J. Korean Math. Soc. **45** (2008), No. 2, pp. 537–558

REGULARITY CRITERION ON WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Consider a weak solution u of the Navier-Stokes equations in the class $L^2\left((0,T);\dot{X}_1(\mathbb{R}^d)^d\right)$. We establish a new approach to treat the regularity problem for the Navier-Stokes equation in term of the multiplier space $\dot{X}_1(\mathbb{R}^d)$.

1. Introduction

Consider the Navier-Stokes equations in $(0,T)\times\mathbb{R}^d$ with $0<T<\infty$ and $d\geq 3$:

$$
(1.1)\qquad \begin{aligned}
\partial_t u + (u.\nabla)u - \Delta u + \nabla p &= 0, &\quad (x,t)&\in\mathbb{R}^d\times(0,\infty),\\
\nabla.u &= 0, &\quad (x,t)&\in\mathbb{R}^d\times(0,\infty),\\
u(x,0) &= a(x), &\quad x&\in\mathbb{R}^d,
\end{aligned}
$$

where $u=u(x,t)$ is the velocity field, $p=p(x,t)$ is the scalar pressure and $a(x)$ with div $a=0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

In their famous paper, Leray [12] and Hopf [6] constructed a weak solution u of (1.1) for arbitrary $a\in L^2_\sigma$. The solution is called the Leray-Hopf weak solution. In the general case the problem on uniqueness and regularity of Leray-Hopf' s weak solutions are still open question. Masuda [14] extended Serrin' s class for uniqueness of weak solutions and made it clear that the class $L^\infty\left((0,T);L^d\left(\mathbb{R}^d\right)\right)$ plays an important role for uniqueness of weak solutions. Kozono-Sohr [8] showed that the uniqueness holds in $L^\infty\left((0,T);L^d\right)$.

Foias [4] and Serrin [16] introduced the class $L^\alpha\left((0,\infty);L^q\right)$ and showed that under the additional assumption

$$u\in L^\alpha\left((0,\infty);L^q\right)\quad\text{for}\quad \frac{2}{\alpha}+\frac{d}{q}=1\quad\text{with}\quad q>d,$$

Received September 16, 2006; Revised July 27, 2007.
2000 *Mathematics Subject Classification.* 35B45, 35B65, 76D05.
Key words and phrases. Navier-Stokes equations, weak solution, regularity, multiplier spaces.

u is the only weak solution.

The purpose of this paper is to improve the criterion on regularity of weak solutions to in the class $L^2\left((0,T);\dot{X}_1(\mathbb{R}^d)^d\right)$. We know that for every $a \in L^2_\sigma(\mathbb{R}^d)$, there is at least one weak solution u of (1.1) satisfying the energy inequality :

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \|a\|_{L^2}^2 .$$

This is the solution obtained by Leray [12] in the class $L^\infty\left((0,T);L^2_\sigma\right) \cap L^2\left((0,T);\dot{H}^1_\sigma\right)$ and satisfying (1.1) in the sense of distributions. The natural regularity obtained from the above energy inequality is that

$$u \in L^\alpha\left((0,T);L^q\left(\mathbb{R}^d\right)\right) \quad \text{for} \quad \frac{2}{\alpha}+\frac{d}{q}=2 \quad \text{with} \quad 2 \leq q \leq \frac{2d}{d-2}.$$

If Leray' s weak solution u satisfies the following

$$u \in L^\alpha\left((0,T);L^q\left(\mathbb{R}^d\right)\right) \quad \text{for} \quad \frac{2}{\alpha}+\frac{d}{q}=1 \quad \text{with} \quad q > d,$$

then u is regular on $(0,T]$. For more facts concerning regularity of weak solutions, we refer to a celebrated paper of Kozono-Sohr [8].

1.1. BMO and Hardy space $\mathcal{H}^1(\mathbb{R}^d)$

We recall that a locally summable function g on $\mathbb{R}^d$ is said to have bounded mean oscillation if

$$\|g\|_{BMO} = \sup_{x,R} \frac{1}{|B(x,R)|} \int_{B(x,R)} \left|g(y) - g_{B(x,R)}\right| dy < \infty,$$

where

$$g_{B(x,R)} = \frac{1}{|B(x,R)|} \int_{B(x,R)} g(y)dy.$$

The class of functions of bounded mean oscillation is denoted by BMO and often is refereed as John-Nirenberg space.

Note that

$$\|g\|_{BMO} = 0 \quad \text{if and only if} \quad g = \text{const}.$$

It is thus natural to consider the quotient space $BMO/\mathbb{R}$ with the norm induced by $\|.\|_{BMO}$. Then $BMO/\mathbb{R}$ is a Banach space, which will also be denoted BMO for simplicity. We easily see that $L^\infty \subset BMO$ with continuous injection. For $f(x) = \log|x|$, we have $f \in BMO$ but $f \notin L^\infty$, so BMO is strictly larger than L^∞.

Next, we recall the definition and some of the main properties of Hardy spaces $\mathcal{H}^p(\mathbb{R}^d)$ introduced by E. Stein and G. Weiss [18] (for more facts on these spaces see C. Fefferman and E. Stein [5]).

Definition 1 ([5]). Let $0 < p < \infty$, and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfy $\int\limits_{\mathbb{R}^n} \varphi dx = 1$. A tempered distribution f belongs to the Hardy space $\mathcal{H}^p(\mathbb{R}^d)$ if

$$f^*(x) = \sup_{t>0} |(\varphi_t * f)(x)| \in L^p(\mathbb{R}^d), \tag{1.2}$$

where $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$.

It is known that if $f \in \mathcal{H}^p(\mathbb{R}^d)$, then (1.2) holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\int\limits_{\mathbb{R}^d} \varphi dx = 1$. The (quasi)-norm of $\mathcal{H}^p(\mathbb{R}^d)$ is defined, up to equivalence, by

$$||f||_{\mathcal{H}^p(\mathbb{R}^d)} = ||f^*(x)||_{L^p(\mathbb{R}^d)} = \left(\int\limits_{\mathbb{R}^d} |f^*(x)|^p \, dx\right)^{\frac{1}{p}}.$$

We known by ([5], [17]) that if $1 \le p < \infty$, then $\mathcal{H}^p$ is a Banach space :

$$\begin{aligned} \mathcal{H}^p(\mathbb{R}^d) &= L^p(\mathbb{R}^d) \quad \text{for} \quad 1 < p < \infty, \\ \mathcal{H}^1(\mathbb{R}^d) &\subset L^1(\mathbb{R}^d) \quad \text{with continuous injection,} \end{aligned}$$

and that $\mathcal{H}^p(\mathbb{R}^d)$, $0 < p < 1$, are quasi-Banach spaces in the quasi-norm $||\cdot||_{\mathcal{H}^p(\mathbb{R}^d)}$.

The crucial fact for our purpose is the boundedness of the Riesz transforms R_j on all of the spaces $\mathcal{H}^p$. Furthermore, an L^1-function f on $\mathbb{R}^d$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$ if and only if its Riesz transforms $R_j f$ all belong to $L^1(\mathbb{R}^d)$ and

$$||f||_{\mathcal{H}^1(\mathbb{R}^d)} \cong ||f||_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} ||R_j f||_{L^1(\mathbb{R}^d)} \quad \text{(equivalent norms).}$$

Notice that all function $f \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy

$$\int\limits_{\mathbb{R}^d} f(x) dx = 0. \tag{1.3}$$

Indeed, the assumption $f \in \mathcal{H}^1(\mathbb{R}^d)$ implies that the Fourier transforms

$$\widehat{f}(\xi) = \int f(x) e^{-ix\xi} dx \quad \text{and} \quad \widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|}\widehat{f}(\xi), \quad (j = 1, \ldots, d),$$

are all continuous on $\mathbb{R}^d$, so $\widehat{f}(0) = 0$, and (1.3) is proved.

A fundamental theorem in the theory of Hardy spaces $\mathcal{H}^1(\mathbb{R}^d)$ developed by C. Fefferman and E. Stein [5] asserts

Theorem 1 (Fefferman). *The dual space of $\mathcal{H}^1(\mathbb{R}^d)$ is BMO. More precisely, L is a continuous linear functional on $\mathcal{H}^1(\mathbb{R}^d)$ if and only if it can be represented* *as*

$$L(f) = \int_{\mathbb{R}^d} fg$$

for some function g in BMO, moreover for any $g \in BMO$ and any $f \in \mathcal{H}^1(\mathbb{R}^d)$ we have

$$\left| \int_{\mathbb{R}^d} fg dx \right| \leq c(d) \|f\|_{\mathcal{H}^1} \|g\|_{BMO} . \tag{1.4}$$

Let $\gamma > 1$. We define the maximal function of f depending on γ,

$$M_\gamma f(x) = \sup_{t>0} \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |f(y)|^\gamma dy \right)^{\frac{1}{\gamma}} .$$

We begin by establishing the following result which is a variant of the Hardy-Littlewood maximal theorem. We need

Lemma 1. *If $\gamma < p \leq \infty$, then*

$$M_\gamma : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \quad \textit{is bounded.}$$

See [17] for the proof.

In [2], Coifman, Lions, Meyer, and Semmes, it was shown that the Hardy spaces can be used to analyze the regularity of the various nonlinear quantities by the compensated compactness theory due to L. Murat [13] and F. Tartar [15]. Since then, theses spaces play an important role in studing the regularity of solutions to partial differential equations. In particular, it was shown that for exponents p, q with $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and vector fields $u \in L^p(\mathbb{R}^d)^d$, $v \in L^q(\mathbb{R}^d)^d$ with div $u = 0$, curl $v = 0$ in the sense of distributions, the scalar product $u.v$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$. Moreover, there exists a positive constant C such that

$$\|u.v\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \|u\|_{L^p} \|v\|_{L^q} .$$

The main purpose of this subsection is to prove two facts about div-curl lemma without assuming any a priori assumptions on exact cancelation, namely the divergence and curl need not be zero, and which lead to div (uv) being in the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$.

The proof will be divided into two parts. In part 1, we consider the case u and v are supported on the ball $|x| \leq R_0$ where $R_0 > 1$ is a positive constant to be determined later, while in part 2, the general case follows by partition of unity. In order to simplify the presentation, we take $p = q = 2$.

The Sobolev space $H_p^1(\mathbb{R}^d)$, $1 \leq p < \infty$, consists of functions $f \in L^p(\mathbb{R}^d)$ such that $|\nabla f| \in L^p(\mathbb{R}^d)$. It is a Banach space with respect to the norm

$$\|f\|_{H_p^1} = \|f\|_{L^p} + \|\nabla f\|_{L^p} .$$

Specifically, we will prove

Theorem 2. *Let* $u \in H_p^1\left(\mathbb{R}^d\right)^d$ *and* $v \in H_q^1\left(\mathbb{R}^d\right)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then there exists a positive constant* $C(d)$ *such that*

$$\|\operatorname{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \le C\left(\|u\|_{L^p}\|\nabla v\|_{L^q} + \|\operatorname{div} u\|_{L^p}\|v\|_{L^q}\right). \tag{1.5}$$

Remark 1. Such inequalities and their generalizations are useful in hydrodynamics. Reader is refereed, in particular to [2], [3].

Theorem 2 is a generalized version of the "div-curl" lemma ([2], Theorem II.1). Observe that when div $u = 0$, Theorem 2 reduces to the classical div-curl lemma [2].

The following result due to [2], shows the importance of the Hardy space theory in estimating the non-linear term $u.\nabla v$ attached to the Navier-Stokes equations and this produces a useful tool for PDE.

Lemma 2. *Let* $1 < p < \infty$, $1 < q < d$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + 1$. *If* $u \in L^p\left(\mathbb{R}^d\right)^d$ *with* $\nabla.u = 0$ *and* $\nabla v \in L^q\left(\mathbb{R}^d\right)$. *Then*

$$u.\nabla v \in \mathcal{H}^r\left(\mathbb{R}^d\right),$$

and

$$\|u.\nabla v\|_{\mathcal{H}^r(\mathbb{R}^d)} \le C\|u\|_{L^p}\|\nabla v\|_{L^q}.$$

Proof. The result is due to [2]; but we give it here a detailed proof for the reader' s convenience. Observe that

$$f = u.\nabla v = \nabla.\left(u \otimes (v - c)\right)$$

for an arbitrary constant vector c. So we get

$$(\varphi_t * f)(x) = t^{-d-1}\int_{B_t(x)} (\nabla\varphi)\left(t^{-1}(x-y)\right)u(y)\left(v(y) - m_B(v)\right)dy,$$

where

$$m_B(v) = \frac{1}{|B_t(x)|}\int_{B_t(x)} v(y)dy.$$

Taking

$$1 < \gamma < \infty, \quad 1 < \beta < d, \quad \text{with} \quad \frac{1}{\gamma} + \frac{1}{\beta} = 1 + \frac{1}{d},$$

and writing

$$\frac{1}{\beta^*} = \frac{1}{\beta} - \frac{1}{d},$$

we see by Poincaré-Sobolev inequality that

$$\begin{aligned}
|(\varphi_t * f)(x)| &\leq \frac{C}{t^{d+1}} \left(\int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\int_{B_t(x)} |v(y) - m_B(v)|^{\beta^*} \, dy \right)^{\frac{1}{\beta^*}} \\
&\leq \frac{C}{t^{d+1}} \left(\int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\int_{B_t(x)} |\nabla v(y)|^\beta \, dy \right)^{\frac{1}{\beta}} \\
&= C \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left(\frac{1}{|B_t(x)|} \int_{B_t(x)} |\nabla v(y)|^\beta \, dy \right)^{\frac{1}{\beta}} \\
&\leq C \, (M_\gamma u)(x). \, (M_\beta(\nabla v))(x).
\end{aligned}$$

We thus obtain

$$\sup_{t>0} |(\varphi_t * f)(x)| \leq C \, (M_\gamma u)(x). \, (M_\beta(\nabla v))(x).$$

Since we can take γ and β so that

$$1 < \gamma < p, \quad 1 < \beta < q < d,$$

it follows from Lemma 1 that

$$\|M_\gamma u\|_{L^p} \leq C \|u\|_{L^p}, \qquad \|M_\beta (\nabla v)\|_{L^q} \leq C \|\nabla v\|_{L^q}.$$

Lemma 2 now follows from Hölder' s inequality :

$$\|f.g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \qquad \left(0 < p < \infty, 0 < q < \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \right).$$

This finishes the proof of the lemma. □

We are now in a position to proof Theorem 2.

Proof. To prove this, we distinguish three cases.

Case A. Let us assume first that

$$\nabla . u = 0.$$

In this case we get

$$\operatorname{div}(vu) = (\nabla v).u + v \operatorname{div} u = u.\nabla v.$$

Then we have $u \in L^p(\mathbb{R}^d)^d$, $\nabla v \in L^q(\mathbb{R}^d)$ with div $u = 0$, curl $(\nabla v) = 0$ in the sense of distributions. It follows from Lemma 2 that

$$u.\nabla v \in \mathcal{H}^1(\mathbb{R}^d)$$

and there exists an absolute constant C such that

$$\|\operatorname{div}(vu)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \|u\|_{L^p} \|\nabla v\|_{L^q}.$$

Case B. We may of course suppose under additional assumptions that u and v are supported on the ball $|x| \leq R_0$. In order to simplify the presentation, we take $p = q = 2$. We shall write Ω for the ball in $\mathbb{R}^d$ of radius R_0 centered at the origin. By $H_0^1(\Omega)$ we denote the closed subspace of $H^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in the H^1 norm. Let

$$g = \operatorname{div} u \in L^2(\mathbb{R}^d).$$

By the classical result (see e.g. [20]) we know that

$$g = \partial_1 g_1 + \cdots + \partial_d g_d,$$

where $g_1, \ldots, g_d$ belong to $H_0^1(\Omega)$. Setting

$$G = (g_1, \ldots, g_d) \quad \text{and} \quad r = u - G.$$

Then it follows that

$$\operatorname{div} r = 0 \quad \text{and} \quad r \in L^2(\Omega).$$

Using Lemma 2 we infer

$$\operatorname{div}(rv) \in \mathcal{H}^1(\mathbb{R}^d).$$

Further we set

$$f = \operatorname{div}(Gv).$$

For this purpose we use Lemma 3 below, it follows that $f \in \mathcal{H}^1(\mathbb{R}^d)$.

Case C. The general case. We call φ a smooth bump function with compact support such that

$$1 = \sum_{k \in \mathbb{Z}^d} \varphi^2(x - k).$$

We have thus, if f and g are two functions,

$$\begin{aligned} f(x)g(x) &= \sum_{k \in \mathbb{Z}^d} f(x)\varphi^2(x-k)g(x) \\ &= \sum_{k \in \mathbb{Z}^d} f_k(x)g_k(x), \end{aligned}$$

where

$$f_k(x) = \varphi(x-k)f(x) \quad \text{and} \quad g_k(x) = \varphi(x-k)g(x).$$

Now set

$$u_k(x) = \varphi(x-k)u(x) \quad \text{and} \quad v_k(x) = \varphi(x-k)v(x)$$

for $k \in \mathbb{Z}^d$. We then have

$$\operatorname{div}(uv) = \sum_{k \in \mathbb{Z}^d} (u_k v_k) = \sum_{k \in \mathbb{Z}^d} w_k, \quad w_k = \operatorname{div}(u_k v_k).$$

We are going to check that

$$\sum_{k \in \mathbb{Z}^d} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^d)} < \infty.$$

To do this, we apply the local version (**Case A**) and it follows

$$\begin{aligned} \|w_k\|_{\mathcal{H}^1(\mathbb{R}^d)} &\leq C\left(\|u_k\|_{L^2} + \|\operatorname{div} u_k\|_{L^2}\right)\left(\|v_k\|_{L^2} + \|\operatorname{div} v_k\|_{L^2}\right) \\ &= \epsilon_k \in l^1\left(\mathbb{Z}^d\right). \end{aligned}$$

Up to now we have proved

$$\|\operatorname{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\left(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2}\right)\left(\|v\|_{L^2} + \|\operatorname{div} v\|_{L^2}\right). \tag{1.6}$$

This automatically yields the estimate

$$\|\operatorname{div}(uv)\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C\left(\|u\|_{L^2}\|\nabla v\|_{L^2} + \|v\|_{L^2}\|\operatorname{div} u\|_{L^2}\right). \tag{1.7}$$

To see this, we may replace u in the inequality above by

$$u = \delta^{\left(\frac{1}{2}-\frac{d}{2}\right)} u\left(\frac{x}{\delta}\right), \qquad \text{whenever } \ 0 < \delta < \infty.$$

and similarly v by

$$v_\delta = \delta^{\left(\frac{1}{2}-\frac{d}{2}\right)} v\left(\frac{x}{\delta}\right), \qquad \text{whenever } \quad 0 < \delta < \infty.$$

Thus the left-hand side of (1.6) fortunately does not change, while at right-hand we get rid the undesirable terms by letting δ either to 0, or to $+\infty$. This completes the proof. □

Now we turn to the proof of Lemma 3. One can show that every function $f \in L^p(\mathbb{R}^d)$, $p \in (1, +\infty]$, with compact support and $\int f dx = 0$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$. In particular,

Lemma 3. *If* $d^* = \frac{d}{d-1}$, $f \in L^{d^*}$, *supp* $f \subset \overline{\Omega}$ *and*

$$\int f dx = 0,$$

then $f \in \mathcal{H}^1(\mathbb{R}^d)$.

Proof. We have

$$f = \operatorname{div}(G)\, v + G.\nabla v$$

and we have to prove that the two terms belong to L^{d^*}. We consider the first term on the right. Since $\nabla v \in L^2$, we have

$$\operatorname{div}(G) \in L^2 \ \text{ and } \ v \in L^q, \quad \text{where } \ \frac{1}{2} - \frac{1}{q} = \frac{1}{d}.$$

Thus,

$$v \operatorname{div}(G) \in L^{d^*}.$$

A similar argument works in the second term and this completes the proof of the lemma. □

1.2. Multipliers and Morrey-Campanato spaces

In this section, we give a description of the multiplier space $\dot{X}_r$ introduced recently by P. G. Lemarié-Rieusset in his work [10] (see also [11]). The space $\dot{X}_r$ of pointwise multipliers which map L^2 into $\dot{H}^{-r}$ is defined in the following way

Definition 2. For $0 \le r < \frac{d}{2}$, we define the homogeneous space $\dot{X}_r$ by

$$\dot{X}_r = \left\{ f \in L^2_{loc} : \ \forall g \in \dot{H}^r \quad fg \in L^2 \right\},$$

where we denote by $\dot{H}^r\left(\mathbb{R}^d\right)$ the completion of the space $\mathcal{D}\left(\mathbb{R}^d\right)$ with respect to the norm $\|u\|_{\dot{H}^r} = \left\| (-\Delta)^{\frac{r}{2}} u \right\|_{L^2}$.

The norm of $\dot{X}_r$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2}.$$

Similarly, we define the nonhomogeneous space X_r for $0 \le r < \frac{d}{2}$ equipped with the norm

$$\|f\|_{X_r} = \sup_{\|g\|_{H^r} \le 1} \|fg\|_{L^2}.$$

We have the homogeneity properties : $\forall x_0 \in \mathbb{R}^d$

$$\|f(x + x_0)\|_{X_r} = \|f\|_{X_r}$$

$$\|f(x + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}$$

$$\|f(\lambda x)\|_{X_r} \le \frac{1}{\lambda^r} \|f\|_{X_r}, \quad 0 < \lambda \le 1$$

$$\|f(\lambda x)\|_{\dot{X}_r} \le \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \lambda > 0.$$

The following imbedding

$$L^{\frac{d}{t}} \subset X_r, \quad 0 \le r < \frac{d}{2}, \quad 0 \le t \le r.$$

$$L^{\frac{d}{r}} \subset \dot{X}_r, \quad 0 \le r < \frac{d}{2}$$

holds.

Example 1. If $u(x) \in \mathcal{D}\left(\mathbb{R}^d\right)$, $\varphi(x) = \left(\sum_{k=1}^{d} |x_k|^{\gamma_k} \right)^{-1}$, $\gamma_k > 0$, $d > 2$, and $\sum_{k=1}^{d} \gamma_k^{-1} = \frac{d}{2}$, then

$$\int_{\mathbb{R}^d} \varphi(x) \left| u(x) \right|^2 dx \le C \int_{\mathbb{R}^d} \left| \nabla u(x) \right|^2 dx$$

and $\varphi \in X_1\left(\mathbb{R}^d\right)$.

Indeed, the inequality

$$\int\limits_{\lambda<|x|<2\lambda} \varphi(x)\,|u(x)|^2\,dx \leq \left[\int\limits_{\lambda<|x|<2\lambda} |u(x)|^{\frac{2d}{d-2}}\,dx\right]^{\frac{d-2}{2}} \cdot \left[\int\limits_{\lambda<|x|<2\lambda} \varphi(x)^{\frac{d}{2}}\,dx\right]^{\frac{2}{d}}$$

and the Sobolev theorem imply that for $\lambda > 0$

$$\int\limits_{\lambda<|x|<2\lambda} \varphi(x)\,|u(x)|^2\,dx \leq C\left[\int\limits_{\lambda<|x|<2\lambda} |\nabla u(x)|^2\,dx + \int\limits_{\lambda<|x|<2\lambda} \frac{|u(x)|^2}{|x|^2}dx\right] \cdot \left[\int\limits_{\lambda<|x|<2\lambda} \varphi(x)^{\frac{d}{2}}\,dx\right]^{\frac{2}{d}},$$

where C does not depend on λ. Let us estimate the integral

$$S(\lambda) = \int\limits_{\lambda<|x|<2\lambda} \varphi(x)^{\frac{d}{2}}\,dx.$$

The domain $\lambda < |x| < 2\lambda$ can be represented as a finite sum of domain $\Omega_{j\lambda}$ such that $|x_j| > \frac{\lambda}{2}$ if $x \in \Omega_{j\lambda}$ for $j = 1, \ldots, d$. Let for instance $|x_1| > \frac{\lambda}{2}$. Then

$$\int\limits_{\Omega_{j\lambda}} \varphi(x)^{\frac{d}{2}}dx \leq \frac{3\lambda}{2} \int\limits_{\lambda<|x|<2\lambda} \frac{dx_1 \cdots dx_d}{\left(\left(\frac{\lambda}{2}\right)^{\gamma_1} + |x_2|^{\gamma_2} + \cdots + |x_d|^{\gamma_d}\right)^{\frac{d}{2}}}.$$

The substitution $x_j = t_j\left(\frac{\lambda}{2}\right)^{\frac{\gamma_1}{\gamma_j}}$ gives the relations

$$\begin{aligned} S(\lambda) &\leq C \int\limits_{\mathbb{R}^{d-1}} \frac{dt_1 \cdots dt_d}{(1 + |t_2|^{\gamma_2} + \cdots + |t_d|^{\gamma_d})^{\frac{d}{2}}} \\ &\leq C, \end{aligned}$$

since the integral is converging. To see this, set $t_s = \tau_s^{\frac{1}{\gamma_s}}$. Then

$$\int\limits_{\mathbb{R}^{d-1}} \frac{dt_1 \cdots dt_d}{(1 + |t_2|^{\gamma_2} + \cdots + |t_d|^{\gamma_d})^{\frac{d}{2}}}$$

$$\leq C \int_{\mathbb{R}^{d-1}} \frac{|\tau|^{\frac{1}{\gamma_2}+\cdots+\frac{1}{\gamma_d}-(d-1)}}{(1+|\tau|)^{\frac{d}{2}}} d\tau_1 \cdots d\tau_d$$

$$\leq C \int_0^\infty \frac{d|\tau|}{(1+|\tau|)^{\frac{1}{\gamma}+1}} < \infty.$$

Therefore,

$$\int_{\lambda<|x|<2\lambda} \varphi(x) |u(x)|^2 dx \leq C_5 \left[\int_{\lambda<|x|<2\lambda} |\nabla u(x)|^2 dx + \int_{\lambda<|x|<2\lambda} \frac{|u(x)|^2}{|x|^2} dx \right].$$

Setting $\lambda = 2^m$, $m \in \mathbb{Z}$ and assuming these inequalities over all m, we obtain that

$$\int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 dx \leq C \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \right).$$

By Hardy' s inequality in $\mathbb{R}^d$, $d \geq 3$

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx, \qquad u(x) \in \mathcal{D}\left(\mathbb{R}^d\right),$$

and hence

$$\int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 dx \leq C \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx.$$

Now we recall the definition of Morrey-Campanato spaces ([7], [19]):

Definition 3. For $1 < p \leq q \leq +\infty$, the Morrey-Campanato space $\mathcal{M}_{p,q}$ is defined by :

(1.8)

$$\mathcal{M}_{p,q} = \left\{ f \in L^p_{loc}\left(\mathbb{R}^d\right) : \|f\|_{\mathcal{M}_{p,q}} = \sup_{x \in IR^d} \sup_{0<R\leq 1} R^{d/q-d/p} \left\| f(y) 1_{B(x,R)}(y) \right\|_{L^p(dy)} < \infty \right\}.$$

Let us define the homogeneous Morrey-Campanato spaces $\dot{\mathcal{M}}_{p,q}$ for $1 < p \leq q \leq +\infty$ by

$$\|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^d} \sup_{R>0} R^{d/q-d/p} \left(\int_{B(x,R)} |f(y)|^p dy \right)^{1/p}. \tag{1.9}$$

It is easy to check the following properties :

$$\|f(\lambda x)\|_{\mathcal{M}_{p,q}} = \frac{1}{\lambda^{\frac{d}{q}}} \|f\|_{\mathcal{M}_{p,q}}, \qquad 0 < \lambda \leq 1,$$

$$\|f(\lambda x)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{\frac{d}{q}}} \|f\|_{\dot{\mathcal{M}}_{p,q}}, \qquad \lambda > 0.$$

We shall assume the following classical results [7].

a) For $1 \leq p \leq p', \; p \leq q \leq +\infty$ and for all function f so that $f \in \dot{\mathcal{M}}_{p,q} \cap L^\infty$:

$$\|f\|_{\dot{\mathcal{M}}_{p',q\frac{p'}{p}}} \leq \|f\|_{L^\infty}^{1-\frac{p}{p'}} \|f\|_{\dot{\mathcal{M}}_{p,q}}^{\frac{p}{p'}} .$$

b) For p, q, p', q' so that $\frac{1}{p} + \frac{1}{p'} \leq 1, \; \frac{1}{q} + \frac{1}{q'} \leq 1, \; f \in \dot{\mathcal{M}}_{p,q}, \; g \in \dot{\mathcal{M}}_{p',q'}$. Then

$$fg \in \dot{\mathcal{M}}_{p",q"} \text{ with } \frac{1}{p} + \frac{1}{p'} = \frac{1}{p"}, \; \frac{1}{q} + \frac{1}{q'} = \frac{1}{q"}.$$

c) For $1 \leq p \leq d$, we have

$$\forall \lambda > 0, \|\lambda f(\lambda x)\|_{\dot{\mathcal{M}}_{p,d}} = \|f\|_{\dot{\mathcal{M}}_{p,d}} .$$

d) If $p' < p$,

$$\dot{\mathcal{M}}_{p,q} \subset \mathcal{M}_{p,q},$$

$$\mathcal{M}_{p,q} \subset \mathcal{M}_{p',q}.$$

e) If $q_2 < q_1$, we have

$$\mathcal{M}_{p,q_1} \subset \mathcal{M}_{p,q_2},$$

$$L^q = \dot{\mathcal{M}}_{q,q} \subset \dot{\mathcal{M}}_{p,q}, \quad p \leq q.$$

We have the following comparison between multipliers and Morrey-Campanato spaces :

Proposition 1. *For* $0 \leq r < \frac{d}{2}$ *, we have*

$$X_r \subseteq \mathcal{M}_{2,\frac{d}{r}},$$

$$\dot{X}_r \subseteq \dot{\mathcal{M}}_{2,\frac{d}{r}}.$$

Proof. Let $f \in X_r$, $0 < R \leq 1$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{D}$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$\begin{aligned} R^{r-\frac{d}{2}} \left(\int_{|x-x_0| \leq R} |f(x)|^2 \, dx \right)^{1/2} &= R^r \left(\int_{|y-\frac{x_0}{R}| \leq 1} |f(Ry)|^2 \, dy \right)^{1/2} \\ &\leq R^r \left(\int_{y \in \mathbb{R}^d} |f(Ry)\phi(y)|^2 \, dy \right)^{1/2} \\ &\leq R^r \|f(Ry)\|_{X_r} \|\phi\|_{H^r} \\ &\leq \|f(y)\|_{X_r} \|\phi\|_{H^r} \\ &\leq C \|f(y)\|_{X_r} . \end{aligned}$$

We observe that the same proof is also valid for homogeneous spaces. □

Additionally, for $2 < p \leq \frac{d}{r}$ and $0 \leq r < \frac{d}{2}$, we have the following inclusion relations :

$$L^{\frac{d}{r}}\left(\mathbb{R}^d\right) \subset L^{\frac{d}{r},\infty}\left(\mathbb{R}^d\right) \subset \dot{\mathcal{M}}_{p,\frac{d}{r}}\left(\mathbb{R}^d\right) \subset \dot{X}_r\left(\mathbb{R}^d\right) \subset \dot{\mathcal{M}}_{2,\frac{d}{r}}\left(\mathbb{R}^d\right),$$

where $L^{p,\infty}$ denotes the usual Lorentz (weak L^p) space. For the definition and basic properties of Lorentz spaces $L^{p,q}$ we refer to [18].

2. Regularity theorem

In this section we give the regularity criterion by velocity to the Leray type weak solution of the Navier-Stokes equation (1.1). Before turning our attention to regularity issues, we start with some prerequisites for our main result. We use the notations

$$D_j = \frac{\partial}{\partial x_j}, \qquad j = 1, \ldots, d$$

means the j^{th} partial derivative and

$$\nabla = (D_1, \ldots, D_d)$$

the gradient.

$$\nabla^2 = (D_j D_k)_{j,k=1}^d$$

means the matrix of the second order derivatives. Let

$$\begin{array}{rcl} u & : & \mathbb{R}^d \to \mathbb{R}^d \\ x & \mapsto & u(x) = (u_1(x), \ldots, u_d(x)) \end{array}$$

be a vector field. Then we set

$$\begin{aligned} \operatorname{div} u &= \nabla . u = D_1 u_1 + \cdots + D_d u_d, \\ \Delta u &= \operatorname{div} \nabla u = \left(D_1^2 + \cdots + D_d^2\right) u, \\ \nabla u &= (D_1, \ldots, D_d)\, u = (D_j u_k)_{j,k=1}^d, \\ \nabla^2 u &= (D_j D_k)_{j,k=1}^d\, u = (D_j D_k u_l)_{j,k,l=1}^d, \end{aligned}$$

and

$$\begin{aligned} u.\nabla u &= (u.\nabla)\, u = (u_1 D_1 + \cdots + u_d D_d)\, u \\ &= (u_1 D_1 u_k + \cdots + u_d D_d u_k)_{k=1}^d \end{aligned}$$

whenever this is meaningful. Further we set

$$\begin{aligned} \operatorname{div}(u\ u) &= D_1(u_1 u) + \cdots + D_d(u_d u) \\ &= (D_1(u_1 u_k) + \cdots + D_d(u_d u_k))_{k=1}^d, \end{aligned}$$

where the matrix $u\ u = u \otimes u = (u_j u_k)_{j,k=1}^d$ means the usual tensor product. We prefer the simple notation $u\ u$.

If div $u = 0$, we call u is divergence free or solenoidal. In this case we get

$$\begin{aligned} u.\nabla u &= D_1(u_1 u) + \cdots + D_d(u_d u) - (u_1 D_1 + \cdots + u_d D_d) u \\ &= D_1(u_1 u) + \cdots + D_d(u_d u) \\ &= \operatorname{div}(u\, u). \end{aligned}$$

Let

$$C_{0,\sigma}^{\infty}(\mathbb{R}^d) = \left\{\varphi \in \left(C_0^{\infty}(\mathbb{R}^d)\right)^d : \operatorname{div}\varphi = 0\right\} \subseteq \left(C_0^{\infty}(\mathbb{R}^d)\right)^d.$$

The subspace

$$L_\sigma^2(\mathbb{R}^d) = \overline{C_{0,\sigma}^{\infty}(\mathbb{R}^d)}^{\|\cdot\|_{L^2}} = \left\{u \in L^2(\mathbb{R}^d)^d : \operatorname{div} u = 0\right\}$$

obtained as the closure of $C_{0,\sigma}^{\infty}$ with respect to L^2-norm $\|\cdot\|_{L^2}$. H_σ^r denotes the closure of $C_{0,\sigma}^{\infty}$ with respect to the norm

$$\|u\|_{H^r} = \|u\|_{L^2} + \left\|(1-\Delta)^{\frac{r}{2}} u\right\|_{L^2} \quad \text{for } r \geq 0.$$

Our definition of Leray-Hopf weak solutions (see e.g., [9], [8]) now reads :

Definition 4 (weak solutions). Let $a \in L_\sigma^2$ and $T > 0$. A measurable function u is called a weak solution of (1.1) on $(0, T)$ if u satisfies the following properties

(1) $u \in L^\infty\left((0,T); L_\sigma^2\right) \cap L^2\left((0,T); \dot{H}_\sigma^1\right)$ for all $T > 0$;

(2) $u(t)$ is continuous in time in the weak topology of L_σ^2 with

$$\langle u(t), \phi\rangle \to \langle a, \phi\rangle \quad \text{as } t \to 0^+$$

for all $\phi \in L_\sigma^2$;

(3) for any $0 \leq s \leq t \leq T$, u satisfies the identity

(2.1)
$$\int_s^t \left\{-\langle u, \partial_\tau \phi\rangle + \langle u.\nabla u, \phi\rangle + \langle \nabla u, \nabla \phi\rangle\right\} d\tau = -\langle u(t), \phi(t)\rangle + \langle u(s), \phi(s)\rangle$$

for all $\phi \in H^1\left((s,t); H_\sigma^1\right)$. Here $\langle \cdot, \cdot\rangle$ denotes the scalar product and $\|\cdot\|_{L^2}$ denotes the norm in $L^2\left(\mathbb{R}^d\right)^d$.

Remark 2. For u and ϕ as above, the integral

$$\int_0^T \langle u.\nabla u, \phi\rangle\, d\tau$$

is well defined since we have by the Sobolev inequality

$$\|u\|_{L^{\frac{2d}{d-2}}} \leq C \|\nabla u\|_{L^2}$$

that

$$\left|\int_0^T \langle u.\nabla u, \phi\rangle \, d\tau\right| \leq \int_0^T \|u\|_{L^{\frac{2d}{d-2}}} \|\nabla u\|_{L^2} \|\phi\|_{L^d} \, d\tau$$

$$\leq C \sup_{0<t<T} \|\phi\|_{L^d} \int_0^T \|\nabla u\|_{L^2}^2 \, d\tau.$$

Existence of weak solutions has been established by Leray in [12] for initial velocity in $L^2_\sigma(\mathbb{R}^d)$. The result is the following

Theorem 3 (Leray - Hopf). *Let $T > 0$. Let $a \in L^2_\sigma(\mathbb{R}^d)$ and*

$$u \in L^\infty\left((0,T); L^2_\sigma\right) \cap L^2\left((0,T); \dot{H}^1_\sigma\right)$$

be a weak solution of the Navier-Stokes equation (1.1) *satisfying the strong type energy inequality:*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \|a\|_{L^2}^2 \qquad \text{for a.a.} \quad 0 \leq t < T. \tag{2.2}$$

We assume that the solution satisfies

$$\|u(t) - a\|_{L^2} \to 0 \quad \text{as} \quad t \to +0.$$

Let us introduced the class $L^s((0,T); L^\gamma)$ with the norm $\|\cdot\|_{L^s((0,T);L^\gamma)}$

$$\|u\|_{L^s((0,T);L^\gamma)} = \left(\int_0^T \|u(t)\|_{L^\gamma}^s \, dt\right)^{\frac{1}{s}}.$$

The classical result on uniqueness and regularity of weak solutions in the class $L^s((0,T); L^\gamma)$ was given by Foias, Serrin and Masuda [4], [16], [14].

Theorem 4 (Foias-Serrin-Masuda). *Let $a \in L^2_\sigma(\mathbb{R}^d)$.*

(i) *Let u and v are two weak solutions of* (1.1) *on $(0,T)$. Suppose that u satisfies*

$$u \in L^s((0,T); L^\gamma) \quad \text{for} \quad \frac{2}{s} + \frac{d}{\gamma} = 1 \quad \text{with} \quad d < \gamma < \infty. \tag{2.3}$$

Assume that v fulfills the energy inequality (2.2) *for $0 \leq t < T$. Then we have $u = v$ on $[0,T)$.*

(ii) *Every weak solution u of* (1.1) *in the class* (2.3) *satisfies*

$$\frac{\partial u}{\partial t}, \ \frac{\partial^{\alpha_1+\cdots+\alpha_d} u}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}} \in C\left((0,T)\times\mathbb{R}^d\right) \tag{2.4}$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq 2$.

Kozono and Taniuchi [9] proved

Theorem 5 (Kozono-Taniuchi). *Let* $a \in L^2_\sigma(\mathbb{R}^d)$.

(i) (uniqueness) *Let* u, v *be two weak solutions of* (1.1) *on* $(0,T)$. *Suppose that*

$$u \in L^2((0,T);BMO)$$

and that v *satisfies the energy inequality* (2.2). *Then we have* $u = v$ *on* $[0,T]$.

(ii) (regularity) *Suppose that* u *is a weak solution satisfying either of the following conditions*

$$u \in L^2((0,T);BMO) \quad or \quad rot\, u \in L^1((0,T);BMO).$$

Then u *is a solution of* (1.1) *in the class*

$$u \in C([\epsilon,T);H^s_\sigma) \cap C^1([\epsilon,T);H^s) \cap C\left([\epsilon,T);H^{s+2}\right), \quad s > \frac{d}{2} - 1 \tag{2.5}$$

for all $0 < \epsilon < T$. *Actually* u *is regular in* $\mathbb{R}^d \times (0,T)$.

Our aim result is to show a new regularity criterion for each of the problems to (1.1).

Theorem 6. *Let* u *be a smooth solution to* (1.1) *in some interval* $[0,T)$ *with initial data* $a \in L^2_\sigma(\mathbb{R}^d)^d$. *Suppose that the solution* u *satisfies*

$$\int_0^T \|\nabla u(\tau)\|^2_{\dot{X}_1(\mathbb{R}^d)} d\tau < \infty.$$

Then the solution

$$u \in C\left((0,T);\dot{H}^1_\sigma(\mathbb{R}^d)^d\right) \cap L^2\left((0,T);\dot{H}^1_\sigma(\mathbb{R}^d)^d \cap H^2(\mathbb{R}^d)^d\right).$$

Moreover,

$$\begin{aligned} &\sup_{0\le t<T} \|\nabla u(t)\|^2_{L^2} + \int_0^T \left\|\nabla^2 u(\tau)\right\|^2_{L^2} d\tau \\ \le\;& C\,\|\nabla u(0)\|^2_{L^2}\left[1 + \exp\left(c\int_0^T \|\nabla u(\tau)\|^2_{\dot{X}_1(\mathbb{R}^d)} d\tau\right)\right]. \end{aligned}$$

The same result holds when the assumption $\nabla u \in L^2\left((0,T);\dot{X}_1(\mathbb{R}^d)^d\right)$ *is replaced by* $u \in L^2\left((0,T);BMO(\mathbb{R}^d)^d\right)$.

Remark 3. Theorem 6 covers the bordeline case $s = 2$ and $\gamma = d$. Our class $L^2\left((0,T);\dot{X}_1(\mathbb{R}^d)^d\right)$ is larger than $L^2\left((0,T);L^d(\mathbb{R}^d)^d\right)$.

To clarify the main part of the result, we recall the known regularity criterion in the following.

Lemma 4 (Beirão da Veiga [1]). *If we assume the following condition on the gradient of velocity for the Leray-Hopf weak solution u :*

$$\int_0^T \|\nabla u(\tau)\|_{L^\gamma}^s \, d\tau < \infty \ , \quad \frac{2}{s} + \frac{d}{\gamma} = 2, \quad \frac{d}{2} < \gamma \leq \infty, \tag{2.6}$$

then the weak solution is smooth on $(0, T]$.

Corollary 1. *If we assume the following condition on the gradient of velocity for the Leray-Hopf weak solution u :*

$$\int_0^T \|\nabla u(\tau)\|_{\dot{X}_1}^2 \, d\tau < \infty,$$

then the weak solution is smooth on $(0, T]$.

The marginale case $q = \infty$ was considered by Kozono and Taniuchi in BMO frame work.

Lemma 5 (Kozono-Taniuchi [9]). *Instead of the condition (2.6), if we assume the following condition on the vorticity of the weak solution u :*

$$\int_0^T \|rot\ u(\tau)\|_{BMO} \, d\tau < \infty \ ,$$

then the weak solution is smooth on $(0, T]$.

The following lemmas play a fundamental role in estimating the nonlinear term.

Lemma 6. *Let $f \in H^1(\mathbb{R}^d)$, $g(x) = (g_i(x))_{i=1}^d$ with $\nabla \cdot g = 0$ and $g \in L^2(\mathbb{R}^d)^d$. Furthers we assume that $\nabla h \in \dot{X}_1(\mathbb{R}^d)$. Then there exists a constant $C(d) > 0$ independent of f, g and h such that*

$$\left| \int_{\mathbb{R}^d} f g . \nabla h dx \right| \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)} \tag{2.7}$$

and

$$\left| \int_{\mathbb{R}^d} \nabla f . g h dx \right| \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)} \, . \tag{2.8}$$

Proof. The proof is easy, due to definition of $\dot{X}_1(\mathbb{R}^d)$. Suppose that $\nabla h \in \dot{X}_1(\mathbb{R}^d)$ and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^d} fg.\nabla h dx\right| &\leq \left(\int_{\mathbb{R}^d} |f|^2 |\nabla h|^2 dx\right)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d)^d} \\ &\leq C \|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |\nabla f|^2 dx\right)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d)^d},\end{aligned}$$

where the constant C is independent of f, g and h. Thus the Lemma is proved in the case of (2.7). The proof is similar in the case of (2.8). □

The same result holds when we replace the assumption $\nabla h \in \dot{X}_1(\mathbb{R}^d)$ by the assumption $h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d)$. Indeed, we known that

$$h(x) = \log|x| \in BMO$$

and

$$|\nabla h|^2 \leq \frac{1}{|x|^2},$$

then by Hardy' s inequality in $\mathbb{R}^d$ $(d \geq 3)$, we have

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx \leq C(d) \int_{\mathbb{R}^d} |\nabla f|^2 dx, \quad \forall f \in H^1(\mathbb{R}^d).$$

This remark suggest that the lemma will also be holds when we replace the $\dot{X}_1(\mathbb{R}^d)-$norm of ∇h by $BMO-$norm of h. In fact, the following is a combination of the compensated compactness results of Coifman, Lions, Meyer and Semmes [2] and the duality of the space BMO, we have :

Lemma 7. *Let $f \in H^1(\mathbb{R}^d)$, $g = (g_i(x))_{i=1}^d$ with $\nabla \cdot g = 0$ and $g \in L^2(\mathbb{R}^d)^d$ and a function $h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d)$. Then there exists a constant $C(d) > 0$ independent of f, g and h such that*

$$|\langle g.\nabla f, h\rangle| \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|h\|_{BMO(\mathbb{R}^d)}. \tag{2.9}$$

Proof. It is an immediate consequence of Lemma 2 and the duality inequality (1.4)

$$\begin{aligned}|\langle g.\nabla f, h\rangle| &\leq C \|g.\nabla f\|_{\mathcal{H}^1(\mathbb{R}^d)} \|h\|_{BMO(\mathbb{R}^d)} \\ &\leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|h\|_{BMO(\mathbb{R}^d)}.\end{aligned}$$

□

Next we recall the following well-known result :

Lemma 8 (Poincaré inequality). *Suppose Q is a unit cube in $\mathbb{R}^d$ of side length ρ and f is C^2 on Q with $\nabla f \in L^2(Q)$. There exists c not depending on f such that*

$$\int_Q |f - m_Q f|^2 \, dy \leq c\, \rho^2 \int_Q |\nabla f(y)|^2 \, dy, \tag{2.10}$$

where $m_Q f = \frac{1}{|Q|} \int_Q f(y)dy$ is the integral mean of f on Q.

Combining this result with Proposition 1 gives :

Proposition 2. *If $f \in H^1(\mathbb{R}^d)$ and $\nabla f \in \dot{X}_1(\mathbb{R}^d)$, then*

$$f \in BMO(\mathbb{R}^d).$$

Proof. Since $\dot{X}_1(\mathbb{R}^d) \subset \dot{\mathcal{M}}_{2,d}(\mathbb{R}^d)$, it follows that

$$\nabla f \in \dot{\mathcal{M}}_{2,d}(\mathbb{R}^d).$$

By the classical Poincaré inequality (2.10), we have

$$\begin{aligned} \int_{B(x,R)} \left| f(y) - m_{B(x,R)} f(y) \right|^2 dy &\leq C\, R^2 \int_{B(x,R)} |\nabla f(y)|^2 \, dy \\ &\leq C\, R^d \, \|\nabla f\|^2_{\dot{\mathcal{M}}_{2,d}} \end{aligned}$$

for every ball $B(x,R)$ of any radius R and there holds

$$\begin{aligned} \|f\|^2_{BMO} &= \sup_{x\in\mathbb{R}^d} \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} \left| f(y) - m_{B(x,R)} f(y) \right|^2 dy \\ &\leq C\, \|\nabla f\|^2_{\dot{\mathcal{M}}_{2,d}} \\ &\leq C\, \|\nabla f\|^2_{\dot{X}_1(\mathbb{R}^d)}. \end{aligned}$$

□

Now we turn into the proof of our Theorem 2.

Proof. Let u be a smooth solution to (1.1) on $[0,T)$. By operating the Laplacian to the equation and then taking a L^2 inner product of the equation with $(-\Delta u)$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|\nabla u\|^2_{L^2} + \left\|\nabla^2 u\right\|^2_{L^2} d\tau &= \langle u.\nabla u, \Delta u\rangle - \langle \nabla p, \Delta u\rangle \\ &= \sum_{j,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l dx + \langle p, \operatorname{div} \Delta u\rangle \\ &= \sum_{j,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l dx, \end{aligned}$$

where we have used

$$\operatorname{div} u = 0 = \operatorname{div} \Delta u.$$

Now, we use integration by parts to have

$$\begin{aligned}&\sum_{j,l=1}^{d}\int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l dx\\ &= -\sum_{j,k,l=1}^{d}\int_{\mathbb{R}^d} D_k u_j D_j u_l D_k u_l dx - \sum_{j,k,l=1}^{d}\int_{\mathbb{R}^d} u_j D_j D_k u_l D_k u_l dx,\end{aligned}$$

or

$$\begin{aligned}\sum_{j,k,l=1}^{d}\int_{\mathbb{R}^d} u_j D_j \left(D_k u_l D_k u_l\right) dx &= \frac{1}{2}\sum_{j=1}^{d}\int_{\mathbb{R}^d} u_j D_j \left|\nabla u\right|^2 dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^d} \operatorname{div} u \left|\nabla u\right|^2 dx = 0.\end{aligned}$$

Then

$$\begin{aligned}\sum_{j,l=1}^{d}\int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l dx &= \sum_{j,k,l=1}^{d}\int_{\mathbb{R}^d} \left(D_k u_j\right)\left(D_j D_k u_l\right) u_l dx\\ &= \left\langle u.\nabla u, \nabla^2 u\right\rangle.\end{aligned}$$

From Lemma 6 with

$$g = \nabla u, \quad \nabla f = \nabla^2 u \quad \text{and} \quad h = u$$

yields directly

$$\left|\left\langle u.\nabla u, \nabla^2 u, \right\rangle\right| \leq C \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)} \left\|\nabla^2 u\right\|_{L^2(\mathbb{R}^d)^d} \left\|\nabla u\right\|_{X_1(\mathbb{R}^d)}.$$

By the Young inequality, we have

$$\begin{aligned}&\left|\int_0^t \left\langle \nabla u.\Delta u, u\right\rangle\right| d\tau\\ &\leq \frac{1}{2}\int_0^t \left\|\nabla^2 u\right\|_{L^2(\mathbb{R}^d)}^2 d\tau + \frac{C}{2}\int_0^t \left\|\nabla u\right\|_{L^2(\mathbb{R}^d)^d}^2 \left\|\nabla u\right\|_{\dot{X}_1(\mathbb{R}^d)}^2 d\tau. \qquad (2.11)\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla u(t)\right\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2 d\tau \leq C\int_0^t \left\|\nabla u\right\|_{L^2()^d}^2 \left\|\nabla u\right\|_{\dot{X}_1(\mathbb{R}^d)}^2 d\tau \qquad (2.12)$$

for all $t > 0$. Since $\nabla u \in L^2\Big((0,T)\,;\dot{X}_1(\mathbb{R}^d)^d\Big)$, it follows from the Gronwall inequality that

$$\sup_{0\le t<T} \|\nabla u(t)\|_{L^2}^2 \le \|\nabla a\|_{L^2}^2 \left(1 + \exp\left\{C\int_0^t \|\nabla u\|^2_{\dot{X}_1(\mathbb{R}^d)}\, d\tau\right\}\right)$$

from which we get the desired result. □

Acknowledgement. The author would like to thank the referee for his / her careful reading of the work and his many helpful suggestions which improve the presentation much.

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