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REGULARITY AND UNIQUENESS FOR THE STATIONARY LARGE EDDY SIMULATION MODEL

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Abstract. In the note we are concerned with higher regularity and uniqueness of solutions to the stationary problem arising from the large eddy simulation of turbulent flows. The system of equations contains a nonlocal nonlinear term, which prevents straightforward application of a difference quotients method. The existence of weak solutions was shown in A. Świerczewska: Large eddy simulation. Existence of stationary solutions to the dynamical model, ZAMM, Z. Angew. Math. Mech. 85 (2005), 593–604 and P. Gwiazda, A. Świerczewska: Large eddy simulation turbulence model with Young measures, Appl. Math. Lett. 18 (2005), 923–929.

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1. Introduction

The equations considered are a dynamical version of the classical Smagorinsky model

$$
\begin{aligned}
v\cdot\nabla v-\operatorname{div}(c(y)|Dv|Dv)-\nu\Delta v+\nabla q&=f\quad\text{in }\Omega,\\
\operatorname{div}v&=0\quad\text{in }\Omega,
\end{aligned}\tag{1}
$$

where $\Omega=(0,L)^3$, $L>0$, is a cube in $\mathbb{R}^3$, ν is a positive constant, $Dv=\frac{1}{2}(\nabla v+\nabla^T v)$, c is a continuous function of $y=(\tilde v,\widetilde{vv},\widetilde{Dv},\widetilde{|Dv|Dv})$ and by $\tilde{}$ we mean a convolution, which will be specified later. Given the external force f we are looking for the velocity $v\colon\Omega\longrightarrow\mathbb{R}^3$ and the pressure $q\colon\Omega\longrightarrow\mathbb{R}$. The above equations arise from large eddy simulation of turbulent flows. The idea of this approach consists in decomposing the velocity into a part containing large flow structures and a part consisting of small scales. These scales are separated by averaging the velocity, the so-called *filtering*, namely convoluting it with an appropriate function—*filter*. The equations for filtered terms are derived from the Navier-Stokes equations. By adding an additional constitutive relation, which models the contribution of small scales into the flow, we may obtain the classical Smagorinsky model, i.e. system (1) with $c \equiv c_s$, $c_s > 0$ being a constant. The improvement of the Smagorinsky model consisting in finding the so-called Smagorinsky constant c_s dynamically is the Germano model, cf. [4], [11]. System (1) is a stationary case of a slight generalization of the Germano model. For more details on derivation of the model we refer to [9], [13]. We will equip (1) with periodic boundary conditions ($i = 1, 2, 3$)

$$\begin{aligned} v(x + Le_i) &= v(x), \\ q(x + Le_i) &= q(x), \end{aligned} \tag{2}$$

where $\{e_i\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$.

In Section 2 we introduce the notation, collect the properties of a *turbulent term* $c(y)|Dv|Dv$ and recall the existence result from [14]. Some conjectures concerning higher regularity are also formulated. Section 3 consists of the proof of $W^{2,2}$-regularity of solutions for more regular data and function c than in the existence result. We will prove the following theorem.

Theorem 1.1. *Suppose that $f \in L^2(\Omega)$ and $c \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)$ satisfies conditions* (C1)–(C2) *below. Then every weak solution $v \in V$ to problem* (1), (2) *satisfies*

$$v \in W^{2,2}(\Omega).$$

The fact of higher regularity enables us to show the uniqueness for small data, namely

Theorem 1.2. *Let $f \in L^2(\Omega)$ with L^2-norm sufficiently small. Let the function $c \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)$ satisfy conditions* (C1)–(C2) *below. Then the weak solution v to* (1), (2) *is unique.*

The proof of this theorem is contained in Section 4. All the notation for the function spaces used in the above theorems appears in Section 2.

2. Preliminaries

2.1. Notation

By $\mathbb{S}^3$ we mean the set of 3×3 symmetric matrices. Let us introduce spaces of divergence free periodic functions. By $C^\infty_{\text{per}}(\mathbb{R}^3)$ we denote the set of functions from $C^\infty(\mathbb{R}^3)$, which are periodic in each ith direction with a period $L > 0$, i.e., $u(x+Le_i) = u(x)$, $i = 1, 2, 3$. Further let

$$\mathcal{V} \equiv \left\{ u\colon\ u \in C^\infty_{\text{per}}(\mathbb{R}^3),\ \ \operatorname{div} u = 0,\ \ \int_\Omega u \,\mathrm{d}x = 0 \right\}$$

and let V be the closure of $\mathcal{V}$ with respect to the norm $\|u\|_V = \left(\int_\Omega |\nabla u|^3 \,\mathrm{d}x\right)^{1/3}$. Its dual space will be denoted by V'. For the dual pairing between V and V' the notation $\langle \cdot, \cdot \rangle$ will be used. All L^p- and $W^{1,p}$- functions are meant to be periodic in each ith direction with period L and with vanishing mean on Ω. We will often use $b(u, v, w)$ to denote the trilinear form

$$b(u, v, w) := \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x.$$

Note that b is well defined, continuous on $V \times V \times V$ and $b(u, v, v) = 0$, $b(u, v, w) = -b(u, w, v)$.

2.2. Filtering and properties of the turbulent term

We choose as filter a non-negative $C^\infty_{\text{per}}(\mathbb{R}^3)$-function φ with a period $L > 0$ such that $\int_\Omega \varphi \,\mathrm{d}x = 1$, where $\Omega = (0, L)^3$. Filtering of v, denoted by $\tilde{v}$, is now equivalent to the standard convolution (over the whole $\mathbb{R}^3$). The filtered values will be defined for all $x \in \mathbb{R}^3$ by

$$\tilde{v}(x) = \int_\Omega v(y)\varphi_\delta(x - y) \,\mathrm{d}y, \quad \varphi_\delta(y) = \frac{1}{\delta^3}\varphi\Big(\frac{y}{\delta}\Big), \quad y \in \mathbb{R}^3$$

where δ is a positive, constant filter width. We recall the facts concerning convolutions which we will use later (see also [8], [2], [1]).

(i) Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. If $1 \leqslant p, q \leqslant \infty$ and $1/r = 1/p + 1/q - 1$, $1 \leqslant r \leqslant \infty$ then $f * g$ exists for a.a. $x \in \mathbb{R}^n$, $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q}.$$

(ii) $\nabla^\alpha \tilde{v}(x) = \int_\Omega \nabla^\alpha \varphi(x-y) v(y) \,\mathrm{d}y$, where $\nabla^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$ with multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

By the turbulent term we mean the operator $c(y)|Dv|Dv$ with the notation for nonlocal (filtered) variables $y = (\tilde{v}, \widetilde{vv}, \widetilde{Dv}, \widetilde{|Dv|Dv})$. The properties of the operator c are the following:

(C1) $c\colon \mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathbb{R}$ is a continuous function with respect to y;

(C2) c satisfies the condition

$$0 < \alpha \leqslant c(y) \leqslant \beta < \infty \tag{3}$$

for all $y \in (\mathbb{R}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)$.

For later use we assemble also the properties of the operator $\eta \mapsto |\eta|\eta$ for $\eta \in \mathbb{S}^3$. There exists a scalar function $U \in C^2(\mathbb{S}^3)$, $U(\eta) = \frac{1}{3}|\eta|^3$ such that for all $\eta, \xi \in \mathbb{S}^3$, $i, j = 1, 2, 3$

$$\frac{\partial U(\eta)}{\partial \eta_{ij}} = |\eta|\eta_{ij} \tag{4}$$

and

$$\frac{\partial^2 U(\eta)}{\partial \eta_{mn} \partial \eta_{rs}}\, \xi_{mn}\xi_{rs} \geqslant |\eta||\xi|^2. \tag{5}$$

Moreover, $|\eta|\eta$ is *strongly monotone*, i.e. there exists a positive constant K_1 such that

$$(|\eta|\eta_{ij} - |\xi|\xi_{ij}) \cdot (\eta_{ij} - \xi_{ij}) \geqslant K_1|\eta - \xi|^3 \tag{6}$$

for all $\eta, \xi \in \mathbb{S}^3$.

2.3. Existence of weak solutions

We start with recalling the definition of weak solutions.

Definition 2.1. A function $v \in V$ is a weak solution to problem (1), (2) if the equation

$$\int_\Omega (v \cdot \nabla v \cdot \varphi + c(y)|Dv|Dv \cdot D\varphi + \nu \nabla v \cdot \nabla \varphi)\,\mathrm{d}x = \langle f, \varphi \rangle \tag{7}$$

is satisfied for all $\varphi \in V$.

Theorem 2.1 (Existence). *Let $f \in V'$ and let c satisfy conditions* (C1)–(C2). *Then there exists a weak solution to* (1), (2).

2.4. Do the solutions have a chance to be more regular?

The equation contains a strongly nonlinear term; thus before applying the difference quotients technique, which will be relatively technical here, we prove an *a priori* estimate for $v \in W^{2,2}(\Omega)$. This allows to inquire whether such regularity can be expected. Therefore let us assume that v is smooth enough, such that all derivatives have classical sense, more precisely $v \in C^3(\bar{\Omega})$.

A priori estimate. In (7) we insert as a test function $-\Delta v$ and obtain

$$-\int_\Omega c(y)|Dv|Dv \cdot D(\Delta v)\,\mathrm{d}x + \nu(\Delta v, \Delta v) - b(v, v, \Delta v) + (f, \Delta v) = 0. \tag{8}$$

We start with the first integral

$$\begin{aligned}-\int_\Omega c(y)|Dv|Dv \cdot D(\Delta v)\,\mathrm{d}x &= \int_\Omega [\nabla_x c(y)]|Dv|Dv \cdot \nabla(Dv)\,\mathrm{d}x \\ &\quad + \int_\Omega c(y)\frac{\partial^2 U(Dv)}{\partial(Dv)^2} \cdot \nabla(Dv) \cdot \nabla(Dv)\,\mathrm{d}x.\end{aligned}$$

Since $c \in W^{1,\infty}$, all the derivatives

$$\frac{\partial c}{\partial \tilde{v}},\ \frac{\partial c}{\partial(\widetilde{vv})},\ \frac{\partial c}{\partial(D\tilde{v})},\ \frac{\partial c}{\partial(\widetilde{|Dv|Dv})}$$

are bounded in the L^∞-norm. Thus recalling that $Dv \in L^3(\Omega)$ and using the properties of convolutions we conclude for

$$\nabla_x c = \left(\frac{\partial c}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial x_i} + \frac{\partial c}{\partial(\widetilde{vv})}\frac{\partial(\widetilde{vv})}{\partial x_i} + \frac{\partial c}{\partial(D\tilde{v})}\frac{\partial(D\tilde{v})}{\partial x_i} + \frac{\partial c}{\partial(\widetilde{|Dv|Dv})}\frac{\partial(\widetilde{|Dv|Dv})}{\partial x_i}\right)_{i=1}^3$$

the existence of a positive constant m such that

$$\|\nabla_x c\|_{L^\infty(\Omega)} \leqslant m. \tag{9}$$

Next, using (5) we obtain

$$\begin{aligned}\int_\Omega c(y)\frac{\partial^2 U(Dv)}{\partial(Dv)^2}\nabla(Dv) \cdot \nabla(Dv)\,\mathrm{d}x &\geqslant \int_\Omega c(y)|Dv||\nabla(Dv)|^2\,\mathrm{d}x \\ &\geqslant \alpha\int_\Omega |Dv||\nabla(Dv)|^2\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_\Omega \nabla_x c(y)|Dv|Dv\cdot\nabla(Dv)\,\mathrm{d}x\right| \\
&\qquad\leqslant \|\nabla_x c\|_{L^\infty(\Omega)}\int_\Omega |Dv|^{3/2}(|Dv|^{1/2}|\nabla(Dv)|)\,\mathrm{d}x \\
&\qquad\overset{\text{Young}}{\leqslant} m\Big(\frac{m}{4\alpha}\int_\Omega |Dv|^3\,\mathrm{d}x+\frac{\alpha}{m}\int_\Omega |Dv||\nabla(Dv)|^2\,\mathrm{d}x\Big) \\
&\qquad\leqslant k\|\nabla v\|^3_{L^3(\Omega)}+\alpha\int_\Omega |Dv||\nabla(Dv)|^2\,\mathrm{d}x.
\end{aligned}$$

Now we estimate all the other terms:

$$\left|\int_\Omega v\cdot\nabla v\cdot\Delta v\,\mathrm{d}x\right|\leqslant\int_\Omega|\nabla v|^3\,\mathrm{d}x+\left|\int_\Omega v\cdot\nabla^2 v\cdot\nabla v\,\mathrm{d}x\right|=\int_\Omega|\nabla v|^3\,\mathrm{d}x.$$

Moreover, in the space of periodic functions we have

$$(\Delta v,\Delta v)=\|\nabla^2 v\|^2_{L^2(\Omega)}.$$

Now we estimate the term containing f and get

$$|(f,\Delta v)|\leqslant\|f\|_{L^2(\Omega)}\|\nabla^2 v\|_{L^2(\Omega)}\overset{\text{Young}}{\leqslant}\frac{1}{2\nu}\|f\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|\nabla^2 v\|^2_{L^2(\Omega)}.$$

All the above information yields the *a priori* estimate

$$\nu\|\nabla^2 v\|^2_{L^2(\Omega)}\leqslant 2(k+1)\|\nabla v\|^3_{L^3(\Omega)}+\frac{1}{\nu}\|f\|^2_{L^2(\Omega)}. \tag{10}$$

Hence v has a uniform estimate in $W^{2,2}(\Omega)$ given bounds for $\|f\|_{L^2(\Omega)}$ and $\|\nabla v\|_{L^3(\Omega)}$. The *a priori* estimate for the latter was provided in [14]:

$$\|v\|^3_V+\nu\|\nabla v\|^2_{L^2}\leqslant k\|f\|^{3/2}_{V'}. \tag{11}$$

Galerkin approximation. It is worth noticing that the second energy estimate (10) is another method for showing the existence of solutions. We can show that for the sequence of Galerkin approximations (v^n) also estimate (10) holds and hence v^n is bounded in $W^{2,2}(\Omega)$. Next we conclude that for a subsequence, $\nabla v^n\to\nabla v$ strongly in $L^p(\Omega)$ and a.e. in Ω. Once we have obtained the a.e. convergence of the gradients we can also conclude

$$c(y^n)|Dv^n|Dv^n\longrightarrow c(y)|Dv|Dv\quad\text{a.e. in }\Omega.$$

We complete the proof by showing uniform integrability of the turbulent term and applying Vitali's Theorem, cf. [12] for the case of non-Newtonian fluid.

For showing higher regularity we use the method of difference quotients. We cannot repeat the proof of higher regularity for a class of non-Newtonian fluids in [10]. The term produced by the gradient of c will demand our special attention. First let us collect general facts concerning this technique, for details see [5], [3], [6].

We denote

$$d_k^h v(x) := \frac{v(x+he_k) - v(x)}{h}, \quad k = 1, \ldots, n,$$

where e_k denotes the kth unit vector and

$$d^h v := (d_1^h v, \ldots, d_n^h v).$$

We consider the case of periodic boundary conditions and all the functions are meant to be periodic. Then, if $v(x)$ is defined in Ω, so is $v(x+he_k)$, and therefore also $d_k^h v$. The following assertions hold:

(i) If $v \in W^{1,p}(\Omega)$ then $d_k^h v \in W^{1,p}(\Omega)$ and $d_k^h \nabla v = \nabla d_k^h v$. The difference quotient also commutes with the symmetric part of the gradient, i.e., $d_k^h Dv = D d_k^h v$, since

$$\begin{aligned} d_k^h D_{ij} v &= \frac{1}{2} d_k^h \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \Big) = \frac{1}{2} \Big(d_k^h \frac{\partial v_i}{\partial x_j} + d_k^h \frac{\partial v_j}{\partial x_i} \Big) \\ &= \frac{1}{2} \Big(\frac{\partial}{\partial x_j} d_k^h v_i + \frac{\partial}{\partial x_i} d_k^h v_j \Big) = D_{ij}(d_k^h v). \end{aligned}$$

(ii) If either u or v have compact support, then

$$\int_\Omega u d_k^h v \, \mathrm{d}x = - \int_\Omega v d_k^{-h} u \, \mathrm{d}x.$$

(iii) $d_k^h(uv)(x) = u(x+he_k) d_k^h v + v(x) d_k^h u$.

Proposition 3.1.

(i) *Let $\Omega = (0,L)^3$ and $1 \leqslant p \leqslant \infty$. Then*

$$\|d^h v\|_{L^p(\Omega)} \leqslant \|\nabla v\|_{L^p(\Omega)} \tag{12}$$

for all $v \in W^{1,p}(\Omega)$ and $h \in \mathbb{R}$.

(ii) *If $v \in L^p(\Omega)$, $1 < p < \infty$ and if there exists a constant k independent of h such that*

$$\|d^h v\|_{L^p(\Omega)} \leqslant k, \tag{13}$$

then $v \in W^{1,p}_{\rm per}(\Omega)$ and $\|\nabla v\|_{L^p(\Omega)} \leqslant k$.

P r o o f of Theorem 1.1. By virtue of (7) it can be shown that for all $\varphi \in V$ the equation for the difference quotients holds, namely for $k = 1, \dots, n$

$$\begin{aligned}
&\int_\Omega \Big(d_k^h v_j(x) \frac{\partial v_i}{\partial x_j}(x) + v_j(x + h e_k) \frac{\partial d_k^h v_i(x)}{\partial x_j} \Big) \varphi_i(x) \, \mathrm{d}x \\
&\qquad + \int_\Omega (d_k^h c(y(x)) |Dv(x)| D_{ij} v(x) + c(y(x + h e_k)) d_k^h (|Dv(x)| D_{ij} v(x))) D_{ij} \varphi(x) \, \mathrm{d}x \\
&\qquad + \nu \int_\Omega d_k^h \Big(\frac{\partial v_i(x)}{\partial x_j} \Big) \frac{\partial \varphi_i(x)}{\partial x_j} \, \mathrm{d}x \\
&\quad = \int_\Omega d_k^h f_i(x) \varphi_i(x) \, \mathrm{d}x.
\end{aligned}$$

Choosing as a test function $\varphi = d_k^h v \in V$ and summing over k one obtains

$$\begin{aligned}
&\int_\Omega \Big(d_k^h v_j(x) \frac{\partial v_i}{\partial x_j}(x) + v_j(x + h e_k) \frac{\partial d_k^h v_i(x)}{\partial x_j} \Big) d_k^h v_i(x) \, \mathrm{d}x \\
&+ \int_\Omega d_k^h c(y(x)) |Dv(x)| D_{ij} v(x) D_{ij} (d_k^h v(x)) \, \mathrm{d}x \\
&+ \int_\Omega c(y(x + h e_k)) d_k^h (|Dv(x)| D_{ij} v(x)) D_{ij} (d_k^h v(x)) \, \mathrm{d}x + \nu \int_\Omega |d_k^h \nabla v(x)|^2 \, \mathrm{d}x \\
&\qquad\qquad = \int_\Omega d_k^h f_i(x) d_k^h v_i(x) \, \mathrm{d}x.
\end{aligned}$$

It is easy to observe that $\int_\Omega v_j (\partial d_k^h v_i / \partial x_j) d_k^h v_i \, \mathrm{d}x = 0$ and the first term on the right-hand side can be estimated with help of Hölder's inequality and condition (12) as

$$\tag{14} \begin{aligned}
\Big| \int_\Omega d_k^h v_j(x) \frac{\partial v_i}{\partial x_j}(x) d_k^h v_i(x) \, \mathrm{d}x \Big| &\leqslant \|d_k^h v\|_{L^3(\Omega)} \|\nabla v\|_{L^3(\Omega)} \|d_k^h v\|_{L^3(\Omega)} \| \\
&\leqslant \|\nabla v\|_{L^3(\Omega)}^3.
\end{aligned}$$

Next we concentrate on the turbulent term. The first term is estimated using Young's inequality. The choice of a constant K appearing in the following estimates will be specified later,

$$\tag{15} \begin{aligned}
&\Big| \int_\Omega d_k^h c(y(x)) |Dv(x)| D_{ij} v(x) D_{ij} (d_k^h v) \, \mathrm{d}x \Big| \\
&\qquad \leqslant \|d_k^h c(y)\|_{L^\infty(\Omega)} \int_\Omega |Dv(x)|^2 |D_{ij} (d_k^h v(x))| \, \mathrm{d}x \\
&\qquad \leqslant \|\nabla_x c\|_{L^\infty(\Omega)} \Big(\frac{1}{4K} \int_\Omega |Dv(x)|^3 \, \mathrm{d}x + K \int_\Omega |Dv(x)| |D(d_k^h v)|^2 \, \mathrm{d}x \Big) \\
&\qquad \leqslant \|\nabla_x c\|_{L^\infty(\Omega)} \Big(\frac{1}{4K} \|\nabla v\|_{L^3(\Omega)}^3 + K \int_\Omega |Dv(x)| |D(d_k^h v)|^2 \, \mathrm{d}x \Big).
\end{aligned}$$

Note that $\|\nabla_x c\|_{L^\infty(\Omega)} < \infty$, cf. (9). We will use the term

$$J := \int_\Omega c(y(x+he_k)) d_k^h(|Dv(x)|D_{ij}v(x)) D_{ij}(d_k^h v)\,\mathrm{d}x$$

to cancel the term $\int_\Omega |Dv(x)||D_{ij}(d_k^h v)|^2\,\mathrm{d}x$ from the right-hand side However, it is not as straightforward as it was in the formal *a priori* estimate. The shifts produce some different terms, therefore, an additional estimate using strong monotonicity of the operator $|Dv|Dv$ has to be used to obtain the desired inequality. Notice that due to (4) we have

$$\begin{aligned}
\text{(16)} \qquad d_k^h(|Dv(x)|D_{ij}v(x)|) &= \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial U(Dv(x)+s(Dv(x+he_k)-Dv(x)))}{\partial D_{ij}v}\,\mathrm{d}s \\
&= \int_0^1 \frac{\partial^2 U(Dv(x)+s(Dv(x+he_k)-Dv(x)))}{\partial(D_{ij}v)\partial(D_{lm}v)}\,\mathrm{d}s \\
&\quad \times \frac{D_{lm}(x+he_k)-D_{lm}v(x)}{h}.
\end{aligned}$$

From (5) and (16) one obtains

$$\begin{aligned}
J &\geqslant \alpha \int_\Omega \int_0^1 |Dv(x)+s(Dv(x+he_k)-Dv(x))|\,\mathrm{d}s\,|D(d_k^h v)|^2\,\mathrm{d}x \\
&\geqslant \alpha \int_\Omega \left|\int_0^1 Dv(x)+s(Dv(x+he_k)-Dv(x))\,\mathrm{d}s\right| |D(d_k^h v)|^2\,\mathrm{d}x \\
&= \frac{1}{2}\alpha \int_\Omega |Dv(x)+Dv(x+he_k)||D(d_k^h v)|^2\,\mathrm{d}x.
\end{aligned}$$

On the other hand, the strong monotonicity (6) implies that

$$\begin{aligned}
J &\geqslant \alpha \int_\Omega d_k^h(|Dv(x)|D_{ij}v(x)|) D_{ij}(d_k^h v)\,\mathrm{d}x \\
&\geqslant \alpha K_1 \int_\Omega \frac{1}{h^2}|Dv(x+he_k)-Dv(x)|^3\,\mathrm{d}x \\
&= \alpha K_1 \int_\Omega |Dv(x+he_k)-Dv(x)||d_k^h Dv|^2\,\mathrm{d}x.
\end{aligned}$$

Thus the above estimates for J yield two inequalities

$$\text{(17)} \qquad J \geqslant \frac{\alpha}{2}\int_\Omega |Dv(x)+Dv(x+he_k)||D(d_k^h v)|^2\,\mathrm{d}x$$

and

$$\text{(18)} \qquad J \geqslant \alpha K_1 \int_\Omega |Dv(x+he_k)-Dv(x)||d_k^h Dv|^2\,\mathrm{d}x.$$

After summing (17) and (18) we obtain a further estimate

$$\begin{aligned}
&\frac{2K_1+1}{\alpha K_1}J \\
&\quad \geqslant \int_\Omega (|Dv(x)+Dv(x+he_k)|+|Dv(x)-Dv(x+he_k)|)\cdot|D(d_k^h v)|^2\,\mathrm{d}x \\
&\quad \geqslant \int_\Omega |Dv(x)+Dv(x+he_k)+Dv(x)-Dv(x+he_k)||D(d_k^h v)|^2\,\mathrm{d}x \\
&\quad = 2\int_\Omega |Dv(x)||D(d_k^h v)|^2\,\mathrm{d}x
\end{aligned}$$

which finally yields

$$J \geqslant \frac{2\alpha K_1}{2K_1+1}\int_\Omega |Dv(x)||D(d_k^h v(x))|^2\,\mathrm{d}x. \tag{19}$$

Now the constant K in inequality (15) can be determined, namely

$$K = \frac{2\alpha K_1}{(2K_1+1)\|\nabla_x c\|_{L^\infty(\Omega)}}. \tag{20}$$

Next we concentrate on the term $\int_\Omega d_k^h f_i d_k^h v_i\,\mathrm{d}x$. Since

$$\|d_k^h f\|_{H^{-1}(\Omega)} = \sup_{\|\varphi\|_{H^1(\Omega)}\leqslant 1} |\langle d_k^h f,\varphi\rangle|$$

and according to Proposition 3.1 one has $\|d_k^{-h}\varphi\|_{L^2(\Omega)} \leqslant \|\nabla\varphi\|_{L^2(\Omega)}$, we estimate

$$\int_\Omega |d_k^h f\varphi|\,\mathrm{d}x = \int_\Omega |f d_k^{-h}\varphi|\,\mathrm{d}x \leqslant \|f\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)}.$$

Thus, finally, with use of Young's inequality we arrive at

$$\begin{aligned}
\int_\Omega |d_k^h f_i d_k^h v_i|\,\mathrm{d}x &\leqslant \|d_k^h f\|_{H^{-1}(\Omega)}\|d_k^h v\|_{H^1(\Omega)} \leqslant k\|f\|_{L^2(\Omega)}\|d_k^h\nabla v\|_{L^2(\Omega)} \\
&\leqslant \frac{1}{2\nu}\|f\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|d_k^h\nabla v\|^2_{L^2(\Omega)}.
\end{aligned} \tag{21}$$

Combining (14), (15), (19), (20) and (21) yields

$$\frac{\nu}{2}\int_\Omega |d_k^h(\nabla v)|^2\,\mathrm{d}x \leqslant \Big(\frac{k\|\nabla_x c\|_{L^\infty(\Omega)}}{4K}+1\Big)\|\nabla v\|^3_{L^3(\Omega)} + \frac{1}{2\nu}\|f\|^2_{L^2(\Omega)}. \tag{22}$$

As was recalled in (11), $v \in V$ and we assumed $c \in W^{1,\infty}$, $f \in L^2(\Omega)$. Hence $d_k^h(\nabla v)$ is uniformly bounded (w.r.t. h) in $L^2(\Omega)$ and Proposition 3.1 allows to conclude that $\nabla v \in W^{1,2}(\Omega)$, thus $v \in W^{2,2}(\Omega)$. $\square$

3.1. Uniqueness

Higher regularity of solutions enables us to prove uniqueness of solutions for a small right-hand side f. The crucial points in estimating the nonlinear turbulent term will be the facts that the solution is in $W^{2,2}(\Omega)$ and that $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$.

Proof of Theorem 1.2. Let v^1, v^2 be two solutions to problem (1), namely they satisfy the equations

$$b(v^1, v^1, \varphi) + \int_\Omega c(y^1)|Dv^1|Dv^1 \cdot D\varphi \,\mathrm{d}x + \nu(\nabla v^1, \nabla\varphi) = (f, \varphi), \tag{23}$$

$$b(v^2, v^2, \varphi) + \int_\Omega c(y^2)|Dv^2|Dv^2 \cdot D\varphi \,\mathrm{d}x + \nu(\nabla v^2, \nabla\varphi) = (f, \varphi) \tag{24}$$

for all $\varphi \in V$ where

$$y^1 = (\widetilde{v^1}, \widetilde{v^1 v^1}, \widetilde{Dv^1}, \widetilde{|Dv^1|Dv^1}), \quad y^2 = (\widetilde{v^2}, \widetilde{v^2 v^2}, \widetilde{Dv^2}, \widetilde{|Dv^2|Dv^2}).$$

Subtracting equation (24) from (23) and choosing as a test function $w = v^1 - v^2$ we obtain

$$\begin{aligned} b(v^1, v^1, w) - b(v^2, v^2, w) + \int_\Omega c(y^1)|Dv^1|Dv^1 \cdot Dw \,\mathrm{d}x \\ - \int_\Omega c(y^2)|Dv^2|Dv^2 \cdot Dw \,\mathrm{d}x + \nu\|\nabla w\|^2_{L^2(\Omega)} = 0. \end{aligned}$$

Notice that the difference of the trilinear forms b can be transformed to

$$b(v^1, v^1, w) - b(v^2, v^2, w) = b(v^1, w, w) + b(v^1, v^2, w) - b(v^2, v^2, w) = b(w, v^2, w)$$

and then estimated by

$$|b(w, v^2, w)| \leqslant \|w\|^2_{L^3(\Omega)}\|\nabla v^2\|_{L^3(\Omega)} \leqslant k_1\|\nabla w\|^2_{L^2(\Omega)}\|\nabla v^2\|_{L^3(\Omega)}.$$

Transforming the difference of the turbulent terms into two integrals, i.e.,

$$\begin{aligned} &\int_\Omega \{c(y^1)|Dv^1|Dv^1 - c(y^2)|Dv^2|Dv^2\} Dw \,\mathrm{d}x \\ &= \int_\Omega c(y^1)(|Dv^1|Dv^1 - |Dv^2|Dv^2) Dw \,\mathrm{d}x + \int_\Omega (c(y^1) - c(y^2))|Dv^2|Dv^2 Dw \,\mathrm{d}x, \end{aligned}$$

we estimate the first using the strict monotonicity (6) and Korn's inequality:

$$\int_\Omega c(y^1)(|Dv^1|Dv^1 - |Dv^2|Dv^2) \cdot Dw \,\mathrm{d}x \geqslant \alpha k_2 \|\nabla w\|^3_{L^3(\Omega)}.$$

As c is Lipschitz continuous, the properties of convolutions allow us to claim that for small data

$$|c(y^1) - c(y^2)| \leqslant k(|\tilde{v}^1 - \tilde{v}^2| + |D\tilde{v}^1 - D\tilde{v}^2|) \leqslant k\|v^1 - v^2\|_{L^3(\Omega)}.$$

Then Hölder's inequality and the embeddings $W^{1,2}(\Omega) \subset L^3(\Omega)$, $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$ yield

$$\begin{aligned}\left|\int_\Omega (c(y^1) - c(y^2))|Dv^2|Dv^2 \cdot Dw \,\mathrm{d}x\right| &\leqslant k\|w\|_{L^3(\Omega)} \int_\Omega |Dv^2|^2 \cdot |\nabla w| \,\mathrm{d}x \\ &\leqslant k\|\nabla w\|_{L^2(\Omega)} \|\nabla v^2\|^2_{L^4(\Omega)} \|\nabla w\|_{L^2(\Omega)} \\ &\leqslant k_3(\|\nabla^2 v^2\|^2_{L^2(\Omega)} + \|\nabla v^2\|^2_{L^2(\Omega)})\|\nabla w\|^2_{L^2(\Omega)}.\end{aligned}$$

Collecting all the above estimates we obtain

$$\begin{aligned}\alpha k_2\|\nabla w\|^3_{L^3(\Omega)} + \nu\|\nabla w\|^2_{L^2(\Omega)} &\leqslant k_3(\|\nabla^2 v^2\|^2_{L^2(\Omega)} + \|\nabla v^2\|^2_{L^2(\Omega)})\|\nabla w\|^2_{L^2(\Omega)} \\ &\quad + k_1\|\nabla w\|^2_{L^2(\Omega)}\|\nabla v^2\|_{L^3(\Omega)}.\end{aligned} \tag{25}$$

From the first and second energy estimate (11) and (10) we know that there exist positive constants k_4, k_5, k_6 such that

$$\begin{gathered}\|\nabla v^2\|^3_{L^3(\Omega)} \leqslant k_4\|f\|^{3/2}_{V'}, \quad \|\nabla^2 v^2\|^2_{L^2(\Omega)} \leqslant k_5(\|f\|^2_{L^2(\Omega)} + \|f\|^{3/2}_{V'}) \\ \text{and} \quad \|\nabla v^2\|^2_{L^2(\Omega)} \leqslant k_6\|f\|^{3/2}_{V'}.\end{gathered}$$

The same estimates hold also for v^1. Thus inserting the latter estimates into (25) we get that

$$\alpha k_2\|\nabla w\|^3_{L^3} + \bigl[\nu - k_1k_4^{1/2}\|f\|^{1/2}_{V'} - k_3k_6\|f\|^{3/2}_{V'} - k_3k_5(\|f\|^2_{L^2} + \|f\|^{3/2}_{V'})\bigr]\|\nabla w\|^2_{L^2} \leqslant 0.$$

Choosing f small enough in the L^2-norm (hence also in V') such that the factor next to $\|\nabla w\|^2_{L^2(\Omega)}$ remains positive we can satisfy the inequality only if $w = 0$, which implies $v^2 = v^1$. Thus the solution is unique. □

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