

Regularity and the Gorenstein property of *L*-convex Polyominoes

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Abstract

We study the coordinate ring of an *L*-convex polyomino, determine its regularity in terms of the maximal number of rooks that can be placed in the polyomino. We also characterize the Gorenstein *L*-convex polyominoes and those which are Gorenstein on the punctured spectrum, and compute the Cohen–Macaulay type of any *L*-convex polyomino in terms of the maximal rectangles covering it. Though the main results are of algebraic nature, all proofs are combinatorial.

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1 Introduction

Commonly, a polyomino is a shape in the Cartesian plane $\mathbb{N} \times \mathbb{N}$ consisting of unit squares which are joined edge-to-edge. Classical examples of polyominoes are Ferrer diagrams, the stack and parallelogram polyominoes. They have a long history in combinatorics. Originally, polyominoes appeared in mathematical recreations, but it turned out that they have applications in various fields, for example, theoretical physics and bio-informatics. Among the most popular topics in combinatorics related to polyominoes one finds enumerating polyominoes of given size, including the asymptotic growth of the numbers of polyominoes, tiling problems, and reconstruction of polyominoes. A very nice introduction to the combinatorics of polyominoes and tilings is given in the monograph [23].

In the last decade, polyominoes have been related to the study of binomial ideals generated by collections of 2-minors of a generic matrix. Ideals of this kind were first introduced and studied by Qureshi [19]. For a polyomino \mathcal{P} , in [19], the ideal $I_{\mathcal{P}}$ generated by the inner minors of \mathcal{P} was considered. For a field K , the K -algebra $K[\mathcal{P}]$ whose relations are given by $I_{\mathcal{P}}$ is called the coordinate ring of \mathcal{P} . Several authors studied the algebraic properties and invariants of $K[\mathcal{P}]$, relating them to the shape of \mathcal{P} .

The study of the ideal of t -minors and related ideals of an $m \times n$ -matrix $X = (x_{ij})$ of indeterminates is a classical subject of commutative algebra and algebraic geometry; see for example the lecture notes [2] and its references to original articles. Several years after the appearance of these lecture notes, a new aspect of the theory was introduced by considering Gröbner bases of determinantal ideals. These studies were initiated by the articles [18], [3] and [22]. More generally and motivated by geometric applications, ideals of t -minors of 2-sided ladders have been studied, see [7], [5], [6] and [11]. For the case of 2-minors, these classes of ideals may be considered as special cases of the ideal $I_{\mathcal{P}}$ of inner 2-minors of a polyomino \mathcal{P} .

One of the most challenging problems in the algebraic theory of polyominoes is the classification of the polyominoes \mathcal{P} whose coordinate ring $K[\mathcal{P}]$ is a domain. It has been shown in [13] and [20] that this is the case if the polyomino is simply connected. In a more recent paper by Mascia, Rinaldo and Romeo [16], it is shown that if $K[\mathcal{P}]$ is a domain then \mathcal{P} should not have a zig-zag walk, and they conjecture that this is also a sufficient condition for $K[\mathcal{P}]$ to be a domain. They verify this conjecture computationally for polyominoes of rank ≤ 14 . It is expected that $K[\mathcal{P}]$ is always reduced.

Additional structural results on $K[\mathcal{P}]$ for special classes of polyominoes were already shown in Qureshi's article [19]. There she proved that $K[\mathcal{P}]$ is a Cohen–Macaulay normal domain when \mathcal{P} is a convex polyomino, and characterized all stack polyominoes for which $K[\mathcal{P}]$ is Gorenstein by computing the class group of this algebra.

In the present paper we focus on so-called L -convex polyominoes. This is a particularly nice class of convex polyominoes which is distinguished by the property

that any two cells of the polyomino can be connected by a path of cells with at most one change of directions. The combinatorics of this class of polyominoes is described in the paper [9] and [4] by Castiglione et al. In Section 1 we recall some of the remarkable properties of L -convex polyominoes referring to the above mentioned papers. In particular, if \mathcal{P} is an L -convex polyomino, then there is a natural bipartite graph $F_{\mathcal{P}}$ whose edges correspond to the cells of \mathcal{P} . By using this correspondence, we show in Proposition 2.3 that there exists a polyomino \mathcal{P}^* which is a Ferrer diagram and such that the bipartite graphs $F_{\mathcal{P}}$ and $F_{\mathcal{P}^*}$ are isomorphic. We call \mathcal{P}^* the Ferrer diagram projected by \mathcal{P} . Similarly there exists a bipartite graph $G_{\mathcal{P}}$ whose edges correspond to the coordinates of the vertices of \mathcal{P} . By using the intimate relationship between $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ it can be shown that $G_{\mathcal{P}}$ and $G_{\mathcal{P}^*}$ are isomorphic as well, see Corollary 2.7. The crucial observation which then follows from these considerations is the result (Theorem 3.1) that $K[\mathcal{P}]$ and $K[\mathcal{P}^*]$ are isomorphic as standard graded K -algebras. Therefore all algebraic invariants and properties of $K[\mathcal{P}]$ are shared by $K[\mathcal{P}^*]$. For many arguments this allows us to assume that \mathcal{P} itself is a Ferrer diagram. Since the coordinate ring of a Ferrer diagram can be identified with the edge ring of a Ferrer graph, results of Corso and Nagel [8] can be used to compute the Castelnuovo-Mumford regularity of $K[\mathcal{P}]$, denoted by $\text{reg}(K[\mathcal{P}])$. It turns out that $\text{reg}(K[\mathcal{P}])$ has a very nice combinatorial interpretation. Namely, for an L -convex polyomino, $\text{reg}(K[\mathcal{P}])$ is equal to maximal number of non-attacking rooks that can be placed on \mathcal{P} , as shown in Theorem 3.3. This is the main result of Section 2.

In Section 3 we study the Gorenstein property of L -convex polyominoes. We first observe that if we remove the rectangle of maximal width from \mathcal{P} , then the result is again an L -convex polyomino. Repeating this process we obtain a finite sequence of L -convex polyominoes, which we call the derived sequence of \mathcal{P} . In Theorem 4.3 we then show that $K[\mathcal{P}]$ is Gorenstein if and only if the bounding boxes of the derived sequence of L -convex polyominoes of \mathcal{P} are all squares. For the proof we use again that $K[\mathcal{P}] \cong K[\mathcal{P}^*]$, and the characterization of Gorenstein stack polyominoes given by Qureshi in [19]. In addition, under the assumption $K[\mathcal{P}]$ is not Gorenstein, we show in Theorem 4.3 that $K[\mathcal{P}]$ is Gorenstein on the punctured spectrum if and only if \mathcal{P} is a rectangle, but not a square. Here we use that the coordinate ring of a Ferrer diagram may be viewed as a Hibi ring. Then we can apply a recent result of Herzog et al [14] which characterizes the Hibi rings which are Gorenstein on the punctured spectrum.

Finally, in Section 4 we compute the Cohen–Macaulay type of $K[\mathcal{P}]$ for an L -convex polyomino \mathcal{P} . Again we use the fact that $K[\mathcal{P}^*]$ may be viewed as a Hibi ring (of a suitable poset Q). The number of generators of the canonical module of $K[\mathcal{P}^*]$, which by definition is the Cohen–Macaulay type, is described by Miyazaki [17] (based on results of Stanley [21] and Hibi [15]). It is the number of minimal strictly order reversing maps on Q . Then somewhat technical counting arguments provide us in Theorem 5.2 with the desired formula.

2 Some combinatorics of L -convex polyominoes

2.1 Polyominoes

In this subsection we recall definitions and notation about polyominoes. If $a = (i, j), b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$, the set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an *interval* of \mathbb{N}^2 . If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$. In particular, a and b are called the *diagonal corners* whereas c and d are called the *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the lower left (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a + (1, 0), a + (0, 1), a + (1, 1)$ and its edges $E(C)$ are

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}.$$

Let \mathcal{P} be a finite collection of cells of \mathbb{N}^2 , and let C and D be two cells of \mathcal{P} . Then C and D are said to be *connected*, if there is a sequence of cells $C = C_1, \dots, C_m = D$ from \mathcal{P} such that C_i and C_{i+1} have a common edge for all $i = 1, \dots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then C_1, \dots, C_m is called a *path* (connecting C and D). A collection of cells \mathcal{P} is called a *polyomino* if any two cells of \mathcal{P} are connected. We denote by $V(\mathcal{P}) = \cup_{C \in \mathcal{P}} V(C)$ the vertex set of \mathcal{P} . A polyomino \mathcal{P}' whose cells belong to \mathcal{P} is called a *subpolyomino* of \mathcal{P} .

A polyomino \mathcal{P} is called *row convex* if for any two of its cells with lower left corners $a = (i, j)$ and $b = (k, j)$, with $k > i$, all cells with lower left corners (l, j) with $i \leq l \leq k$ are cells of \mathcal{P} . Similarly, \mathcal{P} is called *column convex* if for any two of its cells with lower left corners $a = (i, j)$ and $b = (i, k)$, with $k > j$, all cells with lower left corners (i, l) with $j \leq l \leq k$ are cells of \mathcal{P} . If a polyomino \mathcal{P} is simultaneously row and column convex then \mathcal{P} is called *convex*.

Each proper interval $[(i, j), (k, l)]$ in \mathbb{N}^2 can be identified as a polyomino and it is referred to as *rectangular polyomino*, or simply as *rectangle*. A rectangular subpolyomino \mathcal{P}' of \mathcal{P} is called *maximal* if there is no rectangular subpolyomino \mathcal{P}'' of \mathcal{P} that properly contains \mathcal{P}' . A rectangle has size $m \times n$ if it contains m columns and n rows of cells. Given a polyomino \mathcal{P} , the rectangle that contains \mathcal{P} and has the smallest size with this property is called *bounding box* of \mathcal{P} . After a shift of coordinates, we may assume that the bounding box is $[(0, 0), (m, n)]$ for some $m, n \in \mathbb{N}$. In this case, the width of \mathcal{P} , denoted by $w(\mathcal{P})$ is m . Similarly, the height of \mathcal{P} , denoted by $h(\mathcal{P})$ is n .

Moreover, an interval $[a, b]$ with $a = (i, j)$ and $b = (k, \ell)$ is called a *horizontal edge interval* of \mathcal{P} if $j = \ell$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \dots, k - 1$ are edges of cells of \mathcal{P} . If a horizontal edge interval of \mathcal{P} is not strictly contained in any other horizontal edge interval of \mathcal{P} , then we call it a *maximal horizontal edge interval*. Similarly one defines vertical edge intervals and maximal vertical edge intervals of \mathcal{P} .

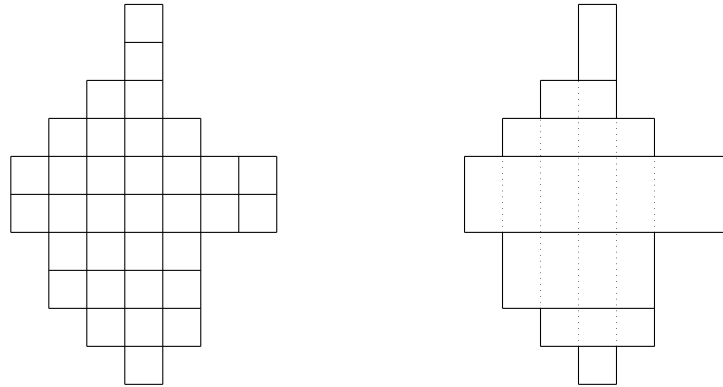
2.2 L -convex polyominoes

Let $\mathcal{C} : C_1, C_2, \dots, C_m$ be a path of cells and (i_k, j_k) be the lower left corner of C_k for $1 \leq k \leq m$. Then \mathcal{C} has a change of direction at C_k for some $2 \leq k \leq m - 1$ if $i_{k-1} \neq i_{k+1}$ and $j_{k-1} \neq j_{k+1}$.

A convex polyomino \mathcal{P} is called k -convex if any two cells in \mathcal{P} can be connected by a path of cells in \mathcal{P} with at most k change of directions. The 1-convex polyominoes are simply called L -convex polyomino. A maximal rectangle \mathcal{R} of size $m \times n$ is said to have unique occurrence in a polyomino \mathcal{P} , if \mathcal{R} is the only *rectangular* subpolyomino of \mathcal{P} with size $m \times n$. The next lemma gives information about the maximal rectangles of an L -convex polyomino.

Lemma 2.1. [9, Section 4, Corollary 1] *A maximal rectangle of an L -convex polyomino \mathcal{P} has a unique occurrence in \mathcal{P} .*

The maximal rectangles of the polyomino in Figure 1 are of sizes 7×2 , 4×5 , 3×6 , 2×7 and 1×10 .



(A) An L -convex polyomino \mathcal{P} . (B) The maximal rectangles of \mathcal{P} .

Figure 1: The maximal rectangles of \mathcal{P}

As a consequence of Lemma 2.1 we have that, given an L -convex polyomino \mathcal{P} , there is a unique maximal rectangle R_w such that $w(\mathcal{P}) = w(R_w)$ and a unique maximal rectangle R_h such that $h(\mathcal{P}) = h(R_h)$.

2.3 The bipartite graphs associated to polyominoes

Let \mathcal{P} be a convex polyomino with bounding box $[(0, 0), (m, n)]$. In \mathcal{P} there are n rows of cells, numbered increasingly from the top to the bottom, and m columns of cells, numbered increasingly from the left to the right. We attach a bipartite graph $F_{\mathcal{P}}$ to the polyomino \mathcal{P} in the following way. Let $V(F_{\mathcal{P}}) = \{X_1, \dots, X_m\} \sqcup \{Y_1, \dots, Y_n\}$ and $\{Y_i, X_j\} \in E(F_{\mathcal{P}})$ if the i -th row of \mathcal{P} intersects the j -th column of \mathcal{P} non-trivially. The unique cell in the intersection of i -th row and j -th column is labelled as C_{ij} . For all $1 \leq i \leq n$, we define the i -th horizontal projection of \mathcal{P} as

the number of cells in the i -th row, and denote it by h_i . Similarly, for all $1 \leq j \leq m$, we define the j -th vertical projection of \mathcal{P} as the number of cells in the j -th column and denote it by v_j . The degree of a vertex v in a graph G , denoted by $\deg v$, is the number of vertices adjacent to v in G . Note that $h_i = \deg Y_i$ and $v_j = \deg X_j$ in the graph $F_{\mathcal{P}}$. In the sequel, we will refer to the vector $H_{\mathcal{P}} = (h_1, h_2, \dots, h_n)$ as the horizontal projections of \mathcal{P} and $V_{\mathcal{P}} = (v_1, v_2, \dots, v_m)$ as the vertical projection of \mathcal{P} . For an L -convex polyomino one has

Theorem 2.2 ([1, Lemma 1,2,3]). *Let \mathcal{P} be an L -convex polyomino, then:*

- (a) \mathcal{P} is uniquely determined by $H_{\mathcal{P}}$ and $V_{\mathcal{P}}$;
- (b) $H_{\mathcal{P}}$ and $V_{\mathcal{P}}$ are unimodal vectors;
- (c) Let j, j' be two different columns of \mathcal{P} such that $v_j \leq v_{j'}$. Then for each row i of \mathcal{P} , we have $C_{ij'} \in \mathcal{P}$ if $C_{ij} \in \mathcal{P}$.
- (d) Let i, i' be two different rows of \mathcal{P} such that $h_i \leq h_{i'}$. Then for each column j of \mathcal{P} , we have $C_{i'j} \in \mathcal{P}$ if $C_{ij} \in \mathcal{P}$.

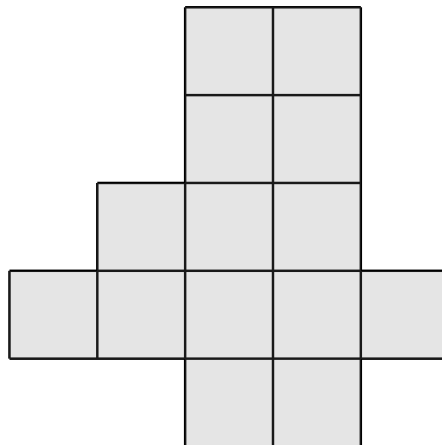


Figure 2: An L -convex polyomino with $H_{\mathcal{P}} = (2, 2, 3, 5, 2)$ and $V_{\mathcal{P}} = (1, 2, 5, 5, 1)$.

A *Ferrer graph* G is a bipartite graph with $V(G) = \{u_1, \dots, u_m\} \sqcup \{v_1, \dots, v_n\}$ such that $\{u_1, v_n\}, \{u_m, v_1\} \in E(G)$ and if $\{u_i, v_j\} \in E(G)$ then $\{u_r, v_s\} \in E(G)$ for all $1 \leq r \leq i$ and for all $1 \leq s \leq j$. Let G be a Ferrer graph and \mathcal{P} be a polyomino such that $H_{\mathcal{P}} = (\deg v_1, \dots, \deg v_n)$, $V_{\mathcal{P}} = (\deg u_1, \dots, \deg u_m)$ and $F_{\mathcal{P}} = G$. Then \mathcal{P} is called a *Ferrer diagram*. Note that a Ferrer diagram is a special type of stack polyomino (after a counterclockwise rotation by 90 degrees). Note that if $[(0, 0), (m, n)]$ is the bounding box of a Ferrer diagram \mathcal{P} , then $(0, 0), (m, n) \in V(\mathcal{P})$.

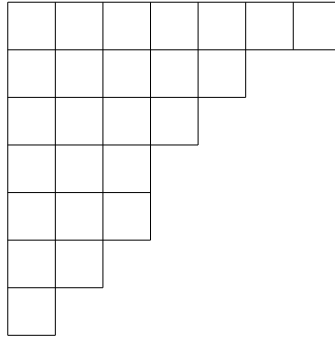


Figure 3: Ferrer diagram

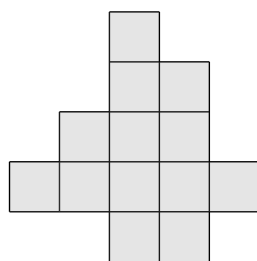
Proposition 2.3. *Let \mathcal{P} be an L -convex polyomino. Then there exists a Ferrer diagram \mathcal{P}^* such that $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$.*

Proof. Let $F_{\mathcal{P}}$ be the bipartite graph associated to \mathcal{P} , with vertex set $V(F_{\mathcal{P}}) = \{X_1, \dots, X_m\} \sqcup \{Y_1, \dots, Y_n\}$. We first prove that after a suitable relabelling of vertices of $F_{\mathcal{P}}$, it can be viewed as a Ferrer graph. Let T_1, T_2, \dots, T_m and U_1, U_2, \dots, U_n be the relabelling of the vertices of $F_{\mathcal{P}}$ such that $\deg T_1 \geq \deg T_2 \geq \dots \geq \deg T_m$ and $\deg U_1 \geq \deg U_2 \geq \dots \geq \deg U_n$. We set $v_i^* = \deg T_i$ for $1 \leq i \leq m$ and $h_j^* = \deg U_j$ for $1 \leq j \leq n$.

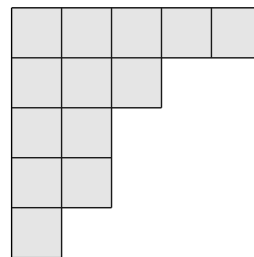
Then $v_1^* = n$ and $h_1^* = m$ which implies that $\{T_1, U_n\}, \{T_m, U_1\} \in E(F_{\mathcal{P}})$. Furthermore, let $\{T_k, U_l\} \in E(F_{\mathcal{P}})$ for some $1 \leq k \leq m$ and $1 \leq l \leq n$. Then for all $1 \leq r \leq k$ and $1 \leq s \leq l$, we have $v_k^* \leq v_r^*$ and $h_l^* \leq h_s^*$. Therefore, by Theorem 2.2.(c), we see that $\{T_r, U_s\} \in E(F_{\mathcal{P}})$ for all $1 \leq r \leq k$ and $1 \leq s \leq l$.

Hence $F_{\mathcal{P}}$ is a Ferrer graph up to relabelling. Let \mathcal{P}^* be the unique polyomino with horizontal and vertical projections $H_{\mathcal{P}^*} = (h_1^*, h_2^*, \dots, h_n^*)$ and $V_{\mathcal{P}^*} = (v_1^*, v_2^*, \dots, v_m^*)$, then \mathcal{P}^* is a Ferrer diagram and $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$. \square

From the proof of the above proposition, one sees that given an L -convex polyomino \mathcal{P} , the Ferrer diagram \mathcal{P}^* such that $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$ is uniquely determined. We refer to \mathcal{P}^* as the Ferrer diagram projected by \mathcal{P} .



(A) L -convex polyomino \mathcal{P}



(B) The Ferrer diagram \mathcal{P}^* projected by \mathcal{P}

Let $r(\mathcal{P}, k)$ be the number of ways of arranging k non-attacking rooks in cells of \mathcal{P} . Recall that, for a graph G with n vertices, a k -matching of G is the set of

k pairwise disjoint edges in G . Let $p(G, k)$ be the number of k matchings of G . It is a fact, for example see [10, page 56], that $r(\mathcal{P}, k) = p(F_{\mathcal{P}}, k)$. Let $r(\mathcal{P})$ denote the maximum number of rooks that can be arranged in \mathcal{P} in non-attacking position, that is $r(\mathcal{P}) = \max_k r(\mathcal{P}, k)$. We have the following

Lemma 2.4. *Let \mathcal{P} be an L-convex polyomino and \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} . Then $r(\mathcal{P}, k) = r(\mathcal{P}^*, k)$. In particular, $r(\mathcal{P}) = r(\mathcal{P}^*)$.*

Proof. From Proposition 2.3, we have $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$ then $p(F_{\mathcal{P}}, k) = p(F_{\mathcal{P}^*}, k)$. Then by using the theorem on [10, page 56], we see that $r(\mathcal{P}, k) = r(\mathcal{P}^*, k)$. \square



Figure 5: Placement of rooks in non-attacking position in \mathcal{P} and \mathcal{P}^* .

As described in [19, Section 2], we can associate another bipartite graph $G_{\mathcal{P}}$ to \mathcal{P} in the following way. Let $\mathcal{I} = [(0, 0), (m, n)]$ be the bounding box of \mathcal{P} . Since \mathcal{P} is convex, in \mathcal{P} there are $m + 1$ maximal vertical edge intervals and $n + 1$ maximal horizontal edge intervals, namely there are $m + 1$ columns and $n + 1$ rows of vertices. We number the rows in an increasing order from left to right and we number the columns in an increasing order from top to bottom. Let H_0, \dots, H_n denote the rows and V_0, \dots, V_m denote the columns of vertices of \mathcal{P} . Set $V(G_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Then $\{x_i, y_j\} \in E(G_{\mathcal{P}})$ if and only if $V_i \cap H_j \neq \emptyset$. To distinguish between $G_{\mathcal{P}}$ and $F_{\mathcal{P}}$, we refer to them as follows:

- The graph $F_{\mathcal{P}}$ is the graph associated to the cells of \mathcal{P} .
- The graph $G_{\mathcal{P}}$ is the graph associated to the vertices of \mathcal{P} .

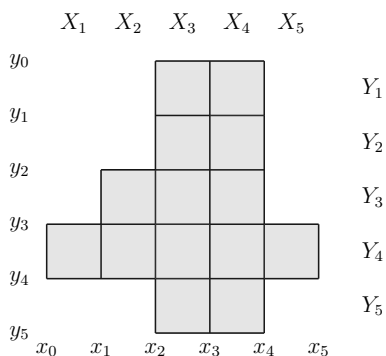
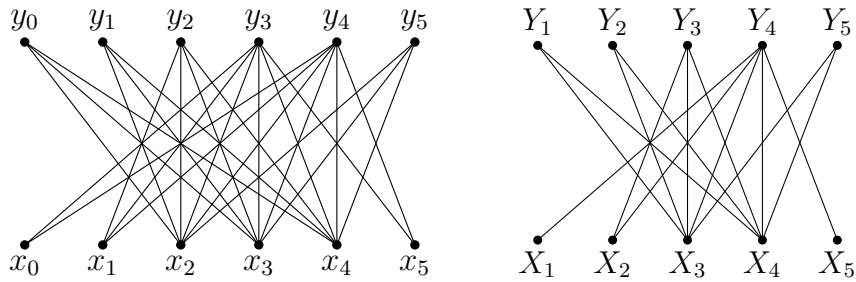


Figure 6: The two labellings on \mathcal{P} of Figure 2



(A) The bipartite graph $G_{\mathcal{P}}$ of the polyomino \mathcal{P} in Figure 2.

(B) The bipartite graph $F_{\mathcal{P}}$ of the polyomino \mathcal{P} in Figure 2.

Figure 7

The relation between $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ is deducible from the following

Observation 2.5. Let \mathcal{P} be an L -convex polyomino and $[(0, 0), (m, n)]$ its bounding box. Then one can interpret $H_{\mathcal{P}} = (h_1, h_2, \dots, h_n)$ and $V_{\mathcal{P}} = (v_1, v_2, \dots, v_m)$ in terms of degrees of vertices of $G_{\mathcal{P}}$ in the following way:

(i) From Theorem 2.2, we know that $V_{\mathcal{P}}$ and $H_{\mathcal{P}}$ are unimodal. Let

$$v_1 \leq v_2 \leq \dots < v_i = n \geq v_{i+1} \geq \dots \geq v_m$$

for some $1 \leq i \leq m$. Then $v_j = \deg x_{j-1} - 1$ for all $1 \leq j < i$ and $v_j = \deg x_j - 1$ for $i \leq j \leq m$. Similarly, by using unimodality of $H_{\mathcal{P}}$, we get

$$h_1 \leq h_2 \leq \dots < h_i = m \geq h_{i+1} \geq \dots \geq h_n$$

for some $1 \leq i \leq m$. Then $h_j = \deg y_{j-1} - 1$ for all $1 \leq j < i$ and $h_j = \deg y_j - 1$ for $i \leq j \leq n$.

(ii) As a consequence of (i), if \mathcal{P} is a Ferrer diagram then

$$v_1 \geq v_2 \geq \dots \geq v_m,$$

$$h_1 \geq h_2 \geq \dots \geq h_n.$$

Let $G_{\mathcal{P}}$ and $F_{\mathcal{P}}$ be the graphs associated to \mathcal{P} as described above with $V(G_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$ and $V(F_{\mathcal{P}}) = \{X_1, \dots, X_m\} \sqcup \{Y_1, \dots, Y_n\}$. Then $v_j = \deg X_j = \deg x_j - 1$ for all $1 \leq j \leq m$, and $h_j = \deg Y_j = \deg y_j - 1$ for all $1 \leq j \leq n$.

Now we obtain

Lemma 2.6. *Let \mathcal{P} be an L -convex polyomino and $G_{\mathcal{P}}$ be the graph associated to the vertices of \mathcal{P} with $V(G_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Then we have the following:*

(a) *if $\deg x_i < \deg x_{i'}$, then $\{x_{i'}, y_j\} \in E(G_{\mathcal{P}})$ whenever $\{x_i, y_j\} \in E(G_{\mathcal{P}})$.*

(b) *if $\deg y_j < \deg y_{j'}$, then $\{x_i, y_{j'}\} \in E(G_{\mathcal{P}})$ whenever $\{x_i, y_j\} \in E(G_{\mathcal{P}})$.*

Proof. Let $H_{\mathcal{P}} = (h_1, h_2, \dots, h_n)$ and $V_{\mathcal{P}} = (v_1, v_2, \dots, v_m)$ be the horizontal and vertical projection of \mathcal{P} .

(a): Let $p = \deg x_i < \deg x_{i'} = q$. Then following Observation 2.5, $h_s = p - 1$ and $h_t = q - 1$ for some $1 \leq s \neq t \leq n$. Then $h_s < h_t$ and the conclusion follows from Theorem 2.2(d).

(b): Let $p = \deg y_i < \deg y_{i'} = q$. Then following Observation 2.5, $v_s = p - 1$ and $v_t = q - 1$ for some $1 \leq s \neq t \leq m$. Then $v_s < v_t$ and the the conclusion follows from Theorem 2.2(c). \square

A result similar to Proposition 2.3 holds also for the graph $G_{\mathcal{P}}$.

Corollary 2.7. *Let \mathcal{P} be an L -convex polyomino, let \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} . Then $G_{\mathcal{P}} \cong G_{\mathcal{P}^*}$.*

Proof. First, we will show that $G_{\mathcal{P}} \cong H$ where H is a Ferrer graph. Let $V(G_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Similar to the proof of Proposition 2.3, we relabel the vertices of $G_{\mathcal{P}}$ as $V(G_{\mathcal{P}}) = \{t_0, \dots, t_m\} \sqcup \{s_0, \dots, s_n\}$ such that $\deg t_0 \geq \deg t_1 \geq \dots \geq \deg t_m$ and $\deg s_0 \geq \deg s_1 \geq \dots \geq \deg s_n$. Let H be the new graph obtained by relabelling of the vertices of $G_{\mathcal{P}}$. Then by using Lemma 2.6, we conclude that H is a Ferrer graph.

Now we will show that $H \cong G_{\mathcal{P}^*}$. This is an immediate consequence of Observation 2.5(ii). Indeed $V_{\mathcal{P}^*} = (\deg t_1 - 1, \dots, \deg t_m - 1)$ and $H_{\mathcal{P}^*} = (\deg s_1 - 1, \dots, \deg s_n - 1)$. \square

3 Regularity of L -convex polyominoes

Let K be a field. We denote by S the polynomial ring over K with variables x_v , where $v \in V(\mathcal{P})$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of \mathcal{P} if $[a, b]$ is a rectangular subpolyomino of \mathcal{P} . Here c, d are the anti-diagonal corners of $[a, b]$. The ideal $I_{\mathcal{P}} \subset S$, generated by all of the inner 2-minors of \mathcal{P} , is called the *polyomino ideal* of \mathcal{P} , and $K[\mathcal{P}] = S/I_{\mathcal{P}}$ is called the *coordinate ring* of \mathcal{P} .

Let G be a graph with vertex set $[n] = \{1, \dots, n\}$ and $T = K[x_1, \dots, x_n]$ be a polynomial ring over K . Then the toric ring $K[G] \subset T$ is generated by those monomials $x_i x_j$ for which $\{i, j\} \in E(G)$.

Theorem 3.1. *Let \mathcal{P} be an L -convex polyomino and let \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} . Then $K[\mathcal{P}]$ and $K[\mathcal{P}^*]$ are isomorphic standard graded K -algebras.*

Proof. Since \mathcal{P} is convex, it is known that $K[\mathcal{P}]$ is isomorphic to the edge ring $K[G_{\mathcal{P}}]$ of the bipartite graph $G_{\mathcal{P}}$ (see [19, Section 2]). By Corollary 2.7, $G_{\mathcal{P}}$ is isomorphic to $G_{\mathcal{P}^*}$. Hence the assertion follows. \square

Theorem 3.2. *Let \mathcal{P} be an L -convex polyomino and let \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} . Moreover, let $H_{\mathcal{P}^*} = (h_1, \dots, h_n)$. Then*

$$\text{reg}(K[\mathcal{P}]) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}.$$

Proof. By Theorem 3.1, we have $K[\mathcal{P}] \cong K[\mathcal{P}^*]$. Therefore, it is enough to show that

$$\text{reg}(K[\mathcal{P}^*]) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}.$$

Recall $G_{\mathcal{P}^*}$ is the bipartite graph associated to the vertices of \mathcal{P}^* . We may assume that $V(G_{\mathcal{P}^*}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Then $\deg y_0 = m + 1 \geq 2$ and $\deg x_0 = n + 1$. Hence, [8, Proposition 5.7] gives

$$\begin{aligned} \text{reg}(K[G_{\mathcal{P}^*}]) &= \min\{n, \deg y_j + (j + 1) - 3 \mid 1 \leq j \leq n\} \\ &= \min\{n, \deg y_j + j - 2 \mid 1 \leq j \leq n\} \end{aligned}$$

We want to rewrite the formula above in terms of the horizontal projection of \mathcal{P}^* . According to Remark 2.5.(2), for any $1 \leq j \leq n$ we have $h_j = \deg y_j - 1$. Hence

$$\{\deg y_j + j - 2 \mid 1 \leq j \leq n\} = \{h_j + j - 1 \mid 1 \leq j \leq n\},$$

and the assertion follows. \square

Let \mathcal{P} be a Ferrer diagram with horizontal projections (h_1, \dots, h_n) . Then, by using a combinatorial argument, it is easy to see that for any $r \leq n$ the number of ways of placing r rooks in non-attacking position in \mathcal{P} is given by

$$\prod_{i=1}^r (h_{r-i+1} - (i - 1)). \tag{1}$$

By using this fact we obtain

Theorem 3.3. *Let \mathcal{P} be an L -convex polyomino. Then*

$$\text{reg}(K[\mathcal{P}]) = r(\mathcal{P}).$$

Proof. From Lemma 2.4 we know that $r(\mathcal{P}) = r(\mathcal{P}^*)$ where \mathcal{P}^* is the Ferrer diagram projected by \mathcal{P} . By Theorem 3.1, it is enough to show that

$$r(\mathcal{P}^*) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}, \tag{2}$$

where (h_1, \dots, h_n) are the horizontal projections of \mathcal{P}^* . It follows from Equation (1) that $r(\mathcal{P}^*)$ is the greatest integer $r \leq n$ such that each factor of $\prod_{i=1}^r (h_{r-i+1} - (i - 1))$ is positive. Hence, for any $i \in \{1, \dots, r\}$ we must have $h_{r-i+1} - (i - 1) > 0$. Fix an integer $i \in \{1, \dots, r\}$. Then we see that

$$h_{r-i+1} - (i - 1) > 0 \Leftrightarrow h_{r-i+1} - i + 1 + r - r > 0 \Leftrightarrow r < h_{r-i+1} + (r - i) + 1.$$

Hence we conclude that $r \leq h_{r-i+1} + (r - i)$. Therefore,

$$\begin{aligned} r(\mathcal{P}^*) &= \max\{r \mid r \leq n \text{ and } r \leq \min\{h_{r-i+1} + (r - i) \mid 1 \leq i \leq r\}\} \\ &= \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\} \end{aligned}$$

as requested. \square

We observe that, by exchanging the role of rows and columns in \mathcal{P}^* , we obtain

$$r(\mathcal{P}^*) = \min\{m, v_j + j - 1 \mid 1 \leq j \leq m\}$$

which is similar to (2).

4 On the Gorenstein property of L -convex polyominoes

Let \mathcal{P} be a L -convex polyomino with width m . Assume that the unique maximal rectangle of \mathcal{P} with width m , has height d . Then for some positive integer s ,

$$H_{\mathcal{P}} = (h_1, \dots, h_s, m, \dots, m, h_{s+d+1}, \dots, h_n)$$

with $h_s, h_{s+d+1} < m$. Let \mathcal{P}_1 be the collection of cells with $n - d$ rows satisfying the following property: C_{ij} is a cell of \mathcal{P} if and only if C_{ij} is a cell of \mathcal{P}_1 for $1 \leq i \leq s$, and for $s + d + 1 \leq i \leq n$, $C_{i-d,j}$ is a cell of \mathcal{P}_1 .

Lemma 4.1. \mathcal{P}_1 is an L -convex polyomino.

Proof. \mathcal{P}_1 could be seen as the polyomino \mathcal{P} from which we remove the maximal rectangle R having width m . Hence, each cell in \mathcal{P}_1 corresponds uniquely to a cell in \mathcal{P} . Let $C, D \in \mathcal{P}_1$. Then we consider the corresponding cells $C', D' \in \mathcal{P}$. We observe that neither C' nor D' is a cell of R . Since \mathcal{P} is L -convex, there exists a path of cells \mathcal{C}' in \mathcal{P} connecting C' and D' with at most one change of direction. If no cell of \mathcal{C}' belongs to R , then \mathcal{C}' determines a path of cells \mathcal{C} of \mathcal{P}_1 with at most one change of direction connecting C and D .

Otherwise, since neither C' nor D' are cells of R , the path \mathcal{C}' crosses R and the induced path $\mathcal{C}' \cap R$ has no change of direction. Therefore, the path \mathcal{C} in \mathcal{P}_1 , obtained by cutting off the induced path $\mathcal{C}' \cap R$ from \mathcal{C}' , is a path of cells with at most one change of direction connecting C and D . \square

If $\mathcal{P}_1 \neq \emptyset$, we may again remove the unique rectangle of maximal width from \mathcal{P}_1 to obtain \mathcal{P}_2 in a similar way. After a finite number of such steps, say t steps, we arrive at \mathcal{P}_t which is a rectangle. Then $\mathcal{P}_{t+1} = \emptyset$. We set $\mathcal{P}_0 = \mathcal{P}$, and call the sequence $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ the *derived sequence* of L -convex polyominoes of \mathcal{P} .

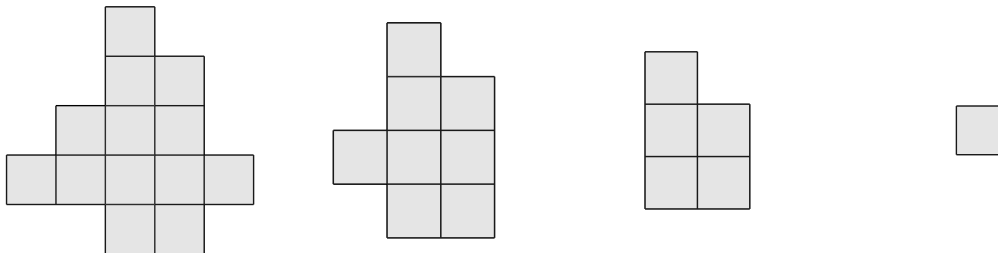


Figure 8: The derived sequence of L -convex polyominoes $\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Lemma 4.2. Let \mathcal{P} be an L -convex polyomino and $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ be the derived sequence of L -convex polyominoes of \mathcal{P} . Let \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} and let $(\mathcal{P}^*)_0, (\mathcal{P}^*)_1, \dots, (\mathcal{P}^*)_{t'}$ be the derived sequence of L -convex polyominoes of \mathcal{P}^* . Then $t' = t$ and for any $0 \leq k \leq t$ the polyomino $(\mathcal{P}^*)_k$ is the Ferrer diagram projected by \mathcal{P}_k . In other words, for all k $(\mathcal{P}^*)_k = (\mathcal{P}_k)^*$.

Proof. For $k = 0$, the assertion is trivial. We show that $(\mathcal{P}^*)_1$ is the Ferrer diagram projected by \mathcal{P}_1 . For this aim, assume that the unique rectangular subpolyomino of \mathcal{P} having width m has height $d \in \mathbb{N}$. Let

$$H_{\mathcal{P}} = (h_1, \dots, h_s, m, \dots, m, h_{s+d+1}, \dots, h_n)$$

with $h_s, h_{s+d+1} < m$ and let

$$V_{\mathcal{P}} = (d, d, \dots, d, v_{r+1}, \dots, v_{r+l}, d, \dots, d)$$

with $v_{r+1}, v_{r+l} > d$.

From Proposition 2.3 it follows that \mathcal{P}^* has a maximal rectangle R^* of width m and height d and

$$H_{\mathcal{P}^*} = (m, \dots, m, h_1^*, \dots, h_{n-d}^*)$$

with $m > h_1^* \geq \dots \geq h_{n-d}^*$ and

$$V_{\mathcal{P}^*} = (v_1^*, \dots, v_l^*, d, \dots, d).$$

with $v_1^* \geq \dots \geq v_l^* > d$. Hence the L -convex polyomino $(\mathcal{P}^*)_1$ is uniquely determined by the projections

$$H_{(\mathcal{P}^*)_1} = (h_1^*, \dots, h_{n-d}^*) \text{ and } V_{(\mathcal{P}^*)_1} = (v_1^* - d, \dots, v_l^* - d).$$

On the other hand, \mathcal{P}_1 is the L -convex polyomino uniquely determined by the projections $H_{\mathcal{P}_1} = (h_1, \dots, h_s, h_{s+d+1}, \dots, h_n)$ and, $V_{\mathcal{P}_1} = (v_{r+1} - d, v_{r+2} - d, \dots, v_{r+l} - d)$. By reordering the two vectors in a decreasing order, we obtain the Ferrer diagram projected by \mathcal{P}_1 which coincides with $(\mathcal{P}^*)_1$. This proves the assertion for $k = 1$. By inductively applying the above argument, the assertion follows for all k . \square

Theorem 4.3. *Let \mathcal{P} be an L -convex polyomino and let $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ be the derived sequence of L -convex polyominoes of \mathcal{P} . Then following conditions are equivalent:*

- (a) \mathcal{P} is Gorenstein.
- (b) For $0 \leq k \leq t$, the bounding box of \mathcal{P}_k is a square.

Proof. By Theorem 3.1, we have $K[\mathcal{P}] \cong K[\mathcal{P}^*]$, where \mathcal{P}^* is the Ferrer diagram projected by \mathcal{P} . Therefore, $K[\mathcal{P}]$ is Gorenstein if and only if $K[\mathcal{P}^*]$ is Gorenstein. Note that \mathcal{P}^* can be viewed as a stack polyomino. Hence it follows from [19, Corollary 4.12] that $K[\mathcal{P}^*]$ is Gorenstein if and only if the bounding box of $(\mathcal{P}^*)_k$ is a square for all $0 \leq k \leq t$. By Lemma 4.2, this is the case if and only if the bounding box of \mathcal{P}_k is a square for all $0 \leq k \leq t$. \square

The following numerical criteria for the Gorensteinness of \mathcal{P} are an immediate consequence of Theorem 4.3.

Corollary 4.4. *Let \mathcal{P} be an L -convex polyomino with vector of horizontal projections $H_{\mathcal{P}} = (h_1, h_2, \dots, h_n)$ of \mathcal{P} and vector of vertical projections $V_{\mathcal{P}} = (v_1, v_2, \dots, v_m)$. We set*

$$\{h_1, \dots, h_n\} = \{g_1 < g_2 < \dots < g_r\} \text{ and } \{v_1, \dots, v_m\} = \{w_1 < w_2 < \dots < w_s\},$$

and let

$$a_i = |\{h_j : h_j = g_i\}| \text{ for } i = 1, \dots, r, \text{ and } b_i = |\{v_j : v_j = w_i\}| \text{ for } i = 1, \dots, s.$$

Then the following conditions are equivalent:

- (a) \mathcal{P} is Gorenstein.
- (b) $g_\ell = \sum_{i=1}^{\ell} a_i$ for $\ell = 1, \dots, r$.
- (c) $w_\ell = \sum_{i=1}^{\ell} b_i$ for $\ell = 1, \dots, s$.

Theorem 4.5. *Let \mathcal{P} be L -convex polyominoes such that $K[\mathcal{P}]$ is not Gorenstein. Then following are equivalent:*

- (a) $K[\mathcal{P}]$ is Gorenstein on the punctured spectrum.
- (b) \mathcal{P} is not a square, and $K[\mathcal{P}]$ has an isolated singularity.
- (c) \mathcal{P} is rectangle, but not a square.

Before we start the proof of the theorem, we note that if \mathcal{P} is a Ferrer diagram, then $K[\mathcal{P}]$ can be viewed as a Hibi ring. Recall for a given finite poset $Q = \{v_1, \dots, v_n\}$ and a field K , the *Hibi ring* over the field K associated to Q , which we denote by $K[Q] \subset K[y, x_1, \dots, x_n]$, is defined as follows. The K -algebra $K[Q]$ is generated by the monomials $yx_I := y \prod_{v_i \in I} x_i$ for every $I \in \mathcal{I}(Q)$, that is

$$K[Q] := K[yx_I | I \in \mathcal{I}(Q)].$$

The algebra $K[Q]$ is standard graded if we set $\deg(yx_I) = 1$ for all $I \in \mathcal{I}(Q)$. Here $\mathcal{I}(Q)$ is the set of poset ideals of Q . The poset ideals of Q are just the subset $I \subset Q$ with the property that if $p \in Q$ and $q \leq p$, then $q \in I$.

Let \mathcal{P} be a Ferrer diagram with maximal horizontal edge intervals $\{H_0, \dots, H_n\}$, numbered increasingly from the bottom to the top, and maximal vertical edge intervals $\{V_0, \dots, V_m\}$, numbered increasingly from the left to the right. We let Q be the poset on the set $\{H_1, \dots, H_n, V_1, \dots, V_m\}$ consisting of two chains $H_1 < \dots < H_n$ and $V_1 < \dots < V_m$ and the covering relations $H_i < V_j$, if H_i intersects V_j in a way such that there is no $0 \leq i' < i$ for which $H_{i'}$ intersects V_j , and j is the smallest integer with this property.

Lemma 4.6. *The standard graded K -algebras $K[Q]$ and $K[\mathcal{P}]$ are isomorphic.*

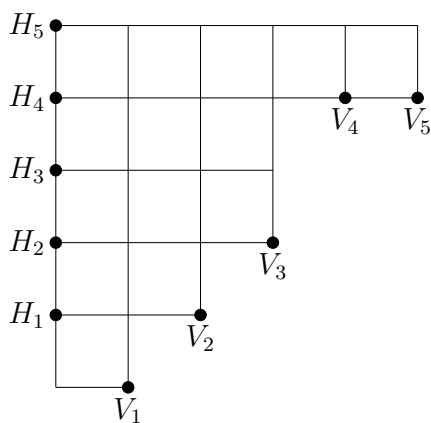
Proof. We may assume that the interval $[(0,0), (m,n)]$ is the bounding box of the Ferrer diagram \mathcal{P} . It follows from the definition of Ferrer diagrams that $(0,0)$ and (m,n) belong to $V(\mathcal{P})$. For any two vertices $a = (i,j)$ and $b = (k,l)$ of \mathcal{P} we define the meet $a \wedge b = (\min\{i,k\}, \min\{j,l\})$ and the join $a \vee b = (\max\{i,k\}, \max\{j,l\})$. With this operations of meet and join, \mathcal{P} is a distributive lattice. An element c of this lattice is called join-irreducible, if $c \neq (0,0)$ and whenever $a \wedge b = c$, then $a = c$ or $b = c$. By Birkhoff's fundamental structure theorem [1], any finite distributive lattice is the ideal lattice of the poset of its join irreducible elements. The join irreducible elements of \mathcal{P} can be described as follows:

1. Every $a_j = (0,j)$ with $1 \leq j \leq n$ is a join irreducible element in \mathcal{P} and $a_1 < a_2 < \dots < a_n$.
2. Let $(i,k) \in V(\mathcal{P})$ with $1 \leq i \leq m$. Then (i,k) is a join irreducible if $(i,k-1) \notin V(\mathcal{P})$. It shows that in each vertical edge interval V_1, \dots, V_m of \mathcal{P} , there is exactly one join irreducible element. We denote by b_i , the unique join irreducible element in V_i with $1 \leq i \leq m$. Then $b_1 < b_2 < \dots < b_m$.

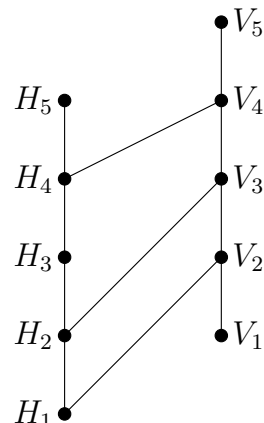
In the poset of join irreducible elements of \mathcal{P} , we have two chains $a_1 < a_2 < \dots < a_n$ and $b_1 < b_2 < \dots < b_m$, and the covering relations $a_j = (0,j) < b_i = (i,k)$ if $j = k$ and b_i is the minimal element with this property. Then, it follows that the poset of join irreducible elements of \mathcal{P} is exactly the poset Q described above. Thus the elements $a \in \mathcal{P}$ are in bijection with the poset ideals of Q . In fact, the poset ideal $I_a \in \mathcal{I}(Q)$ corresponding to a is the set of join irreducible elements $q \in Q$ with $q \leq a$. Thus we have a surjective K -algebra homomorphism

$$\varphi: S = K[x_a: a \in \mathcal{P}] \rightarrow K[Q] = K[yx_{I_a}: I_a \in \mathcal{I}(Q)].$$

As shown by Hibi [15] (see also [12, Theorem 10.1.3]), $\text{Ker}(\varphi)$ is generated by the relations $x_a x_b - x_{a \wedge b} x_{a \vee b}$. This shows that $\text{Ker}(\varphi) = I_{\mathcal{P}}$, as desired. \square



(A) Ferrer diagram



(B) Poset of join-irreducible elements

Let Q be a poset. The Hasse diagram of Q , viewed as a graph, decomposes into connected components. The corresponding posets Q_1, \dots, Q_r are called the *connected components* of Q .

Now for the proof of Theorem 4.5 will use the following results

Theorem 4.7. *Let Q be a finite poset and let Q_1, \dots, Q_r be the connected components of Q .*

- (a) ([15, page 105]) $K[Q]$ is Gorenstein if and only if Q is pure (i.e. all maximal chains in Q have the same length).
- (b) ([14, Corollary 3.5]) $K[Q]$ is Gorenstein on the punctured spectrum if and only if each Q_i is pure.

Proof of Theorem 4.5. Since $K[\mathcal{P}] \cong K[\mathcal{P}^*]$ and since \mathcal{P} is a rectangle if and only if \mathcal{P}^* is a rectangle, we may assume that \mathcal{P} is a Ferrer diagram.

Let Q be the poset such that $K[Q] \cong K[\mathcal{P}]$, and assume that $K[Q]$ is Gorenstein on the punctured spectrum. Then each component of Q is pure, by Theorem 4.7(b). Since we assume that $K[Q]$ is not Gorenstein, Theorem 4.7(a) implies that Q is not connected. It follows from the description of Q in terms of its Ferrer diagram \mathcal{P} that \mathcal{P} has no inner corner. In other words, \mathcal{P} is a rectangle. By Theorem 4.3 it cannot be a square. This yields (a) \Rightarrow (b). The implication (c) \Rightarrow (b) follows from [2, Theorem 2.6], and (b) \Rightarrow (a) is trivial. \square

5 The Cohen–Macaulay type of L -convex polyominoes.

In this section, we give a general formula for the Cohen–Macaulay type of the coordinate ring of an L -convex polyomino. To illustrate our result, we first consider the special case of an L -convex polyomino with just two maximal rectangles.

Proposition 5.1. *Let \mathcal{P} be an L -convex polyomino whose maximal rectangles are R_1 having size $m \times s$ and R_2 having size $t \times n$ with $s < n$ and $t < m$. Let $r = \max\{n, m, n + m - (s + t)\}$. Then*

$$\text{type}(K[\mathcal{P}]) = \begin{cases} \sum_{i=m-t}^{m-(n-s)} \binom{i}{s} \binom{m-i-1}{n-s-1} & \text{if } r = m \\ \sum_{i=m-t}^s \binom{i-1}{m-t-1} \binom{n-i}{t} & \text{if } r = n \\ \binom{n-s}{t} \binom{m-t}{s} & \text{if } r = n + m - (s + t) \end{cases}$$

Proof. Let \mathcal{P}^* be the Ferrer diagram projected by \mathcal{P} and let Q be the poset of the join-irreducible elements associated to \mathcal{P}^* . It consists of the two chains $V_1 < \dots < V_m$ and $H_1 < \dots < H_n$, and the cover relation $H_{n-s} < V_{t+1}$. We have $|Q| = m + n$, and $r = \text{rank } Q + 1$. We compute the number of minimal generators of the canonical module $\omega_{K[\mathcal{P}^*]}$. For this purpose, let \widehat{Q} be the poset obtained from Q by adding the

elements $-\infty$ and ∞ with $\infty > p$ and $-\infty < p$ for all $p \in Q$, and let $\mathcal{T}(\widehat{Q})$ be the set of integer valued functions $\nu : \widehat{Q} \rightarrow \mathbb{Z}_{\geq 0}$ with $\nu(\infty) = 0$ and $\nu(p) < \nu(q)$ for all $p > q$. By using a result of Stanley [21], Hibi shows in [15, (3.3)] that the monomials of the form

$$y^{\nu(-\infty)} \prod_{p \in Q} x_p^{\nu(p)}$$

for $\nu \in \mathcal{T}(\widehat{Q})$ form a K -basis for $\omega_{K[\mathcal{P}^*]}$. By using [17, Corollary 2.4], we have that the number of generators of $\omega_{K[\mathcal{P}^*]}$ is the number of minimal maps $\nu \in \mathcal{T}(\widehat{Q})$ with respect to the order given in [17, Page 5]. In fact, $\nu \leq \mu$ for $\nu, \mu \in \mathcal{T}(\widehat{Q})$ if $\mu - \nu$ is decreasing. We observe that the minimal maps ν necessarily assign the numbers $1, \dots, r$ to the vertices of a maximal chain of Q in reversed order, hence we have to find the possible values for the remaining $|Q| - r = m + n - r$ elements, depending on r . We distinguish three cases:

- (a) $r = m$;
- (b) $r = n$;
- (c) $r = (n - s) + (m - t)$.

In the case (a), the maximal chain is $V_1 < \dots < V_m$. Hence we must take $\nu(V_{m-i+1}) = i$ for $i \in \{1, \dots, m\}$. We have to determine how many vectors (a_1, \dots, a_n) with integers entries $0 < a_1 < \dots < a_n$ satisfy $m - t < a_{s+1} < r - (n - s) = m - (n - s) + 2$, where the left inequality follows from the cover relation, while the right inequality follows from the fact that $a_{s+2} < \dots < a_n < m + 1$ are determined. Therefore, for fixed $i = a_{s+1}$, there are $\binom{i-1}{s}$ ways to choose the values a_1, \dots, a_s in the range $\{1, \dots, i - 1\}$. Moreover, there are $\binom{m-i}{n-s-1}$ ways to choose a_{s+2}, \dots, a_n in the range $\{i + 1, \dots, m\}$. Hence we conclude

$$\text{type}(K[\mathcal{P}]) = \sum_{i=m-t+1}^{m-(n-s)+1} \binom{i-1}{s} \binom{m-i}{n-s-1} = \sum_{i=m-t}^{m-(n-s)} \binom{i}{s} \binom{m-i-1}{n-s-1}.$$

In the case (b), we assign to each element of the chain H_1, \dots, H_n a number in $\{1, \dots, n\}$ in strictly decreasing order. We have to determine how many vectors (b_1, \dots, b_m) with integers entries $0 < b_1 < \dots < b_m$ satisfy $m - t - 1 < b_{m-t} < s + 1$, where the left inequality follows from the fact that $0 < b_1 < \dots < b_{m-t-1}$, while the rightmost inequality follows from the cover relation. Therefore, for $i = b_{m-t}$, there are $\binom{i-1}{m-t-1}$ ways to choose the values b_1, \dots, b_{m-t-1} in the range $\{1, \dots, i - 1\}$. Moreover, there are $\binom{n-i}{t}$ ways to choose b_{m-t+1}, \dots, b_m in the range $\{i + 1, \dots, n\}$. Hence we conclude

$$\text{type}(K[\mathcal{P}]) = \sum_{i=m-t}^s \binom{i-1}{m-t-1} \binom{n-i}{t}.$$

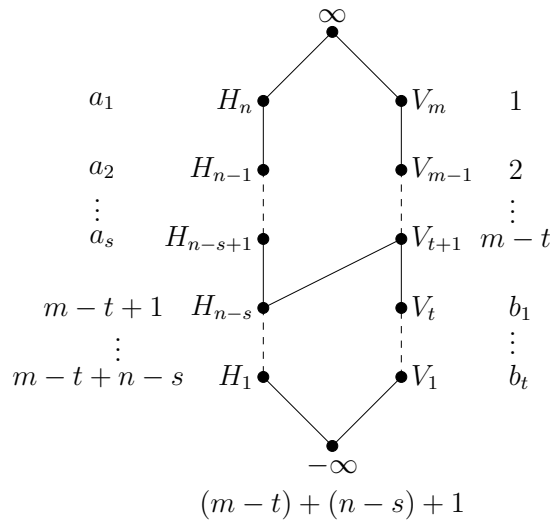


Figure 10: We count the number of minimal maps assigning $1 < \dots < m-t+n-s$ to $V_m > \dots > V_{t+1} > H_{n-s} > \dots > H_1$.

In the case (c), we assign to each element of the chain $H_1, \dots, H_{n-s}, V_{t+1}, \dots, V_m$ a number in $\{1, \dots, (m+n) - (s+t)\}$ in strictly decreasing order. We have to determine how many vectors $(a_1, \dots, a_s, b_1, \dots, b_t)$ with integers entries $0 < a_1 < \dots < a_s$, $m-t < b_1 < \dots < b_t$ satisfy $a_s < m-t+1$ and $b_1 > m-t$ (see Figure 10). Therefore, there are $\binom{m-t}{s}$ ways to choose the values a_1, \dots, a_s in the range $\{1, \dots, m-t\}$ and there are $\binom{n-s}{t}$ ways to choose b_1, \dots, b_t in the range $\{m-t+1, \dots, m-t+n-s\}$. Hence in this case, we conclude

$$\text{type}(K[\mathcal{P}]) = \binom{n-s}{t} \binom{m-t}{s}. \quad \square$$

Now we consider the general case.

Theorem 5.2. *Let \mathcal{P} be an L-convex polyomino whose maximal rectangles are $\{R_i\}_{i=1, \dots, t}$. For $i = 1, \dots, t$, let $c_i \times d_i$ be the size of R_i with $d_1 = n$ and $c_t = m$ and $c_i < c_j$ and $d_i > d_j$ for $i < j$. Let $r = \max\{n, m, \{n+m - (c_i + d_{i+1})\}_{i=1, \dots, t-1}\}$. Then*

$$\text{type}(K[\mathcal{P}]) = \begin{cases} A & \text{if } r = m \\ B & \text{if } r = n \\ A_h B_h & \text{if } r = n + m - (c_h + d_{h+1}) \end{cases}$$

where

$$A = \sum_{i_1, \dots, i_{t-1}} \binom{i_1 - 1}{d_t} \prod_{k=2}^{t-1} \binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1} \binom{m - i_{t-1}}{n - d_2 - 1}$$

with

$$m - c_{t-j} + 1 \leq i_j \leq m - (n - d_{t+1-j}) + 1 \quad \text{for } 1 \leq j \leq t-1,$$

and

$$B = \sum_{i_1, \dots, i_{t-1}} \binom{i_1 - 1}{m - c_{t-1} - 1} \prod_{k=2}^{t-1} \binom{i_k - i_{k-1} - 1}{c_{t-k+1} - c_{t-k} - 1} \binom{m - i_{t-1}}{c_1}$$

with $m - c_{t-1} \leq i_1 \leq d_t$ and

$$i_{j-1} + c_{t-j+1} - c_{t-j} \leq i_j \leq d_{t-j+1} \quad \text{for } 2 \leq j \leq t - 1.$$

Moreover,

$$A_h = \sum_{i_1, \dots, i_{t-h-1}} \binom{i_1 - 1}{d_t} \prod_{k=2}^{t-h-1} \binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1} \binom{m - c_h - i_{t-h-1}}{d_{h+1} - d_{h+2} - 1}$$

with

$$m - c_{t-j} + 1 \leq i_j \leq m - c_h - (d_{h+1} - d_{t-j+1}) + 1 \quad \text{for } 1 \leq j \leq t - h - 1.$$

for $h = 1, \dots, t - 2$ and $A_{t-1} = \binom{m - c_{t-1}}{d_t}$, and

$$B_h = \sum_{i_1, \dots, i_{h-1}} \binom{i_1 - 1}{c_h - c_{h-1} - 1} \prod_{k=2}^{h-1} \binom{i_k - i_{k-1} - 1}{c_{h-k+1} - c_{h-k} - 1} \binom{m - c_h + n - d_{h+1} - i_{h-1}}{c_1}$$

with $m - c_{h-1} \leq i_1 \leq m - c_h + (d_h - d_{h+1})$ and

$$i_{j-1} + (c_{h-j+1} - c_{h-j}) \leq i_j \leq m - c_h + (d_{h-j+1} - d_{h+1}) \quad \text{for } 2 \leq j \leq h - 1,$$

for $h = 2, \dots, t - 1$ and $B_1 = \binom{n - d_2}{c_1}$.

Proof. First observe that in the general case the cover relations are $H_{n-d_{i+1}} < V_{c_i+1}$ for $i = 1, \dots, t - 1$. We just generalize the ideas of Proposition 5.1. We distinguish three cases:

- (a) $r = m$;
- (b) $r = n$;
- (c) $r = (n - d_{h+1}) + (m - c_h)$ for some $k = 1, \dots, t - 1$.

In the case (a), we assign to each element of the chain V_1, \dots, V_m a number in $\{1, \dots, m\}$ in decreasing order. We have to determine how many vectors (a_1, \dots, a_n) with integers entries $0 < a_1 < \dots < a_n < m + 1$ satisfy

$$m - c_{t-k} < a_{d_{t-k+1}+1} < m - (n - d_{t-k+1}) + 2 \quad \text{for } k = 1, \dots, t - 1,$$

where the left inequality follows from the cover relations, while the right inequality follows from the fact that $a_{d_{t-k+1}+2} < a_{d_{t-k+1}+3} < \dots < a_n < m + 1$. Therefore, fixed $i_1 = a_{d_t+1}$ there are $\binom{i_1-1}{d_t}$ ways to choose the values a_1, \dots, a_{d_t} in the range $\{1, \dots, i_1 - 1\}$. Moreover, for $2 \leq k \leq t - 1$ and fixed $i_k = a_{d_{t-k+1}+1}$, there are $\binom{i_k-i_{k-1}-1}{d_{t-k+1}-d_{t-k+2}-1}$ ways to choose the values $a_{d_{t+k+2}+2}, \dots, a_{d_{t+k+1}}$ in the range $\{i_{k-1} + 1, \dots, i_k - 1\}$. Finally, there are $\binom{m-i_{t-1}}{n-d_2-1}$ ways to choose a_{d_2+2}, \dots, a_n in the range $\{i_{t-1} + 1, \dots, m\}$. Hence in this case, we conclude that $\text{type}(K[\mathcal{P}])$ is A . In the case (b), we assign to each element of the chain H_1, \dots, H_n a number in $\{1, \dots, n\}$ in decreasing order. We have to determine how many vectors (b_1, \dots, b_m) with integers entries $0 < b_1 < \dots < b_m$ satisfy

$$m - c_{t-1} - 1 < b_{m-c_{t-1}} < d_t + 1$$

and

$$b_{m-c_{t-k+1}} + (c_{t-k+1} - c_{t-k}) - 1 < b_{m-c_{t-k}} < d_{t-k+1} + 1 \text{ for } k = 2, \dots, t - 1,$$

where the left inequalities follow from the fact that $b_{m-c_{t-k+1}+1} < \dots < b_{m-c_{t-k}-1}$, while the right inequalities follow from the cover relations. Therefore for fixed $i_1 = b_{m-c_{t-1}}$ there are $\binom{i_1-1}{m-c_{t-1}-1}$ ways to choose the values $b_1, \dots, b_{m-c_{t-1}-1}$ in the range $\{1, \dots, i_1 - 1\}$. Moreover, for $2 \leq k \leq t - 1$ and fixed $i_k = b_{m-c_{t-k}}$, there are $\binom{i_k-i_{k-1}-1}{c_{t-k+1}-c_{t-k}-1}$ ways to choose the values $b_{m-c_{t-k+1}+1}, \dots, b_{m-c_{t-k}-1}$ in the range $\{i_{k-1} + 1, \dots, i_k - 1\}$. Finally, there are $\binom{n-i_{t-1}}{c_1}$ ways to choose b_{m-c_1+1}, \dots, b_m in the range $\{i_{t-1} + 1, \dots, n\}$. Hence in this case, we conclude that $\text{type}(K[\mathcal{P}])$ is B . In the case (c), fix $h \in \{1, \dots, t - 1\}$. We assign to each element of the chain $H_1, \dots, H_{n-d_{h+1}}, V_{c_h+1}, \dots, V_m$ a number in $\{1, \dots, (m + n) - (c_h + d_{h+1})\}$ in decreasing order. Let $\tilde{m} = m - c_h, \tilde{n} = n - d_{h+1}$. We have to determine how many vectors $(a_1, \dots, a_{d_{h+1}}, b_1, \dots, b_{c_h})$ with integers entries $0 < a_1 < \dots < a_{d_{h+1}}, \tilde{m} < b_1 < \dots < b_{c_h}$ satisfy

$$m - c_{t-k} < a_{d_{t-k+1}+1} < \tilde{m} - (d_{h+1} - d_{t-k+1}) + 2 \text{ for } k = 1, \dots, t - h - 1$$

$$m - c_{h-1} - 1 < b_{c_h-c_{h-1}} < \tilde{m} + (d_h - d_{h+1}) + 1,$$

$$b_{c_h-c_{h-k+1}} + (c_{h-k+1} - c_{h-k}) - 1 < b_{c_h-c_{h-k}} < \tilde{m} + (d_{h-k+1} - d_{h+1}) + 1 \text{ for } k = 2, \dots, h - 1.$$

For fixed $i_1 = a_{d_t+1}$ there are $\binom{i_1-1}{d_t}$ ways to choose the values a_1, \dots, a_{d_t} in the range $\{1, \dots, i_1 - 1\}$. Moreover, for $2 \leq k \leq t - h - 1$ and fixed $i_k = a_{d_{t-k+1}+1}$, there are $\binom{i_k-i_{k-1}-1}{d_{t-k+1}-d_{t-k+2}-1}$ ways to choose the values $a_{d_{t+k+2}+2}, \dots, a_{d_{t+k+1}}$ in the range $\{i_{k-1} + 1, \dots, i_k - 1\}$. Furthermore, there are $\binom{\tilde{m}-i_{t-h-1}}{d_{h+1}-d_{h+2}-1}$ ways to choose $a_{d_{h+2}+2}, \dots, a_{d_{h+1}}$ in the range $\{i_{t-h-1} + 1, \dots, \tilde{m}\}$.

For fixed $j_1 = b_{c_h-c_{h-1}}$ there are $\binom{j_1-1}{c_h-c_{h-1}-1}$ ways to choose $b_1, \dots, b_{c_h-c_{h-1}-1}$ in the range $\{1, \dots, j_1 - 1\}$. Moreover, for $2 \leq k \leq h$ and fixed $j_k = b_{c_h-c_{h-k}}$

there are $\binom{j_k - j_{k-1} - 1}{c_{h-k+1} - c_{h-k} - 1}$ ways to choose $b_{c_h - c_{h-k+1} + 1}, \dots, b_{c_h - c_{h-k} - 1}$ in the range $\{j_{k-1} + 1, \dots, j_k - 1\}$. Finally, there are $\binom{\tilde{m} + \tilde{n} - j_{h-1}}{c_1}$ ways to choose $b_{c_h - c_1 + 1}, \dots, b_{c_h}$ in the range $\{j_{h-1} + 1, \dots, \tilde{m} + \tilde{n}\}$. Hence in this case, we conclude that $\text{type}(K[\mathcal{P}])$ is $A_h \cdot B_h$. Observe that the formula for the a_i makes sense only if $1 \leq h \leq t - 2$. For $h = t - 1$, we have to choose the numbers

$$a_1, \dots, a_{d_t}$$

among the values $\{1, \dots, m - c_{t-1}\}$, hence $A_{t-1} = \binom{m - c_{t-1}}{d_t}$. Furthermore observe that the formula for the b_i makes sense only if $2 \leq h \leq t - 1$. For $h = 1$, we have to choose the numbers

$$b_1, \dots, b_{c_1}$$

among the values $\{m - c_1 + 1, \dots, (m - c_1) + (n - d_2)\}$, hence $B_1 = \binom{n - d_2}{c_1}$. \square

We observe that Theorem 4.3 can also be deduced from Theorem 5.2. The following example demonstrates Theorem 5.2.

Example 5.3. Let \mathcal{P} be the Ferrer diagram in Figure 11.

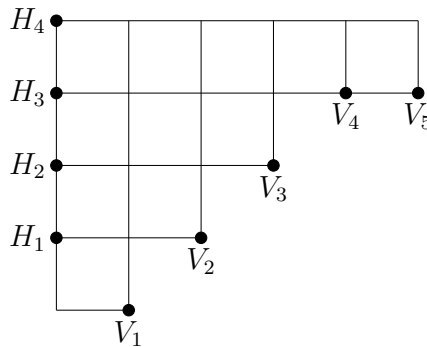


Figure 11

We have $t = 4$ maximal rectangles whose sizes are $\{c_i \times d_i\}_{i=1, \dots, 4}$ with

$$\begin{array}{cccc} c_1 = 1 & c_2 = 2 & c_3 = 3 & c_4 = m = 5 \\ d_1 = n = 4 & d_2 = 3 & d_3 = 2 & d_4 = 1. \end{array}$$

There are 4 maximal chains in the poset Q corresponding to \mathcal{P} containing 5 vertices. For example, the chain V_1, \dots, V_5 and the chain H_1, H_2, V_3, V_4, V_5 , that correspond to the cases $r = m$ and $r = (n - d_3) + (m - c_2)$, hence $h = 2$. We are going to compute A and $A_2 B_2$ as in Theorem 5.2. We have

$$A = \sum_{i_1=3}^3 \sum_{i_2=4}^4 \sum_{i_3=5}^5 \binom{i_1 - 1}{1} \binom{i_2 - i_1 - 1}{2 - 1 - 1} \binom{i_3 - i_2 - 1}{3 - 2 - 1} \binom{5 - i_3}{4 - 3 - 1} = 2,$$

while

$$A_2 = \sum_{i_1=3}^3 \binom{i_1-1}{1} \binom{5-2-i_1}{2-1-1} = 2$$

and

$$B_2 = \sum_{i_1=4}^4 \binom{i_1-1}{2-1-1} \binom{5-i_1}{1} = 1,$$

yielding

$$A_2 B_2 = 2.$$

In conclusion we want to point out that for L -convex polyominoes, important algebraic invariants, like the Castelnuovo-Mumford regularity, the Cohen-Macaulay type, and algebraic properties, like being Gorenstein, are now completely understood and have a nice combinatorial interpretation. It is still a challenge to prove similar results when the polyomino is k -convex for $k > 1$, rather than just L -convex.

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