{x = (x_1, x_2) \in \mathbf{S}^1: -1/\sqrt{2} < x_1 < 1/\sqrt{2}\}$. One can check that in this case the choice $d_r = r/2$ yields $W = \{(x_1, x_2): -1/\sqrt{2} < x_1 < 1/\sqrt{2}, -1/\sqrt{2} < x_2 < 1/\sqrt{2}\}$, $V = \{(x_1, x_2): -1/\sqrt{2} < x_1 < 1/\sqrt{2}$ and either $x_2 < 1/\sqrt{2}$ or $x_2 < -1/\sqrt{2}\}$,

$G^- = \{(x_1, x_2, x_3): x_3 < u(x_1, x_2)\}$, where the graph $x_3 = u(x_1, x_2)$ is *Scherk's surface*; that is

$$u(x_1, x_2) = \frac{\sqrt{2}}{\pi} \log \frac{\cos(\pi x_1/\sqrt{2})}{\cos(\pi x_2/\sqrt{2})}.$$

Note also that the choice $d_r = \frac{3}{4}r$ yields (iii) with

$$V = \{(x_1, x_2) \in \mathbf{U}^2(0, 1): -1/\sqrt{2} < x_1 < 1/\sqrt{2} \quad \text{and either} \quad x_2 < 1/\sqrt{2} \quad \text{or} \quad x_2 < -1/\sqrt{2}\}.$$

The choice $d_r = r/4$ yields (iii) with $V = \{(x_1, x_2) \in \mathbf{U}^2(0, 1): -1/\sqrt{2} < x_1 < 1/\sqrt{2}\}$.

Proof. By Lemma 2.1 and Theorem 1.1, we know that there is a subsequence $\{U_k\} \subset \{U_r\}$ and a $Y \subset \mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}$ such that for each $\varrho > 0$

$$(Y \Delta U_k) \cap (\mathbf{U}^{n+1}(0, \varrho_0) \times (-\varrho, \varrho)) \to 0 \quad \text{as} \quad k \to \infty$$

and such that

$$S = \partial [\![Y]\!] \llcorner (\mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R})$$

is $\tilde{\mathbf{F}}$-minimizing in $\mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}$. Also, since we have the *strict* inequality $\varrho_0 < \lambda_1^{-1}$, we can prove, using a more or less standard barrier argument, that for each compact $K \subset A \cup (\partial \mathbf{U}^{n+1}(0, \varrho_0) \sim \bar{A})$

$$\operatorname{dist}\{G_r, K \times (-d_r + 1, r - d_r - 1)\} \geq c > 0, \tag{44}$$

where c is independent of r. Hence it follows, by using this last fact together with (31) and (32), that

$$\partial [\![Y]\!] \llcorner (\partial \mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}) = [\![A \times \mathbf{R}]\!]. \tag{45}$$

We can assume that Y is open and $\partial Y = \operatorname{spt} S \cup (\bar{A} \times \mathbf{R})$. Taking $x_0 \in \operatorname{reg} S$ we see from Lemma 2.7 and the remarks 2, 3 and 4 following the regularity theorem (Theorem 1.2) that for some $\sigma > 0$, the set $S_\sigma = \operatorname{spt} S \cap \mathbf{U}^{n+2}(x_0, \sigma)$ satisfies

$$S_\sigma \subset \operatorname{reg} S \quad \text{and } \textit{either} \quad \nu_{n+2}^S \equiv 0 \quad \text{on} \quad S_\sigma \quad \textit{or} \quad \nu_{n+2}^S \geq c > 0 \quad \text{on} \quad S_\sigma, \tag{46}$$

where c is a constant.

If we let π denote the projection of $\mathbf{R}^{n+2}$ onto $\mathbf{R}^{n+1}$, defined by $\pi(x_1, ..., x_{n+1}, x_{n+2}) = (x_1, ..., x_{n+1})$, it is then not difficult to check that

$$Y \sim (\pi(\operatorname{sing} S) \times \mathbf{R}) = G^- \cup U, \tag{47}$$

where G^- is of the form (iii)′ (possibly with $W=\varnothing$), and where U is such that

$$(\pi(U)\times\mathbf{R})\cap \operatorname{spt} S=\varnothing.$$

It then easily follows that U is open and

$$U = V\times\mathbf{R}, \tag{49}$$

where

$$V=\pi(U).$$

Then combining (49) and (47), and noting that $\mathcal{L}^{n+2}(\pi(\operatorname{sing} S)\times\mathbf{R})=0$ (because $\mathcal{H}^{n+1}(\operatorname{sing} S\sim \bar{A}\times\mathbf{R})<\infty$ by the regularity theorem (Theorem 1.2)), we deduce

$$[\![Y]\!]=[\![G^-]\!]+[\![V\times\mathbf{R}]\!]. \tag{50}$$

We now consider the two cases $G^-=\varnothing$ and $G^-\neq\varnothing$.

If $G^-=\varnothing$, then $\partial[\![V\times\mathbf{R}]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)\times\mathbf{R}$ is $\tilde{\mathbf{F}}$-minimizing in $\mathbf{U}^{n+1}(0,\varrho_0)\times\mathbf{R}$; hence $\partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)$ is $\mathbf{F}$-minimizing, and we then deduce that $\partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)\in\mathcal{M}'_A(F,\varrho_0)$.

If $G^-\neq\varnothing$, we define, for $r=1,2,\ldots$,

$$Y_r=\{x-re_{n+2}\colon x\in Y\}$$
$$G_r^-=\{x-re_{n+2}\colon x\in G^-\},$$

where $e_{n+2}=(0,\ldots,0,1)\in\mathbf{R}^{n+2}$. Then clearly by (50)

$$[\![Y_r]\!]=[\![G_r^-]\!]+[\![V\times\mathbf{R}]\!].$$

However $G_{r+1}^-\subset G_r^-$ and $\bigcap_{r=1}^{\infty}G_r^-=\varnothing$, hence

$$\mathcal{L}^{n+2}(G_r^-\cap(\mathbf{U}^{n+1}(0,\varrho_0)\times(-\varrho,\varrho)))\to 0 \quad \text{as} \quad r\to\infty$$

for each $\varrho>0$. Thus it follows that

$$\mathcal{L}^{n+2}(Y_r\Delta V\times\mathbf{R})\cap(\mathbf{U}^{n+1}(0,\varrho_0)\times(-\varrho,\varrho)))\to 0 \quad \text{as} \quad r\to\infty,$$

and hence, by Theorem 1.1, $\partial[\![V\times\mathbf{R}]\!]\llcorner(\mathbf{U}^{n+1}(0,\varrho_0)\times\mathbf{R})$ is $\tilde{\mathbf{F}}$-minimizing in $\mathbf{U}^{n+1}(0,\varrho_0)\times\mathbf{R}$. Then, as in the case $G^-=\varnothing$, we deduce $\partial[\![V]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)\in\mathcal{M}'_A(F,\varrho_0)$. This completes the proof of Theorem 2.2.

The next lemma shows that for any $T_1\in\mathcal{M}_A(F,\varrho_0)$, we have $\operatorname{spt} T_1\subset\bigcup\operatorname{spt} T$, where the union is taken over all $T\in\mathcal{M}'_A(F,\varrho_0)$. In the main theorem of I.3 (Theorem 3.1) a much stronger result will be proved; viz. that T_1 can be expressed as a locally finite sum $\Sigma[\![M_i]\!]$, where each M_i is a component of reg T for some $T\in\mathcal{M}'_A(F,\varrho_0)$.

THEOREM 2.3. *If* $T_1 = \partial [\![V_1]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) \in \mathcal{M}_A(F, \varrho_0)$ *and if* $x_0 \in \operatorname{spt} T_1 \cap \mathbf{U}^{n+1}(0, \varrho_0)$, *then the choice* $d_r = u_r(x_0)$ *fulfills condition* (i) *of Theorem* 2.2 *and yields* $T \in \mathcal{M}'_A(F, \varrho_0)$ *such that* $x_0 \in \operatorname{spt} T$.

Proof. We define

$$H_r = -[\![V_1 \times \{r\}]\!] + [\![V_1 \times \{0\}]\!],$$

where the orientations are such that

$$\partial [\![G_r]\!] = \partial ((T_1 \times [\![(0, r)]\!]) + H_r). \tag{51}$$

Then by I.1 (20), writing $S_r = T_1 \times [\![(0, r)]\!] + H_r$, we have

$$\frac{1}{2} \int_{\mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}} |\nu^{[\![G_r]\!]} - \nu^{S_r}|^2 d \|S_r\| \leqslant \tilde{\mathbf{F}}(S_r) - \tilde{\mathbf{F}}([\![G_r]\!]). \tag{52}$$

But $T_1 \times [\![(0, r)]\!]$ is $\tilde{\mathbf{F}}$-minimizing, and by (51)

$$\partial (T_1 \times [\![(0, r)]\!]) = \partial ([\![G_r]\!] - H_r),$$

hence

$$\begin{aligned} \tilde{\mathbf{F}}(S_r) = \tilde{\mathbf{F}}(T_1 \times [\![(0, r)]\!] + H_r) &\leqslant \tilde{\mathbf{F}}(T_1 \times [\![(0, r)]\!]) + \tilde{\mathbf{F}}(H_r) \\ &\leqslant \tilde{\mathbf{F}}([\![G_r]\!] - H_r) + \tilde{\mathbf{F}}(H_r) \leqslant \tilde{\mathbf{F}}([\![G_r]\!]) + \tilde{\mathbf{F}}(H_r) + \tilde{\mathbf{F}}(-H_r). \end{aligned}$$

Hence (52) gives

$$\int_{\mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}} |\nu^{[\![G_r]\!]} - \nu^{S_r}|^2 d \|S_r\| \leqslant c,$$

where c is independent of r. On the other hand

$$\int_{\mathbf{U}^{n+1}(0, \varrho_0) \times \mathbf{R}} |\nu^{[\![G_r]\!]} - \nu^{S_r}|^2 d \|S_r\| \geqslant r \int_{\operatorname{spt} T_1} (\nu_{n+2}^{[\![G_r]\!]})^2 d\mathcal{H}^n, \tag{53}$$

because $\|S_r\| = \|T_1 \times [\![(0, r)]\!]\| + \|H_r\|$ and $\nu_{n+2}^{T_1 \times [\![(0, r)]\!]} = 0$. Thus, since $\nu_{n+2}^{[\![G_r]\!]} = (1 + |Du_r|^2)^{-1/2}$ we have that

$$\int_{\operatorname{spt} T_1} (1 + |Du_r|^2)^{-1} d\mathcal{H}^n \to 0 \quad \text{as} \quad r \to \infty. \tag{54}$$

If we now take any $\sigma \in \varrho_0 - |x_0|$, then we must have

$$\sup_{\mathbf{U}^{n+1}(x_0, \sigma)} (u_r - u_r(x_0)) \to \infty, \quad \sup_{\mathbf{U}^{n+1}(x_0, \sigma)} (u_r(x_0) - u_r) \to \infty. \tag{55}$$

Otherwise, we could deduce from Lemma 2.6 that for some $\sigma' \in (0, \sigma)$ there is a subsequence $\{k\}$ of $\{r\}$ with

$$\sup_{\mathbf{U}^{n+1}(x_0, \sigma)} |Du_k| \leq c,$$

where c is a fixed constant, and this clearly contradicts (54).

In particular, by Lemma 2.2, (55) implies that $u_r(x_0) \to \infty$ and $r - u_r(x_0) \to \infty$; hence we can use Theorem 2.2 with $d_r = u_r(x_0)$, $r = 1, 2, \ldots$ Then by using (55) and I.1 (28) it is clear that the subset U obtained in Theorem 2.2 has the property that

$$\pi^{-1}(x_0) \subset \operatorname{spt} \partial [\![U]\!].$$

It then easily follows that $x_0 \in \operatorname{spt} T$ as required, regardless of which of the alternatives (iii), (iii)′ of Theorem 2.2 holds.

I.3. Main results

Here we intend to use the results of the previous section; A, $B = \partial [\![A]\!]$ are as in I.1 (30), (31). $\mathcal{M}_A(F, \varrho_0)$, $\mathcal{M}'_A(F, \varrho_0)$ are also as introduced in the previous section.

Our aim here is to show that each element $T \in \mathcal{M}_A(F, \varrho_0)$ can be decomposed into a locally finite sum $\Sigma [\![M_i]\!]$, where each M_i is a component of reg S for some $S \in \mathcal{M}'_A(F, \varrho_0)$. In this way, regularity results for $T \in \mathcal{M}_A(F, \varrho_0)$ are inferred from the known regularity results for currents $S \in \mathcal{M}'_A(F, \varrho_0)$. The main results appear in Theorem 3.1.

In the special case of 2-dimensional **F**-minimal currents we can prove that the singular set is *empty*. F. J. Almgren has informed us that he has another proof of this; his method is based partly on the methods of Part II of the present paper and is independent of the results of this section.

The present section will conclude with a uniqueness result (Theorem 3.2).

We will need the following lemma concerning currents in $\mathcal{M}_A(F, \varrho_0)$.

LEMMA 3.1. *Suppose S, $T \in \mathcal{M}_A(F, \varrho_0)$, $F \in \mathcal{F}(\lambda, \varrho_0)$, and define*

$$M = \operatorname{reg} S \cap \operatorname{reg} T.$$

If $M \neq \emptyset$, then M is a C^2 hypersurface with

$$\bar{M} - M \subset \operatorname{sing} S \cup \operatorname{sing} T \tag{1}$$

and with unit normal ν satisfying

$$\nu = \nu^S = \nu^T$$

at each point of M.

Proof. We will eventually show that if $x_0 \in M$, then there is a $\sigma > 0$ such that

$$\mathbf{U}^{n+1}(x_0, \sigma) \cap \operatorname{reg} S = \mathbf{U}^{n+1}(x_0, \sigma) \cap \operatorname{reg} T. \tag{2}$$

This clearly suffices to prove the first assertion of the theorem; the assertion that $\nu = \nu^S = \nu^T$ (i.e., $\nu^S \neq -\nu^T$ on M) will emerge as a consequence of one step in the argument leading to (2).

We beging by letting V_S, V_T denote open subsets of $\mathbf{U}^{n+1}(0, \varrho_0)$ such that (cf. I.1 (24))

$$T = \partial[\![V_T]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0), \quad S = \partial[\![V_S]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0).$$

$\mathbf{U}^{n+1}(0, \varrho_0) \cap \partial V_T = \mathbf{U}^{n+1}(0, \varrho_0) \cap \operatorname{spt} T$, $\mathbf{U}^{n+1}(0, \varrho_0) \cap \partial V_S = \mathbf{U}^{n+1}(0, \varrho_0) \cap \operatorname{spt} S$.

Next we note that

$$[\![V_S]\!] + [\![V_T]\!] = [\![V_S \cup V_T]\!] + [\![V_S \cap V_T]\!], \tag{3}$$

and hence

$$S + T = S' + T', \tag{4}$$

where

$$S' = \partial[\![V_S \cup V_T]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0), \quad T' = \partial[\![V_S \cap V_T]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0).$$

One easily checks that

$$\partial[\![V_S \cup V_T]\!] \llcorner \partial\mathbf{U}^{n+1}(0, \varrho_0) = [\![A]\!]$$

and

$$\partial[\![V_S \cap V_T]\!] \llcorner \partial\mathbf{U}^{n+1}(0, \varrho_0) = [\![A]\!];$$

and hence

$$\partial S' = \partial T' = B. \tag{5}$$

Also, since

$$[\![V_S \cup V_T]\!] + [\![V_S \cap V_T]\!] = \mathbf{E}^{n+1} \llcorner f,$$

where $f \equiv 2$ on $V_S \cap V_T$, $f \equiv 1$ on $(V_S \cup V_T) \sim (V_S \cap V_T)$, and $f \equiv 0$ on $\mathbf{R}^{n+1} \sim (V_S \cup V_T)$. Hence by [7], 4.5.9, (13), we have

$$\|S' + T'\| = \|S'\| + \|T'\|$$

and hence

$$\mathbf{F}(S' + T') = \mathbf{F}(S') + \mathbf{F}(T'). \tag{6}$$

We also have

$$\mathbf{F}(S + T) = \mathbf{F}(S) + \mathbf{F}(T) - \mathbf{F}(S \llcorner L) - \mathbf{F}(T \llcorner L), \tag{7}$$

where

$$L = \{x \in \operatorname{reg} S \cap \operatorname{reg} T : \nu^S(x) = -\nu^T(x)\}.$$

(We note that L is closed relative to both reg S and reg T, and hence is Borel-measurable.)

By combining (4), (6), and (7) we now see that

$$\mathbf{F}(S')+\mathbf{F}(T')+\mathbf{F}(S\llcorner L)+\mathbf{F}(T\llcorner L)\leqslant \mathbf{F}(S)+\mathbf{F}(T).$$

However, using the fact that S, T are $\mathbf{F}$-minimizing together with the fact that $\partial S'=\partial T'=\partial S=\partial T$, we then deduce that

$$\mathcal{H}^n(L)=0 \tag{8}$$

and that S', T' are both $\mathbf{F}$-minimizing in $\mathbf{B}^{n+1}(0,\varrho_0)$.

We can now show that $L=\emptyset$. Suppose on the contrary that we have $x_0\in L$. Since $\nu^S(x_0)=-\nu^T(x_0)$ and since $x_0\in\operatorname{reg} S\cap\operatorname{reg} T$, we can suppose without loss of generality that the coordinate axes have origin at x_0 and are such that, for suitable $\sigma>0$, $\operatorname{reg} S\cap\mathbf{U}^{n+1}(x_0,\sigma)$ and $\operatorname{reg} T\cap\mathbf{U}^{n+1}(x_0,\sigma)$ can be represented in the non-parametric form

$$x_{n+1}=u_1(x_1,\ldots,x_n),\quad x_{n+1}=u_2(x_1,\ldots,x_n), \tag{9}$$

with

$$Du_1(0)=Du_2(0)=0 \tag{10}$$

and with

$$V_S\cap\mathbf{U}^{n+1}(x_0,\sigma)\subset\{x\colon x_{n+1}>u_1(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)\in\text{domain } u_1\} \tag{11}$$

and

$$V_T\cap\mathbf{U}^{n+1}(x_0,\sigma)\subset\{x\colon x_{n+1}<u_2(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)\in\text{domain } u_2\}. \tag{12}$$

Then by (10), (11), (12) we see that

$$\mathcal{L}^{n+1}(\mathbf{U}^{n+1}(x_0,\sigma')\sim(V_S\cup V_T))\leqslant\varepsilon(\sigma')(\sigma')^{n+1}, \tag{13}$$

where $\varepsilon(\sigma')\to 0$ as $\sigma'\to 0$. However, we showed above that $S'=\partial[\![V_S\cup V_T]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)$ is $\mathbf{F}$-minimizing, hence we have that $\partial[\![\mathbf{U}^{n+1}(0,\varrho_0)\sim(V_S\cup V_T)]\!]\llcorner\mathbf{U}^{n+1}(0,\varrho_0)$ is $\mathbf{F}'$-minimizing, where $F'(x,p)=F(x,-p)$. Then (13) contradicts the volume bound of I.1 (35). (Notice that $x_0\in\operatorname{spt}\partial[\![\mathbf{U}^{n+1}(0,\varrho_0)\sim(V_S\cup V_T)]\!]$ because $\mathcal{H}^n(L)=0$.) Thus we deduce $L=\emptyset$ as required.

Next we consider the possibility that $\nu^S(x_0)\neq\nu^T(x_0)$ for some $x_0\in M$. Since we have already proved $\nu^S(x_0)\neq-\nu^T(x_0)$, we can then suppose that the coordinate axes are such that $x_0=0$ and such that for sufficiently small σ $\operatorname{reg} S\cap\mathbf{U}^{n+1}(x_0,\sigma)$ and $\operatorname{reg} T\cap\mathbf{U}^{n+1}(x_0,\sigma)$ can be represented in the form (9) where now

$$V_S\cap\mathbf{U}^{n+1}(x_0,\sigma)=\{x\colon x_{n+1}>u_1(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)\in\text{domain } u_1\} \tag{14}$$

and

$$V_T\cap\mathbf{U}^{n+1}(x_0,\sigma)=\{x\colon x_{n+1}>u_2(x_1,\ldots,x_n),\ (x_1,\ldots,x_n)\in\text{domain } u_2\}. \tag{15}$$

But, again using the fact that $S' = \partial[\![V_S \cup V_T]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0)$ is **F**-minimizing, we then see that the Lipschitz function

$$u^+ = \max\{u_1, u_2\}$$

(defined on the intersection of the domains of u_1 and u_2) must be a weak solution of the Euler-Lagrange equation I.2 (1). However, u^+ is not C^1, and this contradicts Lemma 2.3.

The final possibility is that $\nu^S(x) = \nu^T(x)$ at each point of M. However, non-parametric representations of the form (9), (14), (15), together with Lemma 2.4, then imply that for each $x_0 \in M$, (2) must hold for sufficiently small σ. This completes the proof.

COROLLARY 3.1. *If* $S, T \in \mathcal{M}_A(F, \varrho_0)$ *satisfy*

$$\mathcal{H}^{n-1}((\text{sing } S \cup \text{sing } T) \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0,$$

and if M, M' *are components of* reg S, reg T, *respectively, such that*

$$M \cap M' \neq \emptyset,$$

then

$$M = M' \quad \textit{and} \quad [\![M]\!] \in \mathcal{M}(F, \varrho_0).$$

Also

$$\text{sing } S \cap \mathbf{U}^{n+1}(0, \varrho_0) = (\bigcup (\bar{M} - M)) \cap \mathbf{U}^{n+1}(0, \varrho_0), \tag{16}$$

where the union is taken over all components M *of* reg S.

Proof. $M \cap M'$ is open in M' by the lemma. Then, by the connectedness of M', either $M' \subset M$ or $M' \cap (\bar{M} \sim M) \neq \emptyset$. In the latter case we choose $x_0 \in M' \cap (\bar{M} \sim M)$ and $\sigma > 0$ such that $M' \cap \mathbf{U}^{n+1}(x_0, \sigma)$ is diffeomorphic to $\mathbf{U}^n(0, 1)$. Then by the Poincaré inequality ([7], 4.5.3) we deduce $\mathcal{H}^{n-1}(M' \cap (\bar{M} \sim M)) > 0$, thus contradicting the hypothesis $\mathcal{H}^{n-1}(\text{sing } S \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0$. Thus we must have $M' \subset M$. Similarly, one can prove $M \subset M'$.

Next we note that $[\![M]\!] = \partial[\![V]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0)$, for some Lebesgue-measurable V, by Lemma 1.1. Hence to prove $[\![M]\!] \in \mathcal{M}(F, \varrho_0)$ it remains to prove that $[\![M]\!]$ is **F**-minimizing in $\mathbf{U}^{n+1}(0, \varrho_0)$. To prove this, let K be an arbitrary compact subset of $\mathbf{U}^{n+1}(0, \varrho_0)$, and let $R \in \mathcal{R}_n(\mathbf{R}^{n+1})$ be such that $\partial R = \partial([\![M]\!] \llcorner K)$, spt $R \subset \mathbf{U}^{n+1}(0, \varrho_0)$. Then, using the **F**-minimality of S, we deduce

$$\begin{aligned} \mathbf{F}(R) + \mathbf{F}(S \llcorner K - [\![M]\!] \llcorner K) &\geq \mathbf{F}(R - [\![M]\!] \llcorner K + S \llcorner K) \\ &\geq \mathbf{F}(S \llcorner K) = \mathbf{F}([\![M]\!] \llcorner K) + \mathbf{F}(S \llcorner K - [\![M]\!] \llcorner K). \end{aligned} \tag{17}$$

Note that the last step follows from the fact that

$$\text{spt } (S - [\![M]\!]) \cap \text{spt } [\![M]\!] \subset \text{sing } S,$$

which implies $\mathcal{H}^n(\operatorname{spt}(S-[\![M]\!]) \cap \operatorname{spt}[\![M]\!]=0$. (17) now gives $\mathbf{F}([\![M]\!] \llcorner K) \leqslant \mathbf{F}(R)$ as required.

To prove (16) it suffices to prove $\operatorname{sing} S \cap \mathbf{U}^{n+1}(0, \varrho_0) \subset (\bigcup(\bar{M}-M)) \cap \mathbf{U}^{n+1}(0, \varrho_0)$ since the reverse inclusion is obvious. Then let $x_0 \in \operatorname{sing} S \cap \mathbf{U}^{n+1}(0, \varrho_0)$; then there exists a sequence $\{M_r\}$ of components of reg S with dist $\{x_0, M_r\} \to 0$ as $r \to \infty$. However, since we have shown that $[\![M_r]\!] \in \mathcal{M}(F, \varrho_0)$, it follows from I.1 (28), (33) that there are at most a finite number of distinct terms in the sequence $\{M_r\}$. Hence $x_0 \in \bar{M}$ for some component M of reg S; we then have $x_0 \in \bar{M} \sim M$. ($x_0 \notin M$ because $M \subset \operatorname{reg} S$.)

The following is a consequence of Lemma 1.1 and Theorem 2.3.

LEMMA 3.2. *Suppose $T \in \mathcal{M}_A(F, \varrho_0)$ and N is a component of* reg T. *Then there is $S \in \mathcal{M}'_A(F, \varrho_0)$ and a component M of* reg S *such that $N \subset M$. Furthermore, if $N \neq M$, then*

$$\mathcal{H}^{n-1}(\bar{N}-N) \cap M) > 0 \tag{19}$$

Proof. Take $x_0 \in N$ and let $\sigma > 0$ be small enough to ensure that $N \cap \mathbf{U}^{n+1}(0, \sigma)$ is a connected C^2 hypersurface with $(\bar{N}-N) \cap \mathbf{U}^{n+1}(x_0, \sigma) = \emptyset$. Let $\mathcal{C}$ denote the collection of all $M \cap \mathbf{U}^{n+1}(x_0, \sigma)$, where each M is a component of one of the hypersurfaces in the collection $\{\operatorname{reg} S \colon S \in \mathcal{M}'_A(F, \varrho_0)\}$. Suppose $M \cap N \cap \mathbf{U}^{n+1}(x_0, \sigma) = \emptyset$ for each $M \in \mathcal{C}$. Then by using Lemma 1.1 with $\mathbf{U}^{n+1}(x_0, \sigma)$ in place of $\mathbf{U}^{n+1}(0, \varrho_0)$ and with $\mathcal{C} \cup \{N \cap \mathbf{U}(x_0, \sigma)\}$ in place of $\mathcal{C}$ we deduce

$$\mathcal{H}^{n-1}((\bigcup \bar{M}) \cap N \cap \mathbf{U}^{n+1}(x_0, \sigma)) = 0, \tag{20}$$

where the union is taken over all $M \in \mathcal{C}$. However, by Theorems 2.1, 2.3 and by (16) we have

$$N \subset \bigcup_{M \in \mathcal{C}} \bar{M} \cap N. \tag{21}$$

(20) and (21) are contradictory. Hence we deduce that there is a component M of reg S for some $S \in \mathcal{M}'_A(F, \varrho_0)$ with $M \cap N \neq \emptyset$. But then by Lemma 3.1 $M \cap N$ is open in N. Then by the connectedness of N together with the fact that $\mathcal{H}^{n-1}((\bar{M}-M) \cap \mathbf{U}^{n+1}(0, \varrho)) = 0$, we can deduce that $N \subset M$. (Cf. the first part of the proof of Corollary 3.1.)

To prove (19) take a point $x_0 \in (\bar{N}-N) \cap M$ and choose σ small enough to ensure that $M \cap \mathbf{U}^{n+1}(x_0, \sigma)$ is diffeomorphic to $\mathbf{U}^{n+1}(0, 1)$. Then by the Poincaré inequality [7], 4.5.3, we deduce

$$\mathcal{H}^{n-1}((\bar{N}-N) \cap M \cap \mathbf{U}^{n+1}(x_0, \sigma)) > 0;$$

that is, (19) is proved.

LEMMA 3.3. *If $T \in \mathcal{M}_A(F, \varrho_0)$ and if $x_0 \in \operatorname{sing} T \sim \operatorname{spt} B$, then there exists $S \in \mathcal{M}'_A(F, \varrho_0)$ such that $x_0 \in \operatorname{sing} S \sim \operatorname{spt} B$.*

Proof. By Theorem 2.3 we know $x_0 \in \operatorname{spt} S_0$ for some $S_0 \in \mathcal{M}'_A(F, \varrho_0)$. If $x_0 \in \operatorname{sing} S_0$ we have nothing to prove. We therefore suppose $x_0 \in \operatorname{reg} S_0$ and we define $\nu^0 = \nu^{S_1}(x_0)$.

Since $x_0 \in \operatorname{sing} T \sim \operatorname{spt} B$, we know from remark 2 following Theorem 1.2 that there is an $\varepsilon > 0$ and a sequence $\{x_r\} \subset \operatorname{reg} T$ such that $x_r \to x_0$ as $r \to \infty$ and

$$|\nu^T(x_r) - \nu^0| \geq \varepsilon. \tag{22}$$

For each r we let N_r be a component of reg T such that $x_r \in N_r$; by Lemma 3.2 we have $S_r = \partial[\![V_r]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) \in \mathcal{M}'_A(F, \varrho_0)$ such that $N_r \subset \operatorname{reg} S_r$ and such that $\nu^{S_r} = \nu^T$ on N_r. By I.1 (33) and Theorem 1.1 we have a subsequence $\{S_k\} \subset \{S_r\}$ and $V \subset \mathbf{U}^{n+1}(0, \varrho_0)$ such that

$$\mathcal{L}^{n+1}(V_k \Delta V) \to 0$$

and such that $S = \partial[\![V]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho_0) \in \mathcal{M}'_A(F, \varrho_0)$. (We know $S \in \mathcal{M}'_A(F, \varrho_0)$ by the remarks preceding I.2 (36).) By I.1 (28) and Theorem 1.1 (vii) we can see that $x_0 \in \operatorname{spt} S$. If $x_0 \in \operatorname{sing} S$ there is nothing further to prove. The only other alternative, in view of Lemma 3.1, is that $S \llcorner \mathbf{U}^{n+1}(x_0, \sigma) = S_0 \llcorner \mathbf{U}^{n+1}(x_0, \sigma)$ for some $\sigma > 0$. However, by remark 3 following Theorem 1.2, we then have $\nu^{S_k}(x_k) \to \nu^0$ as $k \to \infty$, thus contradicting (22).

We can now prove the main theorem.

THEOREM 3.1. *Suppose $T \in \mathcal{M}_A(F, \varrho_0)$. Then T can be represented in the form*

$$T = \Sigma[\![M_i]\!]$$

where for each $\varrho \in (0, \varrho_0)$ we have $[\![M_i]\!] \llcorner \mathbf{U}^{n+1}(0, \varrho) = 0$ for all but a finite number N (depending on n, λ, and ϱ/ϱ_0) of i, and where each M_i is a component of reg S *for some $S \in \mathcal{M}'_A(F, \varrho_0)$.*

In particular,

$$\mathcal{H}^{n-2}(\operatorname{sing} T \cap \mathbf{U}^{n+1}(0, \varrho)) < \infty \tag{23}$$

for each $\varrho < \varrho_0$, and

$$\int_{\operatorname{reg} T \cap \mathbf{U}^{n+1}(0, \varrho)} |\delta^T \nu^T|^2 \, d\mathcal{H}^n \leq c \varrho^{n-2}, \tag{24}$$

for each $\varrho \in (0, \varrho_0)$, where c depends only on n, λ, and ϱ/ϱ_0.

Proof. Our aim is to show that each component of reg T is also a component of reg S for some $S \in \mathcal{M}'_A(F, \varrho_0)$. Then Theorem 2.1 together with the area bound I.1 (33) will imply the required results.

Then let N be any component of reg T. By Lemma 3.2 we have $N \subset N_1$ for some component N_1 of reg S, $S \in \mathcal{M}'_A(F, \varrho_0)$. If $N \neq N_1$, we then have by Lemma 3.2 that

$$\mathcal{H}^{n-1}((\bar{N} - N) \cap N_1) > 0. \tag{25}$$

Now let $\mathcal{C}$ be the collection of M such that M is a component of reg S for some $S \in \mathcal{M}'_A(F, \varrho_0)$ and such that $\bar{M} \cap (\bar{N} - N) \cap N_1 \neq \emptyset$. By Theorem 2.1 and Corollary 3.1 we know $\mathcal{H}^{n-1}(\mathbf{U}^{n+1}(0, \varrho_0) \cap (\bar{M} - M)) = 0$ for each $M \in \mathcal{C}$, and $M \cap M' = \emptyset$ for each distinct pair M, $M' \in \mathcal{C}$. Hence we can apply Lemma 1.1. Taking $M_0 = N_1$, this gives

$$\mathcal{H}^{n-1}((\bigcup_{M \in \mathcal{C} \sim \{N_1\}} \bar{M}) \cap N_1) = 0. \tag{26}$$

However, $\bigcup_{M \in \mathcal{C} \sim \{N_1\}} \bar{M} \supset (\bar{N} - N) \cap N_1$ by Lemma 3.3. Hence we see that (25) and (26) are contradictory.

Note that the above theorem asserts in the case $n = 2$ that $\mathcal{H}^0(\text{sing } T \cap \mathbf{U}^3(0, \varrho)) < \infty$ for each $\varrho < \varrho_0$; that is, there are at most a finite collection of singular points of T in $\mathbf{U}^3(0, \varrho)$. We can easily show in this case that there are *no* singular points, because by (24) we have, for each $x_0 \in \text{spt } T \cap \mathbf{U}^3(0, \varrho_0)$,

$$\int_{\text{reg } T \cap \mathbf{U}^3(x_0, \varrho)} |\delta^T \nu^T|^2 \, d\mathcal{H}^2 \to 0 \quad \text{as} \quad \varrho \to 0.$$

But then by Theorem 1.3 (with $n = 2$) and the regularity theorem (Theorem 1.2) we deduce x_0 is a regular point of T. That is, we have the following corollary of Theorem 3.1.

COROLLARY 3.2. *If $T \in \mathcal{M}_A(F, \varrho_0)$ and $n = 2$, then*

$$\text{sing } T \cap \mathbf{U}^3(0, \varrho_0) = \emptyset.$$

Notice that in Part II it will be proved that, for any n, $\mathcal{H}^{n-2}(\text{sing } T \cap \mathbf{U}^{n+1}(0, \varrho_0)) = 0$, and the above corollary could be interpreted as a special case of this general result.

Remark. The above results are all stated for currents $T \in \mathcal{M}_A(F, \varrho_0)$; however, the results apply directly to any $\mathbf{F}$ minimizing $T \in \mathbf{I}_n(\mathbf{R}^{n+1})$ with spt $\partial T \subset \partial \mathbf{U}^{n+1}(0, \varrho_0)$ by virtue of the fact that any such T can be decomposed into a locally finite sum $T = \Sigma T_j$, where each $T_j \in \mathcal{M}_{A_j}(F, \varrho_0)$ for suitable $A_j \subset \partial \mathbf{U}^{n+1}(0, \varrho_0)$.

We conclude Part I with a proof of the following uniqueness theorem.

THEOREM 3.2. *If $T \in \mathcal{M}_A(F, \varrho_0)$, if K is a compact subset of $\mathbf{U}^{n+1}(0, \varrho_0)$, if S is $\mathbf{F}$-minimizing in K, if spt $S \subset K$, and if $\partial S = \partial T \llcorner K$, then $S = T \llcorner K$.*

Remark. A similar theorem can be proved if one takes any integral current T which is $\mathbf{F}$-minimizing in an open set U. (Then it is assumed that K is a compact subset of U.)

Proof of Theorem 3.2. We define $T' = T \llcorner (\mathbf{U}^{n+1}(0, \varrho_0) \sim K) + S$. It is easily checked that then $\partial T' = \partial T$ and $\mathbf{F}(T') = \mathbf{F}(T)$. Then T' is $\mathbf{F}$-minimizing in $\mathbf{U}^{n+1}(0, \varrho_0)$ and it follows that

$T' \in \mathcal{M}_A(F, \varrho_0)$. This is quite easily checked with the aid of [7], 4.5.17, which one can apply to the current $R = [\![A]\!] - T'$. The required uniqueness follows easily from Theorem 3.1 and Corollary 3.1.

PART II

II.1. Terminology

Except where otherwise noted we follow the terminology of part I, [1], or [2]. Throughout part II we assume $n \geq 2$.

(i) $\boldsymbol{Q} = \mathbf{I}_{n+1}(\mathbf{R}^{n+1}) \cap \{[\![A]\!]: A \subset \mathbf{U}^{n+1}(0, 2) \text{ is } \mathcal{L}^{n+1} \text{ measurable}\}$ (recall $[\![A]\!] = \mathbf{E}^{n+1} \llcorner A$ as in I.1) with the $\mathbf{M}$ metric topology, i.e. $\mathbf{M}(Q, R) = \mathbf{M}(Q - R)$ for $Q, R \in \boldsymbol{Q}$. In particular, if $A, B \subset \mathbf{U}^{n+1}(0, 2)$ are $\mathcal{L}^{n+1}$ measurable with $[\![A]\!], [\![B]\!] \in \boldsymbol{Q}$, then A is $\mathcal{L}^{n+1}$ almost equal to $\mathbf{R}^{n+1} \cap \{x: \Theta^{n+1}(\|A\|, x) = 1\}$ and $\mathbf{M}([\![A]\!], [\![B]\!]) = \mathcal{L}^{n+1}[(A \sim B) \cup (B \sim A)]$.

(ii) $\mathcal{S}^* = \mathcal{R}_n(\mathbf{R}^{n+1}) \cap \{\partial Q \llcorner \mathbf{U}^{n+1}(0, 2): Q \in \boldsymbol{Q}\}$. One notes that corresponding to each $S \in \mathcal{S}^*$ there exists a unique $Q \in \boldsymbol{Q}$ for which $S = \partial Q \llcorner \mathbf{U}^{n+1}(0, 2)$. We give $\mathcal{S}^*$ the induced metric $\mathbf{m}$, i.e. $\mathbf{m}(S, T) = \mathbf{M}(Q, R)$ whenever $Q, R \in \boldsymbol{Q}$, $S = \partial Q \llcorner \mathbf{U}^{n+1}(0, 2)$, $T = \partial R \llcorner \mathbf{U}^{n+1}(0, 2)$.

(iii) Whenever $F, G: \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^+$ are parametric functionals such that F, $G | \mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \sim \{0\})$ are of class 2 we define

$$\begin{aligned} &\text{distance}(F, G) \\ &\quad = \sup \{ |F(x, p) - G(x, p)| + |D[F(x, \cdot)](p) - D[G(x, \cdot)](p)| \\ &\qquad + |D^2[F(x, \cdot)](p) - D^2[G(x, \cdot)](p)| : x \in \mathbf{R}^{n+1}, p \in \mathbf{S}^n \}. \end{aligned}$$

We denote by $\mathcal{F}^*$ a fixed subset of the space of all parametric functionals (integrands) $F: \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^+$ for which $F | \mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \sim \{0\})$ is of class 3. We further suppose

(a) the n dimensional area integrand $M = |\cdot|$ is contained in $\mathcal{F}^*$,
(b) $\sup \{ |D^3 F(x, p)| : F \in \mathcal{F}^*, x \in \mathbf{R}^{n+1}, p \in \mathbf{S}^n \} < \infty$,
(c) There exists a positive number c such that for each $F \in \mathcal{F}^*$, cF is positive and elliptic [I.1, (8), (9)],
(d) $\mathcal{F}^*$ is compact in the distance topology.

We further denote by $\mathcal{F}$ the set of all (constant coefficient) integrands of the form $F(x, \cdot)$ corresponding to each $x \in \mathbf{B}^{n+1}(0, 4)$ and each $F \in \mathcal{F}^*$. Clearly $\mathcal{F}$ is compact in the distance topology.

(iv) Corresponding to each $F \in \mathcal{F}^*$ we denote by $\mathcal{S}_F$ the set of all surfaces $S \in \mathcal{S}^*$ such that

(a) $0 \in \operatorname{spt} S$,

(b) $\mathbf{F}(S) \leq \mathbf{F}(T)$ for each $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with $\partial S = \partial T$.

We further set $\mathcal{S} = \cup \{\mathcal{S}_F : F \in \mathcal{F}^*\}$.

(v) We define the Hausdorff dimension functions

$$H: \mathcal{S} \to \mathbf{R}^+, \quad \mathbf{H}: \mathcal{F} \to \mathbf{R}^+$$

by setting $H(S)$ equal to the Hausdorff dimension of $\operatorname{sing} S \cap \mathbf{U}^{n+1}(0, 2)$ for each $S \in \mathcal{S}$ and $\mathbf{H}(F) = \sup H \mid \mathcal{S}_F$ for each $F \in \mathcal{F}$.

(vi) For each $S \in \mathcal{S}$ we define

$$K: \operatorname{reg} S \to \mathbf{R}^+$$

$$K(x) = |\delta^S \nu^S(x)|^2 \quad \text{for each} \quad x \in \operatorname{reg} S.$$

(as noted in I.1, $K(x)$ is the sum of the squares of the principle curvatures of reg S at x). Also we define

$$\mathbf{K}: \mathcal{S} \to \mathbf{R}^+$$

$$\mathbf{K}(S) = \int_{\operatorname{reg} S \cap \mathbf{U}^{n+1}(0,1)} K\, d\|S\| \quad \text{for each} \quad S \in \mathcal{S},$$

and set $K_1 = \sup \mathbf{K} < \infty$ [I, Theorem 3.1] and

$$\begin{aligned} 3K_2 &= \inf \mathbf{K} \mid \{S: 0 \in \operatorname{sing} S\} && \text{in case} \quad \mathcal{S} \cap \{S: 0 \in \operatorname{sing} S\} \neq \varnothing \\ &= 0 && \text{in case} \quad \mathcal{S} \cap \{S: 0 \in \operatorname{sing} S\} = \varnothing. \end{aligned}$$

II.2. Some properties of $\mathcal{S}$, $\mathcal{F}$, $\mathcal{F}^*$

(i) $\mathcal{S} \times \mathcal{F}$ and $\mathcal{S} \times \mathcal{F}^*$ are compact in the $\mathbf{m} \times$ distance topology [7, 4.2.27], [I, Theorem 1.1].

(ii) For each $F \in \mathcal{F}^*$ and each $\delta > 0$ there exists a neighborhood $\mathcal{G}$ of F in $\mathcal{F}^*$ such that $G \in \mathcal{G}$ and $T \in \mathcal{S}_G$ implies $\mathbf{m}(S, T) < \delta$ for some $T \in \mathcal{S}_F$ [7, 5.1.5], [I, Theorem 1.1].

(iii) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $S \in \mathcal{S}$ with $\operatorname{spt} S \subset \mathbf{R}^n \times [-\delta, \delta]$ implies the existence of a function $f: \mathbf{U}^n(0, 2-\varepsilon) \to \mathbf{R}$ such that

$$\operatorname{spt} S \cap \mathbf{U}^n(0, 2-\varepsilon) \times \mathbf{R} = \{(x, y): x \in \mathbf{U}^n(0, 2-\varepsilon), \quad y = f(x)\}$$

and

$$\sup \{|f(x)| + |Df(x)| + |D^2 f(x)| : x \in \mathbf{U}^n(0, 2-\varepsilon)\} < \varepsilon$$

[I, Theorem 1.2, Lemma 2.3].

(iv) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $S, T \in \mathcal{S}$ with $\mathbf{m}(S, T) < \delta$ implies

$$\operatorname{spt} S \cap \mathbf{B}^{n+1}(0, 1) \subset \{x: \operatorname{dist}(x, \operatorname{spt} T) < \varepsilon\}$$

[2, II.3(11)], [I.1(28)].

(v) Corresponding to each $S \in \mathcal{S}$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $T \in \mathcal{S}$ with $\mathbf{m}(S, T) < \delta$ implies

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \{x: \text{dist}(x, \text{sing } S \cap \mathbf{B}^{n+1}(0, 1)) < \varepsilon\}.$$

(vi) In case $s \in [0, n+1]$ and $S \in \mathcal{S}$ with $\mathcal{H}^s[\text{sing } S \cap \mathbf{B}^{n+1}(0, 1)] = 0$, then there exist a positive integer N, points $p_1, p_2, \ldots, p_N \in \text{sing } S \cap \mathbf{B}^{n+1}(0, 1)$, and radii $1/4 > r_1, r_2, \ldots, r_N > 0$ such that

$$\text{sing } S \cap \mathbf{B}^{n+1}(0, 1) \subset \{\mathbf{B}^{n+1}(p_i, r_i/2): i = 1, \ldots, N\}$$

and

$$\Sigma\{(2r_i)^s: i = 1, \ldots, N\} < 1/2.$$

Furthermore there exists $\delta > 0$ such that $T \in \mathcal{S}$ with $\mathbf{m}(S, T) < \delta$ implies

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(p_i, r_i): i = 1, \ldots, N\}.$$

(vii) In case $s \in [0, n+1]$ and $F \in \mathcal{J}$ such that $S \in \mathcal{S}_F$ implies $\mathcal{H}^s(\text{sing } S \cap \mathbf{B}^{n+1}(0, 1)) = 0$ then there exist positive integers M and N, surfaces $S_1, S_2, \ldots, S_M \in \mathcal{S}_F$, points $p(i, j) \in \text{sing } S_i \cap \mathbf{B}^{n+1}(0, 1)$, and radii $1/4 > r(i, j) > 0$ for each $i = 1, \ldots, M$ and $j = 1, \ldots, N$, and $\delta > 0$ such that

(a) $\text{sing } S_i \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{B^{n+1}(p(i, j), r(i, j)/2): j = 1, \ldots, N\}$ for each $i = 1, \ldots, \mathbf{N}$:

(b) $\Sigma\{(2r(i, j))^s: j = 1, \ldots, N\} < 1/2$ for each $i = 1, \ldots, M$;

(c) if $T \in \mathcal{S}$ with $\mathbf{m}(S, T) < \delta$ for some $S \in \mathcal{S}_F$, then there exists $i \in \{1, \ldots, M\}$ such that

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(p(i, j), r(i, j)): j = 1, \ldots, N\}.$$

(d) If $T \in \mathcal{S}$ with $\mathbf{m}(S, T) < \delta$ for some $S \in \mathcal{S}_F$, then there exist $i \in \{1, \ldots, M\}$ and points $q(1), q(2), \ldots, q(N) \in \mathbf{B}^{n+1}(0, 1)$ such that for each $j = 1, \ldots, N$

either

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \cap \mathbf{B}^{n+1}(p(i, j), r(i, j)) = \varnothing$$

or

$q(j) \in \text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \cap \mathbf{B}^{n+1}(p(i, j), r(i, j))$ and hence

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \{\mathbf{B}^{n+1}(q(j), 2r(i, j)): j = 1, \ldots N\}.$$

(viii) $\mathbf{K}$ is lower semicontinuous with $0 \leq 3K_2 < K_1 < \infty$. In case $\mathcal{S} \cap \{S: 0 \in \text{sing } S\} \neq \varnothing$ then $K_2 > 0$ [I, Theorem 1.2, Theorem 1.3, Lemma 2.3]. Also for each $S \in \mathcal{S}$ with $0 \in \text{sing } S$ there exist $\varepsilon > 0$, $\delta > 0$ such that

$$\int_{\text{reg } S \cap \mathbf{U}^{n+1}(0,1) \cap \{x: \text{dist}(x, \text{sing } S) > 2\varepsilon\}} K\, d\|S\| > 2K_2,$$

and $T \in \mathcal{S}$ with $0 \in \text{sing } T$ and $\mathbf{m}(S, T) < \delta$ implies

$$\int_{\text{reg } T \cap \mathbf{U}^{n+1}(0,1) \cap \{x: \text{dist}(x, \text{sing } T) > \varepsilon\}} K\, d\|T\| > K_2.$$

(ix) For each $F \in \mathcal{J}$ there exist a positive integer M, surfaces $S_1, \ldots, S_M \in \mathcal{S}_F$ with $0 \in \operatorname{sing} S_i$ for each i, $\delta > 0$, and $1/2 > \varepsilon > 0$ such that $T \in \mathcal{S}_F$ with $0 \in \operatorname{sing} T$ implies $\mathbf{m}(T, S_i) < \delta$ for some $i = 1, \ldots, M$ and

$$\int_{\operatorname{reg} T \cap \mathbf{U}^{n+1}(0,1) \cap \{x:\, \operatorname{dist}(x, \operatorname{sing} T) > \varepsilon\}} K\, d\|T\| > K_2.$$

II.3

THEOREM. *For each* $S \in \cup\{\mathcal{S}_F: F \in \mathcal{J}\}$, $\mathcal{M}^{*n-2}(\operatorname{sing} S \cap \mathbf{U}^{n+1}(0, 1/2)) = 0$ *and* $\mathcal{H}^{n-2}(\operatorname{sing} S) = 0$; *here* $\mathcal{M}^{*n-2}$ *denotes* $n-2$ *dimensional upper Minkowski content* [7, 3.2.37].

Proof. Let $F \in \mathcal{J}$ and $S \in \mathcal{S}_F$ with $0 \in \operatorname{sing} S$. We will show that the assumption $\mathcal{M}^{*n-2}(\operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1/2)) > 0$ implies

$$\int_{\operatorname{reg} S \cap \mathbf{U}^{n+1}(0,1)} K\, d\|S\| = \infty$$

which is false by [I, Theorem 3.1] as noted in II.1 (6), II.2 (8). The first assertion of the theorem follows, and the second assertion will be clear from the coverings constructed in proving the first.

Let $1/2 > \varepsilon > 0$ be chosen as in II.2 (9).

Suppose then $\mathcal{M}^{*n-2}(\operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1)) > 0$. In that case we can choose $K_3 > 0$ and

$$1/4 > r_1 > \varepsilon r_1 > r_2 > \varepsilon r_2 > r_3 > \varepsilon r_3 > \ldots > 0$$

such that for each $i = 1, 2, 3, \ldots$,

$$[\boldsymbol{\alpha}(3) r^3]^{-1} \mathcal{L}^{n+1}\{x: \operatorname{dist}(x, \operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1/2)) < r_i\} > K_3.$$

Corresponding to each $i = 1, 2, 3, \ldots$ we now choose

$$p(i, 1), p(i, 2), \ldots, p(i, M_i) \in \operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1)$$

such that

$$\operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1) \subset \cup\, \{\mathbf{B}^{n+1}(p(i, j), 2r_i): i = 1, \ldots, M_i\}$$

and

$$\mathbf{B}^{n+1}(p(i, j), r_i) \cap \mathbf{B}^{n+1}(p(i, k), r_i) = \varnothing$$

whenever $j \neq k$. For each i, j we further set

$$A(i, j) = \operatorname{reg} S \cap \mathbf{U}^{n+1}(p(i, j), r_i) \cap \{x: \operatorname{dist}(x, \operatorname{sing} S) > \varepsilon r_i\}.$$

It follows by construction that

$$A(i_1, j_1) \cap A(i_2, j_2) = \emptyset \quad \text{whenever} \quad (i_1, j_1) \neq (i_2, j_2).$$

It follows also from II.2 (8) that

$$\int_{A(i,j)} K\,d\|S\| > U_2 r_i^{n-2}$$

so that

$$\int_{A(i,1)\cup A(i,2)\ldots\cup A(i,M_i)} K\,d\|S\| > M_i K_2 r_i^{n-2}.$$

We note that for each $i = 1, 2, 3, \ldots$

$$\mathbf{R}^{n+1} \cap \{x: \operatorname{dist}(x, \operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1/2)) < r_i\} \subset \cup \{\mathbf{B}^{n+1}(p(i,j), 3r_i): j = 1, \ldots, M_i\}$$

so that

$$\begin{aligned} K_3 &< [\alpha(3) r_i^3]^{-1} \mathcal{L}^{n+1}\{x: \operatorname{dist}(x, \operatorname{sing} S \cap \mathbf{B}^{n+1}(0, 1/2)) < r_i\} \\ &< [\alpha(3) r_i^3]^{-1} M_i \boldsymbol{\alpha}(n+1)(3r_i)^{n+1} = [3^{n+1}\boldsymbol{\alpha}(n+1)/\boldsymbol{\alpha}(3)] M_i r_i^{n-2}. \end{aligned}$$

Combining this last estimate with the previous integral estimate we obtain

$$\int_{\operatorname{reg} S \cap \mathbf{U}^{n+1}(0,1)} K\,d\|S\| \geq \sum \left\{ \int_{A(i,j)} K\,d\|S\| : i = 1, 2, 3, \ldots, j = 1, 2, \ldots, M_i \right\} = \infty.$$

II.4

Remark. The proof of II.3 above uses the estimate $\sup \mathbf{K} < \infty$ which know to hold by [I, Theorem 3.1]. Actually an estimate of the form

$$\sup (\mathbf{K} \,|\, \{F: S \in \mathcal{S}_F \text{ implies } \mathcal{H}^{n-2}(\operatorname{sing} S) = 0\}) < \infty$$

is sufficient since it is then possible to show the subset of $\mathcal{J}$ consisting of those F for which $\mathcal{H}^{n-2}(\operatorname{sing} S) = 0$ for each $S \in \mathcal{S}_F$ is both open and closed in $\mathcal{J}$ (recall that the space of all elliptic integrands is itself convex). This later estimate is implied for a substantial neighborhood of the n dimensional area integrand by a straightforward second variation estimate.

II.5

THEOREM. (1) *Suppose $s \in [0, n+1]$ and $F \in \mathcal{J}$ such that $S \in \mathcal{S}_F$ implies $\mathcal{H}^s(\operatorname{sing} S \cap B^{n+1}(0,1)) = 0$. Then there is a neighborhood $\mathcal{G}$ of F in $\mathcal{J}^*$ such that $G \in \mathcal{G}$ and $T \in \mathcal{S}_G$ implies $\mathcal{H}^s(\operatorname{sing} T \cap \mathbf{B}^{n+1}(0,1)) = 0$.*

(2) $\mathbf{H}$ *is upper semicontinuous.*

Proof. Clearly conclusion (1) implies conclusion (2). We will verify conclusion (1). Let $s \in [0, n+1]$ and $F \in \mathcal{J}$ such that $\mathcal{H}^s(\text{sing } S \cap \mathbf{B}^{n+1}(0, 1)) = 0$ for each $S \in \mathcal{S}_F$. Now, in accordance with II.2(7), choose and fix positive integers M and N, $S_1, \ldots, S_M \in \mathcal{S}_F$, $0 < r(i, j) < 1/4$ for each $i, j = 1, \ldots, N$, and $\delta > 0$ such that $T \in \mathcal{S}$ with $\mathbf{m}(T, S) < \delta$ for some $S \in \mathcal{S}_F$ implies the existence of $i \in \{1, \ldots, M\}$ and $q(1), \ldots, q(N) \in \text{sing } T \cap \mathbf{B}^{n+1}(0, 1)$ such that

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(q(j), 2r(i, j)) : j = 1, \ldots, N\}$$

and

$$\Sigma\{[2r(i, j)]^s : j = 1, \ldots, N\} < (1/2).$$

We now set

$$\mathcal{G} = \mathcal{J} \cap \{G : T \in \mathcal{S}_G \quad \text{implies} \quad \mathbf{m}(T, S) < \delta \quad \text{for some} \quad S \in \mathcal{S}_F\}.$$

As was noted in II.2(2), $\mathcal{G}$ is a neighborhood of F in $\mathcal{J}^*$.

We now fix $G \in \mathcal{G}$ and $T \in \mathcal{S}_G$ and will verify that $\mathcal{H}^s(\text{sing } T \cap \mathbf{B}^{n+1}(0, 1)) = 0$. To do this we will suppose m is a given (fixed) positive integer and will construct

$$Q(j_1, \ldots, j_m) \in \text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \quad \text{and} \quad 0 < R(j_1, \ldots, j_m) < 1/4$$

corresponding to each $(j_1, \ldots, j_m) \in \{1, \ldots, N\}^m$ such that

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(Q(j_1, \ldots, j_m), R(j_1, \ldots, j_m)) : (j_1, \ldots, j_m) \in \{1, \ldots, N\}^m\}$$

and

$$\Sigma\{R(j_1, \ldots, j_m)^s : (j_1, \ldots, j_m) \in \{1, \ldots, N\}^m\} < (1/2)^m.$$

As noted above we can choose $\mathbf{i}(1) \in \{1, \ldots, M\}$, $q(1), \ldots, q(N) \in \text{sing } T \cap \mathbf{B}^{n+1}(0, 1)$ such that

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(q(j_1), 2r(\mathbf{i}(1), j_1)) : j_1 = 1, \ldots, N\}$$

and

$$\Sigma\{[2r(\mathbf{i}(1), j_1)]^s : j_1 = 1, \ldots, N\} < 1/2.$$

In case $m = 1$ we set $Q(j_1) = q(j_1)$ and $R(j_1) = 2r(\mathbf{i}(1), j_1)$ for each $j_1 = 1, \ldots, N$ and we are done.

In case $m > 1$ we now define

$$T(j_1) = [\boldsymbol{\mu}([2r(\mathbf{i}(1), j_1)]^{-1})_\# \circ \boldsymbol{\tau}(q(j_1))_\# T)] \llcorner \mathbf{U}^{n+1}(0, 2) \in \mathcal{S}_G$$

for each $j_1 = 1, \ldots, N$ and, in the same manner as above, choose $\mathbf{i}(1, j_1) \in \{1, \ldots, M\}$ and $q(j_1, 1), \ldots, q(j_1, N) \in \text{sing } T(j_1) \cap \mathbf{B}^{n+1}(0, 1)$ such that

$$\text{sing } T(j_1) \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(q(j_1, j_2), 2r(\mathbf{i}(1, j_1), j_2)) : j_2 = 1, \ldots, N\}$$

and

$$\Sigma\{[2r(\mathbf{i}(1, j_1), j_2)]^s: j_2 = 1, \ldots, N\} < 1/2.$$

In case $m = 2$ we set

$$Q(j_1, j_2) = \boldsymbol{\tau}(-q(j_1))\circ\boldsymbol{\mu}(2r(i(1), j_1))q(j_1, j_2),$$
$$R(j_1, j_2) = [2r(\mathbf{i}(1), j_1)][2r(\mathbf{i}(1, j_1), j_2)]$$

for each $(j_1, j_2) \in \{1, \ldots, N\}^2$, observe that, by construction,

$$\text{sing } T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(Q(j_1, j_2), R(j_1, j_2)): (j_1, j_2) \in \{1, \ldots, N\}^2\}$$

and estimate

$$\begin{aligned}
&\Sigma\{R(j_1, j_2)^s: (j_1, j_2) \in \{1, \ldots, N\}^2\}\\
&\quad = \Sigma\{[2r(\mathbf{i}(1), j_1)]^s[2r(\mathbf{i}(1, j_1), j_2)]^s: (j_1, j_2) \in \{1, \ldots, N\}^2\}\\
&\quad = \Sigma\{[2r(\mathbf{i}(1), j_1)]^s\Sigma\{[2r(\mathbf{i}(1, j_1), j_2)]^s: j_2 = 1, \ldots, N\}: j_1 = 1, \ldots, N\}\\
&\quad < \Sigma\{[2r(\mathbf{i}(1), j_1)]^s(1/2): j_1 = 1, \ldots, N\} < (1/2)^2
\end{aligned}$$

which is the required estimate.

In case $m > 2$ we continue in the same manner to choose in accordance with II.2(7), for each $2 < l \leq m$,

$$T(j_1, j_2, \ldots, j_{l-1}) \in S_G,$$
$$\mathbf{i}(1, j_1, j_2, \ldots, j_{l-1}) \in \{1, \ldots, M\},$$
$$q(j_1, j_2, \ldots, j_l) \in \text{sing } T(j_1, \ldots, j_{l-1}) \cap \mathbf{B}^{n+1}(0, 1),$$
$$0 < r(\mathbf{i}(1, j_1, j_2, \ldots, j_{l-1}), j_l) < 1/4$$

corresponding to each $(j_1, \ldots, j_{l-1}) \in \{1, \ldots, N\}^{l-1}$ and $j_l = 1, \ldots, N$ such that

$$T(j_1, \ldots, j_{l-1}) = [\boldsymbol{\mu}([2r(i(1, j_1, \ldots, j_{l-2}), j_{l-1})]^{-1})_\# \circ \boldsymbol{\tau}(q(j_1, \ldots, j_{l-1}))_\# T(j_1, \ldots, j_{l-2})] \llcorner \mathbf{U}^{n+1}(0, 2)$$

for each $(j_1, \ldots, j_{l-1}) \in \{1, \ldots, N\}^{l-1}$,

$$\text{sing } T(j_1, \ldots, j_{l-1}) \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(q(j_1, \ldots, j_l), r(\mathbf{i}(1, j_1, \ldots, j_{l-1}), j_l)): j_l = 1, \ldots, N\}$$

for each $(j_1, \ldots, j_{l-1}) \in \{1, \ldots, N\}^{l-1}$, and

$$\Sigma\{[2r(\mathbf{i}(1, j_1, \ldots, j_{l-1}), j_l)]^s: j_l = 1, \ldots, N\} < 1/2$$

for $(j_1, \ldots, j_{l-1}) \in \{1, \ldots, N\}^{l-1}$.

We finally define for each $(j_1, \ldots, j_m) \in \{1, \ldots, N\}^m$,

$$Q(j_1, \ldots, j_m) = \boldsymbol{\tau}(-q(j_1)) \circ \boldsymbol{\mu}(2r(\mathbf{i}(1), j_1)) \circ \boldsymbol{\tau}(-q(j_1, j_2)) \circ \boldsymbol{\mu}(2r(\mathbf{i}(1, j_1), j_2)) \circ \boldsymbol{\tau}(-q(j_1, j_2, j_3)) \circ \boldsymbol{\mu}(2r(\mathbf{i}(1, j_1, j_2), j_3)) \circ \ldots \circ \boldsymbol{\tau}(-q(j_1, \ldots, j_{m-1})) \circ \boldsymbol{\mu}(2r(\mathbf{i}(1, j_1, \ldots, j_{m-2}), j_{m-1}))\, q(j_1, \ldots, j_m)$$

and

$$R(j_1, \ldots, j_m) = [2r(\mathbf{i}(1), j_1)][2r(\mathbf{i}(1, j_1), j_2)][2r(\mathbf{i}(1, j_1, j_2), j_3)] \ldots [2r(\mathbf{i}(1, j_1 \ldots, j_{m-1}), j_m)].$$

We have by construction

$$\operatorname{sing} T \cap \mathbf{B}^{n+1}(0, 1) \subset \cup \{\mathbf{B}^{n+1}(Q(j_1, \ldots, j_m), R(j_1, \ldots, j_m)) : (j_1, \ldots j_m) \in \{1, \ldots, N\}^m\}$$

and one readily checks

$$\Sigma\{R(j_1, \ldots, j_m)^s : (j_1, \ldots, j_m) \in \{1, \ldots, N\}^m\} < (1/2)^m.$$

II.6

COROLLARY. *For each $t > 0$ there exists a neighborhood $\mathcal{G}_t$ of the n dimensional area integrand M in $\mathcal{J}$ such that* $\sup \mathbf{H} | \mathcal{G}_t < n - 7 + t$.

Proof. [8].

II.7

THEOREM. *Let $F: \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^+$ be a positive elliptic parametric n dimensional integrand in $\mathbf{R}^{n+1}$ such that $F | \mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \sim \{0\})$ is of class 3 and suppose $S \in \mathcal{R}_n(\mathbf{R}^{n+1})$ such that $\mathbf{F}(S) \leqslant \mathbf{F}(S + T)$ for each $T \in \mathbf{R}_n(\mathbf{R}^{n+1})$ with $\partial T = 0$. Then there exists an open set U in $\mathbf{R}^{n+1}$ such that $\mathcal{H}^{n-2}([\operatorname{spt} S \sim \operatorname{spt} \partial S] \sim U) = 0$ and $\operatorname{spt} S \cap U$ is an n dimensional submanifold of $\mathbf{R}^{n+1}$ of class 2.*

Proof. In view of [I, Theorem 3.1] or the maximum principle of [9, p. 151–152] (more generally [15]) and [7, 4.5.17, 5.3.19] it is sufficient to establish the theorem under the assumption that $S \in \mathcal{S}^*$. Also clearly one can assume $0 \in \operatorname{spt} S$ and $\operatorname{dist}(0, \operatorname{spt} \partial S)$ is large and show the asserted estimate on $\operatorname{sing} S$ near 0. We may furthermore assume that whenever $x, y \in \mathbf{R}^{n+1}$ with $|x|$ and $|y|$ large then $F(x, \cdot) = F(y, \cdot)$. A suitable choice of $\mathcal{J}^*$ is thus

$$\mathcal{J}^* = \{\boldsymbol{\tau}(x)^\# F : x \in \mathbf{B}^{n+1}(0, 1)\}.$$

The theorem then follows from a straightforward adaptation of II.3 and II.5.

II.8

THEOREM. *Suppose $t>0$ and $\mathcal{J}^0$ is a collection of positive elliptic parametric constant coefficient n dimensional integrands $F: \mathbf{R}^{n+1} \to \mathbf{R}^+$ in $\mathbf{R}^{n+1}$ such that $F \in \mathcal{J}^0$ implies $F | \mathbf{R}^{n+1} \sim \{0\}$ is of class 3, $\{F | \mathbf{S}^n: F \in \mathcal{J}^0\}$ is compact in the class 3 topology, and the n dimensional area integrand M is contained in $\mathcal{J}^0$. Then there exists $\varepsilon>0$ and corresponding neighborhood $\mathcal{G} = \mathcal{J}^0 \cap \{F:$ distance $(F, M)<\varepsilon\}$ [II.1 (3)] of M in $\mathcal{J}^0$ with the following property. Suppose $G: \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^+$ is a positive elliptic parametric integrand in $\mathbf{R}^{n+1}$ such that $G | \mathbf{R}^{n+1} \times (\mathbf{R}^{n+1} \sim \{0\})$ is of class 3 and $G(x, \cdot) \in \mathcal{G}$ for each $x \in \mathbf{R}^{n+1}$. Suppose also $S \in \mathcal{R}_n(\mathbf{R}^{n+1})$ such that $\mathbf{G}(S+T) \geq \mathbf{G}(S)$ for each $T \in \mathcal{R}_n(\mathbf{R}^{n+1})$ with $\partial T = 0$. Then there exists an open set U in $\mathbf{R}^{n+1}$ such that $\mathcal{H}^{n-7+t}([\text{spt } S \sim \text{spt } \partial S] \sim U) = 0$ and spt $S \cap U$ is an n dimensional submanifold of $\mathbf{R}^{n+1}$ of class 2.*

Proof. The theorem follows from II.6 and a straightforward adaptation of the arguments of II.5 and II.7.

II.9

Remark. The existence of **F** minimal surfaces S as in II.7 and II.8 is, of course, well known [7, 5.1.6]. Additionally theorems II.7 and II.8 extend immediately from $\mathbf{R}^{n+1}$ to $n+1$ dimensional riemannian manifolds of class 4. Theorems II.7 and II.8 also extend immediately from n dimensional currents to n dimensional flat chains modulo 2 [7, 4.2.26, 5.3.21] (see [2, I.1 (6,11)]) in $\mathbf{R}^{n+1}$ or in manifolds as above. Finally partial boundary regularity estimates for **F** minimal surfaces have been obtained in [10] while the existence of lower bounds on the topological complexity of certain **F** minimal 2 dimensional surfaces in $\mathbf{R}^3$ is shown in [3].

References for Part I and Part II

[1]. ALMGREN, F. J., Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math.*, 87 (1968), 321–391.

[2]. —— Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 165 (1976).

[3]. ALMGREN, F. J., Jr. & THURSTON, W. P., Examples of unknotted curves which bound only surfaces of high genus within their convex hulls. *Ann. of Math.*, 105 (1977), 527–538.

[4]. ALLARD, W. K., On the first variation of a varifold. *Ann. of Math.*, 95 (1972), 417–491.

[5]. BOMBIERI, E., DE GIORGI, E., MIRANDA, M., Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. *Arch. Rational Mech. Anal.*, 32 (1969), 255–267.

[6]. DE GIORGI, E., *Frontiere orientate di misura minima*. Seminario di Mat. della Scuola Normale Superiore, Pisa (1960–61).

[7]. FEDERER, H., *Geometric Measure Theory*. Springer-Verlag New York, 1969.

[8]. —— The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. *Bull. Amer. Math. Soc.*, 76 (1970), 767–771.

[9]. HOPF, E., Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. *Berlin, Sber. Preuss, Akad. Wiss.*, 19 (1927), 147–152.

[10]. HARDT, R. M., On boundary regularity for integral currents and flat chains modulo 2 minimizing the integral of an elliptic integrand. Preprint.

[11]. LADYZHENSKAYA, O. A. & URAL'TSEVA, N. N., Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 23 (1970), 677–703.

[12]. MORREY, C. B., JR., *Multiple Integrals in the Calculus of Variations*. Springer-Verlag, New York, 1966.

[13]. MIRANDA, M., Sulle singolarità della frontiere minimali. *Rend. Sem. Mat. Padova*, (1967), 181–188.

[14]. SERRIN, J., The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A*, 264A (1969), 413–496.

[15]. —— On the strong maximum principle for quasilinear second order differential inequalities. *J. Funct. Anal.*, 5 (1970), 184–193.

[16]. SIMON, L., Interior gradient bounds for non-uniformly elliptic equations. *Indiana Univ. Math. J.*, 25 (1976), 821–855.

[17]. —— Remarks on curvature estimates. *Duke Math. J.*, 43 (1976), 545–553.

[18]. SCHOEN, R. & SIMON, L., A new proof of the regularity theorem for currents minimizing parametric elliptic functionals. To appear.

Received September 1, 1976.