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Regularity in Free Boundary Problems.

D. KINDERLEHRER (*) - L. NIRENBERG (**) (1)

dedicated to Hans Lewy

1. – Introduction.

This paper is concerned with the local regularity of free boundary hypersurfaces, in n dimensional space, for elliptic and parabolic second order partial differential equations. In a free boundary problem, part of the problem is to determine the position and regularity of the free boundary. In order to do this one is usually provided with more boundary conditions at the free boundary than one has for a known boundary. We begin with some simple model examples. In all but Ex. 4, Ω represents a domain in R^n, $n\geq 2$, with the origin on its boundary $\partial\Omega$. All our discussion is purely local, near the origin, and our results have the following nature: assuming *some* degree of regularity of the free boundary and of the solution near it, specifically conditions (I) and (II) below, we prove further regularity. The conditions (I, II) are rather strong and not always satisfied in practice as we indicate. In a neighbourhood of the origin we assume:

(I) The boundary of Ω is a C^1 hypersurface Γ—the « free boundary ».

(II) $u(x)$ is a real function of class C^2 in $\Omega\cup\Gamma$.

When we speak of u or $\Omega\cup\Gamma$ we always mean just near the origin.

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EXAMPLE 1 (arising from problems of variational inequalities): $u(x)$ is a solution in Ω of

$$\Delta u = a(x) \tag{1.1}$$

having zero Cauchy data on Γ: $u = \partial u/\partial \nu = 0$ on Γ, where $\partial u/\partial \nu$ is the normal derivative of u on Γ.

EXAMPLE 2 (arising in axially symmetric problems of cavity flow): u satisfies an elliptic equation like

$$\Delta u = 0 \quad \text{in } \Omega\,, \tag{1.2}$$

$$u = 0 \quad \text{and} \quad g(x, \operatorname{grad} u) = 0 \quad \text{on } \Gamma\,. \tag{1.3}$$

EXAMPLE 3 (arising in cavity flow and water waves): u satisfies

$$\Delta u = 0 \quad \text{in } \Omega\,, \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad |\operatorname{grad} u| = f(x) \quad \text{on } \Gamma\,. \tag{1.5}$$

EXAMPLE 4 (arising in a one phase Stefan problem): Here Ω is a domain in R^{n+1} with variables $x = (x^1, \ldots, x^n)$ and $t \in R$. The origin is on $\partial\Omega$ and near the origin $\partial\Omega$ is a C^1 hypersurface Γ which is not tangent to the hyperplane $t = 0$ at the origin. $u(x^1, \ldots, x^n, t)$ satisfies

$$u_t - \sum_1^n u_{x^i x^i} = a(x, t) \quad \text{in } \Omega \tag{1.6}$$

$$u = |\operatorname{grad} u| = 0 \quad \text{on } \Gamma\,. \tag{1.7}$$

This is the analogue of Example 1 for the heat operator.

In each of these examples we ask the question: Because of the *two* boundary conditions on Γ, in place of the usual single condition, and under suitable assumptions (including regularity) on the given data, can one prove that Γ is regular—say C^∞ or even real analytic?

We have positive results in each of these examples except Ex. 3 for $n > 2$. In this case local regularity is not to be expected. Here is a simple counter-example in the case $f(x) \equiv 1$. The function $u(x) \equiv x^1$ clearly satisfies (1.4) and $|\operatorname{grad} u| \equiv 1$ everywhere. If we take for Γ a cylinder with generators parallel to he x^1 axis:

$$x^n = \sigma(x^2, \ldots, x^{n-1})\,, \qquad \sigma \in C^1\,,$$

we see that $\partial u/\partial \nu = 0$ on Γ. Thus all the conditions are satisfied for any σ in C^1, and no further smoothness of Γ can be inferred. (This argument fails for $n = 2$, and indeed in that case one has positive results in Ex. 3, for if v is the harmonic conjugate of u, with $v(0) = 0$, then v satisfies conditions of Ex. 2—for which positive results are established.) This shows that the problem of regularity of the free boundary in the situation of Ex. 3 with $n > 2$ is not a purely local one. To prove regularity in such a problem (if it exists) one will have to use all the global data in the problem.

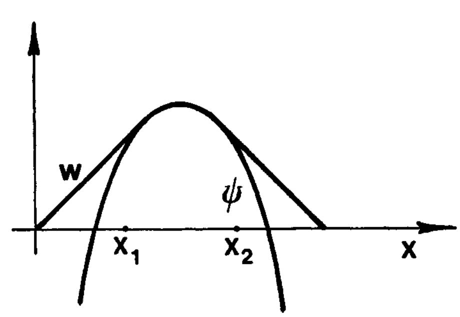
We now present a bit of background for these examples and our results. The situation of Ex. 1 arises in variational inequalities of which the following is a simple model case. Consider functions $w(x)$ defined in a bounded domain G in R^n with smooth boundary, satisfying $w = 0$ on ∂G, and lying above some obstacle, i.e.

$$w \geqslant \psi(x) \quad \text{in } G .$$

Here ψ is, for simplicity, a given smooth function with $\psi < 0$ on ∂G. The problem is to find such a w with minimum energy

$$I[w] = \int_G |\mathrm{grad}\, w|^2\, dx .$$

It is not difficult to show the existence of a minimizing (generalized) solution, and to see that in the region Ω where $w > \psi$ the function w is harmonic. It is natural to ask: how regular is the solution? The main results on existence and regularity are due to Lewy and Stampacchia (see [20] and also [26] and [14] for further material and references). Brézis and Kinderlehrer [3] (see also Gerhardt [13]) have shown that the solution w is in class $C^{1,1}$ i.e., w is in C^1 and its first derivatives are Lipschitz continuous. This result is optimal as the following one dimensional picture illustrates (here ψ is strictly concave)



$w = \psi$ in $x_1 \leqslant x \leqslant x_2$ and w is linear outside; at the points x_1, x_2, the second derivatives of w therefore have jumps.

The next question that one may ask is: how regular is the « free boundary » of Ω: $\partial\Omega \cap G$? In the case $n = 2$ this was initiated by Lewy and Stampacchia [20] (see also Lewy [19]), and our paper grew out of an attempt to extend their results to $n > 2$—in which we have only partially succeeded. For other work in this area we refer to Caffarelli and Rivière [7] and to [14], [15]. Further remarks in the case $n = 2$ are presented here in § 5. The question of regularity of $\partial\Omega \cap G$ is reduced to the situation of Ex. 1 by the following device: Let $u = w - \psi$. Since $u \equiv 0$ in the interior of $G \setminus \Omega$ and $u \in C^{1,1}$ we expect that

$$u = |\text{grad}\, u| = 0 \quad \text{on } \partial\Omega \cap G \tag{1.8}$$

while

$$\Delta u = -\Delta\psi \quad \text{in } \Omega\,. \tag{1.9}$$

This is the situation of Ex. 1 provided $\partial\Omega \cap G$ is a C^1 hypersurface. However this need not be the case. Schaeffer [23] has constructed examples for $n = 2$ in which ψ is real analytic, with $\Delta\psi < 0$, for which $\partial\Omega$ may have singularities. We describe such an example in § 5. Furthermore even if $\partial\Omega \cap G$ is a C^1 hypersurface it is not clear *a priori* that the function w (hence u) has continuous second derivatives in $\bar{\Omega}$. Recently however Caffarelli has established this and condition (I) for solutions of (1.8) and (1.9) in case $\partial\Omega$ is a Lipschitz boundary [5] and in even more general situations [6].

Our result is the following

THEOREM 1. *In Ex.* 1 *assume conditions* (I) *and* (II). *Let* $0 < a(x) \in C^1$ *in a full neighbourhood of the origin. Then*

(i) $\Gamma \in C^{1+\alpha}$ *for every positive* $\alpha < 1$.

(ii) *If, furthermore,* $a \in C^{m+\mu}$, *i.e.* a *is of class* C^m *and its derivatives of order* m *satisfy a Hölder condition with exponent* $\mu < 1$, *then* Γ *is a hypersurface of class* $C^{m+1+\mu}$.

(iii) *If* a *is analytic so is* Γ.

Note that some condition like $a \neq 0$ is needed. For instance if $a \equiv 0$, then $u = 0$ in Ω, and Γ need not be regular.

We shall actually prove a more general result than Theorem 1, in which the equation (1.1) is replaced by a general nonlinear elliptic equation ([2])

$$(1.10) \qquad F(x, u, Du, D^2 u) = 0 \quad \text{in}$$

with F of class C^1 in all the arguments.

THEOREM 1′. *Assume conditions* (I), (II), *that u satisfies* (1.10) *and has zero Cauchy data on* Γ. *Assume also* ([3])

$$(1.11) \qquad F(0, 0, \ldots, 0) \neq 0 .$$

Then (i) $\Gamma \in C^{1+\alpha}$ *for every positive* $\alpha < 1$ *in a neighbourhood of the origin.* (ii) *Furthermore if* $F \in C^{m+\mu}$ *as a function of its arguments,* $m \geq 1$, $0 < \mu < 1$ *then* $\Gamma \in C^{m+1+\mu}$. *Finally* (iii) *if* F *is analytic so is* Γ.

We turn more briefly to the other examples. Ex. 2 and 3 arise in cavitational flow as in the Riabouchinsky model [22] of flow of a liquid past an obstacle which generates a vapour cavity (see also [11] and [12]). In [12], Garabedian, Lewy and Schiffer proved the smoothness of the free boundary in the case of three dimensional flow which is rotationally symmetric about an axis. (Ex. 3 occurs in the case of steady progressing water waves; see for example Stoker [27], §§ 1.3 and 1.4.)

We shall formulate our results for the general equation (1.10) in place of (1.2).

THEOREM 2. *In Ex.* 2 *assume* (I), (II), *that u satisfies* (1.10), *and* (1.3) *and that*

$$|\text{grad}\, u| \neq 0 \qquad \textit{at the origin} .$$

Assume also that the function $g(x, p_1, \ldots, p_n) \in C^2$ *and satisfies*

$$\frac{\partial g}{\partial p_n}(0, \text{grad}\, u(0)) \neq 0 .$$

Then (i) $\Gamma \in C^{2+\alpha}$ *for every positive* $\alpha < 1$, (ii) *If, furthermore,* $F \in C^{m+\mu}$, $g \in C^{m+1+\mu}$, $m \geq 1$, $0 < \mu < 1$ *then* $\Gamma \in C^{m+2+\mu}$. (iii) *If* F *and* g *are analytic so is* Γ.

Theorem 2 applies to the free boundary value problem of a pendant drop hanging from the ceiling; where the edge (free boundary) of the drop

([2]) To say that the equation is «elliptic at u» means the linearization of the operator at u is elliptic, i.e. $F_{u_{x^i x^j}}(x, u, \ldots, D^2 u)$ is a definite symmetric matrix.

([3]) This is just to ensure that the second normal derivative to Γ of u at the origin is not zero.

touches the ceiling, the angle between the lower surface of the drop and the ceiling is given. The equation is close to the minimal surface equation, see e.g. [8]. Under sufficient regularity (i.e. (I) and (II)) analyticity of the free boundary follows. This was brought to our attention by R. Finn.

Theorems 1′ and 2 also prove the regularity of the boundary of the capillary surface in a tube obtained by Simon and Spruck [24].

In addition, Theorem 2 yields a partial extension to higher dimensions of a theorem of Lewy [18] on the analyticity of the curve of contact of a minimal surface S with a given surface Σ in R^3 to which part of the boundary of S is restricted. The natural boundary condition is that S and Σ meet orthogonally. Lewy showed first that $S \cap \Sigma$ is a rectifiable curve and then that it is analytic in case Σ is analytic. Using Theorem 2 one may see that the conclusion of analyticity (C^∞) holds in any dimensions, for a minimal hypersurface S in R^{n+1} whose boundary is partially restricted to a given analytic (C^∞) hypersurface Σ—in case we known that $\Gamma = S \cap \Sigma \in C^1$ and that S is of class C^2 up to and including Γ.

Finally for Ex. 4 we consider in place of (1.6) the general nonlinear parabolic equation

$$(1.12) \qquad u_t - F(x, t, u, D_x u, D_x^2 u) = 0 \quad \text{in } \Omega\,, \quad F \in C^1\,,$$

with F elliptic for each t and $F(0, \ldots, 0) \neq 0$. In place of (II) we assume

(II)′: u and u_{x_i} belong to C^1 in $\Omega \cup \Gamma$, $i = 1, \ldots, n$.

We shall formulate the result only in the C^∞ case though it is clear that it may be extended as in the preceding, to cases of finite differentiability.

THEOREM 3. *In Ex.* 4 *assume* (I), (II)′, *and that* u *is a solution of* (1.12) *satisfying* (1.7). *Here for each* t, F *is elliptic at* u (*uniformly in* t). *If* $F \in C^\infty$, *then* $\Gamma \in C^\infty$.

REMARK. It is natural to ask whether Γ is analytic in case F is. In general the answer is no since analyticity is not a local property for parabolic equations. To prove analyticity one would have to use the full global data of the problem. For the case of one space variable such a result has been recently proved by Friedman [10]. In a subsequent paper we prove such a result in a particular one-phase Stefan problem in n dimensions. In addition we prove the following addendum to Theorem 3:

THEOREM 3′: *If in Theorem* 3 *we also assume that* F *is analytic then, for each* T, $\Gamma \cap \{t = T\}$ *is analytic.*

In a third paper jointly with J. Spruck we treat regularity of free boundaries separating two different media such as arise in certain problems of plasma physics.

The idea of the proofs of our results is simple. In each case we perform a transformation of variable which makes use of the solution to map the region Ω into a region $\tilde{\Omega}$ in such a manner that the image of Γ becomes flat. Then we use the known regularity theory for solutions of nonlinear elliptic and parabolic equations in a domain with given (in this case flat) boundary, under suitable (coercive) boundary conditions. The transformations of variable are of two kinds: either (*a*) a form of the hodograph transformation, and the associated Legendre transform of the function, or (*b*) we introduce the function u as a new independent variable—this is described in § 3. It is convenient to perform hodograph and associated Legendre transforms with respect to *some*, say $x^1, \ldots, x^k$, of the independent variables. These are described in the next section.

2. – Partial hodograph and Legendre transformations.

Recall the familiar hodograph and Legendre transformations: Given a function $u(x)$ whose Hessian matrix $u_{xx} = \{u_{x^i x^j}\}$ is nonsingular, the hodograph transformation is a local mapping of a region into a new region via: $x \mapsto y = \operatorname{grad} u$.

One can then compute the x-derivatives of u in terms of the y-derivatives of the Legendre transform v of u: $v = \sum_1^n x^i y_i - u = \sum x^i u_{x^i} - u$, so that any partial differential equation satisfied by u is transformed into an equation for v. The use of these transformations is well established in problems of fluid dynamics in two dimensions. They occur in Lewy's study [17] of the Monge-Ampère equation. In recent years problems related to porous media (Baiocchi [2]) and fluid dynamics (Brézis and Stampacchia [4]) have been resolved as variational inequalities for new dependent variables obtained from hodograph and Legendre transformations. But these transformations have been little used in higher dimensions.

We are going to use them, rather, a modification, in proving Theorems 1′ and 3. We will carry out these transformations with respect to only one of the variables x^1. For generality and with the belief that they will prove useful, we now describe these transformations in k of the variables, say $x^1, \ldots, x^k$, leaving the others alone. We refer to these as partial hodograph and Legendre transforms. However for technical convenience *we have altered the customary definitions by changes in sign*; the reader is advised to note this change from the usual definitions.

(Modified) Partial hodograph and Legendre transforms: Consider a C^1 function $u(x)$ in a domain in R^n. For fixed $k \leq n$ we assume $u_{x^\alpha} \in C^1$, $\alpha \leq k$, and we wish to make a local C^1 change of variable

$$x \to y = (-u_{x^1}, \ldots, -u_{x^k}, x^{k+1}, \ldots, x^n), \tag{2.1}$$

which we assume to have nonsingular Jacobian, i.e. the $k \times k$ matrix,

$$\{u_{x^\alpha x^\beta}\}, \quad \text{for } \alpha, \beta \leq k, \text{ is nonsingular}.$$

This is called the (modified) hodograph transformation with respect to $x^1, \ldots, x^k$. The associated Legendre transform with respect to $x^1, \ldots, x^k$ is defined as

$$v = \sum_1^k x^\alpha y^\alpha + u = -\sum_1^k x^\alpha u_{x^\alpha} + u. \tag{2.2}$$

The function $v(y)$ satisfies

$$\begin{cases} v_{y^\alpha} = x^\alpha & \text{for } \alpha \leq k \\ v_{y^r} = u_{x^r} & \text{for } r > k. \end{cases} \tag{2.3}$$

So $v \in C^1$ and $v_{y^\alpha} \in C^1$ for $\alpha \leq k$. This is readily seen from the following computation (since $y^\alpha = -u_{x^\alpha}$)

$$\begin{aligned} dv &= \sum_1^k x^\alpha\, dy^\alpha - \sum_1^k u_{x^\alpha}\, dx^\alpha + du \\ &= \sum_1^k x^\alpha\, dy^\alpha + \sum_{r>k} u_{x^r}\, dx^r \\ &= \sum_1^k x^\alpha\, dy^\alpha + \sum_{r>k} u_{x^r}\, dy^r. \end{aligned}$$

The reason for choosing this modified definition of the hodograph and Legendre transforms is the following. If one now performs this same transformation on v with respect to the variable y^{k+1}, the result obtained is the same as applying the transformation on the original function u with respect to the variables $x^1, \ldots, x^{k+1}$, as one easily sees.

Assuming sufficient smoothness we now compute formally the derivatives of u which we denote by subscripts $u_{x^i} = u_i$, $u_{x^i x^j} = u_{ij}$ in terms of the derivatives of $v(y)$, $v_{y^i} = v_i$, $v_{y^i y^j} = v_{ij}$: In what follows the indices i, j run from 1 to n; α, β run from 1 to k; r, s run from $k+1$ to n.

From (2.1) and (2.3) we see that the Jacobian matrix is

$$x_y = \left(\begin{array}{c|c} \overset{k}{v_{\alpha\beta}} & \overset{n-k}{v_{\alpha r}} \\ \hline 0 & I \end{array} \right)_{n-k}^{k} = \left(\begin{array}{c|c} A & B \\ \hline 0 & I \end{array} \right). \tag{2.4}$$

Hence $y_x = x_y^{-1}$ is expressed in terms of the second derivatives of $v(y)$,

$$y_x = \left(\begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline 0 & I \end{array} \right) \tag{2.5}$$

From (2.1) and (2.3) we have

$$u_\alpha = -y^\alpha, \qquad u_r = v_r, \tag{2.6}$$

and therefore we obtain expressions for the second derivatives:

$$\begin{cases} u_{\alpha j} = -y^\alpha_{x^j} = -\text{element of } \alpha\text{-th row, } j\text{-th column of } x_y^{-1}, \quad \alpha \leqslant k, \\ u_{r\beta} = -y^\beta_{x^r} = \sum_{\alpha \leqslant k} v_{r\alpha} y^\alpha_{x^\beta} \qquad \beta \leqslant k < r, \\ u_{rs} = \quad v_{rs} + \sum_{\alpha \leqslant k} v_{r\alpha} y^\alpha_{x^s} \qquad r, s > k. \end{cases} \tag{2.7}$$

Thus a second order differential operator $F(x, u, Du, D^2u)$ is transformed to a second order operator $\tilde{F}$ for v. More generally, this is true for an operator of any order—it is transformed to one of the same order. In particular if $k = n$ the Laplace operator Δu becomes

$$\Delta u = \sum u_{\alpha\alpha} = -\operatorname{tr} y_x = -\operatorname{tr}(x_y^{-1}) = -\operatorname{tr}(v_{yy}^{-1}). \tag{2.8}$$

In the case $k = 1$ the Laplace operator becomes, as one easily verifies from (2.7) and (2.5′) below,

$$\Delta u = \frac{-1}{v_{11}} - \frac{1}{v_{11}} \sum_{r>1} v_{r1}^2 + \sum_{r>1} v_{rr}. \tag{2.8$'$}$$

LEMMA 2.1. *If $F(x, u, Du, D^2u)$ is elliptic so is the transformed operator $\tilde{F}$ for v.*

This seems intuitively obvious since one cannot imagine that changing variables could spoil ellipticity.

PROOF. As we pointed out, our partial hodograph and Legendre transformations can be achieved by a succession of simple ones, each with respect to one variable. Therefore it suffices to prove the lemma for such a simple one, i.e. for $k=1$. In this case (2.5) takes the form

$$y_x = \left(\begin{array}{c|c} \dfrac{1}{v_{11}} & -\dfrac{v_{12}}{v_{11}}, \ \ldots, \ -\dfrac{v_{1n}}{v_{11}} \\ \hline 0 & I \end{array}\right). \tag{2.5$'$}$$

To prove ellipticity of $\tilde{F}$ and v we must show that if we perturb v by δv, the linearized operator

$$\sum_{|\alpha|\leq 2} \tilde{F}_{D^\alpha v}(\ )D^\alpha \delta v$$

is elliptic. This operator has as leading part

$$L\delta v = \sum_{i,j} F_{u_{ij}} \delta u_{ij}$$

where the δu_{ij} is the corresponding linearized change in u_{ij}. From (2.7) and (2.5′) we see that, setting $F_{u_{ij}} = F_{ij}$, and using summation convention,

$$\begin{aligned} L\delta v = {} & F_{11}\delta u_{11} + 2F_{1r}\delta u_{1r} + F_{rs}\delta u_{rs} \\ = {} & -F_{11}\delta y^1_{x^1} - 2F_{1r}\delta y^1_{x^r} + F_{rs}(\delta v_{rs} + \delta v_{r1} y^1_{x^s} + v_{r1}\delta y^1_{x^s}) \\ = {} & F_{11}\frac{\delta v_{11}}{v_{11}^2} + 2F_{1r}\delta\left(\frac{v_{1r}}{v_{11}}\right) + F_{rs}\left(\delta v_{rs} - \frac{v_{1s}}{v_{11}}\delta v_{r1} - v_{r1}\delta\left(\frac{v_{1s}}{v_{11}}\right)\right) \\ = {} & \frac{1}{v_{11}^2}(F_{11} - 2F_{1r}v_{1r} + F_{rs}v_{1r}v_{1s})\delta v_{11} + \\ & + \frac{2}{v_{11}}(F_{1r} - F_{rs}v_{1s})\delta v_{1r} + F_{rs}\delta v_{rs}. \end{aligned}$$

Thus ellipticity of L is equivalent to definiteness of the following quadratic form (recall that r, s go from 2 to n)

$$(F_{11} - 2F_{1r}v_{1r} + F_{rs}v_{1r}v_{1s})\xi_1^2 + 2(F_{1r} - F_{rs}v_{1s})\xi_1\xi_r + F_{rs}\xi_r\xi_s \equiv F_{ij}\zeta_i\zeta_j$$

where

$$\zeta_1 = \xi_1, \qquad \zeta_r = \xi_r - \xi_1 v_{1r} \qquad \text{for } r > 1,$$

as one easily verifies. Since F_{ij} is definite the desired result follows and the lemma is proved.

REMARK. If $u(x^1, \ldots, x^n, t)$ satisfies a parabolic equation

$$u_t = F(x, t, u, D_x u, D_x^2 u) ,$$

with $(F_{u_{ij}})$ positive definite for each t, and if we perform the partial hodograph and Legendre transformations with respect to some of the space variables x we see, as above, that the transform $v(y, t)$ satisfies the equation

$$v_t = \tilde{F}(y, t, v, D_y v, D_y^2 v)$$

and $(\tilde{F}_{v_{ij}})$ is positive definite for each t. Thus v also satisfies a parabolic equation.

3. – Another transformation of variable.

Let $u(x)$ be a C^1 function defined in some open set in R^n, and suppose

$$u_n = u_{x^n} \neq 0$$

in a neighbourhood of a point. Then locally we may make a C^1 change of variables $y = (x^1, \ldots, x^{n-1}, u)$. The derivatives of u may be expressed in terms of the derivatives of

$$x^n = w(y)$$

so that if u satisfies a partial differential equation the equation is transformed into one for w.

In case $u = 0$ on some hypersurface Γ this transformation has the effect of mapping Γ to the flat surface $y^n = 0$. This transformation has been used for this purpose by Friedrichs [11] in a study of jet flow, and we will use it in proving Theorem 2.

Assuming smoothness, let us compute the derivatives of u in terms of derivatives of w. The Jacobian matrix is

$$(3.1) \qquad x_y = y_x^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline u_1, \ldots, u_{n-1} & u_n \end{array}\right)^{-1} = \\ = \left(\begin{array}{c|c} I & 0 \\ \hline -\dfrac{u_1}{u_n}, \; -\dfrac{u_2}{u_n}, \; \ldots, \; -\dfrac{u_{n-1}}{u_n} & \dfrac{1}{u_n} \end{array}\right).$$

Thus

$$(3.2)\qquad \begin{cases} w_\alpha = x^n_{y^\alpha} = -\dfrac{u_\alpha}{u_n} \quad \text{for } \alpha < n \\[2ex] w_n = x^n_{y^n} = \dfrac{1}{u_n} \\[2ex] y^n_{x^n} = u_n = \dfrac{1}{w_n}, \quad y^n_{x^\alpha} = u_\alpha = -\dfrac{w_\alpha}{w_n}, \quad \alpha < n\,. \end{cases}$$

Hence

$$(3.3)\qquad \begin{cases} u_{nn} = -\dfrac{w_{nn}}{w_n^2}\, y^n_{x^n} = -\dfrac{w_{nn}}{w_n^3} \\[2ex] u_{n\alpha} = -\dfrac{w_{n\alpha}}{w_n^2} - \dfrac{w_{nn}}{w_n^2}\, y^n_{x^\alpha} = -\dfrac{w_{n\alpha}}{w_n^2} + \dfrac{w_\alpha w_{nn}}{w_n^3}, \quad \alpha < n \\[2ex] u_{\alpha\beta} = -\dfrac{w_{\alpha\beta}}{w_n} + \dfrac{w_{\alpha n}}{w_n^2}\, w_\beta + \dfrac{w_{\beta n}}{w_n^2}\, w_\alpha - w_\alpha w_\beta \dfrac{w_{nn}}{w_n^3}, \quad \alpha, \beta < n\,. \end{cases}$$

As before we have the following whose proof we omit.

LEMMA 3.1. *An elliptic operator $F(x, u, Du, D^2u)$ is transformed into an operator $\tilde{F}(y, w, Dw, D^2w)$ which is also elliptic.*

In particular the Laplace operator Δu becomes

$$(3.4)\qquad \Delta u = -\frac{1}{w_n}\sum_{\alpha<n} w_{\alpha\alpha} + \frac{2}{w_n^2}\sum_{\alpha<n} w_\alpha w_{\alpha n} - \frac{w_{nn}}{w_n^3}\Big(1 + \sum_{\alpha<n} w_\alpha^2\Big)\,.$$

4. – Proofs of results.

PROOF OF THEOREM 1′ ([4]). We may assume that the positive x^1 axis is the exterior normal to Γ at the origin. Since $\operatorname{grad} u = 0$ on Γ it follows that all the second derivatives of u vanish at the origin except $u_{11}(0) \neq 0$ —in virtue of condition (1.11); say $u_{11}(0) > 0$.

Since $\Gamma \in C^1$ and $u_{x^1} \in C^1(\Omega \cup \Gamma)$ it is easy to see that we may extend u_{x^1} as a C^1 function to a full neighbourhood of the origin. In a neighbourhood

([4]) Our original proofs of Theorem 1′ and 3 made use of the classical hodograph and Legendre transforms with respect to all space variables; the proof presented here uses partial transformations with respect to one variable only. The original proof is presented in [16].

of the origin we now carry out the hodograph and Legendre transformations of § 2 for the function u with respect to the variable x^1.

$$x \to y = (-u_{x^1}, x^2, \ldots, x^n) \quad \text{in } C^1 . \tag{4.1}$$

The hypersurface Γ is mapped into the flat hypersurface $y^1 = 0$ and Ω is mapped into $y^1 > 0$. In $y^1 \geqslant 0$, near the origin, the corresponding Legendre transform

$$v = x^1 y^1 + u = -x^1 u_{x^1} + u \tag{4.2}$$

is of class C^2 (because of (2.3)) and satisfies the boundary conditions

$$v = 0 \quad \text{on } y^1 = 0 . \tag{4.3}$$

By Lemma 2.1 v satisfies an elliptic equation $\tilde{F} = 0$ in the image region $y^1 \geqslant 0$. In case u satisfied (1.1) the equation for v is in fact, as we see from (2.8′),

$$-\frac{1}{v_{11}} - \frac{1}{v_{11}} \sum_{r>1} v_{r1}^2 + \sum_{r>1} v_{rr} = a(v_1, y^2, \ldots, y^n) \quad \text{in } y^1 \geqslant 0 . \tag{4.4}$$

We are now in a position to use known regularity theory for nonlinear elliptic boundary value problems. $\tilde{F}$ is a C^1 function of its arguments. By Theorem 11.1′ in [1], $v \in C^{2+\alpha}$ for every positive $\alpha < 1$ up to the boundary $y^n = 0$. Since Γ is given by the inverse map

$$\Gamma = \text{image of } x = v_y|_{y^1=0} \tag{4.5}$$

we obtain the first assertion (i) of Theorem 1′. Under the conditions for (ii) of the theorem we may now apply Theorem 11.1 of [1] which assures us that $v \in C^{m+2+\mu}$ in $y^1 \geqslant 0$ near the origin. Hence we conclude from (4.5) that $\Gamma \in C^{m+1+\mu}$. Finally if F is analytic, so is $\tilde{F}$, and we find with the aid of the results of § 6.7 in [21] that v is analytic in $y^1 \geqslant 0$. Hence Γ is analytic. Q.e.d.

Remark. The strong assumption (II) enters in a crucial way. It guarantees that the function $v(y)$ which satisfies $\tilde{F}(\cdot) = 0$ in $y^1 \geqslant 0$ belongs to C^2 in $y^1 \geqslant 0$. This is the minimum one needs (at the present state of our knowledge) in order to deduce further regularity for solutions of nonlinear elliptic equations. For $n = 2$ however, stronger results do exist, enabling one, as in [15] to weaken the hypotheses.

We use a similar argument to prove Theorem 3.

PROOF OF THEOREM 3. We may suppose that Γ has the form

$$\Gamma\colon x^1 = \sigma(x^2, \dots, x^n; t) \in C^1, \qquad x^1 < \sigma \text{ in } \Omega,$$

with $\sigma_{x^\alpha}(0) = 0$, $\alpha < n$. As in the preceding case we find that all derivatives $u_{x^i x^j}(0) = 0$ except $u_{x^1 x^1}(0) > 0$ say.

As before, extend u_{x^1} to a full neighbourhood of the origin in R^{n+1} as a C^1 function. In a neighbourhood of the origin we perform the partial hodograph and Legendre transforms of u with respect to the variable x^1, i.e. we introduce new variables

$$y = (-u_{x^1}, x^2, \dots, x^n) \quad \text{and } s = t.$$

The image of $\bar{\Omega}$ near the origin is the set $y^1 \geqslant 0$ near the origin and there the associated Legendre transform $v = x^1 y^1 + u$ then satisfies a nonlinear parabolic equation. On the flat boundary we have the boundary condition

$$v = 0 \quad \text{on} \quad y^1 = 0.$$

Furthermore v and v_{y^j}, $j = 1, \dots, n$, belong to C^1 in $y^1 \geqslant 0$.

We wish now to apply the analogues of the elliptic regularity results Theorems 11.1′, 11.1 of [1]. We do not know a specific reference for these results but they are proved in the same manner as in the elliptic case. In the elliptic theory, Theorem 11.1′ and 11.1 are proved by considering difference quotients in directions parallel to the boundary and applying known estimates for linear elliptic equations—these in turn are based on estimates for linear elliptic equations with constant coefficients. For Theorem 11.1′ one uses the L^p estimates up to the boundary for large p, and for Theorem 11.1 one uses the Schauder estimates up to the boundary. For second order linear parabolic equations analogous estimates have been established enabling one to carry through the same proofs. The appropriate L^p estimates are due to Solonnikov, see for instance Theorem 5.7 of [25], while the appropriate Schauder estimates for second order equations are due to Friedman (see Theorem 4 in § 7 of [9] or Theorem 4.11 in [25]). We may therefore regard Theorem 3 as proved.

PROOF OF THEOREM 2. We may suppose the positive x^n axis is the exterior normal to Γ at the origin. Because of conditions (I) and (II) we may extend u to a full neighbourhood of the origin as a function in C^1. We may suppose $u_{x^n}(0) > 0$. In a neighbourhood of the origin make the transformation

$$x \to y = (x^1, \dots, x^{n-1}, u)$$

described in § 3. This transformation maps the part of $\bar{\Omega}$ near the origin into the region $y^n \leqslant 0$ near the origin. Since $u \in C^2$ in $\Omega \cup \Gamma$ we see that the function

$$x^n = w(y) \quad \text{is in } C^2 \text{ in } y^n \leqslant 0 .$$

By Lemma 3.1 w satisfies an elliptic equation there, while it satisfies the boundary condition:

$$g\left(y^1, \ldots, y^{n-1}, w, -\frac{w_1}{w_n}, \ldots, -\frac{w_{n-1}}{w_n}, \frac{1}{w_n}\right) = 0 \quad \text{on } y^n = 0 .$$

Using the hypothesis $g_{p_n} \neq 0$, it is easy to verify that the conditions of Theorems 11.1′, 11.1 of [1] and § 6.7 in [21] are satisfied. Hence we find in the respective cases (i), (ii), (iii) that near the origin in $y^n \leqslant 0$, $w \in C^{2+\alpha}$ for every positive $\alpha < 1$, $w \in C^{m+2+\mu}$, and finally w is analytic. Since Γ is described by $x^n = w(x^1, \ldots, x^{n-1}, 0)$ the theorem is proved.

5. – Comparison of theorem 1 (iii) with two dimensional results.

We now consider a two dimensional result of Lewy and Stampacchia [20] (see also Lewy [19]). Its essential features may be summarized as follows.

THEOREM. *Suppose that Ω is a simply connected domain in the $z = x_1 + ix_2$ plane whose boundary contains a Jordan arc Γ. Suppose that $a(x_1, x_2)$ is a real positive analytic function in a neighbourhood of Γ and $u \in C^1(\Omega \cup \Gamma) \cap C^2(\Omega)$ satisfies*

$$\begin{cases} \Delta u = a & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ u_{x_i} = 0 , & i = 1, 2 . \end{cases}$$

Then Γ admits an analytic parametrization as the boundary values on $(-1, 1)$ of a conformal mapping

$$\varphi \colon \{|t| < 1, \operatorname{Im} t > 0\} \to \Omega$$

which may be extended across $\operatorname{Im} t = 0$ *as a holomorphic function.*

The case where a is not analytic is treated in [15].

This statement is stronger than Theorem 1 (iii) not only because it is not assumed that u belongs to $C^2(\Omega \cup \Gamma)$, but also because it permits cusp singularities of Γ which are parametrizable as the boundary values of a conformal mapping. These cusps may in fact occur. Schaeffer [23] has recently given examples of this phenomenon (and has also considered the case of an infinitely differentiable $a(x_1, x_2)$). Here we exhibit a simple example which characterizes this behaviour where $a(x_1, x_2)$ is analytic.

Let $t = t_1 + it_2$, $z = x_1 + ix_2$ be complex variables and consider the mapping from

$$G = \{t\colon |t| < 1,\ \operatorname{Im} t > 0\}$$

onto a domain

$$\Omega = \varphi(G)$$

given by

$$z = \varphi(t) = t^2 + it^\mu, \qquad \mu \text{ odd}.$$

This maps $(-1, 1)$ onto the curve

$$\Gamma\colon x_2 = \pm x_1^{\mu/2}, \qquad 0 \leqslant x_1 \leqslant 1.$$

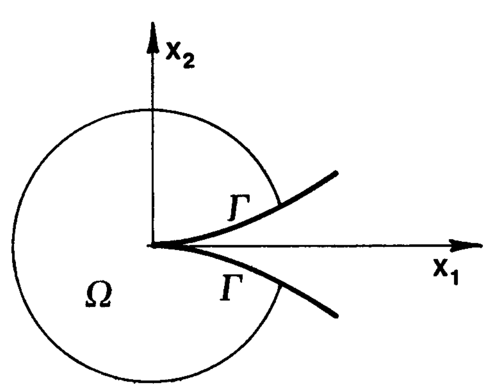
We shall find, for some $\varepsilon > 0$, a function

$$u \in H^{2,\infty}(B_\varepsilon), \qquad B_\varepsilon = \{z\colon |z| < \varepsilon\}$$

such that

$$(5.1) \qquad \begin{cases} \Delta u = 2 \quad \text{and} \quad u > 0 & \text{in } \Omega \cap B_\varepsilon \\ u = u_{x_i} = 0, \quad i = 1, 2, & \text{in } B_\varepsilon \setminus \Omega. \end{cases}$$

More precisely, we shall show that (5.1) may be satisfied if and only if $\mu = 4k + 1$, $k = 1, 2, \ldots$.



In the language of variational inequalities, a function u satisfying (5.1) is a solution of the problem: determine $u \in K$ so that

$$\int_{B_\varepsilon} u_{x_i}(v-u)_{x_i}\,dx \geqslant -2\int_{B_\varepsilon}(v-u)\,dx \quad \text{for all } v \in K$$

where

$$K = \{v \in H^1(B_\varepsilon):\ v \geqslant 0 \text{ in } B_\varepsilon \text{ and } v = u \text{ on } |z| = \varepsilon\}$$

with the property that

$$\Omega \cap B_\varepsilon = \{z:\ u(z) > 0\}.$$

Observe that if u is a solution of (5.1) then, in Ω, $w = u - \frac{1}{2}|z|^2$ is harmonic, so w_z is holomorphic; and $w_z = -\bar{z}$ on Γ. These conditions determine the function u.

Since the curve Γ has an analytic parametrization as the boundary values of a conformal mapping, we know there exists a function $f(z)$ holomorphic in Ω such that

$$f(z) = \bar{z} \quad \text{on } \Gamma.$$

In fact

$$f(z) = F(t) = t^2 - it^\mu \tag{5.2}$$

has this property. To describe the behavior of f we find z as a function of t. It is easy to see that

$$t^2 = z - iz^{\mu/2} + \ldots, \quad 0 \leqslant \arg z^{\frac{1}{2}} \leqslant \pi,$$

so by (5.2), $F(t) = t^2 - it^\mu = -z + 2t^2$, and

$$f(z) = z - 2iz^{\mu/2} + \ldots, \quad z \in \Omega,\ |z| \text{ small}.$$

The function u is then found by integration.

$$u(z) = \begin{cases} \frac{1}{2}|z|^2 - \operatorname{Re}\int_0^z f(\zeta)\,d\zeta, & z \in \Omega \\ 0, & z \in B_\varepsilon - \Omega \end{cases}$$

for ε sufficiently small. Clearly $\Delta u = 2$ in Ω, $u(z) = u(\bar{z})$, and, since $f(z) = \bar{z}$ on Γ,

$$u = u_{x_i} = 0 \quad \text{on } \Gamma, \qquad i = 1, 2.$$

Introducing polar coordinates

$$z = \varrho e^{i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi$$

one easily determines the first term in w,

$$(5.3) \qquad u(z) = x_2^2 - \frac{2}{1+\mu/2}\,\varrho^{(\mu/2)+1}\sin\left(\frac{\mu}{2}+1\right)\theta + \dots, \qquad z \in \Omega,\ |z| < \varepsilon .$$

From (5.3) one sees that

$$u(x_1) < 0 \quad \text{when } x_1 < 0 \quad \text{for } \mu = 4k+3,\ k = 0, 1, \dots$$

$$u(x_1) > 0 \quad \text{when } x_1 < 0 \quad \text{for } \mu = 4k+1,\ k = 1, 2, \dots$$

Therefore the condition $\mu = 4k+1$ is *necessary.*

To see that it is sufficient, we observe, since $\mu/2 + 1 > \frac{5}{2}$,

$$\Delta u = 2 + O(|z|^{\frac{3}{2}}), \qquad z \in \Omega \cap B_\varepsilon$$

$$\Delta u = 0, \qquad z \in B_\varepsilon - \Omega .$$

So on each vertical line $x_1 = a$, $|a| < \varepsilon$, $u(a, x_2)$ is a convex $C^{1,1}$ function. For $a < 0$ it follows from the symmetry of u that $u_{x_2}(0, 0) = 0$; hence $u(a, x_2)$ attains its minimum at $(a, 0)$—which is positive as we have already seen. If $a > 0$ then $u_{x_2}(a, 0) = 0$ since $u = 0$ in a neighbourhood of $(a, 0)$. Hence $\min u = 0$. It follows that

$$\Omega \cap B_\varepsilon = \{z\colon u(z) > 0\} .$$

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