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Regularity of fuzzy convergence spaces

Abstract: (Fuzzy) convergence spaces are extensions of (fuzzy) topological spaces. ⊤-convergence spaces are one of important fuzzy convergence spaces. In this paper, we present an extending dual Fischer diagonal condition, and making use of this we discuss a regularity of ⊤-convergence spaces.

Keywords: Fuzzy set, Fuzzy topology, Fuzzy convergence, Diagonal condition, Regularity

MSC: 54A40, 06D10

1 Introduction

The notion of convergence spaces (refer to [1] for convergence spaces) is introduced by extending the theory of convergence in general topological spaces. For a set X, we denote the power set (resp., the set of filters) on X as $P(X)$ (resp., $\mathcal{F}(X)$). Then *a convergence space* is defined as a pair (X, Q), where $Q \subseteq \mathcal{F}(X) \times X$ is a binary relation satisfying:

(C1) $(\dot{x}, x) \in Q$ for any $x \in X$, where $\dot{x} = \{A \in P(X) | x \in A\}$ is the principal filter generated by x;
(C2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$, $\mathcal{F} \subseteq \mathcal{G}$ and $(\mathcal{F}, x) \in Q$ imply $(\mathcal{G}, x) \in Q$.

If $(\mathcal{F}, x) \in Q$ then we say that $\mathcal{F}$ converges to x, and denote it as $\mathcal{F} \xrightarrow{Q} x$.

A convergence space (X, Q) is called topological whenever $\mathcal{F} \xrightarrow{Q} x$ if and only if $\mathcal{F}$ converges to x w.r.t some topological space. A convergence space (X, Q) is topological if and only if it satisfies the Fischer diagonal condition.

Let J be any set, $\Phi : J \longrightarrow \mathcal{F}(X)$ and $\mathcal{F} \in \mathcal{F}(J)$, where Φ is called a choice function of filters. Then the *Kowalsky compression operator* on $\Phi^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}(\mathcal{F}(X))$ is defined as $K\Phi\mathcal{F} := \bigcup_{A \in \mathcal{F}} \bigcap_{y \in A} \Phi(y)$.

Given a convergence space (X, Q), using Kowalsky compression operator, the *Fischer diagonal condition* is given as follows.

(F) Let J be any set, $\psi : J \longrightarrow X$, $\Phi : J \longrightarrow \mathcal{F}(X)$ such that $\Phi(j) \xrightarrow{Q} \psi(j)$ for each $j \in J$. If $\mathcal{F} \in \mathcal{F}(X)$ satisfies $\psi^{\Rightarrow}(\mathcal{F}) \xrightarrow{Q} x$, then $K\Phi\mathcal{F} \xrightarrow{Q} x$.

If we take $J = X$ and $\psi = id_X$ in (F) then we get the *Kowalsky diagonal condition* (K).

A convergence space (X, Q) is called regular if it satisfies the following dual Fischer diagonal condition.

(DF) Let J be any set, $\psi : J \longrightarrow X$, $\Phi : J \longrightarrow \mathcal{F}(X)$ such that $\Phi(j) \xrightarrow{Q} \psi(j)$ for each $j \in J$. If $\mathcal{F} \in \mathcal{F}(X)$ satisfies $K\Phi\mathcal{F} \xrightarrow{Q} x$, then $\psi^{\Rightarrow}(\mathcal{F}) \xrightarrow{Q} x$.

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For a convergence space generated by a topological space, the convergence space is regular if and only if the corresponding topological space is regular.

In recent years, many kinds of fuzzy convergence spaces, such as stratified L-generalized convergence spaces [2], stratified L-convergence spaces [3], L-ordered convergence spaces [4, 5] and $\top$-convergence spaces [6, 7], were defined and discussed [8–25]. In particular, the regularities of fuzzy convergence spaces were discussed by different kinds of extending dual Fischer diagonal conditions [6, 8, 9, 13, 16–18]. In this paper, we introduce a regularity for $\top$-convergence spaces by an extending dual Fischer diagonal condition.

The contents are arranged as follows. In Section 2, we recall some basic notions. In Section 3, we present the main results: the notion of $\top$-regular $\top$-convergence spaces, and their relationships to regular convergence spaces, and an extension theorem of continuous function. In Section 4, we conclude with a summary.

2 Preliminaries

A commutative quantale is a pair $(L, *)$, where L is a complete lattice with respect to a partial order $\leq$ on it, with the top (resp., bottom) element $\top$ (resp., $\bot$), and $*$ is a commutative semigroup operation on L such that $a * \bigvee_{j \in J} b_j = \bigvee_{j \in J}(a * b_j)$ for all $a \in L$ and $\{b_j\}_{j \in J} \subseteq L$. $(L, *)$ is said to be integral if the top element $\top$ is the unique unit in the sense of $\top * a = a$ for all $a \in L$. $(L, *)$ is said to be meet continuous if the underlying lattice $(L, \leq)$ is a meet continuous lattice, that is, the binary meet operation $\wedge$ distributes over directed joins [26]. In this paper, if not otherwise specified, $L = (L, *)$ is always assumed to be an integral, commutative, and meet continuous quantale.

Since the binary operation $*$ distributes over arbitrary joins, the function $a * (-) : L \longrightarrow L$ has a right adjoint $a \to (-) : L \longrightarrow L$ given by $a \to b = \bigvee\{c \in L : a * c \leq b\}$. We collect here some basic properties of the binary operations $*$ and $\to$ [27, 28]:

(1) $a \to b = \top \Leftrightarrow a \leq b$; (2) $a * b \leq c \Leftrightarrow b \leq a \to c$; (3) $a * (a \to b) \leq b$; (4) $a \to (b \to c) = (a * b) \to c$;
(5) $(\bigvee_{j \in J} a_j) \to b = \bigwedge_{j \in J}(a_j \to b)$; (6) $a \to (\bigwedge_{j \in J} b_j) = \bigwedge_{j \in J}(a \to b_j)$.

We call a function $\mu : X \to L$ an L-fuzzy subset in X. We use L^X to denote the set of all L-fuzzy subsets in X. For any $A \subseteq X$, let $\top_A$ denote the characteristic function of A. The operators $\bigvee, \bigwedge, *$ and $\to$ on L can be translated onto L^X in a pointwise way. That is, for all $\mu_t(t \in T) \in L^X$,

$$\left(\bigvee_{t \in T} \mu_t\right)(x) = \bigvee_{t \in T} \mu_t(x), \quad \left(\bigwedge_{t \in T} \mu_t\right)(x) = \bigwedge_{t \in T} \mu_t(x),$$

$$(\mu * \nu)(x) = \mu(x) * \nu(x), (\mu \to \nu)(x) = \mu(x) \to \nu(x).$$

Let $f : X \longrightarrow Y$ be a function. We define $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$, [27], by $f^{\to}(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ for $\mu \in L^X$ and $y \in Y$, and $f^{\leftarrow}(\nu)(x) = \nu(f(x))$ for $\nu \in L^Y$ and $x \in X$.

Let μ, ν be L-fuzzy subsets in X. The subsethood degree [29–33] of μ, ν, denoted as $S_X(\mu, \nu)$, is defined by $S_X(\mu, \nu) = \bigwedge_{x \in X}(\mu(x) \to \nu(x))$.

Lemma 2.1 ([6, 29, 34–38]). *Let $f : X \longrightarrow Y$ be a function and $\mu_1, \mu_2 \in L^X, \lambda_1, \lambda_2 \in L^Y$. Then*
(1) $S_X(\mu_1, \mu_2) \leq S_Y(f^{\to}(\mu_1), f^{\to}(\mu_2))$,
(2) $S_Y(\lambda_1, \lambda_2) \leq S_X(f^{\leftarrow}(\lambda_1), f^{\leftarrow}(\lambda_2))$.

Definition 2.2 ([27, 39]). *A nonempty subset $\mathbb{F} \subseteq L^X$ is called a $\top$-filter on the set X whenever:*
(TF1) $\bigvee_{x \in X} \lambda(x) = \top$ *for all $\lambda \in \mathbb{F}$;*
(TF2) $\lambda \wedge \mu \in \mathbb{F}$ *for all $\lambda, \mu \in \mathbb{F}$;*
(TF3) *if $\lambda \in L^X$ such that $\bigvee_{\mu \in \mathbb{F}} S_X(\mu, \lambda) = \top$, then $\lambda \in \mathbb{F}$.*
It is easily seen that the condition (TF3) implies a weaker condition (TF3′) $\mu \in \mathbb{F}$ and $\mu \leq \lambda \implies \lambda \in \mathbb{F}$.

The set of all $\top$-filters on X is denoted by $\mathbb{F}_L^{\top}(X)$.

Definition 2.3 ([27]). *A nonempty subset $\mathbb{B} \subseteq L^X$ is called a $\top$-filter base on the set X provided:*
(TB1) $\bigvee_{x \in X} \lambda(x) = \top$ *for all $\lambda \in \mathbb{B}$;*
(TB2) *if $\lambda, \mu \in \mathbb{B}$, then $\bigvee_{\nu \in \mathbb{B}} S_X(\nu, \lambda \wedge \mu) = \top$.*

Each $\top$-filter base $\mathbb{B}$ on X generates a $\top$-filter $\mathbb{F}_\mathbb{B}$ defined by $\mathbb{F}_\mathbb{B} := \{\lambda \in L^X | \bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \top\}$. And for any $\lambda \in L^X$, we have the following equality [23]: $\bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \bigvee_{\mu \in \mathbb{F}_\mathbb{B}} S_X(\mu, \lambda)$.

We list some fundamental facts about $\top$-filters in the following proposition.

Proposition 2.4 ([6, 27]).
(1) *For any $x \in X$, the family $[x]_\top =: \{\lambda \in L^X | \lambda(x) = \top\}$ is a $\top$-filter on X, called the principal $\top$-filter on X generated by x.*
(2) *For any $\{\mathbb{F}_i\}_{i \in I} \subseteq \mathbb{F}_L^\top(X)$, $\bigcap_{i \in I} \mathbb{F}_i$ is also a $\top$-filter.*
(3) *Let $f : X \longrightarrow Y$ be a function. For any $\mathbb{F} \in \mathbb{F}_L^\top(X)$, the family $\{f^\rightarrow(\lambda) | \lambda \in \mathbb{F}\}$ forms a $\top$-filter base on Y, and the $\top$-filter $f^\Rightarrow(\mathbb{F})$ generated by it is called the image of $\mathbb{F}$ under f. For any $\mathbb{G} \in \mathbb{F}_L^\top(Y)$, the family $\{f^\leftarrow(\mu) | \mu \in \mathbb{G}\}$ forms a $\top$-filter base on X if and only if $\bigvee_{x \in X} \mu(f(x)) = \top$ holds for all $\mu \in \mathbb{G}$, and the $\top$-filter $f^\Leftarrow(\mathbb{G})$ (if exists) generated by it is called the inverse image of $\mathbb{G}$ under f.*

Lemma 2.5.
(1) *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}_L^\top(X)$ and $\mathbb{B}$ be a $\top$-filter base of $\mathbb{F}$. Then $\mathbb{B} \subseteq \mathbb{G}$ implies that $\mathbb{F} \subseteq \mathbb{G}$.*
(2) *Let $f : X \longrightarrow Y$ be a function and $\mathbb{F} \in \mathbb{F}_L^\top(X)$. Then $\lambda \in f^\Rightarrow(\mathbb{F})$ if and only if $f^\leftarrow(\lambda) \in \mathbb{F}$.*

Proof. (1) For any $\lambda \in \mathbb{F}$, we have
$$\top = \bigvee_{\mu \in \mathbb{B}} S_X(\mu, \lambda) \leq \bigvee_{\mu \in \mathbb{G}} S_X(\mu, \lambda),$$
which means $\lambda \in \mathbb{G}$, as desired.

(2) Let $\lambda \in f^\Rightarrow(\mathbb{F})$. Then
$$\top = \bigvee_{\mu \in \mathbb{F}} S_Y(f^\rightarrow(\mu), \lambda) \leq \bigvee_{\mu \in \mathbb{F}} S_X(f^\leftarrow f^\rightarrow(\mu), f^\leftarrow(\lambda)) \leq \bigvee_{\mu \in \mathbb{F}} S_X(\mu, f^\leftarrow(\lambda)).$$

It follows that $f^\leftarrow(\lambda) \in \mathbb{F}$. Conversely, let $f^\leftarrow(\lambda) \in \mathbb{F}$. Then $\lambda \geq f^\rightarrow f^\leftarrow(\lambda) \in f^\Rightarrow(\mathbb{F})$, and so $\lambda \in f^\Rightarrow(\mathbb{F})$. $\qquad\square$

Let $\mathbb{F} \in \mathbb{F}_L^\top(X)$, it is easily seen that the set $\iota(\mathbb{F}) = \{A \subseteq X | \top_A \in \mathbb{F}\}$ is a filter on X. Conversely, let $\mathcal{F} \in \mathcal{F}(X)$, then the set $\{\top_A | A \in \mathcal{F}\}$ forms a $\top$-filter base on X and the $\top$-filter generated by it is denoted as $\omega(\mathcal{F})$.

Lemma 2.6. *Let $f : X \longrightarrow Y$, $\mathcal{F} \in \mathcal{F}(X)$, $\mathbb{F} \in \mathbb{F}_L^\top(X)$ and $x \in X$. Then:*
(1) $\iota\omega(\mathcal{F}) = \mathcal{F}$,
(2) $\omega\iota(\mathbb{F}) \subseteq \mathbb{F}$,
(3) $\omega(\dot{x}) = [x]_\top$,
(4) $\iota([x]_\top) = \dot{x}$,
(5) $\iota(f^\Rightarrow(\mathbb{F})) = f^\Rightarrow(\iota(\mathbb{F}))$.

Proof. We prove only (5) and others are easily observed. Indeed,
$$A \in \iota(f^\Rightarrow(\mathbb{F})) \Leftrightarrow f^\leftarrow(\top_A) \in \mathbb{F} \Leftrightarrow \top_{f^\leftarrow(A)} \in \mathbb{F} \Leftrightarrow f^\leftarrow(A) \in \iota(\mathbb{F}) \Leftrightarrow A \in f^\Rightarrow(\iota(\mathbb{F})). \quad \square$$

Definition 2.7 ([6]). *A $\top$-convergence space is a pair (X, q), where $q \subseteq \mathbb{F}_L^\top(X) \times X$ is a binary relation satisfying* (TC1) $([x]_\top, x) \in q$ *for every $x \in X$;* (TC2) *if $(\mathbb{F}, x) \in q$ and $\mathbb{F} \subseteq \mathbb{G}$, then $(\mathbb{G}, x) \in q$.*
  If $(\mathbb{F}, x) \in q$, then we say that $\mathbb{F}$ converges to x, and denote it as $\mathbb{F} \xrightarrow{q} x$.

It is easily seen that a $\top$-convergence space is precisely a convergence space when $L = \{\bot, \top\}$.
  For the categorical theory, we refer to the monograph [40].

3 A regularity for ⊤-convergence spaces

In this section, we shall discuss a regularity for ⊤-convergence spaces by an extending dual Fischer diagonal condition.

Let J be any set, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ and $\mathbb{F} \in \mathbb{F}_L^\top(J)$, where ϕ is called a choice function of ⊤-filters. Then the extending *Kowalsky compression operator* on $\phi^{\Rightarrow}(\mathbb{F}) \in \mathbb{F}_L^\top(\mathbb{F}_L^\top(X))$ is defined as

$$k\phi\mathbb{F} := \bigcup_{A \in \iota(\mathbb{F})} \bigcap_{y \in A} \phi(y).$$

We prove that $k\phi\mathbb{F}$ satisfies (TF1)-(TF3).

(TF1): Let $\lambda \in k\phi\mathbb{F}$. Then there exists an $A \in \iota(\mathbb{F})$ such that for any $y \in A$, $\lambda \in \phi(y)$. It follows by $\phi(y) \in \mathbb{F}_L^\top(X)$ that $\bigvee_{x \in X} \lambda(x) = \top$. Thus the condition (TF1) is satisfied.

(TF2): Let $\lambda, \mu \in k\phi\mathbb{F}$. Then there exist $A, B \in \iota(\mathbb{F})$ such that

$$\lambda \in \bigcap_{y \in A} \phi(y) \text{ and } \mu \in \bigcap_{z \in B} \phi(z).$$

It follows that $A \cap B \in \iota(\mathbb{F})$ and $\lambda \wedge \mu \in \bigcap_{w \in A \cap B} \phi(w)$, and then $\lambda \wedge \mu \in k\phi\mathbb{F}$. Thus the condition (TF2) is satisfied.

(TF3): Let $\lambda \in L^X$ satisfy $\bigvee_{\mu \in k\phi\mathbb{F}} S_X(\mu, \lambda) = \top$. Then for any $\mu \in k\phi\mathbb{F}$, there exists an $A \in \iota(\mathbb{F})$ such that for any $y \in A$, $\mu \in \phi(y)$. By $\mu \in \phi(y)$ we have $\bigvee_{\nu \in \phi(y)} S_X(\nu, \mu) = \top$. Then it follows that for any $y \in A$,

$$\top = \bigvee_{\mu \in k\phi\mathbb{F}} \left(S_X(\mu, \lambda) * \bigvee_{\nu \in \phi(y)} S_X(\nu, \mu) \right) \leq \bigvee_{\nu \in \phi(y)} S_X(\nu, \lambda).$$

That means $\lambda \in \phi(y)$, and so $\lambda \in k\phi\mathbb{F}$. Thus the condition (TF3) is satisfied.

Using Kowalsky compression operator, an extension of the (dual) diagonal condition (F) ((DF)) is given as follows:

(TF) Let J be any set, $\psi : J \longrightarrow X$, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. If $\mathbb{F} \in \mathbb{F}_L^\top(X)$ satisfies $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$, then $k\phi\mathbb{F} \xrightarrow{q} x$.

(TDF) Let J be any set, $\psi : J \longrightarrow X$, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. If $\mathbb{F} \in \mathbb{F}_L^\top(X)$ satisfies $k\phi\mathbb{F} \xrightarrow{q} x$, then $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{q} x$.

If we take $J = X$ and $\psi = id_X$ in (TF) then we get the *Kowalsky diagonal condition* (TK).

Definition 3.1. *A ⊤-convergence space is called ⊤-regular if it satisfies the condition* (TDF).

3.1 ⊤-regularity is a good extension of regularity

Let (X, Q) be a convergence space. We define $\delta(X, Q) = (X, \delta(Q))$ as

$$\forall \mathbb{F} \in \mathbb{F}_L^\top(X), \forall x \in X, \mathbb{F} \xrightarrow{\delta(Q)} x \Leftrightarrow \iota(\mathbb{F}) \xrightarrow{Q} x.$$

Then it is easily seen that $(X, \delta(Q))$ is a ⊤-convergence space. In this subsection, we shall prove that $(X, \delta(Q))$ is ⊤-regular if and only if (X, Q) is regular. In this sense, we say that ⊤-regularity is a good extension of regularity.

Lemma 3.2. *Let $f : X \longrightarrow Y$, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ and $\Phi : J \longrightarrow \mathcal{F}(X)$. Then for any $\mathbb{F} \in \mathbb{F}_L^\top(J)$ and $\mathcal{F} \in \mathcal{F}(J)$, we have:*
(1) $f^{\Rightarrow}(k\phi\mathbb{F}) = k(f^{\Rightarrow} \circ \phi)\mathbb{F}$,
(2) *Take $\Phi_1 = \iota \circ \phi$, then $\iota(k\phi\mathbb{F}) = K\Phi_1\iota(\mathbb{F})$,*
(3) *Take $\phi_1 = \omega \circ \Phi$, then $\iota(k\phi_1\omega(\mathcal{F})) = K\Phi\mathcal{F}$,*
(4) *If $\sigma : J \longrightarrow \mathbb{F}_L^\top(X)$ satisfies $\sigma(j) \subseteq \phi(j)$ for any $j \in J$, then $k\sigma\mathbb{F} \subseteq k\phi\mathbb{F}$.*

Proof. (1) For any $\lambda \in L^Y$, we have

$$\lambda \in f^{\Rightarrow}(k\phi\mathbb{F}) \Leftrightarrow f^{\leftarrow}(\lambda) \in k\phi\mathbb{F} \Leftrightarrow \exists A \in \iota(\mathbb{F}), \text{ s.t } f^{\leftarrow}(\lambda) \in \bigcap_{y \in A} \phi(y)$$

$$\Leftrightarrow \exists A \in \iota(\mathbb{F}), \text{ s.t } \lambda \in \bigcap_{y \in A}(f^{\Rightarrow} \circ \phi)(y) \Leftrightarrow \lambda \in k(f^{\Rightarrow} \circ \phi)\mathbb{F}.$$

(2) It follows by

$$A \in \iota(k\phi\mathbb{F}) \Leftrightarrow \top_A \in k\phi\mathbb{F} \Leftrightarrow \exists B \in \iota(\mathbb{F}), \text{ s.t } \top_A \in \bigcap_{y \in B} \phi(y)$$

$$\Leftrightarrow \exists B \in \iota(\mathbb{F}), \text{ s.t } A \in \bigcap_{y \in B}(\iota \circ \phi)(y) = \bigcap_{y \in B} \Phi_1(y)$$

$$\Leftrightarrow A \in K\Phi_1\iota(\mathbb{F}).$$

(3) It follows by

$$A \in \iota(k\phi_1\omega(\mathcal{F})) \Leftrightarrow \top_A \in k\phi_1\omega(\mathcal{F}) \Leftrightarrow \exists B \in \iota \circ \omega(\mathcal{F}), \text{ s.t } \top_A \in \bigcap_{y \in B} \phi_1(y) = \bigcap_{y \in B}(\omega \circ \Phi)(y)$$

$$\Leftrightarrow \exists B \in \mathcal{F}, \text{ s.t } A \in \bigcap_{y \in B} \Phi(y)$$

$$\Leftrightarrow A \in K\Phi\mathcal{F}.$$

(4) It is obvious. $\qquad\square$

Theorem 3.3. (X, Q) *satisfies (DF) if and only if* $(X, \delta(Q))$ *satisfies (TDF).*

Proof. Let (X, Q) satisfy (DF). Assume that $\psi : J \longrightarrow X$, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{\delta(Q)} \psi(j)$ for each $j \in J$. Take $\Phi = \iota \circ \phi$, it follows by $\phi(j) \xrightarrow{\delta(Q)} \psi(j)$ that

$$\Phi(j) = \iota(\phi(j)) \xrightarrow{Q} \psi(j), \forall j \in J.$$

Assume that $k\phi\mathbb{F} \xrightarrow{\delta(Q)} x$, then by Lemma 3.2 (2) we have $K\Phi\iota(\mathbb{F}) = \iota(k\phi\mathbb{F}) \xrightarrow{Q} x$. It follows by (DF) and Lemma 2.6 (5) we get

$$\iota(\psi^{\Rightarrow}(\mathbb{F})) = \psi^{\Rightarrow}(\iota(\mathbb{F})) \xrightarrow{Q} x,$$

i.e., $\psi^{\Rightarrow}(\mathbb{F}) \xrightarrow{\delta(Q)} x$. Thus the condition (TDF) is satisfied.

Let $(X, \delta(Q))$ satisfy (TDF). Assume that $\psi : J \longrightarrow X$, $\Phi : J \longrightarrow \mathcal{F}(X)$ such that $\Phi(j) \xrightarrow{Q} \psi(j)$ for each $j \in J$. Take $\phi = \omega \circ \Phi$, it follows by Lemma 2.6 (1) that

$$\iota \circ \phi(j) = \iota \circ \omega \circ \Phi(j) = \Phi(j) \xrightarrow{Q} \psi(j),$$

i.e., $\phi(j) \xrightarrow{\delta(Q)} \psi(j)$ for any $j \in J$.

Let $K\Phi\mathcal{F} \xrightarrow{Q} x$. Then by Lemma 3.2 (3) we get $\iota(k\phi\omega(\mathcal{F})) = K\Phi\mathcal{F} \xrightarrow{Q} x$, i.e., $k\phi\omega(\mathcal{F}) \xrightarrow{\delta(Q)} x$. It follows by (TDF) that $\psi^{\Rightarrow}(\omega(\mathcal{F})) \xrightarrow{\delta(Q)} x$, and then by Lemma 2.6 (5), (1) we have

$$\psi^{\Rightarrow}(\mathcal{F}) = \psi^{\Rightarrow}((\iota \circ \omega)(\mathcal{F})) = \iota\psi^{\Rightarrow}(\omega(\mathcal{F})) \xrightarrow{Q} x.$$

Thus the condition (DF) is satisfied. $\qquad\square$

3.2 The category of $\top$-regular $\top$-convergence spaces is a reflective subcategory of $\top$-convergence spaces

In this subsection, we shall prove that the category of $\top$-regular convergence spaces is a reflective subcategory of $\top$-convergence spaces.

A function $f : X \longrightarrow Y$ between two $\top$-convergence spaces (X, q), (Y, p) is called continuous if $f^\Rightarrow(\mathbb{F}) \xrightarrow{p} f(x)$ whenever $\mathbb{F} \xrightarrow{q} x$. The category **T-CS** has as objects all $\top$-convergence spaces and as morphisms the continuous functions.

It is proved in [6] that the category **T-CS** is topological over **SET** in the sense of [40]. Indeed, for a given source $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$, the initial structure, q on X is defined by $\mathbb{F} \xrightarrow{q} x \Leftrightarrow \forall i \in I, f_i^\Rightarrow(\mathbb{F}) \xrightarrow{q_i} f_i(x)$.

Let (X, q) be a $\top$-convergence space, A a subset of X and $i_A : A \longrightarrow X$ the inclusion function. Then the initial $\top$-convergence structure on A w.r.t. the source $i_A : A \longrightarrow (X, q)$ is called the substructure of (X, q) on A, denoted by q_A, where

$$\forall x \in A, \mathbb{F} \in \mathbb{F}_L^\top(A), \mathbb{F} \xrightarrow{q_A} x \Leftrightarrow i_A^\Rightarrow(\mathbb{F}) \xrightarrow{q} x.$$

The pair (X, q_A) is called a subspace of (X, q).

Let X be a nonempty set and let $\top(X)$ denote the set of all $\top$-convergence structures on X. If the identity $\mathrm{id}_X : (X, q) \longrightarrow (X, p)$ is continuous then we say q is finer than p or p is coarser than q, and denote $p \leq q$.

Proposition 3.4. $(\top(X), \leq)$ *forms a complete lattice.*

Proof. For any $\{q_i\}_{i \in I} \subseteq \top(X)$, the supremum q of $\{q_i\}_{i \in I}$ exists and is denoted as $\sup\{q_i\}$. Indeed, $\sup\{q_i\}$ is precisely the initial structures q w.r.t. the source $(X \xrightarrow{\mathrm{id}_X} (X, q_i))_{i \in I}$, i.e., $q = \cap\{q_i\}_{i \in I}$. $\square$

In the following, we denote the full subcategory of **T-CS** consisting of all objects obeying (TDF) as **TDF-CS**.

Theorem 3.5. *The category* **TDF-CS** *is a topological category over* **SET**.

Proof. We need only check that **TDF-CS** has initial structure. Assume that $(X \xrightarrow{f_i} (X_i, q_i))_{i \in I}$ is a source in **T-CS** such that each $(X_i, q_i) \in$ **TDF-CS**. Let q be the initial structure of the above in **TF-CS**, that is

$$\mathbb{F} \xrightarrow{q} x \Leftrightarrow \forall i \in I, f_i^\Rightarrow(\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

We prove below that $(X, q) \in$ **TDF-CS**. Assume that $\psi : J \longrightarrow X$, $\phi : J \longrightarrow \mathbb{F}_L^\top(X)$ such that $\phi(j) \xrightarrow{q} \psi(j)$ for each $j \in J$. It follows that for any $i \in I$, $f_i^\Rightarrow(\phi(j)) \xrightarrow{q_i} f_i(\psi(j))$. Take $\phi_i = f_i^\Rightarrow \circ \phi$ and $\psi_i = f_i \circ \psi$, then

$$\phi_i(j) \xrightarrow{q_i} \psi_i(j), \forall j \in J.$$

Let $k\phi\mathbb{F} \xrightarrow{q} x$. Then by Lemma 3.2 (2) we have $\forall i \in I$,

$$k\phi_i\mathbb{F} = k(f_i^\Rightarrow \circ \phi)\mathbb{F} = f_i^\Rightarrow(k\phi\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

By (X, q_i) satisfies (TDF) we have

$$\psi_i^\Rightarrow(\mathbb{F}) = (f_i \circ \psi)^\Rightarrow(\mathbb{F}) = f_i^\Rightarrow \psi^\Rightarrow(\mathbb{F}) \xrightarrow{q_i} f_i(x).$$

It follows that $\psi^\Rightarrow(\mathbb{F}) \xrightarrow{q} x$. Thus (X, q) satisfies the condition (TDF). $\square$

Let $\top_{DF}(X)$ denote the set of all $\top$-convergence structures on X satisfying (TDF). Then it follows from the above theorem and Proposition 3.4 we get the following corollary.

Corollary 3.6. $(\top_{DF}(X), \leq)$ *forms a complete lattice.*

Theorem 3.7. **TDF-CS** *is a reflective subcategory of* **T-CS**.

Proof. Let $(X, q) \in$ **T-CS**. From Corollary 3.6, the supremum in $(\top(X), \leq)$ of all $s \leq q$ with $s \in \top_{DF}(X)$, denoted as (X, rq), is also in $\top_{DF}(X)$. Indeed, rq is the finest structure coarser than q satisfying $(X, rq) \in$ **TDF-CS**. Hence $\mathrm{id}_X : (X, q) \longrightarrow (X, rq)$ is continuous. Assume that $f(X, q) \longrightarrow (Y, p)$ is continuous, where $(X, p) \in$ **TDF-CS**. Let s denote the initial structure w.r.t. $f : X \longrightarrow (Y, p)$. Then $(X, s) \in$ **TDF-CS** and s is the coarsest structure such that $f : (X, s) \longrightarrow (Y, p)$ is continuous. It follows that $s \leq q$ and so $s \leq rq$. Therefore, $f : (X, rq) \longrightarrow (Y, p)$ is continuous and thus **TDF-CS** is reflective in **T-CS**. $\square$

3.3 Extension of continuous function

In this subsection, based on $\top$-regularity, we shall present an extension theorem of continuous function in the framework of $\top$-convergence space.

Lemma 3.8. *Let (A, q_A) be a subspace of a $\top$-convergence space (X, q). If (X, q) fulfils (TK) then (A, q_A) also fulfils this condition.*

Proof. Assume that $\phi : A \longrightarrow \mathbb{F}_L^\top(A)$ satisfies $\phi(y) \xrightarrow{q_A} y$ for each $y \in A$. Take $\overline{\phi} : X \longrightarrow \mathbb{F}_L^\top(X)$ as

$$\overline{\phi}(y) = i_A^{\rightarrow}(\phi(y)) \text{ if } y \in A \text{ and } \overline{\phi}(y) = [y]_\top \text{ if } y \notin A.$$

It is easily seen that $\overline{\phi}(y) \xrightarrow{q} y, \forall y \in X$.

Let $\mathbb{F} \xrightarrow{q_A} x$, then $i_A^{\rightarrow}(\mathbb{F}) \xrightarrow{q} x$. By (X, q) satisfies (TK) we have $k\overline{\phi}i_A^{\rightarrow}(\mathbb{F}) \xrightarrow{q} x$. We prove below $i_A^{\rightarrow}(k\phi\mathbb{F}) \supseteq k\overline{\phi}i_A^{\rightarrow}(\mathbb{F})$. Indeed,

$$\lambda \in k\overline{\phi}i_A^{\rightarrow}(\mathbb{F}) \Longrightarrow \exists B \in \iota(i_A^{\rightarrow}(\mathbb{F})) \text{ s.t. } \forall y \in B, \lambda \in \overline{\phi}(y)$$
$$\text{by Lemma 2.6 (5)} \Longrightarrow \exists B \in i_A^{\rightarrow}(\iota(\mathbb{F})) \text{ s.t. } \forall y \in B, \lambda \in \overline{\phi}(y)$$
$$\Longrightarrow A \cap B \in \iota(\mathbb{F}) \text{ s.t. } \forall y \in A \cap B, \lambda \in \overline{\phi}(y)$$
$$\Longrightarrow \exists C \in \iota(\mathbb{F}) \text{ s.t. } \forall y \in C, \lambda \in i_A^{\rightarrow}(\phi(y))$$
$$\Longrightarrow \exists C \in \iota(\mathbb{F}) \text{ s.t. } \forall y \in C, i_A^{\leftarrow}(\lambda) \in \phi(y)$$
$$\Longrightarrow i_A^{\leftarrow}(\lambda) \in k\phi\mathbb{F}$$
$$\Longrightarrow \lambda \in i_A^{\rightarrow}(k\phi\mathbb{F}).$$

By $k\overline{\phi}i_A^{\rightarrow}(\mathbb{F}) \xrightarrow{q} x$ we have $i_A^{\rightarrow}(k\phi\mathbb{F}) \xrightarrow{q} x$, i.e., $k\phi\mathbb{F} \xrightarrow{q_A} x$, as desired. Thus (X, q_A) satisfies the condition (TK). $\qquad\square$

Proposition 3.9. *Let (X, q) be a $\top$-convergence space satisfying (TK) and (Y, p) be $\top$-regularity. If A is a nonempty subset of X such that a function $\varphi : (A, q_A) \longrightarrow (Y, p)$ is continuous, then φ has a continuous extension $\overline{\varphi} : (B, q_B) \longrightarrow (Y, p)$, where*

$$B = \{x \in X | C_L^\top(x) \neq \varnothing, \{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\rightarrow}(i_A^{\leftarrow}(\mathbb{F})) \xrightarrow{p} y\} \neq \varnothing\},$$

$$C_L^\top(x) = \{\mathbb{F} \in \mathbb{F}_L^\top(X) | i_A^{\leftarrow}(\mathbb{F}) \text{ exists and } \mathbb{F} \xrightarrow{q} x\}.$$

Proof. (1) we prove that $A \subseteq B$.

For $z \in A$, note that $[z]_\top \xrightarrow{q} z$ and $i_A^{\leftarrow}([z]_\top)$ exists, thus $[z]_\top \in C_L^\top(z)$, which means $C_L^\top(z) \neq \varnothing$. Moreover, for any $\mathbb{F} \in C_L^\top(z)$, we have $i_A^{\leftarrow}(\mathbb{F})$ exists and $\mathbb{F} \xrightarrow{q} z$, then it follows that

$$i_A^{\rightarrow}i_A^{\leftarrow}(\mathbb{F}) \supseteq \mathbb{F} \xrightarrow{q} z \Longrightarrow i_A^{\leftarrow}(\mathbb{F}) \xrightarrow{q_A} z.$$

By the continuity of φ we get that $\varphi^{\rightarrow}(i_A^{\leftarrow}(\mathbb{F})) \xrightarrow{p} \varphi(z)$. Thus $z \in B$, and so $A \subseteq B$.

(2) We extend $\varphi : A \longrightarrow Y$ to $\overline{\varphi} : B \longrightarrow Y$ by $\overline{\varphi}(z) = \varphi(z)$ if $z \in A$ and $\overline{\varphi}(z) = y_z$, if $z \in B - A$, where y_z is some fixed element in $\{y | \forall \mathbb{F} \in C_L^\top(z), \varphi^{\rightarrow}(i_A^{\leftarrow}(\mathbb{F})) \xrightarrow{p} y\}$. Next, we prove that $\overline{\varphi} : (B, q_B) \longrightarrow (Y, p)$ is continuous. We need to check that for any $\mathbb{G} \in \mathbb{F}_L^\top(B)$ and any $z_0 \in B$, that $\mathbb{G} \xrightarrow{q_B} z_0$ implies $\overline{\varphi}^{\rightarrow}(\mathbb{G}) \xrightarrow{p} \overline{\varphi}(z_0)$. We complete it by several steps as follows.

(I) We define a function $\phi_B : B \longrightarrow \mathbb{F}_L^\top(B)$ as $\phi_B(z) = i_B^{\leftarrow}(\mathbb{H}_z)$ for any $z \in B$, where $\mathbb{H}_z \in C_L^\top(z)$. Indeed, by $i_A^{\leftarrow}(\mathbb{H}_z)$ exists and $A \subseteq B$ we get that $i_B^{\leftarrow}(\mathbb{H}_z)$ exists. Thus ϕ_B is well-defined. Note that (X, q) satisfies (TK), it follows by Lemma 3.8 that (X, q_B) also satisfies (TK). Thus by $\mathbb{G} \xrightarrow{q_B} z_0$ and

$$i_B^{\rightarrow}i_B^{\leftarrow}(\mathbb{H}_z) \supseteq \mathbb{H}_z \xrightarrow{q} z \Longrightarrow \phi_B(z) = i_B^{\leftarrow}(\mathbb{H}_z) \xrightarrow{q_B} z,$$

we get that $k\phi_B\mathbb{G} \xrightarrow{q_B} z_0$, i.e., $i_B^{\rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{q} z_0$.

(II) $i_A^{\Leftarrow}(k\phi_B\mathbb{G})$ exists. We need only check that $\bigvee_{z\in A}\lambda(z) = \top$ for any $\lambda \in k\phi_B\mathbb{G}$. Indeed, it follows by $\lambda \in k\phi_B\mathbb{G}$ that there exists an $E \in \iota(\mathbb{G})$ such that $\lambda \in \phi_B(e) = i_B^{\Leftarrow}(\mathbb{H}_e)$ for any $e \in E$. Then

$$\top = \bigvee_{\mu\in\mathbb{H}_e} S_B(i_B^{\leftarrow}(\mu),\lambda) \le \bigvee_{\mu\in\mathbb{H}_e} S_A(i_A^{\leftarrow}i_B^{\leftarrow}(\mu), i_A^{\leftarrow}(\lambda)) \le \bigvee_{\mu\in\mathbb{H}_e} \left((\bigvee_{z\in A}\mu(z)) \to (\bigvee_{z\in A}\lambda(z))\right).$$

Note that $\bigvee_{z\in A}\mu(z) = \top$ since $\mu \in \mathbb{H}_e$ and $i_A^{\leftarrow}(\mathbb{H}_e)$ exists. It follows that $\bigvee_{z\in A}\lambda(z) = \top$, and so $i_A^{\Leftarrow}(k\phi_B\mathbb{G})$ exists.

(III) $i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) = i_A^{\Leftarrow}(k\phi_B\mathbb{G})$. It follows by

$$\lambda \in i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \Leftrightarrow \bigvee_{\mu\in k\phi_B\mathbb{G}} S_A(i_A^{\leftarrow}i_B^{\rightarrow}(\mu),\lambda) = \top \Leftrightarrow \bigvee_{\mu\in k\phi_B\mathbb{G}} S_A(i_A^{\leftarrow}(\mu),\lambda) = \top \Leftrightarrow \lambda \in i_A^{\Leftarrow}(k\phi_B\mathbb{G}).$$

A combination of (I)-(III) we have $i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{q} z_0$ and $i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) = i_A^{\Leftarrow}(k\phi_B\mathbb{G})$ exists. It follows that $i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \in C_L^{\top}(z_0)$.

(IV) $\varphi^{\Rightarrow}i_A^{\Leftarrow}(k\phi_B\mathbb{G}) = \varphi^{\Rightarrow}i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{p} \overline{\varphi}(z_0)$. Indeed, if $z_0 \in A$, then

$$i_A^{\Rightarrow}i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \supseteq i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{q} z_0,$$

which means $i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{q_A} z_0$, then by the continuity of $\varphi : A \longrightarrow Y$ we get $\varphi^{\Rightarrow}i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{p} \varphi(z_0) = \overline{\varphi}(z_0)$.

If $z_0 \in B - A$, then

$$\overline{\varphi}(z_0) = y_{z_0} \in \{y | \forall \mathbb{F} \in C_L^{\top}(z_0), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\},$$

and by $i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \in C_L^{\top}(z_0)$, we conclude that $\varphi^{\Rightarrow}i_A^{\Leftarrow}i_B^{\Rightarrow}(k\phi_B\mathbb{G}) \xrightarrow{p} y_{z_0}$.

(V) Let $\phi_Y : B \longrightarrow \mathbb{F}_L^{\top}(Y)$ be the composition of the following three functions

$$B \xrightarrow{\phi_B} \mathbb{F}_L^{\top}(B) \xrightarrow{i_A^{\Leftarrow}} \mathbb{F}_L^{\top}(A) \xrightarrow{\varphi^{\Rightarrow}} \mathbb{F}_L^{\top}(Y).$$

Note that for any $z \in B$, $(i_A^{\Leftarrow} \circ \phi_B)(z) = i_A^{\Leftarrow}i_B^{\Leftarrow}(\mathbb{H}_z)$ exists since $\lambda \in \mathbb{H}_z$ implies $\bigvee_{w\in A}\lambda(w) = \top$. Therefore, ϕ_Y is well-defined. Next, we check that $k\phi_Y\mathbb{G} \xrightarrow{p} \overline{\varphi}(z_0)$ and $\phi_Y(z) \xrightarrow{p} \overline{\varphi}(z)$ for any $z \in B$.

(i) $k\phi_Y\mathbb{G} \xrightarrow{p} \overline{\varphi}(z_0)$. At first, we prove that $i_A^{\Leftarrow}(k\phi_B\mathbb{G}) \subseteq k(i_A^{\Leftarrow} \circ \phi_B)\mathbb{G}$. Let $\lambda \in i_A^{\Leftarrow}(k\phi_B\mathbb{G})$. Then $\bigvee_{\mu\in k\phi_B\mathbb{G}} S_A(i_A^{\leftarrow}(\mu),\lambda) = \top$. Note that

$$\begin{aligned}\mu \in k\phi_B\mathbb{G} &\Longrightarrow \exists E \in \iota(\mathbb{G}) s.t. \forall e \in E, \mu \in \phi_B(e)\\ &\Longrightarrow \exists E \in \iota(\mathbb{G}) s.t. \forall e \in E, i_A^{\leftarrow}(\mu) \in (i_A^{\Leftarrow} \circ \phi_B)(e)\\ &\Longrightarrow i_A^{\leftarrow}(\mu) \in k(i_A^{\Leftarrow} \circ \phi_B)\mathbb{G}.\end{aligned}$$

It follows that

$$\top = \bigvee_{\mu\in k\phi_B\mathbb{G}} S_A(i_A^{\leftarrow}(\mu),\lambda) \le \bigvee_{i_A^{\leftarrow}(\mu)\in k(i_A^{\Leftarrow}\circ\phi_B)\mathbb{G}} S_A(i_A^{\leftarrow}(\mu),\lambda) \le \bigvee_{\nu\in k(i_A^{\Leftarrow}\circ\phi_B)\mathbb{G}} S_A(\nu,\lambda),$$

which means $\lambda \in k(i_A^{\Leftarrow} \circ \phi_B)\mathbb{G}$. Thus $i_A^{\Leftarrow}(k\phi_B\mathbb{G}) \subseteq k(i_A^{\Leftarrow} \circ \phi_B)\mathbb{G}$. Then by Lemma 3.2 (1) we have

$$\varphi^{\Rightarrow}(i_A^{\Leftarrow}(k\phi_B\mathbb{G})) \subseteq \varphi^{\Rightarrow}(k(i_A^{\Leftarrow} \circ \phi_B)\mathbb{G}) = k(\varphi^{\Rightarrow} \circ i_A^{\Leftarrow} \circ \phi_B)\mathbb{G} = k\phi_Y\mathbb{G},$$

and by $\varphi^{\Rightarrow}i_A^{\Leftarrow}(k\phi_B\mathbb{G}) \xrightarrow{p} \overline{\varphi}(z_0)$, it holds that $k\phi_Y\mathbb{G} \xrightarrow{p} \overline{\varphi}(z_0)$.

(ii) $\phi_Y(z) \xrightarrow{p} \overline{\varphi}(z)$ for any $z \in B$. Note that $i_A^{\Leftarrow}(i_B^{\Rightarrow}(\phi_B(z)))$ exists since

$$i_A^{\Leftarrow}(i_B^{\Rightarrow}(\phi_B(z))) = i_A^{\Leftarrow}(i_B^{\Rightarrow}(i_B^{\Leftarrow}(\mathbb{H}_z))) = i_A^{\Leftarrow}(\mathbb{H}_z).$$

Then by $i_B^{\Rightarrow}(\phi_B(z)) = i_B^{\Rightarrow}(i_B^{\Leftarrow}(\mathbb{H}_z)) \supseteq \mathbb{H}_z \xrightarrow{q} z$ we obtain $i_B^{\Rightarrow}(\phi_B(z)) \in C_L^{\top}(z)$.

If $z \in A$, then

$$i_A^{\Rightarrow}i_A^{\Leftarrow}(\phi_B(y)) = i_A^{\Rightarrow}i_A^{\Leftarrow}(i_B^{\Rightarrow}(i_B^{\Leftarrow}(\mathbb{H}_z))) \supseteq \mathbb{H}_z \xrightarrow{q} z,$$

which means $i_A^{\Leftarrow}(\phi_B(y)) \xrightarrow{q_A} z$ and so $\phi_Y(y) = \varphi^{\Rightarrow}i_A^{\Leftarrow}(\phi_B(y)) \xrightarrow{p} \varphi(z) = \overline{\varphi}(z)$ by the continuity of $\varphi : A \longrightarrow Y$.

If $z \in B - A$, then

$$\overline{\varphi}(z) \in \{y | \forall \mathbb{F} \in C_L^\top(z), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\},$$

and by $i_B^{\Rightarrow}(\phi_B(z)) \in C_L^\top(z)$, we conclude that $\varphi^{\Rightarrow} i_A^{\Leftarrow} i_B^{\Rightarrow}(\phi_B(z)) \xrightarrow{p} \overline{\varphi}(z)$. Note that

$$i_A^{\Leftarrow}(i_B^{\Rightarrow}(\phi_B(z))) = i_A^{\Leftarrow}(\mathbb{H}_z) = i_A^{\Leftarrow} i_B^{\Rightarrow}(\mathbb{H}_z) = i_A^{\Leftarrow}(\phi_B(z)).$$

Thus $\phi_Y(z) = \varphi^{\Rightarrow} i_A^{\Leftarrow}(\phi_B(z)) \xrightarrow{p} \overline{\varphi}(z)$.

It follows from (i), (ii) and that (Y, p) satisfies (TDF) we get that $\overline{\varphi}^{\Rightarrow}(\mathbb{G}) \xrightarrow{p} \overline{\varphi}(z_0)$ for any $\mathbb{G} \xrightarrow{q_B} z_0$. This means that $\overline{\varphi} : (B, q_B) \longrightarrow (Y, p)$ is continuous. $\qquad\square$

A subset B of a $\top$-convergence space (X, q) is said to be dense [6] if for each $x \in X$, there exists a $\top$-filter $\mathbb{F}$ such that $i_B^{\Leftarrow}(\mathbb{F})$ exists and $\mathbb{F}$ converges to x. (X, q) is called $\top$-Hausdorff if for each $\top$-filter $\mathbb{F}$, there exists at most one $x \in X$ such that $\mathbb{F}$ converges to x.

Theorem 3.10 (continuous extension theorem). *Let (X, q) be a $\top$-convergence space satisfying Kowalsky $\top$-diagonal condition (TK), and let (Y, p) be regular and $\top$-Hausdorff. Then for each dense subset A in (X, q), a continuous function $\varphi : (A, q_A) \longrightarrow (Y, p)$ has a unique continuous extension $\overline{\varphi} : (X, q) \longrightarrow (Y, p)$ if and only if $\{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\} \neq \varnothing$ for any $x \in X$.*

Proof. Sufficiency. For any $x \in X$, since A is dense in (X, q) then $C_L^\top(x) \neq \varnothing$, it follows by $\{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\} \neq \varnothing$ and we have that

$$\{x \in X | C_L^\top(x) \neq \varnothing, \{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\} \neq \varnothing\} = X.$$

From Proposition 3.9, we conclude that there exists a continuous extension of φ, defined as $\overline{\varphi} : X \longrightarrow Y$: $\forall x \in X$, $\overline{\varphi}(x) = \varphi(x)$ if $x \in A$ and $\overline{\varphi}(x) = y_x$, if $x \in X - A$, where y_x is some fixed element in $\{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\}$. Note that the set $\{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\}$ has only one element since (Y, p) is $\top$-Hausdorff. This means that $\overline{\varphi}$ is defined uniquely.

Necessity. Assume that φ has a continuous extension $\overline{\varphi} : (X, q) \longrightarrow (Y, p)$. Then we have that $\overline{\varphi}^{\Rightarrow}(\mathbb{F}) \xrightarrow{p} \overline{\varphi}(x)$ for any $\mathbb{F} \xrightarrow{q} x$. Next we check that $\overline{\varphi}^{\Rightarrow}(\mathbb{F}) \subseteq \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F}))$ whenever $i_A^{\Leftarrow}(\mathbb{F})$ exists. Indeed, for any $\lambda \in \mathbb{F}$, it is easily seen that $\varphi^{\rightarrow}(i_A^{\leftarrow}(\lambda)) \leq \overline{\varphi}^{\rightarrow}(\lambda)$ and so $\overline{\varphi}^{\rightarrow}(\lambda) \in \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F}))$. It follows by Lemma 2.5 (1) that $\overline{\varphi}^{\Rightarrow}(\mathbb{F}) \subseteq \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F}))$.

For any $\mathbb{F} \in C_L^\top(x)$, which means $\mathbb{F} \xrightarrow{q} x$ and $i_A^{\Leftarrow}(\mathbb{F})$ exists. From the above statement we observe easily that $\varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} \overline{\varphi}(x)$. Therefore, $\{y | \forall \mathbb{F} \in C_L^\top(x), \varphi^{\Rightarrow}(i_A^{\Leftarrow}(\mathbb{F})) \xrightarrow{p} y\} \neq \varnothing$. $\qquad\square$

4 Conclusions

In this paper, we defined a notion of $\top$-regularity for $\top$-convergence spaces with the use of an extending dual Fischer diagonal condition, which is based on extending Kowalsky compression operator. It is proved that $\top$-regularity is a good extension of regularity, and the category of $\top$-regular $\top$-convergence space is a reflective category of $\top$-convergence spaces. In addition, based on $\top$-regularity, we explored an extension theorem of continuous function.

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