Regularity of gaussian processes

by

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1. Introduction

Let (Ω, Σ, P) denote a complete probability space, that will remain fixed throughout the paper. A (centered) Gaussian random variable X is a real valued measurable function on Ω such that for each real number t,

$$E \exp itX = \exp(-\sigma^2 t^2/2)$$

or that, equivalently, the law of X has a density $(2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$. The law of X is thus determined by $\sigma = (EX^2)^{1/2}$. If $\sigma = 1$, X is called standard normal.

A (centered) Gaussian process is a family $(X_t)_{t \in T}$ of random variables, indexed by some index set T, and such that each finite linear combination $\Sigma \alpha_t X_t$ is Gaussian. The covariance function $\Gamma(u, v) = E(X_u X_v)$ on $T \times T$ determines $E(\Sigma \alpha_t X_t)^2$, so it determines the law of the variables $(X_t)_{t \in T}$. Gaussian processes are thus a very rigid structure. One should expect, at least on philosophical grounds, that they have very nice properties. As of today, this expectation has been entirely fulfilled.

Historically, Gaussian processes, of which Brownian motion is the most important example, first occured as a model of evolution in time of a physical phenomenon. They were then naturally indexed by the real line, or by a subinterval of it. For such a process, the question of continuity arises immediately. We are dealing with an uncountable family of random variables, each of them being defined only a.e., so the very definition of continuity of the process already raises technical problems. These problems are taken care of by the use of a standard tool, the notion of "separable process". We are here hardly concerned with these technicalities, since the prime objective of this paper is to prove quantitative estimates, for which there is no loss of strength to assume T finite. For simplicity, when T is a topological space, let us define here the process $(X_t)_{t\in T}$ to be continuous if there exists a process $(Y_t)_{t\in T}$, with $X_t=Y_t$ a.s. for each t (the exceptional set depending on t) and such that for almost each ω in Ω, the function $t\mapsto Y_t(\omega)$ is continuous on T. (When dealing with a continuous process $(X_t)_{t\in T}$, we will always assume that $t\mapsto X_t(\omega)$ is continuous for almost each ω.) Uniform continuity can be defined in a similar way. The key to the study of continuity is the study of boundedness. We define here the process $(X_t)_{t\in T}$ to be bounded if there is a process $(Y_t)_{t\in T}$ with $X_t=Y_t$ a.s. for each t and such that for almost each ω in Ω, $\sup_{t\in T}|Y_t(\omega)|<\infty$. This is known to be equivalent to the fact that $\sup_D|X_t(\omega)|<\infty$ a.s. for each countable subset D of T.

For simplicity we write

$$E\sup_T X_t=\sup\left\{E\sup_D X_t,\ D \text{ countable subset of } T\right\}$$

and we adopt a similar convention for $E\sup_T|X_t|$. To formulate quantitative estimates, we need a measure of the boundedness of a process $(X_t)_{t\in T}$. Such a measure could be a median of $\sup_T X_t$. A more convenient (but equivalent) measure of boundedness is provided by the important result of H. Landau, L. A. Shepp and X. Fernique ([19], [7]). This result asserts that $(X_t)_{t\in T}$ is bounded if and only if $E\sup_T|X_t|<\infty$; so $E\sup_T|X_t|$ will be a convenient measure of the boundedness of $(X_t)_{t\in T}$. Another closely related measure of boundedness is the quantity $E\sup_T X_t$, which offers the extra advantage that $E\sup_T(X_t+Y)=E\sup_T X_t$ for any Gaussian random variable Y. It is easily seen that for any t_0 in T, we have

$$E\sup_T X_t\le E\sup_T|X_t|\le E|X_{t_0}|+2E\sup_T X_t. \tag{1}$$

The early results on Gaussian processes dealt with processes indexed by a subset of $\mathbf{R}^n$, and tried to take advantage of the special structure of the index set. For example, an important early theorem of X. Fernique [6] was proved using a chaining argument on carefully chosen dyadic partitions of the unit interval. As we first noted, a Gaussian process is determined by its covariance structure, which has no reason to be closely related to the structure of the index set as a subset of $\mathbf{R}^n$. It should then be expected that a more intrinsic point of view would yield better results. This was achieved in the landmark paper of R. M. Dudley [3]. On the index set T, consider the pseudo-distance d given by

$$d(u, v) = \sigma(X_u - X_v) = (E(X_u - X_v)^2)^{1/2} \tag{2}$$

(d will keep this meaning throughout the paper). Denote by N_ε the smallest number of closed d-balls of radius ε that can cover T. We define the metric entropy condition as the finiteness of the entropy integral $\int_0^\infty (\log N_\varepsilon)^{1/2} d\varepsilon$. Note that when ε is larger than the diameter of T, $N=1$, so the integrand is zero. Also, since N_ε is a decreasing function of ε, the issue for the finiteness of the integral is at zero. One major result of R. M. Dudley is that the metric entropy condition implies the boundedness of the process. More precisely

$$E \sup_T X_t \leq K \int_0^\infty (\log N_\varepsilon)^{1/2} d\varepsilon \tag{3}$$

for some universal constant K. We should note here that when T contains one point only, the entropy integral vanishes, so we cannot use $E \sup_T |X_t|$ instead of $E \sup_T X_t$ in (3).

The inequality (3) is in some sense sharp. V. N. Sudakov [25], used a lemma of D. Slepian [24], that is now a cornerstone of the theory, to show that

$$\sup_{\varepsilon > 0} \varepsilon (\log N_\varepsilon)^{1/2} \leq K E \sup_T X_t \tag{4}$$

for some universal constant K. (For simplicity, K will always denote a universal constant, not necessarily the same at each line.)

At this point, we must discuss a simple, well known, but most instructive example. Consider a Gaussian process indexed by $T = \{n; n \geq 1\}$, or, in other words, a jointly Gaussian sequence $(Y_n)_{n \geq 1}$. (We do not assume that the Y_n are independent.) Assume that $\sigma(Y_n) = (EY_n^2)^{1/2} \leq (1 + \log n)^{-1/2}$. Then the sequence (Y_n) is bounded a.s. and in fact

$$E \sup_n |Y_n| \leq K \tag{5}$$

for some universal constant K. The proof is elementary. We note that for a Gaussian r.v. X, with $\sigma = \sigma(X) = (EX^2)^{1/2}$, we have, for $a \geq \sigma$

$$P(\{X \geq a\}) = (2\pi\sigma^2)^{-1/2} \int_a^\infty \exp(-t^2/2\sigma^2) dt$$

$$\leq (2\pi\sigma^2 a^2)^{-1/2} \int_a^\infty \exp(-t^2/2\sigma^2)dt$$
$$\leq \frac{\sigma}{a\sqrt{2\pi}} \exp\left(-\frac{a^2}{2\sigma^2}\right) \leq \exp\left(-\frac{a^2}{2\sigma^2}\right).$$

So, for $a \geq 2$,

$$P(\{\exists n \geq 1, |Y_n| \geq a\}) \leq 2 \sum_{n\geq 1} \exp\left(-\frac{a^2}{2}(1+\log n)\right)$$
$$\leq 2\exp\left(-\frac{a^2}{2}\right) \sum_{n\geq 1} n^{-a^2/2} \leq K \exp\left(-\frac{a^2}{2}\right)$$

and this implies (5).

Let us assume now that the sequence (Y_n) is independent, and that $\sigma(Y_n)=(1+\log n)^{-1/2}$. Let $\varepsilon>0$. Then for $n<n_\varepsilon=\exp(-1+1/2\varepsilon^2)$, we have $\sigma(Y_n)>\varepsilon\sqrt{2}$, so if $m, n<n_\varepsilon$ se have $d(n,m)=\sigma(Y_n-Y_m)>2\varepsilon$. This shows that n, m cannot belong to the same d-ball of radius ε. It follows that $N_\varepsilon \geq n_\varepsilon - 1$, so $\inf_{\varepsilon>0} \varepsilon(\log N_\varepsilon)^{1/2}>0$, and the metric entropy condition fails. This example shows that the boundedness of a Gaussian process is not characterized by the metric entropy condition. A closer inspection will give a clue about the reason of this failure. Each n for $n<n_\varepsilon$ needs a d-ball of radius ε to cover itself alone; on the other hand, the points n for $n>n_\varepsilon$ can be covered by one single d-ball of radius ε. This can be expressed by saying that the numbers N_ε can give exaggerated importance to parts of the space (T, d) that are actually rather thin. They do not take well in account the possible lack of homogeneity of the space (T, d).

One should then try to understand the case where (T, d) has some homogeneity. A typical situation is the case of stationary processes. If G is a locally compact abelian group, a Gaussian process $(X_t)_{t\in G}$ is called stationary if the translations are isometries of (G, d), where d is as usual given by (2). In other words, for t, u, v in G, $d(u, v) = d(u+t, v+t)$. In that case, any two d-balls of G are isometric. Let now T be a compact subset of G, of nonempty interior, and suppose that the covariance of the stationary process $(X_t)_{t\in G}$ is continuous on $G\times G$. In 1974, X. Fernique established (in the case $G=\mathbf{R}^n$) the fundamental fact that the metric entropy condition is necessary and sufficient for the continuity of $(X_t)_{t\in T}$. This result had a considerable influence. For example, it is the main tool used by M. Marcus and G. Pisier in the definitive treatment of random Fourier series [21]. The proof of this result is outlined at the beginning of section 2. The proof of our main result will follow the same general scheme.

For general processes, we need a substitute to the metric entropy condition. The stationary case can give a hint. If G is compact, $(X_t)_{t\in G}$ is stationary, and m denotes the normalized Haar measure of G, it is easily seen that the metric entropy condition is equivalent to

$$\sup_{x\in G}\int_0^\infty\left(\log\left(\frac{1}{m(B(x,\varepsilon))}\right)\right)^{1/2}d\varepsilon<\infty \tag{6}$$

where $B(x,\varepsilon)$ denotes the d-ball, and where the integral is actually independent of x. Condition (6) will prepare the reader to the introduction of majorizing measures. For the simplicity of notations, let us set once and for all $g(t)=(\log(1/t))^{1/2}$ for $0<t\leq 1$. We say that a probability measure m on (T,d) is a majorizing measure if

$$\sup_{x\in T}\int_0^\infty g(m(B(x,\varepsilon)))d\varepsilon<\infty.$$

We note that the integrand is zero when ε is larger than the diameter of T. X. Fernique proved that for a Gaussian process $(X_t)_{t\in T}$, and any probability measure m on (T,d), we have

$$E\sup_T X_t\leq K\sup_x\int_0^\infty g(m(B(x,\varepsilon)))d\varepsilon. \tag{7}$$

It is not hard to show that this contains Dudley's result (3). It should be noted that (7) follows from an earlier work of C. Preston ([23], Lemma 4). However, in his main statements, C. Preston unnecessarily restricts his hypothesis. He was apparently not aware of the power of the present formulation. C. Preston's work itself follows a seminal paper by A. M. Garsia, E. Rodemich, H. R. Rumsey [15].

X. Fernique apparently conjectured as early as 1974 that the existence of majorizing measures might characterize the boundedness of Gaussian processes. (See [10] p. 69). Other researchers however considered these measures as exotic; so Fernique remained very isolated in his efforts; he nevertheless proved a number of important partial results, and his determination eventually motivated the author to attack the problem.

The central result of this paper is the validity of Fernique's conjecture. It will be proved in section 2.

THEOREM 1. *For each bounded Gaussian process* $(X_t)_{t\in T}$ *there exists a probability measure* m *on* (T, d) *such that*

$$\sup_x \int_0^\infty g(m(B(x,\varepsilon)))d\varepsilon \leq K E \sup_T X_t.$$

In section 3, we will explore some of the consequences. We will prove a comparison theorem between Gaussian and subgaussian processes. We will relate the uniform modulus of continuity of (X_t) over (T, d) and the existence of special types of majorizing measures on (T, d). Another consequence of (the proof of) Theorem 1 is the following, that is so unexpected that it does not seem to have been even conjectured earlier.

THEOREM 2. *Let* $(X_t)_{t\in T}$ *be a bounded Gaussian process. Let* $a=E\sup_T|X_t|$, *and let* b *be the d-diameter of* T. *Then there exists a (not necessarily independent) Gaussian sequence* $(Y_n)_{n\geq 1}$ *such that* $\sigma(Y_n)\leq Ka(\log n+a^2/b^2)^{-1/2}$ *and that for each* t *in* T, *we can write*

$$X_t = \sum_n \alpha_n(t)\, Y_n \tag{8}$$

where $\alpha_n(t)\geq 0$, $\Sigma_n\, \alpha_n(t)\leq 1$, *and the series converges a.s. and in* L^2. *Moreover, each* Y_n *is a linear combination of at most two variables of the type* X_t.

The point of this theorem is that this representation of the process implies its boundedness. Actually, (8) implies that

$$|X_t(\omega)| \leq \sup_n |Y_n(\omega)|$$

so the boundedness of (X_t) follows from (4) and the easy fact that $b\leq a\sqrt{2\pi}$. Actually, (8) implies that $E\sup_T|X_t|\leq Ka$. An easy consequence of Theorem 2 is the following solution to the problem of continuity of Gaussian processes.

THEOREM 3. *Let* T *be a compact metric space. Then a Gaussian process* $(X_t)_{t\in T}$ *is continuous if and only if its covariance is continuous and there exists a Gaussian sequence* $(Y_n)_{n\geq 1}$ *such that* $\lim_n (\log n)^{1/2}\sigma(Y_n)=0$ *and that for each* t *in* T, *one can write* $X_t=\Sigma_n\, \alpha_n(t)\, Y_n$, *where the series converges in* L^2, $\alpha_n(t)\geq 0$, $\Sigma_n\, \alpha_n(t)\leq 1$.

As a corollary of Theorem 3, we will be able to describe all Gaussian measures on all separable Banach spaces (Theorem 19).

A natural question raised by Theorem 1 is to better understand the nature of the majorizing measure m. Our proof is not constructive; the existence of m is proved indirectly. The constructions make use of the structure of the metric space (T, d), but they don't relate well this structure to the properties of the process. Consider the case where T is finite and separated by d. Then there is a.s. a unique point $\tau(\omega)$ of T such that $X_{\tau(\omega)}(\omega)=\sup_T X_t(\omega)$. Denote by μ the law of τ. The intuition of X. Fernique was that the majorizing measure should be closely related to μ. We will prove the following result (the left-hand inequality is due to X. Fernique).

THEOREM 4. *Let T, μ be as above, and D be the diameter of T. Then*

$$K^{-1} E \sup_T X_t \leq D + \int_T d\mu(x) \int_0^\infty g(\mu(B(x, \varepsilon)))\, d\varepsilon \leq KE \sup_T X_t. \tag{9}$$

To interpret this theorem, let

$$T' = \left\{ x \in T, \int_0^\infty g(\mu(B(x, \varepsilon)))\, d\varepsilon \leq 2KE \sup_T X_t \right\}.$$

Then $\mu(T') \geq 1/2$, and μ behaves like a majorizing measure on T', with the unessential restriction that it is not supported by T'. The set T' can be much smaller than T; however, as far as the process $\sup_T X_t$ is concerned, T' contains a lot of information since $P(\sup_{T'} X_t = \sup_T X_t) \geq 1/2$. Actually it can be shown that if $U \subset T$ is such that $P(\sup_U X_t = \sup_T X_t) \geq 1/2$, then $E \sup_T X_t \leq KE \sup_U X_t$. So, roughly speaking, Theorem 4 means that μ is a majorizing measure on a subset of T large enough to control $\sup_T X_t$.

As we have seen, $E \sup_T X_t = EX_\tau$, and the law $\mathscr{L}(\tau)$ of τ is μ. Given now any probability measure ν on T, it is natural to consider the functional (introduced by X. Fernique)

$$F(\mathscr{X}, \nu) = \sup_{\mathscr{L}(\eta)=\nu} X_\eta$$

where $\mathscr{X} = (X_t)_{t \in T}$, and the sup is taken over all measurable maps η from Ω to T, of law ν. (So we have $E \sup_T X_t = F(\mathscr{X}, \mu)$.) In section 4, we prove Theorem 4, and we show how to evaluate the functional $F(\mathscr{X}, \nu)$.

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2. Existence of majorizing measures

One essential ingredient of the proof is a very specific property of Gaussian processes, that was discovered by D. Slepian [24]. Close to Slepian's result, but more convenient to use, is the following comparison theorem.

PROPOSITION 5. *Let $(X_t)_{t\in T}$, $(Y_t)_{t\in T}$ be two Gaussian processes indexed by the same set. Assume that for each u, v in T, we have $\sigma(Y_u-Y_v)\leq\sigma(X_u-X_v)$. Then $E\sup_T Y_t\leq E\sup_T X_t$.*

As stated, this theorem was announced by V. N. Sudakov [26]. X. Fernique mentions in [9] that credit is also due to S. Chevet. (We did not have access to Chevet's paper.) The weaker inequality $E\sup_T Y_t\leq 2E\sup_T X_t$ would be sufficient for our purpose. It can be derived easily from Slepian's lemma, as is implicitly proved in an early paper of M. Marcus and L. A. Shepp [20]. A proof of Theorem 5 can be found in [10].

The best way to illustrate the power of Theorem 5 is to prove Sudakov's minoration (4). Let $\varepsilon>0$, and let U be a maximal subset of T with $d(t,u)>\varepsilon$ for t, u in U, $t\neq u$, so $N_\varepsilon\leq\operatorname{card} U$. Let $(Z_t)_{t\in U}$ be an independent standard normal sequence; and let $Y_t=(\varepsilon/\sqrt{2})Z_t$; For t, u in U, $t\neq u$, we have $\sigma(Y_t-Y_u)=\varepsilon\leq\sigma(X_t-X_u)$; so Proposition 5 gives

$$E\sup_U Y_t\leq E\sup_U X_t\leq E\sup_T X_t.$$

An easy estimate shows that $E\sup_U Y_t$ is of order $\varepsilon(\log(\operatorname{card} U))^{1/2}$, and this implies (4).

We now outline the proof of Fernique's theorem that in the stationary case, the continuity of a Gaussian process implies the metric entropy condition. To avoid unessential technicalities, we consider the case where G is compact, $T=G$.

For two subsets A, B of a metric space (T,d), let

$$d(A,B)=\inf\{d(a,b); a\in A, b\in B\}.$$

Assuming that the metric entropy condition fails, Fernique's method allows to construct for $n \geq 1$ numbers N_n, and families $\mathscr{B}_n$ of d-balls with the following properties:

$$\sum_{i \geq 1} 9^{-i} (\log N_i)^{1/2} = \infty. \tag{10}$$

(11) Each B in $\mathscr{B}_n$ has radius 9^{-n}; if B, B' belong to $\mathscr{B}_n$, $B \neq B'$, then $d(B, B') \geq 9^{-n}$.

(12) Each B in $\mathscr{B}_n$ contains N_n balls of $\mathscr{B}_{n+1}$.

(The choice of the number 9 is fairly arbitrary.)

Let S_n be the set of the centers of the balls of $\mathscr{B}_n$. The all-important condition (11) allows to use Theorem 5 to compare $(X_t)_{t \in S_n}$, with a suitable process to get

$$K^{-1} \sum_{i \leq n} 9^{-i} (\log N_i)^{1/2} \leq E \sup_{S_n} X_t \leq E \sup_{T} X_t$$

and this is impossible for n large enough.

One stricking feature of the construction is condition (12). It means that all the balls in $\mathscr{B}_i$ play essentially the same role. This is made possible by the great homogeneity of T, and in particular by the fact that all the d-balls of T of a given radius are isometric. This specific feature cannot carry on in the general case.

Recall that for a metric space T, the diameter of T is the quantity

$$\operatorname{diam} T = \sup \{d(x, y); x, y \in T\}.$$

The main construction of our proof, by induction over n, is the construction of a family $\mathscr{B}_n$ of subsets of T (that are no longer d-balls), that satisfy the following conditions:

(13) Each B in $\mathscr{B}_n$ has diameter $\leq 6^{-n}$;

(14) If B, B' belong to $\mathscr{B}_n$, $B \neq B'$, then $d(B, B') \geq 6^{-n-1}$,

as well as appropriate substitutes for conditions (10) and (12), that we will describe later.

After $\mathscr{B}_n$ has been obtained, we will continue the construction by trying to build inside each set B of $\mathscr{B}_n$ as many sets of $\mathscr{B}_{n+1}$ as possible. To succeed, we must ensure that B is big enough in some sense. So we need an appropriate measure of the size of a metric space.

Given a probability measure m on the metric space (T, d), let

$$\gamma_m(T) = \sup_{x \in T} \int_0^\infty g(m(B(x,\varepsilon)))d\varepsilon$$

$$\gamma(T) = \inf \gamma_m(T)$$

where the infimum is taken over all probability measures m on T. For a subspace A of T, $\gamma(A)$ refers to the quantity associated to the metric space (A,d); that is, $\gamma(A)=\inf\gamma_m(A)$, where the inf is taken over the probability measures supported by A.

From now on, we assume until further notice that all the metric spaces are finite. Recall that a metric space (U,δ) is called ultrametric if for u, v, w in U, we have

$$\delta(u,w) \leq \max(\delta(u,v),\delta(v,w)).$$

A nice feature of ultrametric spaces is that two balls of the same radius are either identical or disjoint. Say that a map f from U onto T is a contraction if $d(f(u),f(v)) \leq \delta(u,v)$ for u, v in U. For a metric space (T,d), consider the quantity

$$\alpha(T) = \inf\{\gamma(U);\ U \text{ is ultrametric and } T \text{ is image of } U \text{ by a contraction}\}.$$

Although $\gamma(T)$ comes first to the mind as a way to measure the size of T, the quantity $\alpha(T)$ is easier to manipulate, and yields stronger results. We first collect some simple facts.

LEMMA 6. (a) $\gamma(T) \leq \alpha(T)$.

(b) *if* $A \subset T$, $\gamma(A) \leq 2\gamma(T)$.

(c) *if* U *is ultrametric,* $A \subset U$, *then* $\gamma(A) \leq \gamma(U)$.

(d) *if* $A \subset T$, *then* $\alpha(A) \leq \alpha(T)$.

(e) $\alpha(T)=\inf\{\gamma(U);$ *U ultrametric,* card $U \leq$ card T, diam $U \leq$ diam T, *T is image of U by a contraction$\}$ and this* inf *is attained.*

(f) diam $T \leq K\gamma(T)$.

Proof. (a) Let f be a contraction from U into T, m a probability measure on U, $\mu=f(m)$. For u in U, $\varepsilon>0$, we have $f^{-1}(B(f(u),\varepsilon)) \supset B(u,\varepsilon)$ since f is a contraction, so $\mu(B(f(u),\varepsilon)) \geq m(B(u,\varepsilon))$ since $\mu=f(m)$. (Here and throughout the paper, when no ambiguity arises, we adopt the convention that $B(x,\varepsilon)$ denotes the ball for the distance on the space that contains x.) Since g is decreasing we get

$$\int_0^\infty g(\mu(B(f(u),\varepsilon)))d\varepsilon \leq \int_0^\infty g(m(B(u,\varepsilon)))d\varepsilon.$$

Since f is onto, we get $\gamma_\mu(T)\leq\gamma_m(U)$, so $\gamma(T)\leq\gamma(U)$ since m is arbitrary, so $\gamma(T)\leq\alpha(T)$ since U, f are arbitrary.

(b) For t in T, take $\varphi(t)$ in A with

$$d(t,\varphi(t)) = d(t,A) = \inf\{d(t,y); y\in A\}.$$

Let m be a probability measure on T, and $\mu=\varphi(m)$, so μ is supported by A. Fix x in A. For t in T, we have $d(t,A)\leq d(t,x)$, so $d(t,\varphi(t))\leq d(t,x)$, so $d(x,\varphi(t))\leq 2d(x,t)$. Since $\mu=\varphi(m)$, it follows that $\mu(B(x,2\varepsilon))\geq m(B(x,\varepsilon))$, so

$$\begin{aligned}\int_0^\infty g(\mu(B(x,\varepsilon)))d\mu &\leq \int_0^\infty g(m(B(x,\varepsilon/2)))d\varepsilon\\ &= 2\int_0^\infty g(m(B(x,\varepsilon)))d\varepsilon.\end{aligned}$$

This shows that $\gamma_\mu(A)\leq 2\gamma_m(T)$, so $\gamma(A)\leq 2\gamma(T)$.

(c) With the notations of the above proof, the ultrametricity gives

$$d(x,\varphi(t)) \leq \max(d(x,t), d(t,\varphi(t)) \leq d(x,t)$$

so $\mu(B(x,\varepsilon))\geq m(B(x,\varepsilon))$ and $\gamma(A)\leq\gamma(U)$ as above.

(d) Let U be ultrametric, and let f be a contraction from U onto T. By (c), we get

$$\alpha(A)\leq\gamma(f^{-1}(A))\leq\gamma(U)$$

so $\alpha(A)\leq\alpha(T)$.

(e) If (U,δ) is ultrametric, and f is a contraction from U onto T, consider the distance δ_1 on U given by

$$\delta_1(u,v) = \inf(\delta(u,v), \operatorname{diam} T).$$

Then (U,δ_1) is ultrametric, f is still a contraction from (U,δ_1) onto T, and $\gamma((U,\delta_1))\leq\gamma((U,\delta))$ by the argument of (a). Also, if $A=f^{-1}(T)$, $\gamma(A)\leq\gamma(U)$ by (c). The last assertion follows by a standard compactness argument.

(f) Take two points u, v in T. Let $\delta=d(u,v)$. The balls $B(u,\delta/3)$, $B(v,\delta/3)$ are disjoint. For a probability measure m on T, one of these balls (say the first) has a measure $\leq\frac{1}{2}$, so

$$\gamma(T) \geq \int_0^{\delta/3} g(B(u, \varepsilon))\, d\varepsilon \geq \delta/3(\log 2)^{1/2}$$

so $d(u, v) \leq K\gamma(T)$. The proof is complete.

The next lemma exhibits a behavior of α that resembles a strong form of subadditivity.

LEMMA 7. *Let T be a finite metric space of diameter D. Suppose that we have a finite covering $A_1, \ldots, A_n$ of T. Then there is a nonempty subset I of $\{1, \ldots, n\}$ such that for i in I we have*

$$\alpha(A_i) \geq \alpha(T) - D(2\log(1+\operatorname{card} I))^{1/2}.$$

Proof. From Lemma 6(e), for $i \leq n$ there exists an ultrametric space (U_i, δ_i) of diameter $\leq D$, a contraction f_i from U_i onto A_i, and a probability measure m_i on U_i such that $\alpha(A_i) = \gamma_{m_i}(U_i)$. Let U be the disjoint sum of the spaces $(U_i)_{i\leq n}$. Define the distance δ on U by $\delta(u, v) = \delta_i(u, v)$ whenever u, v belong to the same U_i, and $\delta(u, v) = D$ otherwise. Then (U, δ) is ultrametric. The map f from U onto T given by $f(u) = f_i(u)$ for u in U_i is a contraction.

There is no loss of generality to assume that $\alpha(A_i) \geq \alpha(A_j)$ for $1 \leq i \leq j \leq n$.

Let $\eta_i = (i+1)^{-2}$, so

$$\sum_{i\geq 1} \eta_i = \sum_{n\geq 2} n^{-2} \leq \int_1^{\infty} t^{-2}\, dt = 1.$$

Consider the positive measure m' on U given by $m' = \Sigma_{i\leq n} \eta_i m_i$. We have $\|m'\| \leq 1$, so there is a probability m on U with $m' \leq m$. Take x in U, and let i with $x \in U_i$. Since for $0 < a, b \leq 1$, we have $g(ab) \leq g(a) + g(b)$, we have

$$\begin{aligned} g(m(B(x, \varepsilon))) &\leq g(m'(B(x, \varepsilon))) \leq g(\eta_i m_i(B(x, \varepsilon))) \\ &\leq g(\eta_i) + g(m_i(B(x, \varepsilon))). \end{aligned}$$

It follows that

$$\int_0^{\infty} g(m(B(x, \varepsilon)))d\varepsilon = \int_0^{D} g(m(B(x, \varepsilon)))\, d\varepsilon$$

$$\leq Dg(\eta_i)+\int_0^\infty g(m_i(B(x,\varepsilon)))\,d\varepsilon$$

$$\leq Dg(\eta_i)+\alpha(A_i).$$

It follows that

$$\alpha(U)\leq\gamma_m(U)\leq\sup_{i\leq n}(Dg(\eta_i)+\alpha(A_i)),$$

So there exists $1\leq i\leq n$ such that

$$\alpha(A_i)\geq\alpha(U)-Dg(\eta_i)=\alpha(U)-D(2\log(1+i))^{1/2}.$$

Taking $I=\{1,\ldots,i\}$, this concludes the proof.

COROLLARY 8. *If* $T=T_1\cup T_2$, $\alpha(T)\leq\max(\alpha(T_1),\alpha(T_2))+2\operatorname{diam}T$.

Proof. Take $n=2$ in Lemma 7, and note that $(2\log 3)^{1/2}\leq 2$.

At this point, we should mention that we never attempt to find sharp numerical constants, but always use crude, but simple, bounds.

The main obstacle to the construction is the separation condition (14). It implies, in particular, that we cannot in general cover an element B of $\mathscr{B}_n$ by elements of $\mathscr{B}_{n+1}$. Thus, we have to take great care that the piece of B we will disregard is not too big. This seems to be difficult to achieve by using the functional α alone, so we will introduce an auxiliary functional that somehow measures the rate of decrease of α.

For i in $\mathbf{N}$, $A\subset T$, let

$$\beta_i(A)=\alpha(A)-\sup_{x\in A}\alpha(A\cap B(x,6^{-i-1})).$$

The next lemma starts to address the problem of condition (14).

LEMMA 9. *Let* (T,d) *be a metric space,* $i\in\mathbf{N}$. *Then we can find non-empty subsets* $B\subset A\subset T$ *satisfying the following conditions*

(15) $\operatorname{diam}A\leq 6^{-i}$;

(16) $d(B,T\setminus A)\geq 6^{-i-1}$;

(17) *One of the following holds:*

(a) $\alpha(B)+\alpha(T)\geq 2\alpha(A)$;
(b) $\alpha(B)+\beta_i(B)\geq\alpha(T)$.

Proof. First case. There exists x in T such that

$$\alpha(B(x,6^{-i-1}))+\alpha(T)\geq 2\alpha(B(x,2\cdot 6^{-i-1})).$$

Then we set

$$B=B(x,6^{-i-1}),\quad A=B(x,2\cdot 6^{-i-1}).$$

and (a) of condition (17) holds.

Second case. For each t in T, we have

$$\alpha(B(t,6^{-i-1}))+\alpha(T)\leq 2\alpha(B(t,2\cdot 6^{-i-1})).$$

Let x be such that $\alpha(B(x,2\cdot 6^{-i-1}))$ is maximal. Set

$$B=B(x,2\cdot 6^{-i-1}),\quad A=B(x,3\cdot 6^{-i-1})$$

so (15) and (16) hold. For any t in T, we have

$$\alpha(B(t,6^{-i-1}))+\alpha(T)\leq 2\alpha(B)$$

so

$$\alpha(B)+(\alpha(B)-\alpha(B(t,6^{-i-1})))\geq\alpha(T).$$

Since

$$\alpha(B(t,6^{-i-1}))\geq\alpha(B)\cap B(t,6^{-i-1}))$$

this shows that $\alpha(B)+\beta_i(B)\geq\alpha(T)$. The proof is complete.

We now have the tools to perform the essential step.

PROPOSITION 10. *Let (T,d) be a metric space of diameter $\leq 6^{-i}$. Then we can find a non-empty index set I, and for k in I, a set $B_k\subset T$ such that the following conditions hold:*

(18) *each set B_k has a diameter $\leq 6^{-i-1}$;*
(19) *if k, $l\in I$, $k\neq l$, then $d(B_k,B_l)\geq 6^{-i-2}$;*

(20) *for k in I,*

$$\alpha(B_k)+\beta_{i+1}(B_k)\geq \alpha(T)+\beta_i(T)-6^{-i+1}(2+(\log(\operatorname{card} I))^{1/2}).$$

Condition (20) will be easily used when successive applications of the proposition are made. It is to obtain a relation of this type that we introduced the quantities β_i.

Proof. By induction over k, we construct subsets A_k, B_k, T_k of T that satisfy the following conditions:

(21) $T_k=T\setminus\bigcup_{p<k}A_p$; $\operatorname{diam} A_k\leq 6^{-i-1}$; $B_k\subset A_k\subset T_k$; $d(B_k, T_k\setminus A_k)\geq 6^{-i-2}$;

(22) For each k, one of the following holds:

(a) $\alpha(B_k)+\alpha(T_k)\geq 2\alpha(A_k)$;

(b) $\alpha(B_k)+\beta_{i+1}(B_k)\geq\alpha(T_k)$.

The construction starts with $T_1=T$. It is straightforward by applying Lemma 9 to the space T_k at step k. We stop the construction at the first integer m for which $\alpha(T_m)<\alpha(T)-2\cdot 6^{-i}$, so in particular $\alpha(T_k)\geq\alpha(T)-2\cdot 6^{-i}$ for $k<m$.

Let $T'=\bigcup_{p<m}A_p$, so $T=T'\cup T_m$. It follows from Corollary 8 that $\alpha(T')\geq\alpha(T)-2\cdot 6^{-i}$.

We now apply Lemma 7 to the covering of T' by the sets $(A_k)_{k<m}$. We get a non-empty subset I of $\{1,\ldots,m-1\}$ such that for each k in I, we have

$$\alpha(A_k)\geq\alpha(T')-6^{-i}(2\log(1+\operatorname{card} I))^{1/2}.$$

Using the easy inequality $(\log(1+n))^{1/2}\leq 1+(\log n)^{1/2}$, we get

$$\alpha(A_k)\geq\alpha(T)-2\cdot 6^{-i}(2+\log(\operatorname{card} I))^{1/2}. \tag{23}$$

Since A_k is of diameter $\leq 6^{-i-1}$, we have

$$\beta_i(T)\leq\alpha(T)-\alpha(A_k)\leq 2\cdot 6^{-i}(2+\log(\operatorname{card} I))^{1/2}. \tag{24}$$

For a given k in J, suppose first that (a) holds in (22). Then from (24)

$$\begin{aligned}\alpha(B_k)&\geq 2\alpha(A_k)-\alpha(T_k)\geq 2\alpha(A_k)-\alpha(T)\\ &\geq\alpha(T)-2(\alpha(T)-\alpha(A_k))\\ &\geq\alpha(T)+\beta_i(T)-3(\alpha(T)-\alpha(A_k))\\ &\geq\alpha(T)+\beta_i(T)-6^{-i+1}(2+(\log(\operatorname{card} I))^{1/2}).\end{aligned}$$

so (20) holds since $\beta_{i+1}(B_k)\geq 0$.

Suppose now that (b) holds in (21). Then from (24), we get

$$\begin{aligned}\alpha(B_k)+\beta_{i+1}(B_k) &\geq \alpha(T_k) \geq \alpha(T)-2\cdot 6^{-i-1}\\ &\geq \alpha(T)+\beta_i(T)-4\cdot 6^{-i}(2+(\log(\operatorname{card} I))^{1/2}).\end{aligned}$$

It remains only to prove (19). For $1\leq k<l<m$, we have $B_l\subset T_l\subset T_{k+1}\subset T_k\setminus A_k$, so (21) implies that $d(B_k, B_l)\geq 6^{-i-2}$. The proof is complete.

Let U be an ultrametric space. For x in U, i in $\mathbf{N}$, let $N_i(x)$ be the number of disjoint balls of radius 6^{-i-1} that are contained in $B(x, 6^{-i})$. Define

$$\xi_x(U)=\sum_{i\in\mathbf{Z}} 6^{-i-1}(\log N_i(x))^{1/2}; \tag{25}$$

$$\xi(U)=\inf_{x\in U} \xi_x(U). \tag{26}$$

We note that if $\operatorname{diam} U\leq 6^{-j}$, and $B(x, 6^{-k})=\{x\}$, we have

$$\xi_x(U)=\sum_{j\leq i<k} 6^{-i-1}(\log N_i(x))^{1/2}.$$

We can now perform the main construction.

THEOREM 11. *There exists a universal constant K with the following property. For each (finite) metric space (T, d), there exists a ultrametric space (U, δ) and a map φ: $U\to T$ such that the following conditions hold:*

$$\alpha(T)\leq K\xi(U); \tag{27}$$

$$\delta(u, v)\leq d(\varphi(u), \varphi(v))\leq 36\delta(u, v), \quad \forall u, v\in U. \tag{28}$$

Proof. Let j be the largest integer with $6^{-j}\geq\operatorname{diam} T$.

Consider two points u, v of T with $d(u, v)=\operatorname{diam} T$. The space $U=(\{u, v\}, d)$ is ultrametric, and the canonical injection φ from U in T satisfies (28). The balls $B(u, 6^{-j-1})$ and $B(v, 6^{-j-1})$ are disjoint; so we have

$$\xi(U)\geq\xi_u(U)\geq 6^{-j-1}(\log 2)^{1/2}\geq(\log 2)^{1/2}\operatorname{diam} T/6.$$

We can assume that $K\geq 6^4(\log 2)^{-1/2}$. Then (27) holds unless $\alpha(T)\geq 6^3\operatorname{diam} T$, so it remains to prove the theorem in that case only. By induction over $i\geq j$, we construct families $\mathcal{B}_i$ of subsets of T that satisfy the following conditions.

(29) for B in $\mathscr{B}_i$, $\operatorname{diam} B \leqslant 6^{-i}$;

(30) for B, B' in $\mathscr{B}_i$, $B \neq B'$, we have $d(B, B') \geqslant 6^{-i-1}$;

(31) for B in $\mathscr{B}_i$, $i > j$, there is B' in $\mathscr{B}_{i-1}$ with $B \subset B'$;

(32) for B in $\mathscr{B}_i$, if

$$N = \operatorname{card}\{B' \in \mathscr{B}_{i+1}, B' \subset B\},$$

we have $N \geqslant 1$, and for each B' in $\mathscr{B}_{i+1}$ with $B' \subset B$, we have

$$\alpha(B') + \beta_{i+1}(B') \geqslant \alpha(B) + \beta_i(B) - 6^{-i+1}(2 + (\log N)^{1/2}).$$

The construction starts with $\mathscr{B}_j = \{T\}$. Each step is performed by application of Proposition 10 to each element of $\mathscr{B}_i$. We stop the construction at some k large enough that any two different points of T are at distance $> 6^{-k}$, so each B of $\mathscr{B}_k$ consists exactly of one point. Let $U = \bigcup \{B; B \in \mathscr{B}_k\}$. For u, v in U, let $\delta(u, v) = 6^{-i-2}$, where $i = \sup\{l; \exists B$ in $\mathscr{B}_l$, $u, v \in B\}$. Since $u, v \in B$ for some B in $\mathscr{B}_i$, we have $d(u, v) \leqslant 6^{-i}$ from (29). Also, there exist two different elements B_1, B_2 of $\mathscr{B}_{i+1}$ such that $u \in B_1, v \in B_2$, so (30) shows that $d(u, v) \geqslant 6^{-i-2}$. Denote by φ the canonical injection from U into T. We have proved (28).

Fix now x in U, and for $j \leqslant i \leqslant k$ denote $B_i(x)$ the element of $\mathscr{B}_i$ that contains x. Let $N_i(x)$ be the number of elements of $\mathscr{B}_{i+1}$ that are contained in $B_i(x)$. From condition (32) we have, for $j \leqslant i \leqslant k$

$$\alpha(B_{i+1}(x)) + \beta_{i+1}(B_{i+1}(x)) \geqslant \alpha(B_i(x)) + \beta_i(B_i(x)) - 6^{-i+1}(2 + (\log N_i(x))^{1/2}). \tag{33}$$

We note that $\alpha(B_j(x)) = \alpha(T)$, $\beta_j(B_j(x)) \geqslant 0$. Also, since $B_k(x) = \{x\}$, we have $\alpha(B_k(x)) = 0$, $\beta_k(B_k(x)) = 0$. Summation of the inequalities (33) for $j \leqslant i < k$ gives

$$\alpha(T) \leqslant \sum_{j \leqslant i < k} 6^{-i+1}(2 + (\log N_i(x))^{1/2}).$$

Since $6^{-j} \leqslant 6 \operatorname{diam} T$, we have

$$\sum_{j \leqslant i} 2 \cdot 6^{-i+1} \leqslant \frac{12}{5} 6^{-j+1} \leqslant \frac{1}{2} 6^3 \operatorname{diam} T \leqslant \alpha(T)/2,$$

so

$$\alpha(T) \leqslant 2 \cdot 6^4 \sum_{j \leqslant i < k} 6^{-i-3}(\log N_i(x))^{1/2}. \tag{34}$$

We note now that for each element B of $\mathcal{B}_i$, $B\cap U$ is a δ-ball of radius 6^{-i-2}, so (34) means that $\alpha(T)\leq 2\cdot 6^4\xi_x(U)$, so $\alpha(T)\leq 2\cdot 6^4\xi(U)$. The proof is complete.

We can now prove the existence of majorizing measures when T is finite.

THEOREM 12. *Let $(X_t)_{t\in T}$ be a Gaussian process indexed by a finite set T, and provide T with the canonical distance d. Then $\alpha(T)\leq KE\sup_T X_t$, where K is a universal constant.*

Proof. Let U,φ be as given by the application of Theorem 11 to the space (T,d). It is enough to show that $\xi(U)\leq KE\sup_{u\in U}X_{\varphi(u)}$. A first approach would be to show that $\xi(U)\leq K\gamma(U)$, and to use a theorem of X. Fernique ([11] Théorème 3–3) which in the present case gives $\gamma(U)\leq KE\sup_{u\in U}X_{\varphi(u)}$. It will, however, be simpler to give a direct proof. We note that for u,v in U, $\sigma(X_{\varphi(u)},X_{\varphi(v)})=d(\varphi(u),\varphi(v))\geq d(u,v)$ so the theorem is a consequence of the following result, that we single out for future reference.

PROPOSITION 13. *Let (U,δ) be a finite ultrametric space. Then for each Gaussian process $(X_u)_{u\in U}$ such that $\sigma(X_u,X_v)\geq\delta(u,v)$ whenever $u,v\in U$, we have $\xi(U)\leq KE\sup_U X_u$, where K is a universal constant.*

Proof. Let j in $\mathbf{N}$ be the largest with $\operatorname{diam} U\leq 6^{-j}$. For $i>j$, let $\mathcal{B}_i$ be the collection of the balls of U of radius 6^{-i}. Let $\mathcal{B}=\bigcup_{i>j}\mathcal{B}_i$. Consider an independent family $(Y_B)_{B\in\mathcal{B}}$ of standard normal r.v. For u in U, $i>j$, we write for simplicity $Y_{u,i}=Y_{B(u,6^{-i})}$. For u in U, let $Z_u=\Sigma_{i>j}6^{-i}Y_{u,i}$.

Let u,v in U, and let k be the largest such that $\delta(u,v)\leq 6^{-k}$. Then $B(u,6^{-i})=B(v,6^{-i})$ for $i\leq k$, so

$$Z_u-Z_v=\sum_{i>k}6^{-i}(Y_{u,i}-Y_{v,i}).$$

It follows that

$$\begin{aligned}\sigma(Z_u-Z_v)&\leq\sum_{i>k}6^{-i}\leq\frac{6}{5}\cdot 6^{-k-1}\leq\frac{6}{5}\delta(u,v)\\&\leq\frac{6}{5}\sigma(X_u-X_v).\end{aligned}$$

Proposition 5 shows that it is enough to show that $\xi(U)\leqslant A\, E\sup_U Z_u$ for some constant A. Let A be a number such that for a finite independent family $(Y_i)_{i\leqslant N}$ of standard normal r.v. we have $(\log N)^{1/2}\leqslant A\, E\sup_{i\leqslant N} Y_i$. By induction over n, we prove the following statement:

(H_n) If U has a diameter $\leqslant 6^{-j}$, if for each x in U, $B(x,6^{-k})=\{x\}$ and if $k-j\leqslant n$, then $\xi(U)\leqslant A\, E\sup_U Z_u$.

For $n=0$, U contains only one point, so $\xi(U)=0$ and (H_0) holds. Let us assume now that (H_n) holds, and let us prove (H_{n+1}). We enumerate $\mathscr{B}_{j+1}$ as $\{B_1,\ldots,B_q\}$. For $p\leqslant q$, let

$$\Omega_p=\left\{\forall i\leqslant q,\, i\neq p,\, Y_{B_p}>Y_{B_i}\right\}.$$

For u in U, define

$$Z'_u=\sum_{i>j+1}6^{-i}\,Y_{u,i}=Z_u-6^{-j-1}\,Y_{u,j+1}.$$

For $k\leqslant q$, consider a measurable map τ_k from Ω to B_k that satisfies $Z'_{\tau_k}=\sup_{u\in B_k}Z'_u$. (For a measurable map τ from Ω to U, we define Z_τ by $Z_\tau(\omega)=Z_{\tau(\omega)}(\omega)$.) Define now a measurable map τ from Ω to U by $\tau(\omega)=\tau_k(\omega)$ for ω in Ω_k. We have

$$\begin{aligned} E\sup_U Z_u \geqslant E Z_\tau &= \sum_{k\leqslant q}E(1_{\Omega_k}Z_{\tau_k})\\ &\geqslant \sum_{k\leqslant q}E(1_{\Omega_k}(6^{-j-1}\,Y_{B_k}+Z'_{\tau_k}))\\ &= 6^{-j-1}\sum_{k\leqslant q}E(1_{\Omega_k}Y_{B_k})+\sum_{k\leqslant q}E(1_{\Omega_k}Z'_{\tau_k}). \end{aligned}$$

Now

$$\sum_{k\leqslant q}E(1_{\Omega_k}Y_{B_k})=E\sup_{k\leqslant q}Y_{B_k}\geqslant A^{-1}(\log q)^{1/2}.$$

The independence of the variables $(Y_B)_{B\in\mathscr{B}}$ shows that 1_{Ω_k} and Z'_{τ_k} are independent, so

$$E(1_{\Omega_k}Z'_{\tau_k})=P(\Omega_k)\,E Z'_{\tau_k}=\frac{1}{q}EZ'_{\tau_k},$$

so we get

$$A\,E\,Z'_{\tau_k} = A\,E \sup_{B_k} Z'_u \geqslant \xi(B_k).$$

The definition of ξ makes it clear that for each k,

$$\xi(B_k)+6^{-j-1}(\log q)^{1/2} \geqslant \xi(U)$$

so the proof is complete.

In the case where T is finite, Theorem 1 follows from Theorem 12 and from the fact, proved in Lemma 6, that $\gamma(T)\leqslant\alpha(T)$. There is, however, a definite loss of information when using $\gamma(T)$ instead of $\alpha(T)$. The following result is more precise than Theorem 1, and essentially contains all the strength of Theorem 12. (Metric spaces are no longer always finite.)

THEOREM 14. *Consider a bounded Gaussian process $(X_t)_{t\in T}$. Then there exists a probability measure m on (T,d) such that for each t in T*

$$\int_0^{\operatorname{diam} T} g(\sup\{m(\{u\});d(t,u)\leqslant\varepsilon\})d\varepsilon \leqslant K\,E \sup_T X_t. \tag{35}$$

Proof. Theorem 12 shows that for each finite subset V of T,

$$\alpha(V) \leqslant K\,E \sup_V X_t \leqslant K\,E\sup_T X_t.$$

It is hence enough to show that if we set $\alpha=\sup\{\alpha(V);V\subset T,\ V \text{finite}\}$ there is a probability measure m on T such that for each t in T,

$$\int_0^{\operatorname{diam} T} g(\sup\{m(\{u\});d(t,u)\leqslant\varepsilon\})\,d\varepsilon \leqslant K\alpha.$$

We start by an elementary observation, that we will use routinely in the rest of the paper. If $h(t)$ is a positive decreasing function, we have

$$\sum_{i\in\mathbf{Z}} 2^{-i-1}h(2^{-i}) \leqslant \int_0^\infty h(\varepsilon)\,d\varepsilon \leqslant \sum_{i\in\mathbf{Z}} 2^{-i}h(2^{-i}). \tag{36}$$

Denote by j the largest integer with $2^{-j}\geqslant \operatorname{diam} T$.

Suppose now that the process $(X_t)_{t\in T}$ is bounded, and for $i\geqslant j$ let T_i be a finite subset of T such that each point of T is at distance $\leqslant 2^{-i}$ of a point of T_i. Consider a map φ_i from T to T_i, such that $d(t,\varphi_i(t))\leqslant 2^{-i}$. For each k, we know that $\alpha(T_k)\leqslant\alpha$. So there exists an ultrametric space (U_k,δ_k), a contraction f_k from U_k onto T_k, and a probability m_k on U_k such that for each u in U_k,

$$\int_0^\infty g(m_k(B(u,\varepsilon)))\,d\varepsilon\leqslant\alpha$$

so from (36) we have

$$\sum_{i>j}2^{-i}g(m_k(B(u,2^{-i})))\leqslant 2\alpha. \tag{37}$$

To each ball B of U, we associate a point $v(B)$ in B. Let $\mathscr{B}_i$ be the family of balls of radius 2^{-i} of U_k. Denote by μ_i^k the probability measure on T that, for each B in $\mathscr{B}_i$, gives mass $m_k(B)$ to the point $\varphi_i(f_k(v(B)))$. We note that μ_i^k is supported by T_i. Fix t in T. Choose t' in T_k with $d(t,t')\leqslant 2^{-k}$. Take u in U_k such that $f_k(u)=t'$. For each i, we have $\delta_k(u,v(B(u,2^{-i})))\leqslant 2^{-i}$, so $d(t',f_k(v(B(u,2^{-i}))))\leqslant 2^{i}$, so

$$d(t,\varphi_i(f_k(v(B(u,2^{-i})))))\leqslant 2^{-i+1}+2^{-k}.$$

We set

$$t_i^k=\varphi_i(f_k(v(B(u,2^{-i}))))$$

so $d(t,t_i^k)\leqslant 2^{-i+1}+2^{-k}$, and $\mu_i^k(\{t_i^k\})\geqslant m_k(B(u,2^{-i}))$. If follows from (37) that we have

$$\sum_{i>j}2^{-i}g(\mu_i^k(\{t_i^k\}))\leqslant 2\alpha. \tag{38}$$

Let $\mathscr{U}$ be an ultrafilter on $\mathbf{N}$. Since t_i^k belongs to the finite set T_i, the limit $t_i=\lim_{k\to\mathscr{U}}t_i^k$ exists, and $d(t,t_i)\leqslant 2^{-i+1}$. Since μ_i^k is supported by the finite set T_i, the limit $\mu_i=\lim_{k\to\mathscr{U}}\mu_i^k$ exists, and (38) implies that for each t in U we have

$$\sum_{i>j}2^{-i}g(\mu_i(\{t_i\}))\leqslant 2\alpha. \tag{39}$$

Let $m=\Sigma_{i>j}2^{j-i}\mu_i$, so m is a probability on T.

We note that

$$g(m(\{t_i\}))\leqslant g(2^{j-i}\mu_i(\{t_i\}))$$
$$\leqslant g(2^{j-i})+g(\mu_i(\{t_j\}))$$

so from (39) we get

$$\sum_{i>j} 2^{-i} g(m(\{t_i\}))\leqslant \sum_{i>j} 2^{-i} g(2^{j-i})+2\alpha.$$

Now

$$\sum_{i>j} 2^{-i} g(2^{j-i}) = \sum_{i>j} 2^{-i}((i-j)\log 2)^{1/2} \leqslant K2^{-j} \leqslant K \operatorname{diam} T.$$

From lemma 6(f), diam $T\leqslant K\alpha$, so we get

$$\sum_{i>j} 2^{-i} g(m(\{t_i\}))\leqslant K\alpha.$$

For $\varepsilon>2^{-i+1}$ we have $\sup\{m(\{u\}); d(u,t)\leqslant\varepsilon\}\geqslant m(\{t_i\})$ and the result follows from (36) again.

3. Applications

Our first application is a comparison theorem between processes.

THEOREM 15. *Let $(X_t)_{t\in T}$ be a Gaussian process, and (Y_t) be any other centered process indexed by the same set. Assume that for each θ in* $\mathbf{R}$, *we have*

$$E\exp\theta(Y_u-Y_v)\leqslant E\exp\theta(X_u-X_v)=\exp\left(\frac{\theta^2}{2}d^2(u,v)\right). \tag{40}$$

Then we have $E\sup_T Y_t\leqslant KE\sup_T X_t$ where K is a universal constant.

When (Y_t) is also Gaussian, $E\exp\theta(Y_u-Y_v)=\exp(\theta^2\sigma(Y_u-Y_v)^2/2)$ so (40) reduces to the inequality $\sigma(Y_u-Y_v)\leqslant\sigma(X_u-X_v)$. In that case, Theorem 15 reduces to Proposition 5 (but with an unspecified constant).

Proof. Theorem 1 shows that there is a probability measure m on (T, d) such that

$$\sup_{x\in T}\int_0^\infty g(m(B(x,\varepsilon)))\,d\varepsilon < K\,E\sup_T X_t.$$

It follows from (40) that for each $t\geq 0$, each θ, we have

$$\exp(t\theta)\,P\{Y_u - Y_v > td(u,v)\} \leq \exp(\theta^2/2).$$

Taking $\theta = t$,

$$P\{Y_u - Y_v > td(u,v)\} \leq \exp(-t^2/2).$$

As is well known to specialists, (and is shown e.g. by the proof given in [10]), this inequality implies that (Y_t) satisfies the majorizing measure bound (7). This completes the proof.

Remarks. (1) The above proof is very indirect; it would be desirable to have a more direct argument.

(2) M. Marcus pointed out that, by standard techniques, one can deduce from Theorem 15 the fact that if $(X_t)_{t\in T}$ is continuous on (T, d), then the process $(Y_t)_{t\in T}$ satisfies the central limit theorem.

We now prove a version of Theorem 1 that is adapted to the study of continuity. For completeness, we prove the following simple and well known fact.

PROPOSITION 16. *Consider a bounded Gaussian process $(X_t)_{t\in T}$. Consider a metric τ on T such that the canonical metric d is τ-uniformly continuous. Then the process $(X_t)_{t\in T}$ is τ-uniformly continuous if and only if $\lim_{n\to 0}\varphi_\tau(\eta)=0$, where φ_τ is given by*

$$\varphi_\tau(\eta) = E\sup_{\tau(u,v)<\eta}(X_u - X_v). \tag{41}$$

Proof. We prove necessity. For each ω we have

$$\lim_{\eta\to 0}\sup_{\tau(u,v)\leq\eta}|X_u(\omega) - X_v(\omega)| = 0$$

so the fact that $\lim_{\eta\to 0}\varphi_\tau(\eta)=0$ follows from dominated convergence.

Conversely, since d is τ-uniformly continuous, we can find a sequence

(η_n) with $\varphi_\tau(\eta_n) \leq 2^{-n}$, and $\tau(u, v) < \eta_n \Rightarrow d(u, v) < 2^{-n}$. We borrow from Proposition 18 (to be proved later) the fact that $\Sigma_n P(A_n) < \infty$, where

$$A_n = \left\{ \sup_{\tau(u,v) \leq \eta_n} |X_u(\omega) - X_v(\omega)| > K2^{-n/2} \right\}.$$

It follows that almost all ω belong to at most finitely many sets A_n; so $(X_t)_{t \in T}$ is uniformly continuous for τ. This completes the proof.

For a probability measure m on (T, d), we write

$$\gamma_m(\eta) = \sup_{x \in T} \int_0^\eta g(m(B(x, \varepsilon)))\, d\varepsilon \tag{42}$$

$$\alpha_m(\eta) = \sup_{x \in T} \int_0^\eta g(\sup \{m(\{u\}); d(u, x) \leq \varepsilon\})\, d\varepsilon. \tag{43}$$

Recall that for a given metric space (T, d), we denote by N_ε the smallest number of ε-balls that can cover T.

THEOREM 17. *Consider a bounded Gaussian process* $(X_t)_{t \in T}$. *Then*

(a) *For any probability measure m on T, we have* $\varphi_d(\eta) \leq K\gamma_m(\eta)$, *where* φ_d *is given by* (41).

(b) *Define*

$$\beta(\eta) = \sup_{x \in T} E\,(\sup_{d(x,u) < \eta} |X_u - X_x|). \tag{44}$$

Then there exists a probability measure m on (T, d) *such that for each* $\eta > 0$,

$$\alpha_m(\eta) \leq K\beta(\eta) + \eta(\log(2N_\eta \log_2^2(2D/\eta)))^{1/2}. \tag{45}$$

In particular, the process $(X_t)_{t \in T}$ *is bounded and uniformly continuous on* (T, d) *if and only of* (T, d) *is totally bounded and there exists a probability measure m on* (T, d) *such that* $\lim_{\eta \to 0} \gamma_m(\eta) = 0$.

Proof. (a) (Due to X. Fernique.) Let

$$U = \{(x, y) \in T \times T; d(x, y) < \eta\}.$$

We provide $T\times T$ and its subspace U with the distance d' given by

$$d'((x,y),(x',y')) = \sigma(X_x - X_y - (X_{x'} - X_{y'})).$$

For (x,y) in $T\times T$, we have

$$B((x,y),\varepsilon) \supset B(x,\varepsilon/2)\times B(y,\varepsilon/2)$$

where the first ball is a d'-ball. It follows that

$$\begin{aligned}\int_0^{2\eta} g(m\otimes m(B((x,y),\varepsilon)))\,d\varepsilon &\leqslant 2\sup_{z\in T}\int_0^{2\eta} g(m(B(z,\varepsilon/2)))\,d\varepsilon\\ &\leqslant 4\sup_{z\in T}\int_0^{\eta} g(m(B(z,\varepsilon)))\,d\varepsilon = 4\gamma_m(\eta).\end{aligned}$$

The method of Lemma 6(b) shows that for any finite subset V of U we have $\gamma(V)\leqslant 8\gamma_m(\eta)$. The conclusion follows then from (7) and an easy limit argument.

(b) Let $D=\operatorname{diam} T$. For $n>0$, let $a_n=2^{-n}D$. For each $n>0$ we consider a family $B_{n,1},\ldots,B_{n,p(n)}$ of d-balls of radius a_n that covers T, where $p(n)=N_{a_n}$. So, for $i\leqslant p(n)$, if we denote by $u(n,i)$ the center of $B_{n,i}$, we have

$$E\sup_{B_{n,i}} X_t = E\sup_{t\in B_{n,i}} X_t - X_{u(n,i)} \leqslant E\sup_{t\in B_{n,i}} |X_t - X_{u(n,i)}| \leqslant \beta(a_n).$$

Denote by $d_{n,i}$ the diameter of $B_{n,i}$ (that can be smaller than $2a_n$). Theorem 14 shows that there is a probability measure $m_{n,i}$ on $B_{n,i}$ such that for each x in $B_{n,i}$ we have

$$\int_0^{d_{n,i}} g(\sup\{m_{n,i}(\{u\}); d(x,u)\leqslant\varepsilon\})\,d\varepsilon \leqslant K\beta(a_n). \tag{46}$$

Let $m'_{n,i}=\frac{1}{2}(\delta_{u(n,i)}+m_{n,i})$, and let

$$m = \sum_{\substack{n>0\\ i\leqslant p(n)}} n^{-2}p(n)^{-1}m'_{n,i},$$

so $\|m\|\leqslant 1$. Fix x in T, $0\leqslant\eta\leqslant D$. Let n be the smallest integer $\geqslant 1$ with $a_n\leqslant\eta$, so $\eta\leqslant 2a_n$. We note that if $x\in B_{n,i}$

$$\sup\{m(\{u\}); d(x,u)\leqslant\varepsilon\} \geqslant \frac{1}{2}n^{-2}p(n)^{-1}\sup\{m_{n,i}(\{u\}); d(x,u)\leqslant\varepsilon\}.$$

Also, for $\varepsilon \geqslant \inf(d_{i,n}, a_n)$,

$$\sup\{m(\{u\}); d(x,u) \leqslant \varepsilon\} \geqslant \frac{1}{2} n^{-2} p(n)^{-1}.$$

So we have from (46) that

$$\alpha_m(\eta) \leqslant \int_0^{d_{n,i}} g(\sup\{m_{n,i}(\{u\}); d(x,u) \leqslant \varepsilon) \, d\varepsilon + \eta g(n^{-2} p(n)^{-1}/2).$$

Since $a_n \leqslant \eta \leqslant 2a_n$, we obtain (45).

We now prove the last assertion of the theorem. If $(X_t)_{t\in T}$ is uniformly continuous, $\lim_{\eta\to 0} \beta(\eta)=0$, by dominated convergence. Let m be a probability measure on T such that (45) holds. To prove that $\lim_{\eta\to\infty} \alpha_m(\eta)=0$, it is enough to show that $\lim_{\varepsilon\to 0} \varepsilon(\log N_\varepsilon)^{1/2}=0$. This is known [25], but we give the simple proof for completeness. Let $\alpha>0$. Let η be small enough that $\beta(\eta)<\alpha$. Fix a finite subset A of T such that each element of T is within distance η of an element of A. Let $\varepsilon>0$. For each x in A, it follows from (3) that there is a subset A_x of T such that card $A_x \leqslant \exp(K^2\alpha^2/\varepsilon^2)$, and that for t in T, with $d(x,t)<\eta$, there is y in A_x with

$$d(t,y) = \sigma(X_t - X_x - (X_y - X_x)) \leqslant \varepsilon.$$

Let $B = \bigcup_{x\in A} A_x$. Each point of T is within distance ε of an element of B, and card $B \leqslant \operatorname{card} A \exp(K^2\alpha^2/\varepsilon^2)$. This shows that $\limsup \varepsilon(\log N_\varepsilon)^{1/2} \leqslant K\alpha$ for each $\alpha>0$, and finishes the proof.

We now prove the converse. Consider $\eta>0$ such that $\gamma_m(\eta)<\infty$. Then (7) shows that (X_t) is bounded on each ball of radius η. Since T is totally bounded, T can be covered by finitely many such balls; so $(X_t)_{t\in T}$ is bounded. We complete the proof by using (a) and Proposition 16 with $\tau=d$.

We now prove Theorem 2. We will prove at the same time that if we assume $\lim_{\eta\to 0} \varphi_d(\eta)=0$, where $\varphi_d(\eta)$ is given by (41), we can force the sequence (Y_n) to have the additional property that $\lim_{n\to\infty} (\log n)^{1/2} \sigma(Y_n)=0$.

Let j be the largest integer with $2^{-j} \geqslant \operatorname{diam} T$. It follows from Theorem 14 and (36) that there is a probability measure m on T such that for each t on T, we have

$$\sum_{i\geqslant j} 2^{-i} g(\sup\{m(\{u\}); d(t,u) \leqslant 2^{-i}\}) \leqslant Ka. \tag{47}$$

If $\lim_{\eta\to 0}\varphi_d(\eta)=0$, we can moreover assume from Theorem 17 that

$$\lim_{k\to\infty}\sup_{t\in T}\sum_{i\geq k}2^{-i}g(\sup\{m(\{u\});d(t,u)\leq 2^{-i}\})=0. \tag{48}$$

For t in T, $i\geq j$, we pick t_i such that $d(t_i,t)\leq 2^{-i}$ and

$$m(\{t_i\})=\sup\{m(\{u\});d(t,u)\leq 2^{-i}\}.$$

We can assume that t_j does not depend on t.

From (47), we see that $2^{-i}g(m(\{t_i\}))\leq Ka$. This shows that for each t in T, t_i belongs to the finite set

$$A_i=\{u\in T;m(\{u\})\leq\exp(-2^{2i}(Ka)^2)\}.$$

Let $\alpha_i=2^{-i+j-1}\exp(-(a/b)^2)$. Using the fact that $\operatorname{diam}T\leq b\leq(\pi/2)^{1/2}a$ we find from (47) that we have

$$\sum_{i\geq j}2^{-i}g(\alpha_i m(\{t_i\}))\leq K_1 a \tag{49}$$

for some universal constant K_1. If $\lim_{\eta\to 0}\varphi_d(\eta)=0$, we moreover have

$$\lim_{k\to\infty}\sup_{t\in T}\sum_{i\geq k}2^{-i}g(\alpha_i m(\{t_i\}))=0 \tag{50}$$

For each t in T, $i\geq j$, we define

$$a_{t,i}=2^{-i}(g(\alpha_i m(\{t_i\}))+g(\alpha_{i+1}m(\{t_{i+1}\}))).$$

From (49), we get

$$\sum_{i\geq j}a_{t,i}\leq 3K_1 a. \tag{51}$$

Moreover, if $\lim_{\eta\to 0}\varphi_d(\eta)=0$, we have from (50)

$$\lim_{k\to\infty}\sup_{t\in T}\sum_{i\geq k}a_{t,i}=0 \tag{52}$$

Define, for $i\geq j$,

$$z_{t,i}=6K_1 a(a_{t,i})^{-1}(X_{t_{i+1}}-X_{t_i}). \tag{53}$$

Let Z_i be the set of all $z_{t,i}$ for t in T. Since t_i, t_{i+1} belong to the finite set A_{i+1}, Z_i is finite.

Let Z be the union of the sets Z_i for $i>j$. Fix $\varepsilon>0$. We note that

$$\sigma(X_{t_{i+1}}-X_{t_i}) \leq d(t, t_{i+1})+d(t, t_i) \leq 3 \cdot 2^{-i-1}$$

so if $\sigma(z_{t,i}) \geq \varepsilon$, we have $a_{t,i} \leq 9 \cdot 2^{-i} K_1 a/\varepsilon$, so

$$g(\alpha_i m(\{t_i\}))+g(\alpha_{i+1} m(\{t_{i+1}\})) \leq 9K_1 a/\varepsilon.$$

This implies

$$m(\{t_i\}), m(\{t_{i+1}\}) \geq 2^{i-j+1} \exp\left(-\left(\frac{9K_1 a}{\varepsilon}\right)^2+\left(\frac{a}{b}\right)^2\right).$$

Since m is a probability, this shows that there are at most

$$2^{-i+j-1} \exp\left(\left(\frac{9K_1 a}{\varepsilon}\right)^2-\frac{a^2}{b^2}\right)$$

possible choices for either t_i or t_{i+1}; so

$$\operatorname{card}\{z \in Z_i; \sigma(z) \geq \varepsilon\} \leq 2^{-2i+2j-2} \exp\left(2\left(\left(\frac{9K_1 a}{\varepsilon}\right)^2-\frac{a^2}{b^2}\right)\right).$$

It follows that

$$\operatorname{card}\{z \in Z; \sigma(z) \geq \varepsilon\} \leq \exp\left(\left(\frac{18K_1 a}{\varepsilon}\right)^2-\frac{a^2}{b^2}\right).$$

We can index Z as a sequence $(Y_n)_{n\geq 1}$ such that $\sigma(Y_n)$ does not increase. For each n,

$$n \leq \operatorname{card}\{z \in Z; \sigma(z) \geq \sigma(Y_n)\} \leq \exp\left(\left(\frac{18K_1 a}{\sigma(Y_n)}\right)^2-\frac{a^2}{b^2}\right)$$

This implies that $\sigma(Y_n) \leq K_2 a(a^2/b^2+\log n)^{-1/2}$. For t in T, we have

$$X_t-X_{t_j}=\sum_{i\geq j} X_{t_{i+1}}-X_{t_i}.$$

From (53), we have

$$X_t-X_{t_j}=\sum_{i\geq j} (6K_1 a)^{-1} a_{t,i} z_{t,i}.$$

From (51), this implies that $X_t - X_{t_j} = \Sigma_{n\geq 1} \alpha_n(t) Y_n$ where $\Sigma \alpha_n(t) \leq 1/2$. Since the sequence Y_n is bounded a.s., it is clear that this series converges a.s. And we have $X_t = X_{t_j} + \Sigma_{n\geq 1} \alpha_n(t) Y_n$. Since $a(a^2/b^2 + \log n)^{-1/2} \leq Ka(a^2/b^2 + \log(n+1))^{-1/2}$ for $n\geq 1$, and $\sigma(X_{t_j}) \leq b$, this completes the proof of Theorem 2. When (52) holds, there is a sequence $b_i \to \infty$ such that

$$\sup_{t\in T} \sum_{i\geq j} a_{t,i} b_i \leq 4K_1 a$$

so we get

$$X_t - X_{t_j} = \sum_{i\geq j} (6K_1 a)^{-1} a_{t,i} b_i (z_{t,i} b_i^{-1}). \tag{54}$$

For each n, let $i(n)$ such that $Y_n \in Z_{i(n)}$. Since each Z_i is finite we have $\lim_{n\to\infty} i(n) = \infty$ so $\lim_{n\to\infty} b_{i(n)} = \infty$. From (51) and (53) we get

$$X_t - X_{t_j} = \sum \alpha_n(t) Y'_n$$

where $\Sigma \alpha_n(t) \leq 2/3$, and $Y'_n = Y_n b_{i(n)}^{-1}$, so $\lim_n (\log n)^{1/2} \sigma(Y'_n) = 0$. The proof is complete.

To prove Theorem 3, we first need the following result of R. M. Dudley and J. Feldman [3], [5]: If (T, τ) is metric compact, a Gaussian process $(X_t)_{t\in T}$ is continuous on (T, τ) if and only if the convariance of (X_t) is τ-continuous and the process is continuous on (T, d). We give the simple argument for completeness. Suppose, first, that $(X_t)_{t\in T}$ is τ-continuous. Then the convariance is τ-continuous by dominated convergence. For $\eta \geq 0$, let

$$A_\eta = \{(x, y) \in T\times T; d(x, y) \leq \eta\}.$$

This is a τ-closed subset of $T\times T$, and $\bigcap_{\eta>0} A_\eta = A_0$. Fix $\varepsilon>0$. By compactness, there is $\eta>0$ and a finite set $B \subset A_0$ such that whenever $(x, y) \in A_\eta$, there is $(x', y') \in B$ with $\tau(x, x')$, $\tau(y, y') \leq \varepsilon$. We have

$$|X_x - X_y| \leq |X_x - X_{x'}| + |X_{x'} - X_{y'}| + |X_{y'} - X_y|.$$

Since $(x', y') \in A_0$, we have $X_{x'} = X_{y'}$ a.s., so $E|X_{x'} - X_{y'}| = 0$. It follows that $\varphi_d(\eta) \leq 2\varphi_\tau(\varepsilon)$, where φ_τ, φ_d are given by (35). Proposition 16 implies that $\lim_{\varepsilon\to 0}\varphi_\tau(\varepsilon)=0$, so $\lim_{\eta\to 0}\varphi_d(\eta)=0$, so $(X_t)_{t\in T}$ is uniformly continuous on (T,d). The converse is obvious since the identity map $(X,\tau)\to(X,d)$ is continuous when the covariance is continuous.

We now prove Theorem 3. Only sufficiency remains to prove. By Proposition 16, it is enough to show that $\lim_{\eta\to 0}\varphi_\tau(\eta)=0$. Fix $\varepsilon>0$. Let $k>0$ be such that $\sigma(Y_n)\leqslant\varepsilon(1+\log n)^{-1/2}$ for $n\geqslant k$. Let H (resp. G) be the closed linear span in $L^2(P)$ of the sequence (Y_n) (resp. $(Y_n)_{n\leqslant k}$). Since G is finite-dimensional, there exists $\alpha>0$ such that $E\sup_{X\in A}X\leqslant\varepsilon$, where

$$A=\{X\in G;\sigma(X)\leqslant\alpha\}.$$

Let γ be small enough that $d(u,v)<\alpha$ for $\tau(u,v)<\gamma$. Denote by P the orthogonal projection of H onto G. Since $\sigma(Y_n-PY_n)\leqslant\sigma(Y_n)$, (4) implies that

$$E\sup_{n>k}(Y_n-PY_n)<K\varepsilon.$$

For u,v in T with $d(u,v)<\alpha$, we have $P(X_u-X_v)\in A$. Also

$$X_u-X_v=\sum_{n\geqslant 1}\alpha_n Y_n$$

where $\Sigma_{n\geqslant 1}|\alpha_n|\leqslant 2$, so, since $Y_n=PY_n$ for $n\leqslant k$, we get

$$X_u-X_v-P(X_u-X_v)=\sum_{n>k}\alpha_n(Y_n-PY_n).$$

This shows that

$$E\sup_{\tau(u,v)<\alpha}\leqslant E\sup_A|X|+2E\sup_n|Y_n-PY_n|<K\varepsilon,$$

and finishes the proof.

It might be interesting to point out that Theorem 2 can be interpreted as a theorem of geometry in the finite dimensional Hilbert space. Consider such a space H of dimension n. Denote by σ the normalized measure on the unit ball. For a subset A of H, consider

$$V(A)=\int\sup_{y\in A}|\langle x,y\rangle|d\sigma(x).$$

This quantity has been studied in geometry under the name mixed volume, and plays an increasing role in the local theory of Banach spaces. Fix an orthonormal basis $(e_i)_{i\leq n}$ of H. For t in H, let $X_t=\sum_{i\leq n}\langle t, e_i\rangle g_i$, where $(g_i)_{i\leq n}$ is an independent sequence of standard normal random variables.

The distribution of the sequence $(g_i)_{i\leq n}$ is rotation invariant; the central limit theorem implies that $\sum_{i\leq n} g_i^2$ is concentrated around n; this implies that

$$K^{-1}n^{1/2}V(A)\leq E\sup_{t\in A}|X_t|\leq Kn^{1/2}V(A).$$

Define now

$$C(A)=\inf\left\{a>0; \exists (y_n)_{n\geq 1} \text{ in } H,\ \|y_n\|\leq a(1+\log n)^{-1/2},\ A\subset \operatorname{conv}\{y_n\}\right\}.$$

Then Theorem 2 can be reformulated as follows:

$$K^{-1}n^{-1/2}C(A)\leq V(A)\leq Kn^{-1/2}C(A).$$

A version of the following concentration result with sharp constants is a well known consequence of Borell's inequality [2]. The present version is, however, a simple consequence of Theorem 2, and is sufficient for many purposes.

PROPOSITION 18. *Consider a bounded process $(X_t)_{t\in T}$. Let $a=E\sup_T(X_t)$, $b=\sup_T\sigma(X_t)$. Then for each $u\geq 1$,*

$$P(\{\sup_T|X_t|\geq K(a+ub)\})\leq K\exp(-u^2). \tag{55}$$

In particular for $A\subset\Omega$, $\theta=P(A)$,

$$E(1_A \sup_T|X_t|)\leq K(\theta a+\theta b g(\theta)). \tag{56}$$

Proof. As mentioned after the statement of Theorem 2, this theorem implies that

$$\sup_T|X_t|\leq \sup_n|Y_n|.$$

where $(Y_n)_{n\geq 1}$ is Gaussian and $\sigma(Y_n)\leq Ka(a^2/b^2+\log n)^{-1/2}\leq Kb$. We have, for $s\geq b$

$$P(\{\sup_n |Y_n| > Ks\}) \leq \sum_{n\geq 1} 2 \exp\left(-\frac{s^2}{a^2}\left(\frac{a^2}{b^2}+\log n\right)\right)$$
$$\leq \sum_{n\geq 1} 2n^{-s^2/a^2} \exp(-s^2/b^2).$$

Taking $s=2a+ub$, we get (55), since $\Sigma_{n\geq 1} 2n^{-4} \leq 3$. To prove (56), we write

$$E(1_A \sup |X_t|) = \int_0^\infty P(A \cap \{\sup_T |X_t| > s\})\, ds = I_1+I_2,$$

where I_1 is the integral over $\{s\leq K(a+bg(\theta))\}$, and I_2 the integral over $\{s\geq K(a+bg(\theta))\}$. We have $I_1 \leq \theta K(a+bg(\theta))$. We have

$$I_2 \leq Kb \int_{g(\theta)}^\infty P(\{\sup_T |X_t| > K(a+bu)\})\, du$$
$$\leq K^2 b \int_{g(\theta)}^\infty \exp(-u^2\, du) \leq K_1 b\theta$$

for some universal constant K_1. This completes the proof.

As a consequence of Theorem 3, we can give a description of all Gaussian measures on separable Banach spaces. Consider a separable Banach space E. A Borel probability μ on E is called a (centered) Gaussian measure if the law of each continuous linear functional on E is Gaussian. Let $c_0=c_0(\mathbf{N})$.

THEOREM 19. *If μ is a Gaussian measure on a separable Banach space E, there exists a Gaussian sequence (Y_n) such that $\lim_{n\to\infty} \sigma(Y_n)(\log n)^{1/2}=0$, a closed linear subspace Z of c_0 such that $(Y_n)\in Z$ a.s., and a bounded linear operator U: $Z\to E$ such that μ is the law of $U((Y_n))$.*

Proof. Since E is separable, the unit ball E_1^* of its dual is metrizable for the weak* topology $\sigma(E^*, E)$. Denote by (z_k) a weak* dense sequence in E_1^*. For x in E, we have $\|x\|=\sup_k z_k(x)$. Consider each y in E_1^* as a Gaussian random variable on (E, μ). This defines a weak* continuous Gaussian process, since for each x in E, the map $y\mapsto y(x)$ is weak* continuous. Denote by H the closed linear span of E^* in $L^2(\mu)$. From Theorem 3, there is a sequence Y_n in E^* such that $\lim_{n\to\infty} \sigma(Y_n)(\log n)^{1/2}=0$, such that for each k, $z_k=\Sigma_n \alpha_{k,n} Y_n$, where $\alpha_{k,n}\geq 0$, $\Sigma\, \alpha_{k,n}\leq 1$, and the series converges a.s. Denote by G the set of x in E for which all these series converges and for which $\lim_n Y_n(x)=0$. It is a Borel linear subspace of E, and $\mu(G)=1$. Consider the operator $V: G \to c_0$ given by $G(x)=(Y_n(x))$. For x in G, we have

$$\|x\| \leq \sup_k z_k(x) \leq \sup_n Y_n(x)$$

so $\|x\| \leq \|V(x)\|$. For x in $V(G)$, denote $U(x)=V^{-1}(x)$. It follows that $\|U(x)\| \leq \|x\|$. If Z denotes the closure of $V(G)$, then U extends by continuity to an operator from Z to E, denoted U again, and $(Y_n(x)) \in Z$ a.s. For x in G, we have $UV(x)=x$, so μ is the law of $U((Y_n))$. The proof is complete.

A linear operator A between two Banach spaces E, F is called of type 2 if for some constant C, and for each sequence $(x_i)_{i \leq n}$ of E we have

$$E \sum_{i \leq n} g_i A(x_i) \leq C \left(\sum_{i \leq n} \|x_i\|^2 \right)^{1/2} \tag{56'}$$

where (g_i) is an independent sequence of standard normal random variables. The type 2 constant $T_2(A)$ of A is the smallest C that satisfies (56′).

Consider any metric space (T, τ). Denote by $\mathrm{Lip}(T)$ the space of lipschitz functions on T, provided with the norm

$$\|f\|_{\mathrm{Lip}} = (\operatorname{diam} T)^{-1} \|f\|_\infty + \sup_{\substack{t, u \in T \\ t \neq u}} \left| \frac{f(t)-f(u)}{\tau(t, u)} \right|.$$

It has been proved by B. Heinkel [16], who adapted a result of C. Jain and M. Marcus [18] that the canonical injection $A: \mathrm{Lip}(T) \to C(T)$ satisfies $T_2(A) \leq K\gamma(T)$ (where $\gamma(T)$ is the functional considered in section 2). We can now prove the converse.

THEOREM 20. *For some universal constant K, we have*

$$K^{-1}\gamma(T) \leq T_2(A) \leq K\gamma(T).$$

Proof. We denote by D the diameter of T. We first prove Heinkel's result that $T_2(A) \leq K\gamma(T)$. Consider $x_1, \ldots, x_n$ in $\mathrm{Lip}(T)$. For t in T, let $X_t = \Sigma_{i \leq n} g_i x_i(t)$. Since $g_1, \ldots, g_n$ are independent, we have

$$\sigma(X_t) = \left(\sum_{i \leq n} x_i^2(t) \right)^{1/2} \leq \left(\sum_{i \leq n} \|x_i\|_\infty^2 \right)^{1/2} \leq D \left(\sum_{i \leq n} \|x_i\|_{\mathrm{Lip}}^2 \right)^{1/2} \tag{57}$$

and for t, u in T, we have

$$\sigma(X_t - X_u) = \left(\sum_{i\le n}(x_i(t)-x_i(u))^2\right)^{1/2} \le \tau(t,u)\left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{1/2}. \tag{58}$$

To avoid confusion between the τ metric and the metric associated to X_t, we write

$$B_X(x,\varepsilon) = \{y \in T; \sigma(X_y - X_x) \le \varepsilon\}$$

$$B(x,\varepsilon) = \{y \in T; \tau(y,x) \le \varepsilon\}$$

so we have

$$B_X(x,\varepsilon) \supset B\left(x, \varepsilon\left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{-1/2}\right).$$

For a probability m on T, we have

$$\begin{aligned}\int_0^\infty g(m(B_X(x,\varepsilon)))\,d\varepsilon &\le \int_0^\infty g\left(m\left(B\left(x,\varepsilon\left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{1/2}\right)\right)\right)d\varepsilon\\ &\le \left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{1/2}\int_0^\infty g(m(B(x,\varepsilon)))\,d\varepsilon\end{aligned}$$

so (7) implies

$$E\sup_T X_t \le K\left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{1/2}\gamma(T).$$

Heinkel's result follows from Lemma 6(f) and (57), (1).

We now prove the converse. Consider a subspace U of T, and the canonical injection B from $\mathrm{Lip}(U)$ into $C(U)$. We first show that $T_2(B) \le T_2(A)$. Let $(x_i)_{i\le n}$ be a sequence in $\mathrm{Lip}(U)$. As well known there is a sequence $(y_i)_{i\le n}$ in $\mathrm{Lip}(T)$ such that x_i is the restriction of y_i to x_i and $\|y_i\|_{\mathrm{Lip}} \le \|x_i\|_{\mathrm{Lip}}$. So we have

$$\begin{aligned}E\left\|\sum_{i\le n} g_i x_i\right\|_\infty \le E\left\|\sum_{i\le n} g_i y_i\right\|_\infty &\le T_2(A)\left(\sum_{i\le n}\|y_i\|_{\mathrm{Lip}}^2\right)^{1/2}\\ &\le T_2(A)\left(\sum_{i\le n}\|x_i\|_{\mathrm{Lip}}^2\right)^{1/2}.\end{aligned}$$

This shows that $T_2(B) \le T_2(A)$. In the proof of Theorem 14 we have shown that

$$\gamma(T) \leqslant K \sup \{\alpha(V);\ V \subset T,\ V \text{ finite}\}$$

so it is enough to prove that when T is finite, we have $\alpha(T) \leqslant KT_2(A)$. In that case, Theorem 11 shows that it is enough to prove that when T is ultrametric and finite, we have $\xi(T) \leqslant KT_2(A)$. (Here again, $\alpha(T)$ and $\xi(T)$ are the functionals defined in section 2.)

For $j>0$, denote by $\mathscr{B}_j$ the family of balls of T of diameter $4^{-j}D$. Since T is finite, there exists m such that $B(x, D4^{-m}) = \{x\}$ for x in T. Let $\mathscr{B} = \bigcup_{j \leqslant m} \mathscr{B}_j$. For

$$\varepsilon = (\varepsilon_B)_{B \in \mathscr{B}} \in \mathscr{E} = \{0,1\}^{\mathscr{B}},$$

we define

$$f_\varepsilon = \sum_{1<j\leqslant m} \sum_{B \in \mathscr{B}_j} D4^{-j} \varepsilon_B 1_B.$$

We note that $\|f_\varepsilon\|_\infty \leqslant \Sigma_{j>0} D4^{-j} \leqslant D/3$. Consider now u, v in T, and let k be the largest integer with $D4^{-k} \geqslant \tau(u,v)$. If $B \in \mathscr{B}_l$ for some $l \leqslant k$, we have $1_B(u) = 1_B(v)$. It follows that

$$|f_\varepsilon(u) - f_\varepsilon(v)| \leqslant \sum_{j>k} D4^{-j} \leqslant 4^{-k}D/3 \leqslant 4\tau(u,v)/3.$$

This shows that $\|f_\varepsilon\|_{\mathrm{Lip}} \leqslant 5/3 \leqslant 2$. The definition of k shows that the two balls

$$B_1 = B(u, D4^{-k-1});\quad B_2 = B(v, D4^{-k-1})$$

are different. Since they belong to $\mathscr{B}_{k+1}$, we have

$$\begin{aligned}|f_\varepsilon(u) - f_\varepsilon(v)| &\geqslant D4^{-k-1}|\varepsilon_{B_1}(u) - \varepsilon_{B_2}(v)| - \sum_{j>k+1} D4^{-j} \\ &\geqslant D4^{-k-1}|\varepsilon_{B_1}(u) - \varepsilon_{B_2}(v)| - D4^{-k-1}/3.\end{aligned}$$

Since $|\varepsilon_{B_1}(u) - \varepsilon_{B_2}(v)|$ is zero or one, we have

$$|f_\varepsilon(u) - f_\varepsilon(v)| \geqslant 2 \cdot 4^{-k-1} D |\varepsilon_{B_1}(u) - \varepsilon_{B_2}(v)|/3. \tag{59}$$

Set $N = 2^{\operatorname{card} \mathscr{B}}$. It follows from (59) that

$$\left(\sum_{\varepsilon \in \mathscr{E}} N^{-1} |f_\varepsilon(u) - f_\varepsilon(v)|^2\right)^{1/2} \geqslant 2^{1/2} \cdot 4^{-k-1} D/3 \geqslant \tau(u,v)/12. \tag{60}$$

We set $h_\varepsilon = 12N^{-1/2}f_\varepsilon$. We have

$$\left(\sum_\varepsilon \|h_\varepsilon\|^2_{\mathrm{Lip}}\right)^{1/2} \leqslant 24. \tag{61}$$

Denote by $(g_\varepsilon)_{\varepsilon\in\mathscr{E}}$ an independent family of standard normal r.v. For t in T, let $X_t = \Sigma_{\varepsilon\in\mathscr{E}} g_\varepsilon h_\varepsilon(t)$. It follows from (60) that for t, u in T, we have

$$\sigma(X_t - X_u) = \left(\sum_{\varepsilon\in\mathscr{E}} (h_\varepsilon(t) - h_\varepsilon(u))^2\right)^{1/2} \geqslant \tau(u, v).$$

Proposition 13 shows that $K^{-1}\xi(U) \leqslant E\sup_T X_t$. On the other hand,

$$\begin{aligned} E\sup_T X_t \leqslant E\sup_T |X_t| &= E\left\|\sum_{\varepsilon\in\mathscr{E}} g_\varepsilon h_\varepsilon\right\|_\infty \\ &\leqslant T_2(A)\left(\sum_{\varepsilon\in\mathscr{E}} \|h_\varepsilon\|^2_{\mathrm{Lip}}\right)^{1/2} \leqslant 24T_2(A) \end{aligned}$$

from (61). The proof is complete.

4. Evaluation of Fernique's functional

The very definition of $F(\mathscr{X}, \mu)$ when the process $\mathscr{X} = (X_t)_{t\in T}$ is indexed by an infinite set raises non-trivial technical problems [14]. To avoid these problems, we will from now on assume T to be finite. Since our results will be quantitative estimates, this is an unimportant restriction.

Theorem 12 will be one of our tools. We first illustrate how it will be used, by proving the right-hand inequality of Theorem 4.

LEMMA 21. *Let* $\xi = (2e)^{-1/2}$. *Then for* $0 < a, b \leqslant 1$, we have

$$ag(a) \leqslant ag(b) + \xi b.$$

Proof. Take $t, u \geqslant 0$ with $a = \exp(-t^2)$, $b = \exp(-u^2)$. We have to prove that $t\exp(-t^2) \leqslant u\exp(-t^2) + \xi\exp(-u^2)$, or equivalently that $t \leqslant u + \xi\exp(t^2 - u^2)$. This is true for $t \leqslant u$. For $t \geqslant u$, let $v = t - u$; we want to show that $v \leqslant \xi\exp(2uv + v^2)$. But $\exp(2uv) \geqslant 1$ and $\xi = \sup_v v\exp(-v^2)$. The proof is complete.

LEMMA 22. (a) *Let U be a finite ultrametric space. Then for each probability ν on U, we have*

$$\int_U d\nu(x) \int_0^\infty g(\nu(B(x,\varepsilon)))\, d\varepsilon \leq K\gamma(U).$$

(b) *Let T be a finite metric space and μ a probability on T. Then*

$$\int_T d\mu(x) \int_0^\infty g(\mu(B(x,\varepsilon)))\, d\varepsilon \leq K\alpha(T).$$

Proof. (a) Let m be a probability measure on U such that $\gamma_m(U)=\gamma(U)$, so for each x in U, we have

$$\int_0^\infty g(m(B(x,\varepsilon)))\, d\varepsilon \leq \gamma(U). \tag{62}$$

Let $D=\operatorname{diam} U$.

From (36) and (62) follows that for x in U, we have

$$\sum_{i\geq 0} 2^{-i-1} D g(m(B(x, D2^{-i}))) \leq \gamma(U),$$

and hence

$$\int_U \sum_{i\geq j} 2^{-i-1} D g(m(B(x, D2^{-i})))\, d\nu(x) \leq \gamma(U). \tag{63}$$

Denote by $\mathcal{B}_i$ the family of balls of U of radius $D2^{-i}$. For any $i\geq 0$, we have

$$\int_U g(m(B(x, D2^{-i})))\, d\nu(x) = \sum_{B\in\mathcal{B}_i} g(m(B))\, \nu(B).$$

It follows from (63) that

$$\sum_{i\geq 0} \sum_{B\in\mathcal{B}_i} 2^{-i-1} D g(m(B))\, \nu(B) \leq \gamma(U).$$

From Lemma 21, we see that

$$\sum_{i\geq 0} \sum_{B\in\mathcal{B}_i} 2^{-i-1} D\nu(B)\, g(\nu(B)) \leq \gamma(U) + \sum_{i\geq 0} \xi 2^{-i-1} D$$

$$\leq \gamma(U) + \xi D.$$

This means that

$$\int_U \sum_{i\geq 0} 2^{-i-1} D g(\nu(B(x, D2^{-i})))\, d\nu(x) \leq \gamma(U)+\xi D.$$

The result follows from this inequality, (36) and Lemma 6(f).

(b) Let U be an ultrametric space, and f be a contraction from U onto T. Let ν be a probability measure on U such that $f(\nu)=\mu$. Since f is a contraction, for u in U we have

$$\int_0^\infty g(\mu(B(f(u),\varepsilon)))\, d\varepsilon \leq \int_0^\infty g(\nu(B(u,\varepsilon)))\, d\varepsilon.$$

By (a) we get

$$\begin{aligned}\int_T d\mu(x) \int_0^\infty g(\mu(B(x,\varepsilon)))\, d\varepsilon &\leq \int_U d\nu(u) \int_0^\infty g(\nu(B(u,\varepsilon)))\, d\varepsilon \\ &\leq K\gamma(U)\end{aligned}$$

and the result follows from the definition of $\alpha(T)$.

The right hand inequality of Theorem 4 follows by combining (b) of Lemma 22 with Theorem 12. X. Fernique has found good upper bounds for $F(\mathscr{X},\mu)$. It is somehow surprising that we will be able to find good lower bounds by combining Lemma 22 with a fairly crude minoration result for $F(\mathscr{X},\mu)$ (Proposition 26 below). The arguments are easy, but somewhat lengthy. The following is a consequence of the integrability result of H. Landau, L. A. Shepp and X. Fernique ([7], [19]).

LEMMA 23. *There exists $\zeta>0$, $K_1<\infty$ such that for each Gaussian process $(X_t)_{t\in T}$ and each $a>0$ we have*

$$P(\{\sup_T |X_t| \geq a\}) \leq 2\zeta \;\Rightarrow\; E\sup_T |X_t| < K_1 a.$$

We need the following immediate consequence:

LEMMA 24. *For any Gaussian process $(X_t)_{t\in T}$, we have*

$$P(\{\sup_T X_t \geq a\}) \leq \zeta \;\Rightarrow\; E\sup_T |X_t| < K_1 a.$$

Proof. By symmetry, $\sup_T -X_t = -\inf_T X_t$ has the same law as $\sup_T X_t$. It follows that

$$P(\{-\sup_T X_t \geq a\}) \leq P(\{-\inf_T X_t \geq a\}) = P(\{\sup_T X_t \geq a\})$$

so

$$P(\{\sup_T |X_t| \geq a\}) \leq 2P(\{\sup_T X_t \geq a\})$$

and the result follows from Lemma 23.

Consider a Gaussian process $\mathscr{X}=(X_t)_{t\in T}$, and Y a Gaussian random variable. Let $X'_t=X_t+Y$. Since $EY=0$, we see that $F(\mathscr{X},\mu)=F(\mathscr{X}',\mu)$ for any probability μ on T. In the study of $F(\mathscr{X},\mu)$, it is natural to single out the variable $\Sigma_{t\in T}\mu(\{t\})X_t$ that we will denote $\int X_t\,d\mu(t)$. The preceding remark shows that to study $F(\mathscr{X},\mu)$, there is no loss of generality to assume $\int X_t\,d\mu(t)=0$. From now on, all maps and sets are understood to be measurable, even when we do not mention it explicitly.

LEMMA 25. *Consider a Gaussian process $(X_t)_{t\in T}$. Let A be a (measurable) subset of Ω, $(\alpha_t)_{t\in T}$ be positive numbers such that $\Sigma_{t\in T}\alpha_t=P(A)$. Assume $\Sigma_{t\in T}\alpha_t X_t=0$ a.s. Then there exists a measurable map τ from A to T such that $E(X_\tau 1_A)\geq 0$ and that $\alpha_t=P(\{\tau^{-1}(t)\})$ for each t in T.*

Proof. Consider the set C of families $f=(f_t)_{t\in T}$ of measurable functions such that

$$\forall t\in T,\quad 0\leq f_t\leq 1_A,\quad Ef_t=\alpha_t;\quad \sum_{t\in T} f_t=1_A.$$

This is a weak* compact convex set of $L^\infty(\Omega)^T$. For f in C, we define

$$\varphi(f)=E\Bigl(\sum_{t\in T} X_t f_t 1_A\Bigr).$$

This defines a weak* continuous affine functional on C. Define $g=(g_t)_{t\in T}$, where

$$g_t=\frac{\alpha_t}{P(A)}1_A,$$

so $g\in C$. We have

$$\varphi(g)=E\Bigl(\sum_{t\in T}\frac{1}{P(A)}\alpha_t X_t 1_A\Bigr)=0$$

since $\Sigma_T \alpha_t X_t = 0$ a.s. This shows that $\sup_C \varphi \geq 0$. It follows that there is an extreme point of C at which φ is ≥ 0. It is routine to show that, since P has no atom, each extreme point of C is of the type $(1_{A_t})_{t\in T}$ for $A_t \subset A$, $P(A_t) = \alpha_t$. The proof is complete.

We can now prove the essential minoration result.

PROPOSITION 26. *Consider a Gaussian process $\mathscr{X} = (X_t)_{t\in T}$ and a probability measure μ on T. Assume that $\int X_t \, d\mu(t) = 0$. Then we can find a partition $(Z_i)_{i\geq 0}$ of T such that*

$$\forall i \geq 0, \quad E \sup_{Z_i} |X_t| \leq 2^i F(\mathscr{X}, \mu), \tag{64}$$

$$\sum_{i\geq 0} 2^i \mu(Z_i) \leq K F(\mathscr{X}, \mu). \tag{65}$$

Proof. By homogeneity, we can assume that $F(\mathscr{X}, \mu) = 1$. We can assume that the constant ζ of Lemma 24 is small enough that for each Gaussian r.v. Y we have

$$P(A) \leq \zeta \quad \Rightarrow \quad |E(Y 1_A)| \leq \frac{(1-\zeta)}{2K_1} E|Y|, \tag{66}$$

where K_1 is the constant of Lemma 24.

By induction over k, we construct $t_1, \ldots, t_k$ in T, disjoint subsets $B_1, \ldots, B_k$ of Ω such that if we write

$$T_i = T \setminus \{t_1, \ldots, t_{i-1}\}, \quad d_i = E \sup_{T_i} |X_t|$$

the following conditions hold

$$\forall i \leq k, \quad P(B_i) = \mu(\{t_i\}); \quad \sum_{i\leq k} P(B_i) < \zeta \tag{67}$$

$$\forall i \leq k, \quad 1_{B_i} X_{t_i} \geq d_i 1_{B_i}. \tag{68}$$

We proceed to the first step. We have $T_1 = T$, $d_1 = E \sup_T |X_t|$. Let

$$A_1' = \{\sup_T X_t \geq d_1/K_1\},$$

so $P(A_1') \geq \zeta$ from Lemma 24. Consider $A_1 \subset A_1'$ such that $P(A_1) = \zeta$. Consider a measurable map τ_1 from A_1 to T such that on A_1 we have $X_{\tau_1} \geq d_1/K_1$.

First case. For some t_1 in T, we have $P(\{\tau_1=t_1\})>\mu(\{t_1\})$. We choose $B_1\subset\{\tau_1=t_1\}$ such that $P(B_1)=\mu(\{t_1\})$, so $P(B_1)<\zeta$. The induction continues.

Second case. For each t in T, $P(\{\tau_1=t\})\leqslant\mu(\{t\})$. The induction stops.

We now proceed to the kth step. We have $T_k=T\setminus\{t_1,\ldots,t_{k-1}\}$, $d_k=E\sup_{T_k}|X_t|$. Let

$$A_k'=\{\sup_{T_k}X_t\geqslant d_k/K_1\},$$

so $P(A_k')\geqslant\zeta$ from Lemma 24. Consider $A_k\subset A_k'\setminus\bigcup_{i<k}B_i$ such that $P(A_k)=\zeta-\Sigma_{j<k}P(B_j)$. Consider a measurable map τ_k from A_k to T_k such that on A_k we have $X_{\tau_k}\geqslant d_k/K_1$.

First case. For some t_k in T_k, we have $P(\{\tau_k=t_k\})>\mu(\{t_k\})$. We choose $B_k\subset\{\tau_k=t_k\}$ such that $P(B_k)=\mu(\{t_k\})$, so $\Sigma_{i\leqslant k}P(B_i)<\zeta$. The induction continues.

Second case. For each t in T_k, $P(\{\tau_k=t\})\leqslant\mu(\{t\})$. The induction stops.

This completes the induction. Since T is finite, the induction stops at some step k. With the notations above, for $t\in T_k$, let $C_t=\{\tau_k=t\}$. Let $B=\bigcup_{i<k}B_i\cup\bigcup_{t\in T_k}C_t$. We have $P(B)=\zeta$. Define the map τ from B to T by $\tau=t_i$ on B_i for $i<k$, and $\tau=\tau_k$ on $C=\bigcup_{t\in T_k}C_t$. We have $X_\tau\geqslant d_i/K_1$ on B_i for $i<k$, and $X_\tau\geqslant d_k/K_1$ on C. Define

$$u=\sum_{i<k}d_i\mu(\{t_i\})+d_kP(C). \tag{69}$$

We have $E(X_\tau 1_B)\geqslant u/K_1$. For t in T, let $\beta_t=P(\{\tau=t\})$, so for $i<k$, we have $\beta_{t_i}=\mu(\{t_i\})$ and for $t\in T_k$ we have $\beta_t=P(C_t)$. For t in T_i, we have $E|X_t|\leqslant E\sup_{T_i}|X_t|=d_i$. It follows from (69) that $E|\Sigma_{t\in T}\beta_tX_t|\leqslant u$. For t in T, define $\alpha_t=\mu(\{t\})-\beta_t$, so $\Sigma_{t\in T}\alpha_t=1-\zeta$. Define $Y=\Sigma_{t\in T}\alpha_tX_t$. Since $\int X_t\,d\mu(t)=0$, we have $Y=-\Sigma_{t\in T}\beta_tX_t$, so $E|Y|\leqslant u$. Let $A=\Omega\setminus B$, so $P(A)=1-\zeta$. From (66) we get

$$E|Y1_A|\leqslant u(1-\zeta)/2K_1. \tag{70}$$

Since $\Sigma_{t\in T}\alpha_t=1-\zeta$, we have $\Sigma_{t\in T}\alpha_t(X_t-(1-\zeta)^{-1}Y)=0$. It follows from Lemma 25 that there is a map τ' from A to T such that

$$\forall t\in T,\quad P(\{\tau'=t\})=\alpha_t;\quad E((X_{\tau'}-(1-\zeta)^{-1}Y)\,1_A)\geqslant 0.$$

It follows from (70) that

$$E(X_{\tau'}1_A)\geqslant -u/2K_1. \tag{71}$$

Define the map τ'' from Ω to T by $\tau''=\tau$ on B and $\tau''=\tau'$ on A. Then $\mathscr{L}(\tau'')=\mu$. We have

$$E(X_{\tau''}) = E(X_\tau 1_B)+E(X_{\tau'} 1_A) \geqslant \frac{u}{K_1} - \frac{u}{2K_1} \geqslant \frac{u}{2K_1}.$$

Since $E(X_{\tau''}) \leqslant F(\mathscr{X},\mu)=1$, we have $u \leqslant 2K_1$. The sequence $(d_i)_{i\leqslant k}$ decreases; so (69) implies that $u \geqslant \zeta d_k$. Let l be the smallest integer for which $2^l \geqslant d_k$. For $i<l$, we set $Z_i=\varnothing$. We set $Z_l=T_k$. For $j>l$, we set $Z_j=\{t_i;\ i<k,\ 2^{j-1}<d_i\leqslant 2^j\}$. We have

$$\begin{aligned}\sum_{j\geqslant l}\mu(Z_j)\,2^j &\leqslant 2\sum_{i<k}\mu(\{t_i\})\,d_i+2d_k \leqslant 2u+2d_k\\ &\leqslant 4K_1+4K_1/\zeta.\end{aligned}$$

The proof is complete.

For simplicity, we now set $\sigma(t)=\sigma(X_t)$.

COROLLARY 27. *Under the hypothesis of Proposition* 26, *we have*

$$\textstyle\int_T \sigma(t)\,d\mu(t) \leqslant KF(\mathscr{X},\mu).$$

The following result, with a better constant, is due to X. Fernique ([14], Proposition 2-3-3). X. Fernique obtains this result as a consequence of a difficult comparison theorem. We shall give here a simple direct argument.

PROPOSITION 28. *Let $\mathscr{X}=(X_t)_{t\in T}$ be a Gaussian process, μ be a probability measure on T. Let $J=\int \sigma(t)\,d\mu(t)$. Then we have*

$$\int_T \sigma(t)\,(\log(1+\sigma(t)/J))^{1/2}\,d\mu(t) \leqslant F(\mathscr{X},\mu)+KJ. \tag{72}$$

Proof. We first note that if X is standard normal, a crude estimate yields $\lim_{u\to\infty} P(\{X\geqslant u\})\exp(u^2)=\infty$, so there is a constant $b>0$ such that for $u\geqslant b$, we have

$$P(\{X \geqslant (\log(1+u))^{1/2}\}) \geqslant 1/u. \tag{73}$$

By homogeneity, we can assume $J=\int_T \sigma(t)\,d\mu(t)=1$. We enumerate T as $\{t_1,\dots,t_n\}$ in such a way that $\sigma(t_1)\geqslant\dots\geqslant\sigma(t_n)$. There is nothing to prove if $\sigma(t_1)\leqslant b$. Otherwise, let q be the largest integer $\leqslant n$ for which $\sigma(t_q)\geqslant b$. For $i\leqslant q$, we have

$$\sigma(t_i)\sum_{j\leqslant i}\mu(\{t_j\}) \leqslant \sum_{j\leqslant i}\sigma(t_j)\,\mu(\{t_j\}) \leqslant \int_T \sigma(t)\,d\mu(t) = 1$$

so

$$\sum_{j\leqslant i}\mu(\{t_j\})\leqslant 1/\sigma(t_i). \tag{74}$$

On the other hand, from (73) we have

$$P(\{X_{t_i}\geqslant\sigma(t_i)\,(\log(1+\sigma(t_i)))^{1/2}\})\geqslant 1/\sigma(t_i). \tag{75}$$

From (74) and (75), we see by a straightforward induction argument on $i\leqslant q$ that we can construct disjoint sets $(B_i)_{i\leqslant q}$ such that $P(B_i)=\mu(\{t_i\})$ and

$$X_{t_i}\geqslant\sigma(t_i)\,(\log(1+\sigma(t_i)))^{1/2}$$

on B_i. Let $B=\bigcup_{i\leqslant q}B_i$. Let τ' be the map from B to T such that for $i\leqslant q$, $\tau'=t_i$ on B_i. So we have

$$E(X_{\tau'}1_B)\geqslant\sum_{i\leqslant q}\sigma(t_i)\,(\log(1+\sigma(t_i)))^{1/2}\mu(\{t_i\}). \tag{76}$$

For $q<i\leqslant n$, let $\alpha_i=\mu(\{t_i\})$. Let

$$Y=\Bigl(\sum_{q<i\leqslant n}\alpha_i\Bigr)^{-1}\Bigl(\sum_{q<i\leqslant n}\alpha_iX_i\Bigr)$$

so $\sigma(Y)\leqslant b$ and $\Sigma_{q<i\leqslant n}\alpha_i(X_{t_i}-Y)=0$. Let $A=\Omega\setminus B$. From Lemma 25, there is a map τ from A to T such that for $q<i\leqslant n$, we have $P(\{\tau=t_i\})=\alpha_i$ and $E((X_\tau-Y)\,1_A)\geqslant 0$, so

$$E(X_\tau 1_A)\geqslant E(Y1_A)\geqslant -E|Y|\geqslant -b(2/\pi)^{1/2}.$$

Consider the map τ'' from Ω to T given by $\tau''=\tau'$ on B and $\tau''=\tau$ on A. We have $\mathscr{L}(\tau'')=\mu$, so $E(X_{\tau''})\leqslant F(\mathscr{X},\mu)$. Now,

$$E(X_{\tau''})=E(X_{\tau'}1_B)+E(X_\tau 1_A)$$

so from (76)

$$F(\mathscr{X},\mu)\geqslant\sum_{i\leqslant q}\mu(\{t_i\})\,\sigma(t_i)\,(\log(1+\sigma(t_i)))^{1/2}-b(2/\pi)^{1/2}.$$

The result follows, since

$$\sum_{q<i\leq n} \mu(\{t_i\})\,\sigma(t_i)\,(\log(1+\sigma(t_i)))^{1/2} \leq b(\log(1+b))^{1/2}.$$

The proof is complete.

We are now ready to prove our first minoration result.

THEOREM 29. *Let $\mathscr{X}=(X_t)_{t\in T}$ be a Gaussian process, where T is finite, and μ be a probability on T. Then there exists an ultrametric space U, a contraction f from U onto T, a probability ν on U with $f(\nu)=\mu$, such that if for x in U we set*

$$a(x) = \inf\{\varepsilon>0;\ \nu(B(x,\varepsilon)) \geq 1/2\}$$

we have

$$\int_U d\nu(x) \int_0^{a(x)} g(\nu(B(x,\varepsilon)))\, d\varepsilon \leq KF(\mathscr{X},\mu). \tag{77}$$

Proof. We already noted that there is no loss of generality to assume $\int X_t\, d\mu(t)=0$. By homogeneity, we can assume that $J=\int \sigma(t)\, d\mu(t)=1$. Let $F=F(\mathscr{X},\mu)$. Corollary 27 shows that $1=J\leq KF$. Proposition 26 shows that one can write $T=\bigcup_{i\geq 0} Z_i$, where $E\sup_{Z_i}|X_t|\leq 2^i F$ and $\Sigma_{i\geq 0} 2^i\mu(Z_i)\leq KF$. Let

$$D_0 = \{t;\ \sigma(t)<1\} \quad \text{and} \quad D_n=\{t;\ 2^{n-1}\leq\sigma(t)<2^n\}, \quad \text{for } n\geq 1.$$

Proposition 28 shows that $\Sigma_{n\geq 0} 2^n n^{1/2}\mu(D_n)\leq K+F$, and since $KF\geq 1$, this is $\leq KF$ (for a new constant K). For $i, n\geq 0$, we set $T_{i,n}=Z_i\cap D_n$, $a_{i,n}=\mu(T_{i,n})$. We note that

$$\sum_{i,n\geq 0} 2^i a_{i,n} \leq KF \tag{78}$$

$$\sum_{i,n\geq 0} 2^n n^{1/2} a_{i,n} \leq KF. \tag{79}$$

From Theorem 12, for each $i, n\geq 0$, there is an ultrametric space $(U_{i,n}, \delta_{i,n})$ of diameter $\leq \operatorname{diam} T_{i,n}\leq 2^{n+1}$ such that

$$\gamma(U_{i,n}) \leq KE\sup_{T_{i,n}} X_t \leq 2^i KF$$

and a contraction $f_{i,n}$ from $U_{i,n}$ onto $T_{i,n}$. Denote by $\nu_{i,n}$ a probability on $U_{i,n}$ such that for $A\subset T$, we have $a_{i,n}f_{i,n}(\nu_{i,n})(A)=\mu(A\cap T_{i,n})$. From Lemma 22 (a) we have

$$\int_{U_{i,n}} d\nu_{i,n}(x)\int_0^\infty g(\nu_{i,n}(B(x,\varepsilon)))\,d\varepsilon \leqslant 2^i KF. \tag{80}$$

Let U be the disjoint sum of the spaces $U_{i,n}$. For x, y in $U_{i,n}$, we set $\delta(x,y)=\delta_{i,n}(x,y)$. If $x\in U_{i,n}$, $y\in U_{j,m}$, $(i,n)\neq(j,m)$, we set $\delta(x,y)=2^{1+\sup(m,n)}$. The space U is ultrametric. The map f given by $f(u)=f_{i,n}(u)$ for u in $U_{i,n}$ is a contraction from U onto T. Define $\nu=\sum_{i,n\geqslant 0} a_{i,n}\nu_{i,n}$. Then $f(\nu)=\mu$.

We are going to evaluate the integral

$$\begin{aligned} I &= \int_U d\nu(x)\int_0^{a(x)} g(\nu(B(x,\varepsilon)))\,d\varepsilon \\ &= \sum_{i,n\geqslant 0} a_{i,n}\int_U d\nu_{i,n}(x)\int_0^{a(x)} g(\nu(B(x,\varepsilon)))\,d\varepsilon. \end{aligned}$$

Since $\int\sigma(t)\,d\mu(t)=1$, we have $\mu(D_0\cup D_1)\geqslant 1/2$. For x in $U_{i,n}$, $\varepsilon>2^{n+2}$, for $j\geqslant 0$ we have $U_{j,m}\subset B(x,\varepsilon)$ whenever $m=0,1$. This shows that

$$\nu(B(x,\varepsilon))\geqslant \sum_{\substack{m=0,1\\ j\geqslant 0}} \nu(U_{j,m}) = \sum_{\substack{m=0,1\\ j\geqslant 0}} a_{j,m} = \mu(D_0\cup D_1)\geqslant 1/2,$$

so $a(x)\leqslant 2^{n+2}$. It follows that

$$I\leqslant \sum_{i,n\geqslant 0} a_{i,n}\int_U d\nu_{i,n}(x)\int_0^{2^{n+2}} g(\nu(B(x,\varepsilon)))\,d\varepsilon.$$

Since

$$g(\nu(B(x,\varepsilon)))\leqslant g(a_{i,n})+g(\nu_{i,n}(B(x,\varepsilon))),$$

we have $I\leqslant I_1+S$, where

$$I_1 = \sum_{i,n\geqslant 0} a_{i,n}\int_U d\nu_{i,n}(x)\int_0^{2^{n+2}} g(\nu_{i,n}(B(x,\varepsilon)))\,d\varepsilon$$

$$S = \sum_{i,n\geqslant 0} 2^{n+2}a_{i,n}g(a_{i,n}).$$

From (79) and (80) we see that $I_1 \leqslant KF$. We note the almost obvious fact that $(i+n)^{1/2} \leqslant 2n^{1/2}+2^{i-n}$ for $i, n \geqslant 0$. From Lemma 21, we get

$$\begin{aligned} a_{i,n}\, g(a_{i,n}) &\leqslant a_{i,n}(i+n)^{1/2}+\zeta e^{-i-n} \\ &\leqslant 2a_{i,n} n^{1/2}+a_{i,n} 2^{i-n}+\zeta e^{-i-n}. \end{aligned}$$

It then follows from (79) and (80) that $S \leqslant KF$ so $I \leqslant KF$. The proof is complete.

We are now going to state the main result of this section. To simplify the statement, we say that a quantity Q_1 dominates a quantity Q_2 if for some universal constant K we have $Q_2 \leqslant KQ_1$. We say that Q_1 and Q_2 are equivalent if each dominates the other.

THEOREM 30. *Consider a Gaussian process $X=(X_t)_{t\in T}$ where T is finite, and a probability measure μ on T. Assume $\int X_t\, d\mu(t)=0$. We set $\sigma(t)=\sigma(X_t)$, $J=\int_T \sigma(t)\, d\mu(t)$,*

$$I=\int_T \sigma(t)\,(1+(\log(1+\sigma(T)/I))^{1/2})\, d\mu(t).$$

Then the following quantities are equivalent:

(1) $F(\mathscr{X},\mu)$;

(2) $Q_1=I+\inf\{\int \varphi(t)\, d\mu(t);\ X_t=\varphi(t)\, Z_t,\ E\sup_T |Z_t| \leqslant 1\}$;

(3) $Q_2=I+\inf\{\int_T d\mu(t) \int_0^{\sigma(t)} g(\nu(B(t,\varepsilon)))\, d\varepsilon\}$

where the inf *is taken over all probability measures ν on T;*

(4) $Q_3=I+\int_T d\mu(t) \int_0^{\sigma(t)} g(\mu(B(t,\varepsilon)))\, d\varepsilon$;

(5) $Q_4=I+\inf\{\int_U d\nu(x) \int_0^{a(x)} g(\nu(B(x,\varepsilon)))\, d\varepsilon\}$

where $a(x)=\inf\{\varepsilon>0;\ \nu(B(x,\varepsilon)) \geqslant 1/2\}$, and the infimum is taken over all ultrametric spaces U, all contractions f from U to T, all probabilities ν on U such that $f(\nu)=\mu$.

The fact that Q_2 dominates $F(\mathscr{X},\mu)$ is due to X. Fernique (with a slightly different formulation). Since the diameter D of T dominates I, it implies the left-hand side inequality of Theorem 4.

Proof. We set $J=\int \sigma(t)\, d\mu(t)$. We prove that $F(\mathscr{X},\mu)$ dominates Q_4. From Corollary 27, $F(\mathscr{X},\mu)$ dominates J, so Proposition 28 implies that $F(\mathscr{X},\mu)$ dominates I. The rest follows from Theorem 29. We prove that Q_4 dominates Q_3. Consider an ultrametric space U, a contraction f from U onto T, a probability ν on U with $f(\nu)=\mu$ and

$$I+\int_U d\nu(x)\int_0^{a(x)} g(\nu(B(x,\varepsilon)))\,d\varepsilon \leqslant 2Q_4,$$

where $a(x)=\inf\{\varepsilon>0,\ \nu(B(x,\varepsilon))\geqslant 1/2\}$. Since f is a contraction, and $\mu=f(\nu)$, for x in U we have

$$\mu(B(f(x),\varepsilon))\geqslant \nu(B(x,\varepsilon)). \tag{81}$$

If $t=f(x)$, for $\varepsilon>a(x)$, we have $\mu(B(f(x),\varepsilon))\geqslant 1/2$, so we get from (81)

$$\int_0^{\sigma(t)} g(\mu(B(t,\varepsilon)))\,d\varepsilon \leqslant \int_0^{a(x)} g(\nu(B(x,\varepsilon)))\,d\varepsilon+\sigma(t)\,g(1/2)$$

and this implies

$$\int_T d\mu(t)\int_0^{\sigma(t)} g(\mu(B(t,\varepsilon)))\,d\varepsilon \leqslant \int_U d\nu(x)\int_0^{a(x)} g(\nu(B(x,\varepsilon)))\,d\varepsilon+g(1/2)\,J,$$

and the result follows since I dominates J. It is obvious that Q_3 dominates Q_2. We now prove that Q_2 dominates Q_1. Fix a probability measure ν on T with

$$I+\int_T d\mu(t)\int_0^{\sigma(t)} g(\nu(B(t,\varepsilon)))\,d\varepsilon \leqslant 2Q_2.$$

Set

$$D_0=\{t\in T;\sigma(t)\leqslant 2J\}\quad\text{and}\quad D_n=\{t;2^nJ<\sigma(t)\leqslant 2^{n+1}J\},\ \text{for } n\geqslant 1. \tag{82}$$

Since $\sigma(t)\geqslant 2^nJ$ on D_n for $n\geqslant 1$, we have

$$J(\log 2)^{1/2}\sum_{n\geqslant 1} 2^n n^{1/2}\mu(D_n)\leqslant I. \tag{83}$$

For $i\geqslant 1$, $n\geqslant 0$, let

$$V_{i,n}=\{t\in D_n;\ J2^{i-1}\leqslant \int_0^{\sigma(t)} g(\nu(B(t,\varepsilon)))\,d\varepsilon<2^iJ\}.$$

For $n\geqslant 0$, let

$$V_{0,n}=\{t\in D_n;\int_0^{\sigma(t)} g(\nu(B(t,\varepsilon)))\,d\varepsilon<J\}.$$

We have

$$\int_U d\mu(t) \int_0^{\sigma(t)} g(\nu(B(t,\varepsilon)))\, d\varepsilon \geqslant \sum_{\substack{i\geqslant 1\\ n\geqslant 0}} J2^{i-1}\mu(V_{i,n}).$$

Since I dominates J, we see that Q_2 dominates

$$J \sum_{\substack{i\geqslant 0\\ n\geqslant 0}} 2^i \mu(V_{i,n}).$$

Let $c_{i,n} = J \sup(2^i, 2^n n^{1/2})$. Then from (83), Q_2 dominates

$$\sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,n}\mu(V_{i,n}).$$

For t in $V_{i,n}$, we set $\psi(t) = c_{i,n}$. It follows that Q_2 dominates

$$\int_T \psi(t)\, d\mu(t) = \sum_{\substack{i\geqslant 0\\ n\geqslant 0}} c_{i,n}\mu(V_{i,n}).$$

Let $Y_t = X_t \psi(t)^{-1}$. To finish the proof, it is enough to show that $E\sup_T |Y_t| \leqslant K$ for some universal constant K. (We then set $\varphi = K\psi, Z_t = K^{-1}Y_t$.) Since $\sigma(t) \leqslant 2^{n+1}J$ on D_n, we have $\sigma(Y_t) \leqslant 2$ for each t. The proof of Lemma 6(b) shows that for each $n, i \geqslant 0$, there is a probability $\nu_{i,n}$ on $V_{i,n}$ such that for t in $V_{i,n}$

$$\int_0^{2\sigma(t)} g(\nu_{i,n}(B(t,\varepsilon)))\, d\varepsilon \leqslant 2^{i+1}J. \tag{84}$$

We denote by A the set $\{0\} \cup \{Y_t;\, t \in T\}$. We provide A with the distance induced by L^2. We write, for a in A, $B(a,\varepsilon) = \{b \in A;\, \|a-b\| \leqslant \varepsilon\}$. We denote by $\eta_{i,n}$ the image on A of $\mu_{i,n}$ under the map $t \mapsto Y_t$. We consider the probability η on A given by

$$\eta = \sum_{n,i\geqslant 0} 2^{-i-n-3}\eta_{i,n} + \frac{1}{2}\delta_0.$$

Given a in A, we estimate

$$\int_0^\infty g(\eta(B(a,\varepsilon)))\, d\varepsilon = \int_0^4 g(\eta(B(a,\varepsilon)))\, d\varepsilon.$$

For $\varepsilon \geqslant \|a\|$, we have $\eta(B(a,\varepsilon))>1/2$, so $\int_{\|a\|}^{4} g(\eta(B(a,\varepsilon)))\,d\varepsilon \leqslant 4g(1/2)$. If $\|a\| \neq 0$, let $a=Y_t$ for some t in $V_{i,n}$, $n, i \geqslant 0$. For t in $V_{i,n}$, we have $Y_t = X_t c_{i,n}^{-1}$, so we have

$$\eta(B(a,\varepsilon)) \geqslant 2^{-i-n-3}\eta_{i,n}(B(a,\varepsilon)) \geqslant 2^{-i-n-3}\nu_{i,n}(B(t, c_{i,n}\varepsilon)).$$

Since $\|a\| = c_{i,n}^{-1}\sigma(t)$, we have, from (84)

$$\int_0^{\|a\|} g(\nu(B(a,\varepsilon)))\,d\varepsilon \leqslant \|a\| g(2^{-i-n-3}) + \int_0^{c_{i,n}^{-1}\sigma(t)} g(\nu_{i,n}(B(t, c_{n,i}\varepsilon))\,d\varepsilon$$
$$\leqslant c_{i,n}^{-1} J 2^{n+1}(n+i+3)^{1/2} + c_{i,n}^{-1} 2^{i+1} J.$$

Using the fact that $(n+i)^{1/2} \leqslant 2n^{1/2} + 2^{i-n}$, we find that $\int_0^\infty g(\eta(B(a,\varepsilon)))\,d\varepsilon \leqslant K$. It follows from (7) that $E\sup_T Y_t \leqslant K$, so from (1) we have $E\sup_T |Y_t| \leqslant K$ since $\sigma(Y_t) \leqslant 2$ for each t. This completes the proof.

We finally prove that Q_1 dominates $F(\mathscr{X},\mu)$. Let φ be such that $X_t = \varphi(t) Z_t$, where $E\sup_T |Z_t| \leqslant 1$. Let D_0, D_n be given by (82). Let

$$V_{i,n} = D_n \cap \{t;\, J2^{i-1} \leqslant \varphi(t) < J2^i\} \quad \text{for } i \geqslant 1,$$

$$V_{0,n} = D_n \cap \{t;\, \varphi(t) < J\}$$

so $E\sup_{V_{i,n}} X_t \leqslant 2^i JK$ and diam $V_{i,n} \leqslant 2^{n+2}J$. We note that I dominates $J\Sigma_{i,n\geqslant 0} 2^n n^{1/2}\mu(V_{i,n})$ and that $\int \varphi(t)\,d\mu(t)$ dominates $J\Sigma_{i\geqslant 1, n\geqslant 0} 2^{-i-1}\mu(V_{i,n})$. Consider now a map τ from Ω to T such that $\mathscr{L}(\tau)=\mu$. Let $\Omega_{i,n} = \{\tau \in V_{i,n}\}$, so $P(\Omega_{i,n}) = \mu(V_{i,n})$. From Proposition 18, we get

$$E(X_\tau 1_{\Omega_{i,n}}) \leqslant E\sup_{V_{i,n}}(X_t 1_{\Omega_{i,n}})$$
$$\leqslant KJ(2^i\mu(V_{i,n}) + 2^n\mu(V_{i,n})\, g(\mu(V_{i,n}))).$$

The argument of the last few lines of Theorem 29 shows that Q_1 dominates $\Sigma_{i,n\geqslant 0} E(X_\tau 1_{\Omega_{i,n}}) = E(X_\tau)$. The proof is complete.

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