

Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation

Peter Constantin^{a,*}, Jiahong Wu^b

^a *Department of Mathematics, University of Chicago, 5734 S. University Avenue Chicago, IL 60637, USA*

^b *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA*

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Abstract

We present a regularity result for weak solutions of the 2D quasi-geostrophic equation with supercritical ($\alpha < 1/2$) dissipation $(-\Delta)^\alpha$: If a Leray–Hopf weak solution is Hölder continuous $\theta \in C^\delta(\mathbb{R}^2)$ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$.

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1. Introduction

We discuss the surface 2D quasi-geostrophic (QG) equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2, \quad t > 0, \quad (1.1)$$

where $\alpha > 0$ and $\kappa \geq 0$ are parameters, and the 2D velocity field $u = (u_1, u_2)$ is determined from θ by the stream function ψ via the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2} \psi, \partial_{x_1} \psi), \quad (-\Delta)^{1/2} \psi = -\theta. \quad (1.2)$$

Using the notation $\Lambda \equiv (-\Delta)^{1/2}$ and $\nabla^\perp \equiv (\partial_{x_2}, -\partial_{x_1})$, the relations in (1.2) can be combined into

$$u = \nabla^\perp \Lambda^{-1} \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \quad (1.3)$$

where \mathcal{R}_1 and \mathcal{R}_2 are the usual Riesz transforms in \mathbb{R}^2 . The 2D QG equation with $\kappa > 0$ and $\alpha = \frac{1}{2}$ arises in geophysical studies of strongly rotating fluids (see [5,16] and references therein) while the inviscid QG equation ((1.1) with $\kappa = 0$) was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air (see [7,10,16]).

* Corresponding author.

E-mail addresses: const@cs.uchicago.edu (P. Constantin), jiahong@math.okstate.edu (J. Wu).

The problem at the center of the mathematical theory concerning the 2D QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case $\alpha > \frac{1}{2}$, the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [8,17]). In contrast, when $\alpha \leq \frac{1}{2}$, the issue of global existence and uniqueness is more difficult and has still unanswered aspects. Recently this problem has attracted a significant amount of research ([1–6,9,11–15,18–24]). In Constantin, Córdoba and Wu [6], we proved in the critical case ($\alpha = \frac{1}{2}$) the global existence and uniqueness of classical solutions corresponding to any initial data with L^∞ -norm comparable to or less than the diffusion coefficient κ . In a recently posted preprint in arXiv [14], Kiselev, Nazarov and Volberg proved that smooth global solutions exist for any C^∞ periodic initial data, by removing the L^∞ -smallness assumption on the initial data of [6]. Caffarelli and Vasseur (arXiv reference [1]) establish the global regularity of the Leray–Hopf type weak solutions (in $L^\infty((0, \infty); L^2) \cap L^2((0, \infty); \dot{H}^{1/2})$) of the critical 2D QG equation with $\alpha = \frac{1}{2}$ in general \mathbb{R}^n .

In this paper we present a regularity result of weak solutions of the dissipative QG equation with $\alpha < \frac{1}{2}$ (the supercritical case). The result asserts that if a Leray–Hopf weak solution θ of (1.1) is in the Hölder class C^δ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$. The proof involves representing the functions in Hölder space in terms of the Littlewood–Paley decomposition and using Besov space techniques. When θ is in C^δ , it also belongs to the Besov space $\dot{B}_{p,\infty}^{\delta(1-2/p)}$ for any $p \geq 2$. By taking p sufficiently large, we have $\theta \in C^{\delta_1} \cap \dot{B}_{p,\infty}^{\delta_1}$ for $\delta_1 > 1 - 2\alpha$. The idea is to show that $\theta \in C^{\delta_2} \cap \dot{B}_{p,\infty}^{\delta_2}$ with $\delta_2 > \delta_1$. Through iteration, we establish that $\theta \in C^\gamma$ with $\gamma > 1$. Then θ becomes a classical solution.

The results of this paper can be easily extended to a more general form of the quasi-geostrophic equation in which $x \in \mathbb{R}^n$ and u is a divergence-free vector field determined by θ through a singular integral operator.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

2. Besov spaces and related tools

This section provides the definition of Besov spaces and several related tools. We start with a some notation. Denote by $\mathcal{S}(\mathbb{R}^n)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. \hat{f} denotes the Fourier transform of f , namely

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

The fractional Laplacian $(-\Delta)^\alpha$ can be defined through the Fourier transform

$$(-\Delta)^\alpha f = |\xi|^{2\alpha} \hat{f}(\xi).$$

Let

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^n} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \dots \right\}.$$

Its dual \mathcal{S}'_0 is given by

$$\mathcal{S}'_0 = \mathcal{S}' / \mathcal{S}_0^\perp = \mathcal{S}' / \mathcal{P},$$

where \mathcal{P} is the space of polynomials. In other words, two distributions in \mathcal{S}' are identified as the same in \mathcal{S}'_0 if their difference is a polynomial.

It is a classical result that there exists a dyadic decomposition of \mathbb{R}^n , namely a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\text{supp } \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn} \Phi_0(2^j x)$$

and

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where

$$A_j = \{\xi \in \mathbb{R}^n: 2^{j-1} < |\xi| < 2^{j+1}\}.$$

As a consequence, for any $f \in \mathcal{S}'_0$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{2.1}$$

For notational convenience, set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \dots \tag{2.2}$$

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ is defined by

$$\mathring{B}^s_{p,q} = \{f \in \mathcal{S}'_0: \|f\|_{\mathring{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\mathring{B}^s_{p,q}} = \begin{cases} \left(\sum_j (2^{js} \|\Delta_j f\|_{L^p})^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js} \|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

For Δ_j defined in (2.2) and $S_j \equiv \sum_{k < j} \Delta_k$,

$$\Delta_j \Delta_k = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j - k| \geq 3.$$

The following proposition lists a few simple facts that we will use in the subsequent section.

Proposition 2.2. Assume that $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.

- (1) If $1 \leq q_1 \leq q_2 \leq \infty$, then $\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}$.
- (2) (Besov embedding) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2})$, then $\mathring{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \subset \mathring{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$.
- (3) If $1 < p < \infty$, then

$$\mathring{B}^s_{p,\min(p,2)} \subset \mathring{W}^{s,p} \subset \mathring{B}^s_{p,\max(p,2)},$$

where $\mathring{W}^{s,p}$ denotes a standard homogeneous Sobolev space.

We will need a Bernstein type inequality for fractional derivatives.

Proposition 2.3. Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.

- (1) If f satisfies

$$\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^n: |\xi| \leq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C_1 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}.$$

- (2) If f satisfies

$$\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^n: K_1 2^j \leq |\xi| \leq K_2 2^j\} \tag{2.3}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^n)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C_2 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},$$

where C_1 and C_2 are constants depending on α, p and q only.

The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of L^p estimates (see [21,4]).

Proposition 2.4. *Assume either $\alpha \geq 0$ and $p = 2$ or $0 \leq \alpha \leq 1$ and $2 < p < \infty$. Let j be an integer and $f \in \mathcal{S}'$. Then*

$$\int_{\mathbb{R}^n} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geq C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

for some constant C depending on n, α and p .

3. The main theorem and its proof

Theorem 3.1. *Let θ be a Leray–Hopf weak solution of (1.1), namely*

$$\theta \in L^\infty([0, \infty); L^2(\mathbb{R}^2)) \cap L^2([0, \infty); \dot{H}^\alpha(\mathbb{R}^2)). \tag{3.1}$$

Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If

$$\theta \in L^\infty([t_0, t]; C^\delta(\mathbb{R}^2)), \tag{3.2}$$

then

$$\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).$$

Proof. First, we notice that (3.1) and (3.2) imply that

$$\theta \in L^\infty([t_0, t]; \dot{B}_{p,\infty}^{\delta_1}(\mathbb{R}^2)),$$

for any $p \geq 2$ and $\delta_1 = \delta(1 - \frac{2}{p})$. In fact, for any $\tau \in [t_0, t]$,

$$\begin{aligned} \|\theta(\cdot, \tau)\|_{\dot{B}_{p,\infty}^{\delta_1}} &= \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \\ &\leq \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}} \\ &\leq \|\theta(\cdot, \tau)\|_{C^\delta}^{1-\frac{2}{p}} \|\theta(\cdot, \tau)\|_{L^2}^{\frac{2}{p}}. \end{aligned}$$

Since $\delta > 1 - 2\alpha$, we have $\delta_1 > 1 - 2\alpha$ when

$$p > p_0 \equiv \frac{2\delta}{\delta - (1 - 2\alpha)}.$$

Next, we show that

$$\theta \in L^\infty([t_0, t]; \dot{B}_{p,\infty}^{\delta_1} \cap C^{\delta_1})$$

implies

$$\theta(\cdot, t) \in \dot{B}_{p,\infty}^{\delta_2} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$ to be specified. Let j be an integer. Applying Δ_j to (1.1), we get

$$\partial_t \Delta_j \theta + \kappa \Lambda^{2\alpha} \Delta_j \theta = -\Delta_j (u \cdot \nabla \theta). \tag{3.3}$$

By Bony’s notion of paraproduct,

$$\begin{aligned} \Delta_j (u \cdot \nabla \theta) &= \sum_{|j-k| \leq 2} \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) + \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta) \\ &\quad + \sum_{k \geq j-1} \sum_{|k-l| \leq 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta). \end{aligned} \tag{3.4}$$

Multiplying (3.3) by $p|\Delta_j\theta|^{p-2}\Delta_j\theta$, integrating with respect to x , and applying the lower bound

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geq C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

of Proposition 2.4, we obtain

$$\frac{d}{dt} \|\Delta_j\theta\|_{L^p}^p + C\kappa 2^{2\alpha j} \|\Delta_j\theta\|_{L^p}^p \leq I_1 + I_2 + I_3, \tag{3.5}$$

where I_1, I_2 and I_3 are given by

$$\begin{aligned} I_1 &= -p \sum_{|j-k|\leq 2} \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot \Delta_j(S_{k-1}u \cdot \nabla \Delta_k\theta) \, dx, \\ I_2 &= -p \sum_{|j-k|\leq 2} \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot \Delta_j(\Delta_k u \cdot \nabla S_{k-1}\theta) \, dx, \\ I_3 &= -p \sum_{k\geq j-1} \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot \sum_{|k-l|\leq 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l\theta) \, dx. \end{aligned}$$

We first bound I_2 . By Hölder’s inequality

$$I_2 \leq C \|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^p} \|\nabla S_{k-1}\theta\|_{L^\infty}.$$

Applying Bernstein’s inequality, we obtain

$$\begin{aligned} I_2 &\leq C \|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^p} \sum_{m\leq k-1} 2^m \|\Delta_m\theta\|_{L^\infty} \\ &\leq C \|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} \sum_{m\leq k-1} 2^{(m-k)(1-\delta_1)} 2^{m\delta_1} \|\Delta_m\theta\|_{L^\infty}. \end{aligned}$$

Thus, for $1 - \delta_1 > 0$, we have

$$I_2 \leq C \|\Delta_j\theta\|_{L^p}^{p-1} \|\theta\|_{C^{\delta_1}} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k}.$$

We now estimate I_1 . The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest. That is, we rewrite I_1 as

$$\begin{aligned} I_1 &= -p \sum_{|j-k|\leq 2} \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot [\Delta_j, S_{k-1}u \cdot \nabla] \Delta_k\theta \, dx - p \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot (S_j u \cdot \nabla \Delta_j\theta) \, dx \\ &\quad - p \sum_{|j-k|\leq 2} \int |\Delta_j\theta|^{p-2} \Delta_j\theta \cdot (S_{k-1}u - S_j u) \cdot \nabla \Delta_j \Delta_k\theta \, dx \\ &= I_{11} + I_{12} + I_{13}, \end{aligned}$$

where we have used the simple fact that $\sum_{|k-j|\leq 2} \Delta_k \Delta_j\theta = \Delta_j\theta$, and the brackets $[\]$ represent the commutator, namely

$$[\Delta_j, S_{k-1}u \cdot \nabla] \Delta_k\theta = \Delta_j(S_{k-1}u \cdot \nabla \Delta_k\theta) - S_{k-1}u \cdot \nabla \Delta_j \Delta_k\theta.$$

Since u is divergence free, I_{12} becomes zero. I_{12} can also be handled without resort to the divergence-free condition. In fact, integrating by parts in I_{12} yields

$$I_{12} = \int |\Delta_j\theta|^p \nabla \cdot S_j u \, dx \leq \|\Delta_j\theta\|_{L^p}^p \|\nabla \cdot S_j u\|_{L^\infty}.$$

By Bernstein’s inequality,

$$\begin{aligned}
 |I_{12}| &\leq \|\Delta_j \theta\|_{L^p}^p \sum_{m \leq j-1} 2^m \|\Delta_m u\|_{L^\infty} \\
 &= \|\Delta_j \theta\|_{L^p}^p 2^{(1-\delta_1)j} \sum_{m \leq j-1} 2^{(1-\delta_1)(m-j)} 2^{m\delta_1} \|\Delta_m u\|_{L^\infty}.
 \end{aligned}$$

For $1 - \delta_1 > 0$,

$$|I_{12}| \leq C \|\Delta_j \theta\|_{L^p}^p 2^{(1-\delta_1)j} \|u\|_{C^{\delta_1}} \leq C \|\Delta_j \theta\|_{L^p}^{p-1} 2^{(1-2\delta_1)j} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}} \|u\|_{C^{\delta_1}}.$$

We now bound I_{11} and I_{13} . By Hölder’s inequality,

$$|I_{11}| \leq p \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta\|_{L^p}.$$

To bound the commutator, we have by the definition of Δ_j

$$[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta = \int \Phi_j(x-y) (S_{k-1}(u)(x) - S_{k-1}(u)(y)) \cdot \nabla \Delta_k \theta(y) dy.$$

Using the fact that $\theta \in C^{\delta_1}$ and thus

$$\|S_{k-1}(u)(x) - S_{k-1}(u)(y)\|_{L^\infty} \leq \|u\|_{C^{\delta_1}} |x - y|^{\delta_1},$$

we obtain

$$\|[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta\|_{L^p} \leq 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} 2^k \|\Delta_k \theta\|_{L^p}.$$

Therefore,

$$|I_{11}| \leq Cp \|\Delta_j \theta\|_{L^p}^{p-1} 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p}.$$

The estimate for I_{13} is straightforward. By Hölder’s inequality,

$$\begin{aligned}
 |I_{13}| &\leq p \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|S_{k-1} u - S_j u\|_{L^p} \|\nabla \Delta_j \theta\|_{L^\infty} \\
 &\leq Cp \|\Delta_j \theta\|_{L^p}^{p-1} 2^{(1-\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p}.
 \end{aligned}$$

We now bound I_3 . By Hölder’s inequality and Bernstein’s inequality,

$$\begin{aligned}
 |I_3| &\leq p \|\Delta_j \theta\|_{L^p}^{p-1} \left\| \Delta_j \nabla \cdot \left(\sum_{k \geq j-1} \sum_{|l-k| \leq 1} \Delta_l u \Delta_k \theta \right) \right\|_{L^p} \\
 &\leq p \|\Delta_j \theta\|_{L^p}^{p-1} 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}.
 \end{aligned} \tag{3.6}$$

Inserting the estimates for I_1 , I_2 and I_3 in (3.5) and eliminating $p \|\Delta_j \theta\|_{L^p}^{p-1}$ from both sides, we get

$$\begin{aligned}
 \frac{d}{dt} \|\Delta_j \theta\|_{L^p} + C\kappa 2^{2\alpha j} \|\Delta_j \theta\|_{L^p} &\leq C 2^{(1-2\delta_1)j} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}} \|u\|_{C^{\delta_1}} + C 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p} \\
 &\quad + C \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} + C 2^{(1-\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} \\
 &\quad + C 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}.
 \end{aligned} \tag{3.7}$$

The terms on the right can be further bounded as follows.

$$\begin{aligned}
 C2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p} &= C2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} 2^{(k-j)(1-\delta_1)} \\
 &\leq C2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}},
 \end{aligned}$$

$$\begin{aligned}
 C\|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} &= C2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(k-j)(1-2\delta_1)} \\
 &\leq C2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\dot{B}_{p,\infty}^{\delta_1}},
 \end{aligned}$$

$$\begin{aligned}
 C2^{(1-\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} &= C2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(j-k)\delta_1} \\
 &\leq C2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\dot{B}_{p,\infty}^{\delta_1}}
 \end{aligned}$$

and

$$\begin{aligned}
 C2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p} &= C2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-2\delta_1(k-j)} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} \\
 &\leq C2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}}.
 \end{aligned}$$

We can write (3.7) in the following integral form

$$\begin{aligned}
 \|\Delta_j \theta(t)\|_{L^p} &\leq e^{-C\kappa 2^{2\alpha j}(t-t_0)} \|\Delta_j \theta(t_0)\|_{L^p} \\
 &\quad + C \int_{t_0}^t e^{-C\kappa 2^{2\alpha j}(t-s)} 2^{(1-2\delta_1)j} (\|\theta\|_{C^{\delta_1}} \|u\|_{\dot{B}_{p,\infty}^{\delta_1}} + \|u\|_{C^{\delta_1}} \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}}) ds.
 \end{aligned}$$

Multiplying both sides by $2^{(2\alpha+2\delta_1-1)j}$ and taking the supremum with respect to j , we get

$$\begin{aligned}
 \|\theta(t)\|_{\dot{B}_{p,\infty}^{2\delta_1+2\alpha-1}} &\leq \sup_j \{e^{-C\kappa 2^{2\alpha j}(t-t_0)} 2^{(\delta_1+2\alpha-1)j}\} \|\theta(t_0)\|_{\dot{B}_{p,\infty}^{\delta_1}} \\
 &\quad + C\kappa^{-1} \sup_j \{(1 - e^{-C\kappa 2^{2\alpha j}(t-t_0)})\} \max_{s \in [t_0, t]} \|\theta(s)\|_{\dot{B}_{p,\infty}^{\delta_1}} \|\theta(s)\|_{C^{\delta_1}}.
 \end{aligned}$$

Here we have used the fact that

$$\|u\|_{C^{\delta_1}} \leq \|\theta\|_{C^{\delta_1}} \quad \text{and} \quad \|u\|_{\dot{B}_{p,\infty}^{\delta_1}} \leq \|\theta\|_{\dot{B}_{p,\infty}^{\delta_1}}.$$

Therefore, we conclude that if

$$\theta \in L^\infty([t_0, t]; \dot{B}_{p,\infty}^{\delta_1} \cap C^{\delta_1}),$$

then

$$\theta(\cdot, t) \in \dot{B}_{p,\infty}^{2\delta_1+2\alpha-1}. \tag{3.8}$$

Since $\delta_1 > 1 - 2\alpha$, we have $2\delta_1 + 2\alpha - 1 > \delta_1$ and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,

$$\dot{B}_{p,\infty}^{2\delta_1+2\alpha-1} \subset \dot{B}_{\infty,\infty}^{\delta_2},$$

where

$$\delta_2 = 2\delta_1 + 2\alpha - 1 - \frac{2}{p} = \delta_1 + \left(\delta_1 - \left(1 - 2\alpha + \frac{2}{p} \right) \right).$$

We have $\delta_2 > \delta_1$ when

$$p > p_1 \equiv \frac{2}{\delta_1 - (1 - 2\alpha)}.$$

Noting that

$$\dot{B}_{\infty,\infty}^{\delta_2} \cap L^\infty = C^{\delta_2},$$

we conclude that, for $p > \max\{p_0, p_1\}$,

$$\theta(\cdot, t) \in \dot{B}_{p,\infty}^{\delta_2} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$. The above process can then be iterated with δ_1 replaced by δ_2 . A finite number of iterations allow us to obtain that

$$\theta(\cdot, t) \in C^\gamma$$

for some $\gamma > 1$. The regularity in the spatial variable can then be converted into regularity in time. We have thus established that θ is a classical solution to the supercritical QG equation. Higher regularity can be proved by well-known methods. \square

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