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REGULARITY CONDITIONS VIA QUASI-RELATIVE INTERIOR IN CONVEX PROGRAMMING*

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Abstract. We give some new regularity conditions for Fenchel duality in separated locally convex vector spaces, written in terms of the notion of quasi interior and quasi-relative interior, respectively. We provide also an example of a convex optimization problem for which the classical generalized interior-point conditions given so far in the literature cannot be applied, while the one given by us is applicable. By using a technique developed by Magnanti, we derive some duality results for the optimization problem with cone constraints and its Lagrange dual problem, and we show that a duality result recently given in the literature for this pair of problems has self-contradictory assumptions.

Key words. convex programming, Fenchel duality, Lagrange duality, quasi-relative interior

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1. Introduction.

Usually there is a so-called duality gap between the optimal objective values of a primal convex optimization problem and its dual problem. A challenge in convex analysis is to give sufficient conditions which guarantee strong duality, the situation when the optimal objective values of the two problems are equal and the dual problem has an optimal solution. Several generalized interior-point conditions were given in the past in order to eliminate the above-mentioned duality gap. Along the classical interior, some generalized interior notions were used, such as the core [14], the intrinsic core [9], or the strong quasi-relative interior [2], in order to give regularity conditions which guarantee strong duality. For an overview of these conditions we invite the reader to consult [8], [16] (see also [17] for more on this subject).

Unfortunately, for infinite-dimensional convex optimization problems, also in practice, it can happen that the duality results given in the past cannot be applied because, for instance, the interior of the set involved in the regularity condition is empty. This is the case, for example, when we deal with the positive cones

$$l_+^p = \{x = (x_n)_{n\in\mathbb{N}} \in l^p : x_n \geq 0 \ \forall n \in \mathbb{N}\}$$

and

$$L_+^p(T,\mu) = \{u \in L^p(T,\mu) : u(t) \geq 0, \text{ a.e.}\}$$

of the spaces l^p and $L^p(T,\mu)$, respectively, where (T,μ) is a σ-finite measure space and $p \in [1,\infty)$. Moreover, also the strong quasi-relative interior (which is the weakest generalized interior notion from the one mentioned above) of these cones is empty. For this reason, for a convex set, Borwein and Lewis introduced the notion of a quasi-relative interior [3], which generalizes all of the above-mentioned interior notions. They proved that the quasi-relative interiors of l_+^p and $L_+^p(T,\mu)$ are nonempty.

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In this paper, we start by considering the primal optimization problem with the objective function being the sum of two proper convex functions defined on a separated locally convex vector space, to which we attach its Fenchel dual problem, stated in terms of the conjugates of the two functions. We give a new regularity condition for Fenchel duality based on the notion of a quasi-relative interior of a convex set using a separation theorem given by Cammaroto and Di Bella in [4]. Further, two stronger regularity conditions are also given. We provide an appropriate example for which our duality results are applicable, while the other generalized interior-point conditions given in the past fail, justifying the theory developed in this paper. Then we state duality results for the case when the objective function of the primal problem is the sum of a proper convex function with the composition of another proper convex function with a continuous linear operator. Let us notice that for this case Borwein and Lewis in [3] also gave some conditions by means of the quasi-relative interior, but they considered a more restrictive case, namely, that the codomain of the linear operator is finite-dimensional. We consider the more general case, when both of the spaces are infinite-dimensional.

In 1974 Magnanti proved that "Fenchel and Lagrange duality are equivalent" in the sense that the classical Fenchel duality result can be deduced from the classical Lagrange duality result, and vice versa (see [13]). By using this technique we derive some Lagrange duality results for the convex optimization problem with cone constraints, written in terms of the quasi-relative interior. Let us notice that another condition for Lagrange duality, stated also in terms of the quasi-relative interior, was given recently by Cammaroto and Di Bella in [4]. We show that this result has self-contradictory assumptions. Let us mention that also in [11] some regularity conditions, in terms of the quasi-relative interior, have been introduced. However, most of these conditions require the interior of a cone to be nonempty, and this fails for many optimization problems as we pointed out above.

The paper is structured as follows. In the next section we give some definitions and results which will be used later in the paper. Section 3 is devoted to the theory of Fenchel duality. We give here the announced regularity conditions written in terms of the quasi-relative interior. By using an idea due to Magnanti we derive in section 4 some duality results for the optimization problem with cone constraints and its Lagrange dual problem.

2. Preliminary notions and results. Consider X, a separated locally convex vector space, and X^*, its topological dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Further, let $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x$, for all $x \in X$, be the *identity function* of X. The *indicator function* of $C \subseteq X$, denoted by δ_C, is defined as $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$,

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. We also denote by $\widehat{\mathrm{epi}}(f) = \{(x, r) \in X \times \mathbb{R} : (x, -r) \in \mathrm{epi}(f)\}$ the symmetric of $\mathrm{epi}(f)$ with respect to the x-axis. For a given real number α, $f - \alpha : X \to \overline{\mathbb{R}}$ is, as usual, the function defined by $(f - \alpha)(x) = f(x) - \alpha$ for all $x \in X$. Given two functions $f : M_1 \to M_2$ and $g : N_1 \to N_2$, where M_1, M_2, N_1, N_2 are nonempty sets, we define the function $f \times g : M_1 \times N_1 \to M_2 \times N_2$ by $f \times g(m, n) = (f(m), g(n))$ for all $(m, n) \in M_1 \times N_1$.

The *Fenchel–Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \ \forall x^* \in X^*.$$

For a subset C of X we denote by $\operatorname{co} C$, $\operatorname{aff} C$, $\operatorname{cl} C$, and $\operatorname{int} C$ its *convex hull*, *affine hull*, *closure*, and *interior*, respectively. The set $\operatorname{cone} C := \bigcup_{\lambda \geq 0} \lambda C$ is the *cone generated by* C. The following property, the proof of which we omit since it presents no difficulty, will be used throughout the paper: If C is convex, then

$$\operatorname{cone} \operatorname{co}(C \cup \{0\}) = \operatorname{cone} C. \tag{1}$$

The *normal cone* of C at $x \in C$ is defined as $N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in C\}$.

DEFINITION 2.1 (see [3]). *Let C be a convex subset of X. The* quasi-relative interior *of C is the set*

$$\operatorname{qri} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) \textit{ is a linear subspace of } X\}.$$

We give the following useful characterization of the quasi-relative interior of a convex set.

PROPOSITION 2.2 (see [3]). *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qri} C$ if and only if $N_C(x)$ is a linear subspace of X^*.*

In the following we consider another interior notion for a convex set, which is close to the one of a quasi-relative interior.

DEFINITION 2.3. *Let C be a convex subset of X. The* quasi interior *of C is the set*

$$\operatorname{qi} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) = X\}.$$

The following characterization of the quasi interior of a convex set was given in [6], where the space X was considered a reflexive Banach space. One can prove that this property is true even in a separated locally convex vector space.

PROPOSITION 2.4. *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qi} C$ if and only if $N_C(x) = \{0\}$.*

Proof. Assume first that $x \in \operatorname{qi} C$, and take an arbitrary element $x^* \in N_C(x)$. One can easily see that $\langle x^*, z\rangle \leq 0$ for all $z \in \operatorname{cl} \operatorname{cone}(C - x)$. Thus $\langle x^*, z\rangle \leq 0$ for all $z \in X$, which is nothing else than $x^* = 0$.

In order to prove the opposite implication we consider an arbitrary $\bar{x} \in X$ and prove that $\bar{x} \in \operatorname{cl} \operatorname{cone}(C - x)$. By assuming the contrary, by a separation theorem (see, for instance, Theorem 1.1.5 in [17]), one has that there exists $x^* \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, z\rangle < \alpha < \langle x^*, \bar{x}\rangle \ \forall z \in \operatorname{cl} \operatorname{cone}(C - x).$$

Let $y \in C$ be fixed. For all $\lambda > 0$ it holds that $\langle x^*, y - x\rangle < \frac{1}{\lambda}\alpha$, and this implies that $\langle x^*, y - x\rangle \leq 0$. As this inequality is true for every arbitrary $y \in C$, we obtain that $x^* \in N_C(x)$. But this leads to a contradiction, and in this way the conclusion follows. ∎

It follows from the definitions above that $\operatorname{qi} C \subseteq \operatorname{qri} C$ and $\operatorname{qri}\{x\} = \{x\}$ for all $x \in X$. Moreover, if $\operatorname{qi} C \neq \emptyset$, then $\operatorname{qi} C = \operatorname{qri} C$. Although this property is given in [12] in the case of a real normed space, it holds also in an arbitrary separated locally convex vector space, as follows by the properties given above. If X is a finite-dimensional space, then $\operatorname{qi} C = \operatorname{int} C$ (cf. [12]) and $\operatorname{qri} C = \operatorname{ri} C$ (cf. [3]), where $\operatorname{ri} C$ is the *relative interior* of C.

Useful properties of the quasi-relative interior are listed below. For the proof of (i)–(viii) we refer to [1] and [3].

PROPOSITION 2.5. *Let us consider C and D two convex subsets of X, $x \in X$, and $\alpha \in \mathbb{R}$. Then:*

(i) $\operatorname{qri} C + \operatorname{qri} D \subseteq \operatorname{qri}(C + D)$;
(ii) $\operatorname{qri}(C \times D) = \operatorname{qri} C \times \operatorname{qri} D$;
(iii) $\operatorname{qri}(C - x) = \operatorname{qri} C - x$;
(iv) $\operatorname{qri}(\alpha C) = \alpha \operatorname{qri} C$;
(v) $t \operatorname{qri} C + (1 - t)C \subseteq \operatorname{qri} C$, $\forall t \in (0, 1]$, *and hence* $\operatorname{qri} C$ *is a convex set;*
(vi) *if C is an affine set, then* $\operatorname{qri} C = C$;
(vii) $\operatorname{qri}(\operatorname{qri} C) = \operatorname{qri} C$.

If $\operatorname{qri} C \neq \emptyset$, *then:*

(viii) $\operatorname{cl} \operatorname{qri} C = \operatorname{cl} C$;
(ix) $\operatorname{cl} \operatorname{cone} \operatorname{qri} C = \operatorname{cl} \operatorname{cone} C$.

Proof. (ix) The inclusion $\operatorname{cl} \operatorname{cone} \operatorname{qri} C \subseteq \operatorname{cl} \operatorname{cone} C$ is obvious. We prove that $\operatorname{cone} C \subseteq \operatorname{cl} \operatorname{cone} \operatorname{qri} C$. Consider $x \in \operatorname{cone} C$ arbitrary. There exist $\lambda \geq 0$ and $c \in C$ such that $x = \lambda c$. Take $x_0 \in \operatorname{qri} C$. By applying property (v) we get $tx_0 + (1 - t)c \in \operatorname{qri} C$ for all $t \in (0, 1]$, so $\lambda t x_0 + (1 - t)x = \lambda(tx_0 + (1 - t)c) \in \operatorname{cone} \operatorname{qri} C$ for all $t \in (0, 1]$. By passing to the limit as $t \searrow 0$ we obtain $x \in \operatorname{cl} \operatorname{cone} \operatorname{qri} C$, and hence the desired conclusion follows. □

The next lemma plays an important role in this paper.

LEMMA 2.6. *Let A and B be nonempty convex subsets of X such that* $\operatorname{qri} A \cap B \neq \emptyset$. *If* $0 \in \operatorname{qi}(A - A)$, *then* $0 \in \operatorname{qi}(A - B)$.

Proof. Take $x \in \operatorname{qri} A \cap B$, and let $x^* \in N_{A-B}(0)$ be arbitrary. We get $\langle x^*, a - b\rangle \leq 0$, for all $a \in A$, for all $b \in B$. This implies that

$$\langle x^*, a - x\rangle \leq 0 \ \forall a \in A, \tag{2}$$

that is, $x^* \in N_A(x)$. As $x \in \operatorname{qri} A$, $N_A(x)$ is a linear subspace of X^*, and hence $-x^* \in N_A(x)$, which is nothing else than

$$\langle x^*, x - a\rangle \leq 0 \ \forall a \in A. \tag{3}$$

The relations (2) and (3) give us $\langle x^*, a' - a''\rangle \leq 0$, for all $a', a'' \in A$, so $x^* \in N_{A-A}(0)$. Since $0 \in \operatorname{qi}(A - A)$ we have $N_{A-A}(0) = \{0\}$ (cf. Proposition 2.4), and we get $x^* = 0$. As x^* was arbitrary chosen we obtain $N_{A-B}(0) = \{0\}$, and, by using again Proposition 2.4, the conclusion follows. □

Next we give useful separation theorems in terms of the notion of a quasi-relative interior.

THEOREM 2.7. *Let C be a convex subset of X and $x_0 \in C$. If $x_0 \notin \operatorname{qri} C$, then there exists $x^* \in X^*, x^* \neq 0$, such that*

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle \ \forall x \in C.$$

Vice versa, if there exists $x^* \in X^*$, $x^* \neq 0$, such that

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle \ \forall x \in C$$

and

$$0 \in \operatorname{qi}(C - C),$$

then $x_0 \notin \operatorname{qri} C$.

Proof. Suppose that $x_0 \notin \operatorname{qri} C$. According to Proposition 2.2, $N_C(x_0)$ is not a linear subspace of X^*, and hence there exists $x^* \in N_C(x_0)$, $x^* \neq 0$. By using the definition of the normal cone, we get that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$ for all $x \in C$.

Conversely, assume that there exists $x^* \in X^*$, $x^* \neq 0$, such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$ for all $x \in C$ and $0 \in \operatorname{qi}(C - C)$. We obtain

$$\langle x^*, x - x_0\rangle \leq 0 \ \forall x \in C, \tag{4}$$

that is, $x^* \in N_C(x_0)$. If we suppose that $x_0 \in \operatorname{qri} C$, then $N_C(x_0)$ is a linear subspace of X^*, and hence $-x^* \in N_C(x_0)$. By combining this with (4) we get $\langle x^*, x - x_0\rangle = 0$ for all $x \in C$. The last relation implies $\langle x^*, x\rangle = 0$ for all $x \in C - C$, and from here one has further that $\langle x^*, x\rangle = 0$ for all $x \in \operatorname{cl}\operatorname{cone}(C - C) = X$. But this can be the case just if $x^* = 0$, which is a contradiction. In conclusion, $x_0 \notin \operatorname{qri} C$. □

Remark 2.8. In [5], [6] a similar separation theorem in the case when X is a real normed space is given. For the second part of the above theorem the authors require that the following condition must be fulfilled:

$$\operatorname{cl}(T_C(x_0) - T_C(x_0)) = X,$$

where

$$T_C(x_0) = \big\{y \in X : y = \lim_{n\to\infty} \lambda_n(x_n - x_0), \lambda_n > 0 \ \forall n \in \mathbb{N},$$
$$x_n \in C \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x_0\big\}$$

is called the *contingent cone* to C at $x_0 \in C$. In general, we have the following inclusion: $T_C(x_0) \subseteq \operatorname{cl}\operatorname{cone}(C - x_0)$. If the set C is convex, then $T_C(x_0) = \operatorname{cl}\operatorname{cone}(C - x_0)$ (cf. [10]). As $\operatorname{cl}(\operatorname{cl} E + \operatorname{cl} F) = \operatorname{cl}(E + F)$, for arbitrary sets E, F in X and $\operatorname{cone} A - \operatorname{cone} A = \operatorname{cone}(A - A)$, if A is a convex subset of X such that $0 \in A$, the condition $\operatorname{cl}(T_C(x_0) - T_C(x_0)) = X$ can be reformulated as follows: $\operatorname{cl}\operatorname{cone}(C - C) = X$ or, equivalently, $0 \in \operatorname{qi}(C - C)$. Indeed, we have

$$\operatorname{cl}[\operatorname{cl}\operatorname{cone}(C - x_0) - \operatorname{cl}\operatorname{cone}(C - x_0)] = X \Leftrightarrow \operatorname{cl}[\operatorname{cone}(C - x_0) - \operatorname{cone}(C - x_0)] = X$$
$$\Leftrightarrow \operatorname{cl}\operatorname{cone}(C - C) = X \Leftrightarrow 0 \in \operatorname{qi}(C - C).$$

This means that Theorem 2.7 is a generalization to the case of separated locally convex vector spaces of the separation theorem given in [5], [6] in the framework of real normed spaces.

The condition $x_0 \in C$ in Theorem 2.7 is essential (see [6]). However, if x_0 is an arbitrary element of X, we can also give a separation theorem by using the following result due to Cammaroto and Di Bella (Theorem 2.1 in [4]). The mentioned authors use this theorem in order to prove their strong duality result (Theorem 2.2 in [4]). Unfortunately, as we will show in section 4, this result has self-contradictory assumptions.

Theorem 2.9 (see [4]). *Let S and T be nonempty convex subsets of X with* $\operatorname{qri} S \neq \emptyset$, $\operatorname{qri} T \neq \emptyset$, *and such that* $\operatorname{cl}\operatorname{cone}(\operatorname{qri} S - \operatorname{qri} T)$ *is not a linear subspace of X. Then there exists $x^* \in X^*$, $x^* \neq 0$, such that $\langle x^*, s\rangle \leq \langle x^*, t\rangle$ for all $s \in S$, $t \in T$.*

The following result is a direct consequence of Theorem 2.9.

Corollary 2.10. *Let C be a convex subset of X such that* $\operatorname{qri} C \neq \emptyset$ *and* $\operatorname{cl}\operatorname{cone}(C - x_0)$ *is not a linear subspace of X, where $x_0 \in X$. Then there exists $x^* \in X^*, x^* \neq 0$, such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$ for all $x \in C$.*

Proof. We take, in Theorem 2.9, $S := C$ and $T := \{x_0\}$. Then we apply Proposition 2.5 (iii) and (ix) to obtain the conclusion. □

3. Fenchel duality. In this section we give some new Fenchel duality results stated in terms of the quasi interior and quasi-relative interior, respectively.

Consider the convex optimization problem

$$(P_F) \ \inf_{x \in X} \{f(x) + g(x)\},$$

where X is a separated locally convex vector space and $f, g : X \to \overline{\mathbb{R}}$ are proper convex functions such that $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$. The Fenchel dual problem to (P_F) is the followiing:

$$(D_F) \ \sup_{x^* \in X^*} \{-f^*(-x^*) - g^*(x^*)\}.$$

We denote by $v(P_F)$ and $v(D_F)$ the optimal objective values of the primal and the dual problem, respectively. Weak duality always holds; that is, $v(D_F) \leq v(P_F)$. For strong duality, the case when $v(P_F) = v(D_F)$ and (D_F) has an optimal solution, several generalized interior-point regularity conditions were given in the literature. In order to recall them we need the following generalized interior notions. For a convex subset C of X we have:

- $\operatorname{core} C := \{x \in C : \operatorname{cone}(C - x) = X\}$, the *core* of C [14], [17];
- $\operatorname{icr} C := \{x \in C : \operatorname{cone}(C - x)$ is a linear subspace$\}$, the *intrinsic core* of C [1], [9], [17];
- $\operatorname{sqri} C := \{x \in C : \operatorname{cone}(C - x)$ is a closed linear subspace$\}$, the *strong quasi-relative interior* of C [2], [17].

We have the following inclusions:

$$\operatorname{core} C \subseteq \operatorname{sqri} C \subseteq \operatorname{qri} C \text{ and } \operatorname{core} C \subseteq \operatorname{qi} C \subseteq \operatorname{qri} C.$$

If X if finite-dimensional, then $\operatorname{qri} C = \operatorname{sqri} C = \operatorname{icr} C = \operatorname{ri} C$ [3], [8] and $\operatorname{core} C = \operatorname{qi} C = \operatorname{int} C$ [12], [14].

Consider now the following regularity conditions:

(i) $0 \in \operatorname{int}(\operatorname{dom}(f) - \operatorname{dom}(g))$;
(ii) $0 \in \operatorname{core}(\operatorname{dom}(f) - \operatorname{dom}(g))$ (cf. [14]);
(iii) $0 \in \operatorname{icr}(\operatorname{dom}(f) - \operatorname{dom}(g))$ and $\operatorname{aff}(\operatorname{dom}(f) - \operatorname{dom}(g))$ is a closed linear subspace (cf. [8]);
(iv) $0 \in \operatorname{sqri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ (cf. [15]).

Let us notice that all of these conditions guarantee strong duality if we suppose the additional hypotheses that the functions f and g are lower semicontinuous and X is a Fréchet space. Between the above conditions we have the following relation: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) [8].

Trying to give a similar regularity condition for strong duality by means of the notion of a quasi-relative interior of a convex set, a natural question arises: Is the condition $0 \in \operatorname{qri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ sufficient for strong duality? The following example (which can be found in [8]) gives us a negative answer, and this means that we need additional assumptions in order to guarantee Fenchel duality (see Theorem 3.5).

Example 3.1. As in [8], we consider $X = l^2$, the Hilbert space consisting of all sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$. Consider also the sets

$$C = \{x \in l^2 : x_{2n-1} + x_{2n} = 0 \ \forall n \in \mathbb{N}\},$$

$$S = \{x \in l^2 : x_{2n} + x_{2n+1} = 0 \ \forall n \in \mathbb{N}\}.$$

The sets C and S are closed linear subspaces of l^2 and $C \cap S = \{0\}$. Define the functions $f, g : l^2 \to \overline{\mathbb{R}}$ by $f = \delta_C$ and $g(x) = x_1$ if $x \in S$ and $+\infty$ otherwise. One can see that f and g are proper, convex, and lower semicontinuous functions with $\mathrm{dom}(f) = C$ and $\mathrm{dom}(g) = S$. As was shown in [8], $v(P_F) = 0$ and $v(D_F) = -\infty$, so we have a duality gap between the optimal objective values of the primal problem and its Fenchel dual. Moreover, $S - C$ is dense in l^2; thus $\mathrm{cl}\,\mathrm{cone}(\mathrm{dom}(f) - \mathrm{dom}(g)) = \mathrm{cl}(C - S) = l^2$. The last relation implies that $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$, hence $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$.

Let us notice that if $v(P_F) = -\infty$, by the weak duality follows that also strong duality holds. This is the reason why we suppose in the following that $v(P_F) \in \mathbb{R}$.

LEMMA 3.2. *The following relation is always true:*

$$0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g)) \Rightarrow (0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))].$$

Proof. One can see that $\widehat{\mathrm{epi}}(g - v(P_F)) = \{(x,r) \in X \times \mathbb{R} : r \leq -g(x) + v(P_F)\}$. Let us prove first that $(0,1) \in \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$. Since $\inf_{x \in X}[f(x) + g(x)] = v(P_F) < v(P_F) + 1$, there exists $x' \in X$ such that $f(x') + g(x') < v(P_F) + 1$. Then $(0,1) = (x', v(P_F) + 1 - g(x')) - (x', -g(x') + v(P_F)) \in \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$.

Now let $(x^*, r^*) \in N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$. We have

$$\langle x^*, x - x' \rangle + r^*(\mu - \mu' - 1) \leq 0 \ \forall (x,\mu) \in \mathrm{epi}(f) \ \forall (x',\mu') \in \widehat{\mathrm{epi}}(g - v(P_F)). \tag{5}$$

For $(x,\mu) := (x_0, f(x_0))$ and $(x',\mu') := (x_0, -g(x_0) + v(P_F) - 2)$ in (5), where $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ is fixed, we get $r^*(f(x_0) + g(x_0) - v(P_F) + 1) \leq 0$, and hence $r^* \leq 0$. As $\inf_{x \in X}[f(x) + g(x)] = v(P_F) < v(P_F) + 1/2$, there exists $x_1 \in X$ such that $f(x_1) + g(x_1) < v(P_F) + 1/2$. By taking now $(x,\mu) := (x_1, f(x_1))$ and $(x',\mu') := (x_1, -g(x_1) + v(P_F) - 1/2)$ in (5) we obtain $r^*(f(x_1) + g(x_1) - v(P_F) - 1/2) \leq 0$, and so $r^* \geq 0$. Thus $r^* = 0$, and (5) gives: $\langle x^*, x - x' \rangle \leq 0$ for all $x \in \mathrm{dom}(f)$ for all $x' \in \mathrm{dom}(g)$. Hence $x^* \in N_{\mathrm{dom}(f) - \mathrm{dom}(g)}(0)$. Since $N_{\mathrm{dom}(f) - \mathrm{dom}(g)}(0)$ is a linear subspace of X^* (cf. Proposition 2.2), we have $\langle -x^*, x - x' \rangle \leq 0$ for all $x \in \mathrm{dom}(f)$ for all $x' \in \mathrm{dom}(g)$, and so $-(x^*, r^*) = (-x^*, 0) \in N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$, showing that $N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$ is a linear subspace of $X^* \times \mathbb{R}$. Hence, by applying again Proposition 2.2, we get $(0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. ◻

PROPOSITION 3.3. *Assume that* $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$. *Then* $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ *is a linear subspace of* $X^* \times \mathbb{R}$ *if and only if* $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0) = \{(0,0)\}$.

Proof. The sufficiency is trivial. Now let us suppose that the set $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$. Take $(x^*, r^*) \in N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$. Then

$$\langle x^*, x - x' \rangle + r^*(\mu - \mu') \leq 0 \ \forall (x,\mu) \in \mathrm{epi}(f) \ \forall (x',\mu') \in \widehat{\mathrm{epi}}(g - v(P_F)). \tag{6}$$

Let $x_0 \in \mathrm{dom}\, f \cap \mathrm{dom}(g)$ be fixed. By taking $(x,\mu) := (x_0, f(x_0)) \in \mathrm{epi}(f)$ and $(x',\mu') := (x_0, -g(x_0) + v(P_F) - 1/2) \in \widehat{\mathrm{epi}}(g - v(P_F))$ in the previous inequality, we get $r^*(f(x_0) + g(x_0) - v(P_F) + 1/2) \leq 0$, implying $r^* \leq 0$. As the set $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$, the same argument applies also for $(-x^*, -r^*)$, implying $-r^* \leq 0$. In this way we get $r^* = 0$. The inequality (6) and the relation $(-x^*, 0) \in N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ imply that

$$\langle x^*, x - x' \rangle = 0 \ \forall (x,\mu) \in \mathrm{epi}(f) \ \forall (x',\mu') \in \widehat{\mathrm{epi}}(g - v(P_F)),$$

which is nothing else than $\langle x^*, x - x'\rangle = 0$ for all $x \in \mathrm{dom}(f)$ for all $x' \in \mathrm{dom}(g)$, and thus $\langle x^*, x\rangle = 0$ for all $x \in \mathrm{dom}(f) - \mathrm{dom}(g)$. Since x^* is linear and continuous, the last relation implies that $\langle x^*, x\rangle = 0$ for all $x \in \mathrm{cl}\,\mathrm{cone}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))] = X$; hence $x^* = 0$, and the conclusion follows. □

Remark 3.4. (a) By (1) one can see that $\mathrm{cl}\,\mathrm{cone}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}] = \mathrm{cl}\,\mathrm{cone}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. Hence one has the following sequence of equivalences: $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R} \Leftrightarrow (0,0) \in \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Leftrightarrow \mathrm{cl}\,\mathrm{cone}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ is a linear subspace of $X \times \mathbb{R} \Leftrightarrow \mathrm{cl}\,\mathrm{cone}(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F)))$ is a linear subspace of $X \times \mathbb{R}$. The relation $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0) = \{(0,0)\}$ is equivalent to $(0,0) \in \mathrm{qi}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ (cf. Proposition 2.4), so in the case $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$ the conclusion of the previous proposition can be reformulated as follows:

$$\mathrm{cl}\,\mathrm{cone}(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \text{ is a linear subspace of } X \times \mathbb{R} \Leftrightarrow$$
$$(0,0) \in \mathrm{qi}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$$

or, equivalently,

$$(0,0) \in \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Leftrightarrow$$
$$(0,0) \in \mathrm{qi}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}].$$

(b) One can prove that the primal problem (P_F) has an optimal solution if and only if $(0,0) \in \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$. This means that if we suppose that the primal problem has an optimal solution and $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$, then the conclusion of the previous proposition can be rewritten as follows: $N_{(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F)))}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$ if and only if $N_{(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F)))}(0,0) = \{(0,0)\}$ or, equivalently,

$$(0,0) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))] \Leftrightarrow (0,0) \in \mathrm{qi}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))].$$

We give now the first strong duality result for (P_F) and its Fenchel dual (D_F). Let us notice that for the functions f and g we suppose just convexity properties, and we do not use any closedness type of condition.

THEOREM 3.5. *Suppose that* $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$, $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$, *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$, *and* (D_F) *has an optimal solution.*

Proof. Lemma 3.2 ensures that $(0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$, and hence $\mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))] \neq \emptyset$. The condition $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$, together with the relation $\mathrm{cl}\,\mathrm{cone}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}] = \mathrm{cl}\,\mathrm{cone}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$, implies that $\mathrm{cl}\,\mathrm{cone}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$ is not a linear subspace of $X \times \mathbb{R}$. We apply Corollary 2.10 with $C := \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$ and $x_0 = (0,0)$. Thus there exists $(x^*, \lambda) \in X^* \times \mathbb{R}$, $(x^*, \lambda) \neq (0,0)$ such that

$$\langle x^*, x\rangle + \lambda\mu \geq \langle x^*, x'\rangle + \lambda\mu' \ \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)) \ \forall (x', \mu') \in \mathrm{epi}(f). \tag{7}$$

We claim that $\lambda \leq 0$. Indeed, if $\lambda > 0$, then for $(x, \mu) := (x_0, -g(x_0) + v(P_F))$ and $(x', \mu') := (x_0, f(x_0) + n), n \in \mathbb{N}$, where $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ is fixed, we obtain from (7): $\langle x^*, x_0\rangle + \lambda(-g(x_0) + v(P_F)) \geq \langle x^*, x_0\rangle + \lambda(f(x_0) + n)$ for all $n \in \mathbb{N}$. By passing to the limit as $n \to +\infty$ we obtain a contradiction. Next we prove that $\lambda < 0$. Suppose that $\lambda = 0$. Then from (7) we have $\langle x^*, x\rangle \geq \langle x^*, x'\rangle$ for all $x \in \mathrm{dom}(g)$ for all $x' \in \mathrm{dom}(f)$, and hence $\langle x^*, x\rangle \leq 0$ for all $x \in \mathrm{dom}(f) - \mathrm{dom}(g)$. By using the second part of Theorem 2.7, we obtain $0 \notin \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$, which contradicts the hypothesis. Thus we must have $\lambda < 0$, and so we obtain from (7):

$$\left\langle \frac{1}{\lambda}x^*, x \right\rangle + \mu \leq \left\langle \frac{1}{\lambda}x^*, x' \right\rangle + \mu', \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)), \forall (x', \mu') \in \mathrm{epi}(f).$$

Let be $r \in \mathbb{R}$ such that

$$\mu' + \langle x_0^*, x'\rangle \geq r \geq \mu + \langle x_0^*, x\rangle \ \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)) \ \forall (x', \mu') \in \mathrm{epi}(f),$$

where $x_0^* := \frac{1}{\lambda}x^*$. The first inequality shows that $f(x) \geq \langle -x_0^*, x\rangle + r$ for all $x \in X$, that is, $f^*(-x_0^*) \leq -r$. The second one gives us $-g(x) + v(P_F) + \langle x_0^*, x\rangle \leq r$ for all $x \in X$; hence $g^*(x_0^*) \leq r - v(P_F)$, and so we have $-f^*(-x_0^*) - g^*(x_0^*) \geq r + v(P_F) - r = v(P_F)$. This implies that $v(D_F) \geq v(P_F)$. As the opposite inequality is always true, we get $v(P_F) = v(D_F)$, and x_0^* is an optimal solution of the problem (D_F). ∎

The above theorem combined with Remark 3.4(b) gives us the following result.

COROLLARY 3.6. *Suppose that the primal problem* (P_F) *has an optimal solution,* $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$, $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$, *and* $(0,0) \notin \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. *Then* $v(P_F) = v(D_F)$, *and* (D_F) *has an optimal solution.*

Remark 3.7. The condition $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$ implies that

$$0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g)) \Leftrightarrow 0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g)).$$

Indeed, denote that $C := \mathrm{dom}(f) - \mathrm{dom}(g)$. Obviously $0 \in \mathrm{qi}\, C$ implies that $0 \in \mathrm{qri}\, C$. Suppose now that $0 \in \mathrm{qri}\, C$, and let $x^* \in N_C(0)$ be arbitrary. We have $\langle x^*, x\rangle \leq 0$ for all $x \in C$. Since $N_C(0)$ is a linear subspace of X^*, we obtain $\langle x^*, x\rangle = 0$ for all $x \in C$. We get further $\langle x^*, x\rangle = 0$ for all $x \in \mathrm{cl\,cone}(C - C) = X$, which implies that $x^* = 0$. Thus $N_C(0) = \{0\}$, and the conclusion follows.

Some stronger versions of Theorem 3.5 and Corollary 3.6, respectively, follow.

THEOREM 3.8. *Suppose that* $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri\,co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$, *and* (D_F) *has an optimal solution.*

Proof. We have $\mathrm{dom}(f) - \mathrm{dom}(g) \subseteq (\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))$, so the condition $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$ implies that $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$ and $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$. Then we apply Theorem 3.5 to obtain the conclusion. ∎

COROLLARY 3.9. *Suppose that the primal problem* (P_F) *has an optimal solution,* $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$, *and* $(0,0) \notin \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. *Then* $v(P_F) = v(D_F)$, *and* (D_F) *has an optimal solution.*

THEOREM 3.10. *Suppose that* $\mathrm{dom}(f) \cap \mathrm{qri\,dom}(g) \neq \emptyset$, $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(g))$, *and* $(0,0) \notin \mathrm{qri\,co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$, *and* (D_F) *has an optimal solution.*

Proof. We apply Lemma 2.6 with $A := \mathrm{dom}(g)$ and $B := \mathrm{dom}(f)$. We get $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(f))$ or, equivalently, $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$. We obtain the result by applying Theorem 3.8. ∎

Corollary 3.11. *Suppose that the primal problem (P_F) has an optimal solution, $\operatorname{dom}(f) \cap \operatorname{qri} \operatorname{dom}(g) \neq \emptyset$, $0 \in \operatorname{qi}(\operatorname{dom}(g) - \operatorname{dom}(g))$, and $(0,0) \notin \operatorname{qri}[\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))]$. Then $v(P_F) = v(D_F)$, and (D_F) has an optimal solution.*

Remark 3.12. (a) We introduced above three new regularity conditions for Fenchel duality. As one can easily see from the proof of these results, the relation between these conditions is the following one: The regularity condition given in Theorem 3.10 (Corollary 3.11) implies the one given in Theorem 3.8 (Corollary 3.9), which implies the one given in Theorem 3.5 (Corollary 3.6).

(b) If we renounce the condition $(0,0) \notin \operatorname{qri} \operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}]$, or, respectively, $(0,0) \notin \operatorname{qri}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)))$, in the case when the primal problem has an optimal solution, then the duality results given above may fail. By using again Example 3.1 we show that these conditions are essential in our theory. Let us notice that for the problem in Example 3.1 the conditions $0 \in \operatorname{qi}[(\operatorname{dom}(f) - \operatorname{dom}(g)) - (\operatorname{dom}(f) - \operatorname{dom}(g))]$ and $0 \in \operatorname{qri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ are fulfilled. We prove in the following that in the aforementioned example we have $(0,0) \in \operatorname{qri}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)))$. Note that the scalar product on l^2, $\langle \cdot, \cdot \rangle : l^2 \times l^2 \to \mathbb{R}$, is given by $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$ for all $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in l^2$. Also, for $k \in \mathbb{N}$, we denote by $e^{(k)}$ the element in l^2 which has on the kth position 1 and on the other positions 0, that is, $e_n^{(k)} = 1$, if $n = k$ and $e_n^{(k)} = 0$, for all $n \in \mathbb{N} \setminus \{k\}$. We have $\operatorname{epi}(f) = C \times [0, \infty)$. Further, $\widehat{\operatorname{epi}}(g - v(P_F)) = \{(x, r) \in l^2 \times \mathbb{R} : r \leq -g(x)\} = \{(x, r) \in l^2 \times \mathbb{R} : x = (x_n)_{n \in \mathbb{N}} \in S, r \leq -x_1\} = \{(x, -x_1 - \varepsilon) \in l^2 \times \mathbb{R} : x = (x_n)_{n \in \mathbb{N}} \in S, \varepsilon \geq 0\}$. Then $A := \operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)) = \{(x - x', x_1' + \varepsilon) : x \in C, x' = (x_n')_{n \in \mathbb{N}} \in S, \varepsilon \geq 0\}$. Take $(x^*, r) \in N_A(0,0)$, where $x^* = (x_n^*)_{n \in \mathbb{N}}$. We have

$$\langle x^*, x - x' \rangle + r(x_1' + \varepsilon) \leq 0 \ \forall x \in C \ \forall x' = (x_n')_{n \in \mathbb{N}} \in S \ \forall \varepsilon \geq 0. \tag{8}$$

By taking in (8) $x' = 0$ and $\varepsilon = 0$ we get $\langle x^*, x \rangle \leq 0$ for all $x \in C$. As C is a linear subspace of X we have

$$\langle x^*, x \rangle = 0 \ \forall x \in C. \tag{9}$$

Since $e^{(2k-1)} - e^{(2k)} \in C$, for all $k \in \mathbb{N}$, the relation (9) implies that

$$x_{2k-1}^* - x_{2k}^* = 0 \ \forall k \in \mathbb{N}. \tag{10}$$

From (8) and (9) we obtain

$$\langle -x^*, x' \rangle + r(x_1' + \varepsilon) \leq 0 \ \forall x' = (x_n')_{n \in \mathbb{N}} \in S \ \forall \varepsilon \geq 0. \tag{11}$$

By taking $\varepsilon = 0$ and $x' := me^1 \in S$ in (11), where $m \in \mathbb{Z}$ is arbitrary, we get $m(-x_1^* + r) \leq 0$ for all $m \in \mathbb{Z}$, and thus $r = x_1^*$. For $\varepsilon = 0$ in (11) we obtain $-\sum_{n=1}^{\infty} x_n^* x_n' + r x_1' \leq 0$ for all $x' \in S$. By taking into account that $r = x_1^*$, we get $-\sum_{n=2}^{\infty} x_n^* x_n' \leq 0$ for all $x' \in S$. As S is a linear subspace of X it follows that $\sum_{n=2}^{\infty} x_n^* x_n' = 0$ for all $x' \in S$, but, since $e^{(2k)} - e^{(2k+1)} \in S$ for all $k \in \mathbb{N}$, the above relation shows that

$$x_{2k}^* - x_{2k+1}^* = 0 \ \forall k \in \mathbb{N}. \tag{12}$$

By combining (10) with (12) we get $x^* = 0$ (since $x^* \in l^2$). Because $r = x_1^*$, we also have $r = 0$. Thus $N_A(0,0) = \{(0,0)\}$, and Proposition 2.4 gives us the desired conclusion.

(c) Since in all of the strong duality results given above one must have that $0 \in \operatorname{qi}[(\operatorname{dom}(f)-\operatorname{dom}(g))-(\operatorname{dom}(f)-\operatorname{dom}(g))]$, in view of Remark 3.4, the condition $(0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]$ (respectively, $(0,0) \notin \operatorname{qri}[\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))]$) is equivalent to $(0,0) \notin \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]$ (respectively, $(0,0) \notin \operatorname{qi}[\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))]$).

(d) We have the following relation:

$$(0,0) \in \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}] \Rightarrow 0 \in \operatorname{qi}(\operatorname{dom}(f)-\operatorname{dom}(g)).$$

Indeed, $(0,0) \in \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}] \Leftrightarrow \operatorname{cl}\operatorname{cone}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}] = X\times\mathbb{R}$; hence $\operatorname{cl}\operatorname{cone}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))) = X\times\mathbb{R}$. Since $\operatorname{cl}\operatorname{cone}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))) \subseteq \operatorname{cl}\operatorname{cone}(\operatorname{dom}(f)-\operatorname{dom}(g))\times\mathbb{R}$, this implies that $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(f)-\operatorname{dom}(g)) = X$, that is, $0 \in \operatorname{qi}(\operatorname{dom}(f)-\operatorname{dom}(g))$. Hence

$$0 \notin \operatorname{qi}(\operatorname{dom}(f)-\operatorname{dom}(g)) \Rightarrow (0,0) \notin \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}].$$

Nevertheless, in the regularity conditions given above one cannot substitute the condition $(0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]$ by the "nice-looking" one $0 \notin \operatorname{qi}(\operatorname{dom}(f)-\operatorname{dom}(g))$, since in all three strong duality theorems the other hypotheses we consider imply that $0 \in \operatorname{qi}(\operatorname{dom}(f)-\operatorname{dom}(g))$ (cf. Remark 3.7).

Example 3.13. Consider again the space $X = l^2$ equipped with the norm $\|\cdot\| : l^2 \to \mathbb{R}$, $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2$ for all $x = (x_n)_{n\in\mathbb{N}} \in l^2$. We define the functions $f, g : l^2 \to \overline{\mathbb{R}}$ by

$$f(x) = \begin{cases} \|x\| & \text{if } x \in x_0 - l^2_+, \\ +\infty & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \langle c, x\rangle & \text{if } x \in l^2_+, \\ +\infty & \text{otherwise,} \end{cases}$$

where $l^2_+ = \{(x_n)_{n\in\mathbb{N}} \in l^2 : x_n \geq 0, \forall n \in \mathbb{N}\}$ is the positive cone, $x_0 = (\frac{1}{n})_{n\in\mathbb{N}}$, and $c = (\frac{1}{2^n})_{n\in\mathbb{N}}$. Note that $v(P_F) = \inf_{x\in l^2}\{f(x)+g(x)\} = 0$, and the infimum is attained at $x = 0$. We have $\operatorname{dom}(f) = x_0 - l^2_+ = \{(x_n)_{n\in\mathbb{N}} \in l^2 : x_n \leq \frac{1}{n}, \forall n \in \mathbb{N}\}$ and $\operatorname{dom}(g) = l^2_+$. Since $\operatorname{qri} l^2_+ = \{(x_n)_{n\in\mathbb{N}} \in l^2 : x_n > 0, \forall n \in \mathbb{N}\}$ (cf. [3]), we get $\operatorname{dom}(f)\cap\operatorname{qri}\operatorname{dom}(g) = \{(x_n)_{n\in\mathbb{N}} \in l^2 : 0 < x_n \leq \frac{1}{n}, \forall n \in \mathbb{N}\} \neq \emptyset$. Also, $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(g)-\operatorname{dom}(g)) = l^2$, so $0 \in \operatorname{qi}(\operatorname{dom}(g)-\operatorname{dom}(g))$. Further, $\operatorname{epi}(f) = \{(x,r) \in l^2\times\mathbb{R} : x \in x_0 - l^2_+, \|x\| \leq r\} = \{(x, \|x\|+\varepsilon) \in l^2\times\mathbb{R} : x \in x_0 - l^2_+, \varepsilon \geq 0\}$ and $\widehat{\operatorname{epi}}(g-v(P_F)) = \{(x,r) \in l^2\times\mathbb{R} : r \leq -g(x)\} = \{(x,r) \in l^2\times\mathbb{R} : r \leq -\langle c,x\rangle, x \in l^2_+\} = \{(x, -\langle c,x\rangle-\varepsilon) : x \in l^2_+, \varepsilon \geq 0\}$. We get $\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)) = \{(x-x', \|x\|+\varepsilon+\langle c,x'\rangle+\varepsilon') : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon,\varepsilon' \geq 0\} = \{(x-x', \|x\|+\langle c,x'\rangle+\varepsilon) : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon \geq 0\}$.

In the following we prove that $(0,0) \notin \operatorname{qri}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))$. By assuming the contrary we would have that the set $\operatorname{cl}(\operatorname{cone}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))))$ is a linear subspace. Since $(0,1) \in \operatorname{cl}(\operatorname{cone}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))))$ (take $x = x' = 0$ and $\varepsilon = 1$) we must have that also $(0,-1)$ belongs to this set. On the other hand, one can easily see that for all (x,r) belonging to $\operatorname{cl}(\operatorname{cone}(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F))))$ it holds that $r \geq 0$. This leads to the desired contradiction.

Hence the conditions of Corollary 3.11 are fulfilled, and thus strong duality holds. Let us notice that the regularity conditions given in Corollaries 3.6 and 3.9 are also fulfilled (see Remark 3.12(a)).

On the other hand, l^2 is a Fréchet space (being a Hilbert space), the functions f and g are lower semicontinuous, and, as $\operatorname{sqri}(\operatorname{dom}(f) - \operatorname{dom}(g)) = \operatorname{sqri}(x_0 - l^2_+) = \emptyset$, none of the constraint qualifications (i)–(iv) presented in the beginning of this section can be applied for this optimization problem.

As for all $x^* \in l^2$ it holds that

$$g^*(x^*) = \begin{cases} 0 & \text{if } x^* \in c - l^2_+, \\ +\infty & \text{otherwise} \end{cases}$$

and (see Theorem 2.8.7 in [17])

$$f^*(-x^*) = \inf_{x_1^*+x_2^*=-x^*} \{\|\cdot\|^*(x_1^*) + \delta^*_{x_0 - l^2_+}(x_2^*)\} = \inf_{\substack{x_1^*+x_2^*=-x^*,\\ \|x_1^*\|\le 1, x_2^* \in l^2_+}} \{\langle x_2^*, x_0\rangle\},$$

the optimal objective value of the Fenchel dual problem is

$$v(D_F) = \sup_{x^* \in X^*} \{-f^*(-x^*) - g^*(x^*)\}$$
$$= \sup_{\substack{x_2^* \in l^2_+ - c - x_1^*,\\ \|x_1^*\|\le 1, x_2^* \in l^2_+}} \{\langle -x_2^*, x_0\rangle\} = \sup_{x_2^* \in l^2_+} \{\langle -x_2^*, x_0\rangle\} = 0,$$

and $x_2^* = 0$ is the optimal solution of the dual.

In the following, by using the results introduced above, we give regularity conditions for the following convex optimization problem:

$$(P_A) \inf_{x \in X} \{f(x) + (g \circ A)(x)\},$$

where X and Y are separated locally convex vector spaces with their topological dual spaces X^* and Y^*, respectively, $A : X \to Y$ is a linear continuous mapping, $f : X \to \overline{\mathbb{R}}$, and $g : Y \to \overline{\mathbb{R}}$ are proper convex functions such that $A(\operatorname{dom}(f)) \cap \operatorname{dom}(g) \neq \emptyset$. The Fenchel dual problem to (P_A) is (cf. [17])

$$(D_A) \sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\},$$

where $A^* : Y^* \to X^*$ is the *adjoint operator* of A, defined in the usual way: $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for all $(y^*, x) \in Y^* \times X$. We denote by $v(P_A)$ and $v(D_A)$ the optimal objective values of the primal and the dual problem, respectively. We suppose also that $v(P_A) \in \mathbb{R}$. In the following theorem the set

$$A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) = \{(Ax, r) \in Y \times \mathbb{R} : f(x) \le r\}$$

is the image of $\operatorname{epi}(f)$ through the operator $A \times \operatorname{id}_{\mathbb{R}}$.

THEOREM 3.14. *Suppose that* $0 \in \operatorname{qi}[(A(\operatorname{dom}(f)) - \operatorname{dom}(g)) - (A(\operatorname{dom}(f)) - \operatorname{dom}(g))]$, $0 \in \operatorname{qri}(A(\operatorname{dom}(f)) - \operatorname{dom}(g))$, *and* $(0, 0) \notin \operatorname{qri} \operatorname{co}[(A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))) \cup \{(0, 0)\}]$. *Then* $v(P_A) = v(D_A)$, *and* (D_A) *has an optimal solution.*

Proof. Let us introduce the following functions: $F, G : X \times Y \to \overline{\mathbb{R}}$, $F(x, y) = f(x) + \delta_{\{x \in X : Ax = y\}}(x)$, and $G(x, y) = g(y)$. The functions F and G are proper and convex, and $\inf_{(x,y) \in X \times Y}[F(x, y) + G(x, y)] = \inf_{x \in X}\{f(x) + (g \circ A)(x)\} = v(P_A)$. Moreover, $\operatorname{dom}(F) = \operatorname{dom}(f) \times A(\operatorname{dom}(f))$ and $\operatorname{dom}(G) = X \times \operatorname{dom}(g)$, so $\operatorname{dom}(F) \cap \operatorname{dom}(G) \neq \emptyset$. Further,

$$\operatorname{dom}(F) - \operatorname{dom}(G) = X \times (A(\operatorname{dom}(f)) - \operatorname{dom}(g)).$$

By combining the last relation with the hypotheses, we obtain $(0,0) \in \operatorname{qi}[(\operatorname{dom}(F) - \operatorname{dom}(G)) - (\operatorname{dom}(F) - \operatorname{dom}(G))]$ and $(0,0) \in \operatorname{qri}(\operatorname{dom}(F) - \operatorname{dom}(G))$. Since $\operatorname{epi}(F) = \{(x, Ax, r) : f(x) \leq r\}$ and $\widehat{\operatorname{epi}}(G - v(P_A)) = \{(x, y, r) : r \leq -G(x, y) + v(P_A)\} = X \times \widehat{\operatorname{epi}}(g - v(P_A))$, we obtain

$$\operatorname{epi}(F) - \widehat{\operatorname{epi}}(G - v(P_A)) = X \times (A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))),$$

and this means that $(0,0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(F) - \widehat{\operatorname{epi}}(G - v(P_A))) \cup \{(0,0,0)\}]$. Theorem 3.5 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x,y) + G(x,y)] = \max_{(x^*,y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\}.$$

On the other hand, $F^*(x^*, y^*) = f^*(x^* + A^*y^*)$ for all $(x^*, y^*) \in X^* \times Y^*$, and

$$G^*(x^*, y^*) = \begin{cases} g^*(y^*) & \text{if } x^* = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, $\max_{(x^*,y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\} = \max_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}$, and the conclusion follows. □

Corollary 3.15. *Suppose that the primal problem* (P_A) *has an optimal solution,* $0 \in \operatorname{qi}[(A(\operatorname{dom}(f)) - \operatorname{dom}(g)) - (A(\operatorname{dom}(f)) - \operatorname{dom}(g))]$, $0 \in \operatorname{qri}(A(\operatorname{dom}(f)) - \operatorname{dom}(g))$, *and* $(0,0) \notin \operatorname{qri}[A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))]$. *Then* $v(P_A) = v(D_A)$, *and* (D_A) *has an optimal solution.*

Theorem 3.16. *Suppose that* $0 \in \operatorname{qi}(A(\operatorname{dom}(f)) - \operatorname{dom}(g))$ *and* $(0,0) \notin \operatorname{qri}\operatorname{co}[(A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))) \cup \{(0,0)\}]$. *Then* $v(P_A) = v(D_A)$, *and* (D_A) *has an optimal solution.*

Proof. By considering the functions F and G from the proof of Theorem 3.14, we have $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(F) - \operatorname{dom}(G)) = X \times \operatorname{cl}\operatorname{cone}(A(\operatorname{dom}(f)) - \operatorname{dom}(g)) = X \times Y$, and thus $(0,0) \in \operatorname{qi}(\operatorname{dom}(F) - \operatorname{dom}(G))$. Also we have $(0,0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(F) - \widehat{\operatorname{epi}}(G - v(P_A))) \cup \{(0,0,0)\}]$. Theorem 3.8 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x,y) + G(x,y)] = \max_{(x^*,y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\},$$

and the conclusion follows. □

Corollary 3.17. *Suppose that the primal problem* (P_A) *has an optimal solution,* $0 \in \operatorname{qi}(A(\operatorname{dom}(f)) - \operatorname{dom}(g))$, *and* $(0,0) \notin \operatorname{qri}[A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))]$. *Then* $v(P_A) = v(D_A)$, *and* (D_A) *has an optimal solution.*

Theorem 3.18. *Suppose that* $A(\operatorname{dom}(f)) \cap \operatorname{qri}\operatorname{dom}(g) \neq \emptyset$, $0 \in \operatorname{qi}(\operatorname{dom}(g) - \operatorname{dom}(g))$ *and* $(0,0) \notin \operatorname{qri}\operatorname{co}[(A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))) \cup \{(0,0)\}]$. *Then* $v(P_A) = v(D_A)$, *and* (D_A) *has an optimal solution.*

Proof. Consider again the functions F and G defined as in the proof of Theorem 3.14. We have $\operatorname{dom}(F) \cap \operatorname{qri}\operatorname{dom}(G) = (\operatorname{dom}(f) \times (A(\operatorname{dom}(f))) \cap (X \times \operatorname{qri}\operatorname{dom}(g))) = \operatorname{dom}(f) \times (A(\operatorname{dom}(f)) \cap \operatorname{qri}\operatorname{dom}(g)) \neq \emptyset$. Also, $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(G) - \operatorname{dom}(G)) = X \times \operatorname{cl}\operatorname{cone}(\operatorname{dom}(g) - \operatorname{dom}(g)) = X \times Y$, and hence $(0,0) \in \operatorname{qi}(\operatorname{dom}(G) - \operatorname{dom}(G))$. Moreover, $(0,0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(F) - \widehat{\operatorname{epi}}(G - v(P_A))) \cup \{(0,0,0)\}]$. Theorem 3.10 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x,y) + G(x,y)] = \max_{(x^*,y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\},$$

and the conclusion follows. □

COROLLARY 3.19. *Suppose that the primal problem* (P_A) *has an optimal solution,* $A(\operatorname{dom}(f)) \cap \operatorname{qri}\operatorname{dom}(g) \neq \emptyset$*,* $0 \in \operatorname{qi}(\operatorname{dom}(g) - \operatorname{dom}(g))$*, and* $(0,0) \notin \operatorname{qri}[A \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f)) - \widehat{\operatorname{epi}}(g - v(P_A))]$*. Then* $v(P_A) = v(D_A)$*, and* (D_A) *has an optimal solution.*

4. Lagrange duality. By using an approach due to Magnanti (cf. [13]), in this section we derive from the results in the previous section some duality results concerning the Lagrange dual problem. We work in the following setting. Let X be a real linear topological space and S a nonempty subset of X. Let Y be a separated locally convex space partially ordered by a convex cone C. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function $(f,g) : S \to \mathbb{R} \times Y$, defined by $(f,g)(x) = (f(x), g(x))$, for all $x \in S$, is convexlike with respect to the cone $\mathbb{R}_+ \times C \subseteq \mathbb{R} \times Y$; that is, the set $(f,g)(S) + \mathbb{R}_+ \times C$ is convex. Let us notice that this property implies that the sets $f(S) + [0,\infty)$ and $g(S) + C$ are convex (the reverse implication does not always hold). Consider the optimization problem

$$(P_L) \quad \inf_{\substack{x \in S \\ g(x) \in -C}} f(x),$$

where the constraint set $T = \{x \in S : g(x) \in -C\}$ is assumed to be nonempty. The Lagrange dual problem associated to (P_L) is

$$(D_L) \quad \sup_{\lambda \in C^*} \inf_{x \in S} [f(x) + \langle \lambda, g(x) \rangle],$$

where $C^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in C\}$ is the *dual cone* of C. Let us denote by $v(P_L)$ and $v(D_L)$ the optimal objective values of the primal and the dual problem, respectively. A regularity condition for strong duality between (P_L) and (D_L) was proposed in Theorem 2.2 in [4]. We show first that this theorem has self-contradictory assumptions. To this end we prove the following lemma.

LEMMA 4.1. *Suppose that* $\operatorname{cl}(C - C) = Y$ *and there exists* $\overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$*. Then the following assertions are true:*

(a) $0 \in \operatorname{qi}(g(S) + C)$*;*

(b) $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) + C)]$ *is a linear subspace of* Y*.*

Proof. (a) We apply Lemma 2.6 with $A := -C$ and $B := g(S) + C$. The condition $\operatorname{cl}(C - C) = Y$ implies that $0 \in \operatorname{qi}(A - A)$, while the Slater-type condition $g(\overline{x}) \in -\operatorname{qri} C$ ensures that $g(\overline{x}) \in \operatorname{qri} A \cap B$. Hence, by Lemma 2.6 we obtain $0 \in \operatorname{qi}(A - B)$, that is, $0 \in \operatorname{qi}(-g(S) - C)$, which is nothing else than $0 \in \operatorname{qi}(g(S) + C)$.

(b) From (a) it follows that $0 \in \operatorname{qri}(g(S) + C)$. By applying Proposition 2.5(vii) we get $0 \in \operatorname{qri}(\operatorname{qri}(g(S) + C))$, which is nothing else than $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) + C)]$ is a linear subspace of Y. ☐

In order to get strong duality between (P_L) and (D_L) in Theorem 2.2 in [4] the authors ask that the following hypotheses are fulfilled: $\operatorname{cl}(C - C) = Y$, there exists $\overline{x} \in S$ such that $g(\overline{x}) \in -\operatorname{qri} C$, $\operatorname{qri}(g(S) + C) \neq \emptyset$, and $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) + C)]$ is not a linear subspace of Y. The previous lemma proves that these assumptions are in contradiction.

Next we prove some Lagrange duality results written in terms of the quasi interior and quasi-relative interior, respectively. As in the previous section, we may suppose that $v(P_L)$ is a real number.

Consider the following convex set:

$$\mathcal{E}_{v(P_L)} = \{(f(x) + \alpha - v(P_L), g(x) + y) : x \in S, \alpha \geq 0, y \in C\} \subseteq \mathbb{R} \times Y.$$

Let us notice that the set $-\mathcal{E}_{v(P_L)}$ is in analogy with the *conic extension*, a notion used by Giannessi in the theory of image space analysis (see [7]). One can easily prove that the primal problem (P_L) has an optimal solution if and only if $(0,0) \in \mathcal{E}_{v(P_L)}$. Let us introduce the functions $f_1, f_2 : \mathbb{R} \times Y \to \overline{\mathbb{R}}$,

$$f_1(y_0, y_1) = \begin{cases} y_0 & \text{if } (y_0, y_1) \in \mathcal{E}_{v(P_L)} + (v(P_L), 0), \\ +\infty & \text{otherwise,} \end{cases}$$

and $f_2 = \delta_{\mathbb{R}\times(-C)}$. It holds that

$$\operatorname{dom}(f_1) - \operatorname{dom}(f_2) = \mathbb{R} \times (g(S) + C). \tag{13}$$

Moreover, as pointed out by Magnanti (cf. [13]), we have

$$\inf_{(y_0,y_1)\in\mathbb{R}\times Y} \{f_1(y_0, y_1) + f_2(y_0, y_1)\} = \inf_{\substack{x\in S \\ g(x)\in -C}} f(x) = v(P_L) \tag{14}$$

and

$$\sup_{(y_0^*,y_1^*)\in\mathbb{R}\times Y^*} \{-f_1^*(-y_0^*, -y_1^*) - f_2^*(y_0^*, y_1^*)\} = \sup_{\lambda\in C^*} \inf_{x\in S} [f(x) + \langle \lambda, g(x)\rangle] = v(D_L). \tag{15}$$

With this approach, we can derive from the strong duality results given for Fenchel duality corresponding strong duality results for Lagrange duality.

THEOREM 4.2. *Suppose that* $0 \in \operatorname{qi}[(g(S) + C) - (g(S) + C)]$, $0 \in \operatorname{qri}(g(S) + C)$, *and* $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$. *Then* $v(P_L) = v(D_L)$, *and* (D_L) *has an optimal solution.*

Proof. The hypotheses of the theorem and (13) imply that the conditions $(0,0) \in \operatorname{qi}[(\operatorname{dom}(f_1)-\operatorname{dom}(f_2))-(\operatorname{dom}(f_1)-\operatorname{dom}(f_2))]$ and $(0,0) \in \operatorname{qri}(\operatorname{dom}(f_1)-\operatorname{dom}(f_2))$ are fulfilled. Further, $\operatorname{epi}(f_1) = \{(y_0, y_1, r) \in \mathbb{R}\times Y\times\mathbb{R} : (y_0, y_1) \in \mathcal{E}_{v(P_L)}+(v(P_L), 0), y_0 \le r\} = \{(f(x)+\alpha, g(x)+y, r) : x \in S, \alpha \ge 0, y \in C, f(x)+\alpha \le r\}$, and $\widehat{\operatorname{epi}}(f_2-v(P_L)) = \{(y_0, y_1, r) \in \mathbb{R} \times Y \times \mathbb{R} : r \le -f_2(y_0, y_1) + v(P_L)\} = \{(y_0, y_1, r) \in \mathbb{R} \times Y \times \mathbb{R} : y_0 \in \mathbb{R}, y_1 \in -C, r \le v(P_L)\} = \mathbb{R} \times (-C) \times (-\infty, v(P_L)]$. Thus $\operatorname{epi}(f_1) - \widehat{\operatorname{epi}}(f_2 - v(P_L)) = \operatorname{epi}(f_1) + \mathbb{R} \times C \times [-v(P_L), +\infty) = \{(f(x) + \alpha + a, g(x) + y, r - v(P_L) + \varepsilon) : x \in S, \alpha \ge 0, a \in \mathbb{R}, y \in C, \varepsilon \ge 0, f(x) + \alpha \le r\} = \{(f(x) + \alpha + a, g(x) + y, f(x) + \alpha + \varepsilon - v(P_L)) : x \in S, \alpha \ge 0, a \in \mathbb{R}, y \in C, \varepsilon \ge 0\}$, and this means that

$$\operatorname{epi}(f_1) - \widehat{\operatorname{epi}}(f_2 - v(P_L)) = \mathbb{R} \times \{(g(x) + y, f(x) + \alpha - v(P_L)) : x \in S, \alpha \ge 0, y \in C\}.$$

Thus $(0,0,0) \in \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f_1) - \widehat{\operatorname{epi}}(f_2 - v(P_L))) \cup \{(0,0,0)\}]$ if and only if $(0,0) \in \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$. Now we can apply Theorem 3.5 for f_1 and f_2, and we obtain

$$\inf_{(y_0,y_1)\in\mathbb{R}\times Y} \{f_1(y_0, y_1) + f_2(y_0, y_1)\} = \max_{(y_0^*,y_1^*)\in\mathbb{R}\times Y^*} \{-f_1^*(-y_0^*, -y_1^*) - f_2^*(y_0^*, y_1^*)\}.$$

By (14) and (15) the conclusion follows. ∎

COROLLARY 4.3. *Suppose that the primal problem* (P_L) *has an optimal solution,* $0 \in \operatorname{qi}[(g(S) + C) - (g(S) + C)]$, $0 \in \operatorname{qri}(g(S) + C)$, *and* $(0,0) \notin \operatorname{qri}\mathcal{E}_{v(P_L)}$. *Then* $v(P_L) = v(D_L)$, *and* (D_L) *has an optimal solution.*

Further, like for Fenchel duality, other Lagrange duality results can be stated.

THEOREM 4.4. *Suppose that* $0 \in \operatorname{qi}(g(S)+C)$ *and* $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)}\cup\{(0,0)\})$. *Then* $v(P_L) = v(D_L)$, *and* (D_L) *has an optimal solution.*

Proof. This is a direct consequence of the previous theorem since $g(S)+C \subseteq (g(S)+C)-(g(S)+C)$, and so the condition $0 \in \operatorname{qi}(g(S)+C)$ implies that $0 \in \operatorname{qi}[(g(S)+C)-(g(S)+C)]$ and $0 \in \operatorname{qri}(g(S)+C)$. □

COROLLARY 4.5. *Suppose that the primal problem* (P_L) *has an optimal solution,* $0 \in \operatorname{qi}(g(S)+C)$, *and* $(0,0) \notin \operatorname{qri}\mathcal{E}_{v(P_L)}$. *Then* $v(P_L)=v(D_L)$, *and* (D_L) *has an optimal solution.*

THEOREM 4.6. *Suppose that* $\operatorname{cl}(C-C)=Y$ *and there exists* $\overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$, *then* $v(P_L)=v(D_L)$, *and* (D_L) *has an optimal solution.*

Proof. The condition $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$ implies that $(0,0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f_1)-\widehat{\operatorname{epi}}(f_2-v(P_L))) \cup \{(0,0,0)\}]$ (cf. the proof of Theorem 4.2). Further, we have $\operatorname{dom}(f_1) \cap \operatorname{qri}\operatorname{dom}(f_2) = [\mathcal{E}_{v(P_L)}+(v(P_L),0)] \cap \operatorname{qri}(\mathbb{R}\times(-C)) = [\mathcal{E}_{v(P_L)}+(v(P_L),0)] \cap [\mathbb{R}\times(-\operatorname{qri} C)]$. From the Slater-type condition we get that $(f(\overline{x}),g(\overline{x})) \in [\mathcal{E}_{v(P_L)}+(v(P_L),0)] \cap [\mathbb{R}\times(-\operatorname{qri} C)]$, and hence $\operatorname{dom}(f_1) \cap \operatorname{qri}\operatorname{dom}(f_2) \neq \emptyset$. Moreover, $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(f_2)-\operatorname{dom}(f_2)) = \operatorname{cl}\operatorname{cone}[\mathbb{R}\times(C-C)] = \mathbb{R}\times\operatorname{cl}(C-C) = \mathbb{R}\times Y$, and hence $(0,0) \in \operatorname{qi}(\operatorname{dom}(f_2)-\operatorname{dom}(f_2))$. By Theorem 3.10 for f_1 and f_2 we obtain

$$\inf_{(y_0,y_1)\in\mathbb{R}\times Y}\{f_1(y_0,y_1)+f_2(y_0,y_1)\} = \max_{(y_0^*,y_1^*)\in\mathbb{R}\times Y^*}\{-f_1^*(-y_0^*,-y_1^*)-f_2^*(y_0^*,y_1^*)\},$$

and by using again (14) and (15) the conclusion follows. □

COROLLARY 4.7. *Suppose that the primal problem* (P_L) *has an optimal solution,* $\operatorname{cl}(C-C)=Y$, *and there exists* $\overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0,0) \notin \operatorname{qri}\mathcal{E}_{v(P_L)}$, *then* $v(P_L)=v(D_L)$, *and* (D_L) *has an optimal solution.*

Remark 4.8. Let us notice that from the above results one can derive duality theorems for the case when, in the set of constraints, one has also equalities defined by affine functions. Indeed, consider the optimization problem

$$(P_L^{aff}) \quad \inf_{\substack{x\in S\\ g(x)\in -C\\ h(x)=0}} f(x),$$

where $h: X \to Z$ is an affine mapping and Z is a real normed space (the hypotheses regarding the functions f and g remain the same as in the beginning of this section). The Lagrange dual problem associated to (P_L^{aff}) is

$$(D_L^{aff}) \quad \sup_{\substack{\lambda\in C^*\\ \mu\in Z^*}} \inf_{x\in S}[f(x)+\langle\lambda,g(x)\rangle+\langle\mu,h(x)\rangle],$$

where Z^* is the topological dual space of Z.

By using Theorems 4.2 and 4.4 one can formulate Lagrange duality theorems for (P_L^{aff}) and (D_L^{aff}) by noticing that the primal problem can be reformulated as

$$\inf_{\substack{x\in S\\ g(x)\in -C\\ h(x)=0}} f(x) = \inf_{\substack{x\in S\\ u(x)\in -(C\times\{0\})}} f(x),$$

where $u: S \to Y \times Z$, $u(x)=(g(x),h(x))$. For the optimization problem with equality and cone constraints some regularity conditions have been given in [5] by using the notion of a quasi-relative interior. Along them in the strong duality theorem (Theorem 3.1 in [5]) a "separation assumption," called by the authors *Assumption S*, is imposed. Unfortunately, this assumption is not only a sufficient condition for having strong duality, as claimed in the paper, but actually an equivalent formulation of this situation (this makes the other regularity conditions inoperative). More than that, in the proof of Theorem 3.1 in [5] a mistake occurred, namely, in the relation after inequality (8) when trying to prove the "nonverticality" of the separating hyperplane.

The approach we propose above offers a viable alternative for dealing with Lagrange duality for this class of optimization problems.

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REFERENCES

[1] J. M. Borwein and R. Goebel, *Notions of relative interior in Banach spaces*, J. Math. Sci. (N. Y.), 115 (2003), pp. 2542–2553.

[2] J. M. Borwein, V. Jeyakumar, A. S. Lewis, and H. Wolkowicz, *Constrained Approximation via Convex Programming*, preprint, University of Waterloo, 1988.

[3] J. M. Borwein and A. S. Lewis, *Partially finite convex programming, part* I: *Quasi relative interiors and duality theory*, Math. Program., 57 (1992), pp. 15–48.

[4] F. Cammaroto and B. Di Bella, *Separation theorem based on the quasirelative interior and application to duality theory*, J. Optim. Theory Appl., 125 (2005), pp. 223–229.

[5] P. Daniele and S. Giuffrè, *General infinite dimensional duality theory and applications to evolutionary network equilibrium problems*, Optim. Lett., 1 (2007), pp. 227–243.

[6] P. Daniele, S. Giuffrè, G. Idone, and A. Maugeri, *Infinite dimensional duality and applications*, Math. Ann., 339 (2007), pp. 221–239.

[7] F. Giannessi, *Constrained Optimization and Image Space Analysis, Vol.* 1. *Separation of Sets and Optimality Conditions*, Math. Concepts Methods Sci. Engrg. 49, Springer, New York, 2005.

[8] M. S. Gowda and M. Teboulle, *A comparison of constraint qualifications in infinite-dimensional convex programming*, SIAM J. Control Optim., 28 (1990), pp. 925–935.

[9] R. B. Holmes, *Geometric Functional Analysis*, Springer, Berlin, 1975.

[10] J. Jahn, *Introduction to the Theory of Nonlinear Optimization*, Springer, Berlin, 1996.

[11] V. Jeyakumar and H. Wolkowicz, *Generalizations of Slater's constraint qualification for infinite convex programs*, Math. Program., 57 (1992), pp. 85–101.

[12] M. A. Limber and R.K. Goodrich, *Quasi interiors, Lagrange multipliers, and L^p spectral estimation with lattice bounds*, J. Optim. Theory Appl., 78 (1993), pp. 143–161.

[13] T. L. Magnanti, *Fenchel and Lagrange duality are equivalent*, Math. Program., 7 (1974), pp. 253–258.

[14] R. T. Rockafellar, *Conjugate Duality and Optimization*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics 16, Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[15] B. Rodrigues, *The Fenchel duality theorem in Fréchet spaces*, Optimization, 21 (1990), pp. 13–22.

[16] C. Zălinescu, *A comparison of constraint qualifications in infinite-dimensional convex programming revisited*, J. Aust. Math. Soc. Ser. B, 40 (1999), pp. 353–378.

[17] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.