REGULARITY CONDITIONS FOR EXACT CREDIBILITY

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ABSTRACT

Conditions under which the natural conjugate prior is not zero on its boundary are given, correcting an argument about conditions for exact credibility given in another paper.

Keywords: Credibility Theory, Bayesian Forecasting, Exponential Families, Natural Conjugate Priors, Slowly Varying Functions

BACKGROUND

In a previous article [2], it was stated incorrectly that the Bayesian forecasting credibility formula was exact for all simple exponential families with their natural conjugate priors. The purpose of this note is to clarify the conditions under which it is true.

Using slightly more general measure notation, let the likelihood be:

$$P\{dx \mid \theta\} = \frac{e^{-\theta x} A\{dx\}}{c(\theta)}, \quad (x \in X) \tag{1}$$

where, if the density exists, $A\{dx\} = a(x)\, dx$.

The risk parameter θ ranges over the natural parameter space Θ for which the norm

$$c(\theta) = \int_X e^{-\theta x} A\{dx\} \tag{2}$$

is finite; $c(\theta)$ is analytic in the interior of Θ. The natural conjugate prior is then:

$$u(\theta) = \frac{[c(\theta)]^{-n_0} e^{-\theta x_0}}{d(n_0, x_0),}, \quad (\theta \in \Theta). \tag{3}$$

General conditions on the hyperparameters, n_0, x_0, under which $d(n_0, x_0)$ exists are not known, but in credibility forecasts we

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require $n_0 > 0$, and $(x_0/n_0) \varepsilon C(X)$, where $C(X)$ is the convex hull of the support of A.

Let the first two conditional moments be:

$$m(\theta) = \int xP\{dx \mid \theta\} = \frac{-c'(\theta)}{c(\theta)}, \tag{4}$$

$$v(\theta) + (m(\theta))^2 = \int x^2 P\{dx \mid \theta\} = \frac{c''(\theta)}{c(\theta)}, \tag{5}$$

which are finite in the interior of Θ.

For the mixed collective distribution,

$$P(x) = \int_\Theta P(x \mid \theta)\, u(\theta)\, d\theta = E_\theta P(x \mid \theta), \tag{6}$$

the corresponding moments are:

$$m = \int_\Theta m(\theta)\, u(\theta)\, d\theta = E_\theta m(\theta) \tag{7}$$

and

$$v = E_\theta\, v(\theta) + V_\theta m(\theta). \tag{8}$$

If we integrate (7) by parts using (3), we get:

$$m = \frac{x_0 + u(\theta) \mid_\theta}{n_0}, \tag{9}$$

where $u(\theta) \mid_\Theta$ is the difference in $u(\theta)$ across the boundaries of Θ. Finiteness of m implies that this limit exists.

If the *u-condition on the boundary* (UCB)

$$u(\theta) \mid_\Theta \equiv 0 \tag{10}$$

holds, then:

$$m = \frac{x_0}{n_0}, \tag{11}$$

and this implies the credibility forecast:

$$E\{\xi_{n+1} \mid \underline{x}\} = \frac{x_0 + \Sigma x_t}{n_0 + n} \tag{12}$$

after n independent samples of values $\underline{x} = (x_1, x_2, \ldots, x_n)$ are drawn, and the prior $u(\theta)$ is updated [2]. Given the previously mentioned conditions on the hyper-parameters, the main problem in proving (12) is then the conditions under which UCB holds.

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Now Θ is a convex set on the real line, and cases where it is only a single point are uninteresting; therefore, Θ is either $(-\infty, +\infty)$; $(\theta_0, +\infty)$; $(-\infty, \theta_1)$; or (θ_0, θ_1). These intervals may possibly be closed at the finite endpoints θ_0 and/or θ_1, where the analytic continuation of $c(\theta)$ is finite; it turns out that this is the exceptional case of interest.

At infinite endpoints, there is no difficulty, since $u(+\infty)$ must be zero, in order to be normed. On the other hand, from (3), we see that $c(\theta)$ must be infinite at finite boundaries for UCB to hold. In all cases examined in [2], Θ was $(-\infty, +\infty)$, or $c(\theta_0) = \infty$, so that (12) held in those cases. It was also *incorrectly* argued that this was always so, based upon the nature of the endpoint θ_0 which limited further expansion of Θ. However, counterexamples exist for which $c(\theta_0)$ is well defined.

COUNTEREXAMPLES

In a letter to the author, R. B. Miller and A. Banerjee have pointed out that the pair,

$$c(\theta) = e^{-\sqrt{2\theta}},\ \Theta = [0, \infty);\ A(x) = 2\left[1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right],\ X = (0, \infty) \tag{13}$$

violate the UCB, since $\lim_{\theta \to 0} c(\theta) = 1$, and

$$u(\theta)\,|_\Theta = -u(0) = \frac{-1}{d(n_0, x_0)} = \frac{-x_0}{1 + \dfrac{n_0}{\sqrt{x_0}}\,\dfrac{\Phi(n_0/\sqrt{x_0})}{\phi(n_0/\sqrt{x_0})}}. \tag{14}$$

(Φ and ϕ are the unit normal cdf and density, respectively.) This gives a distinctly uncredible formula, when used in (9)!

A a matter of fact, one can see that choosing $A(x)$ to be any of the *stable distributions with parameter* α *on* $(0, \infty)$ will give a counter-example, since for these [1]:

$$c(\theta) = e^{-\theta^\alpha},\ (0 < \alpha < 1),\ \Theta = [0, \infty). \tag{15}$$

Various further generalizations are, for example:

$$c(\theta) = e^{-f(\theta^\alpha)}, \qquad (0 < \alpha < 1) \tag{16}$$

where f is any positive function for which $f(0) < \infty$, and whose derivative is "completely monotone" [1].

There is one drawback to these stable law counterexamples, however. If we integrate $\int v(\theta)\, u(\theta)\, d\theta$ by parts, and assume $m < \infty$, then we find:

$$(m(\theta)\, u(\theta)\,|_{\Theta} < \infty) <=> (v < \infty). \qquad (17)$$

But, for (15), $m(\theta) \to \infty$ $(\theta \to 0)$, so that these counterexamples all have finite collective mean, m, but *infinite* collective variance, v.

Regularity Conditions

For simplicity, assume X is one sided, say $A(0) = 0$. Then Θ is $(-\infty, \infty)$, (θ_0, ∞), or $[\theta_0, \infty)$. The last case is the one of interest. Let $A_0\{dx\} = e^{-\theta_0 x} A\{dx\}$. We may then state:

The (positive) members of the exponential family which violate the UCB as those for which the measures $A^0(x)$ satisfy:

$$c(\theta_0) = \int_X A^0\{dx\} = \lim_{x\to\infty} A^0(x) = A^0(\infty) < \infty, \qquad (18)$$

but for which

$$\int_X e^{\varepsilon x} A^0\{dx\} = \infty \quad \text{for} \quad \varepsilon > 0. \qquad (19)$$

It turns out there are many measures A, other than those described in the last section, which violate the UCB, including some for which v, or any higher moment of the collective is finite. The following beautiful counterexample was suggested by L. Lecam:

$$A\{dx\} = e^{-\sqrt{2x}}\, dx, \qquad X = [0, \infty)$$

$$c(\theta) = \frac{1}{\theta}\left\{1 - \frac{\theta^{-1/2}[1 - \Phi(\theta^{-1/2})]}{\phi(\theta^{-1/2})}\right\}, \qquad \Theta = [0, \infty), \ (\theta_0 = 0). \qquad (20)$$

By using a well-behaved asymptotic expansion of the normal, we find:

$$c(\theta) \sim 1 - 1\cdot 3\theta + 1\cdot 3\cdot 5\theta^2 - 1\cdot 3\cdot 5\cdot 7\theta^3 + \ldots \ (\theta \to 0). \qquad (21)$$

In other words, UCB is violated, (19) holds for any $\varepsilon > 0$, but *all moments*

$$\int x^n A\{dx\} < \infty \qquad (22)$$

are finite, as are those of the collective distribution (6)!

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If A^0 reached its limit (18) as fast as an exponential, then the domain could be extended, since the integral (19) would exist for small enough ε. Therefore, another characterization of the counterexamples is:

The UCB (10) is not satisfied for measures A^0 which approach a finite limit more slowly than any exponential. (23)

Also, we point out that transforms, such as (2), can have many different kinds of behaviour on the boundary of region of analyticity [3]. The UCB counterexamples correspond to nonmeremorphic functions $c(\theta)$, where analytic continuation on the boundary is possible.

Similar remarks apply to the other tail of A, when Θ is limited on the right by a value θ_1.

Exact Credibility

Most exponential families of modelling interest satisfy UCB easily. Many one-sided distributions, such as the Negative Binomial and Gamma, with fixed shape parameters, have $A^0(x) \to \infty$, which implies $c(\theta) \to \infty$. In this category are also approximations which give $c(\theta)$ as the sum of rational functions of θ.

Other exponential-type distributions, such as the Bernoulli, Normal, and the Poisson, and all distributions where $C(X)$ is finite, have $\Theta = (-\infty, \infty)$, which automatically implies exact credibility. Any real-data distribution can be placed in this category by truncation.

Even if $C(X)$ is semi-infinite, and $A^0(\infty)$ is finite, it may approach this limit fast enough to qualify. Dubious densities can be converted to this case by multiplying by $e^{-\varepsilon x^2}$, where ε is as small as desired.

Note also that, pragmatically speaking, one must be able to find $c(\theta)$ in order to deduce the form of the prior. From $c(\theta)$ one can directly find Θ; if $c(\theta)$ diverges on its finite boundaries, UCB is satisfied.

Finally, even if one has a counterexample, (9) still applies. For example, if $\Theta = [\theta_0, \infty)$, then:

$$E\{\xi_{n+1} \mid \underline{x}\} = \frac{x_0 + \Sigma x_t}{n_0 + n} - \frac{(c(\theta_0))^{-(n_0+n)}\, e^{-\theta_0(x_0 + \Sigma x_t)}}{d(n_0 + n, x_0 + \Sigma x_t)}, \tag{24}$$

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and since $u(\theta)$ is unimodal, with increasing mass concentrated at the mode with increasing n, the correction term approaches zero with probability 1.

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