LE MATEMATICHE
Vol. LXI (2006) - Fasc. II, pp. 287-300

REGULARITY OF SOLUTIONS OF THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION

DAGMAR MEDKOVÁ

Let u be a solution of the Neumann problem for the Laplace equation in G with the boundary condition g. It is shown that $u \in L^q(\partial G)$ (equivalently, $u \in B^{q,2}_{1/q}(G)$ for $1 < q \leq 2$, $u \in L^q_{1/q}(G)$ for $2 \leq q < \infty$) if and only if the single layer potential corresponding to the boundary condition g is in $L^q(\partial G)$. As a consequence we give a regularity result for some nonlinear boundary value problem.

If G is a bounded domain in R^m with C^1 boundary then the boundary integral equation method enables not only to prove the existence of solutions of the Dirichlet and Neumann problems for the Laplace equation but also to study its regularity (see [5], [3], [4]). If u is a solution of the Dirichlet problem with the boundary condition from $L^p(\partial G)$ then u is an L^p-solution of the Dirichlet problem, i.e. the nontangential maximal function of u is in $L^p(\partial G)$ and the boundary condition is the nontangential limit of u (or equivalently, $u \in B^{p,2}_{1/p}(G)$ for $1 < p \leq 2$, $u \in L^p_{1/p}(G)$ for $2 \leq p < \infty$ and the boundary condition is the trace of u). If u is

Entrato in redazione il 17 gennaio 2006.

2000 *Mathematical Subject Classifications*: Primary 35B65; Secundary 31B10.

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan N. AV0Z10190503.

a solution of the Neumann problem with the boundary condition from $L^p(\partial G)$ then u is an L^p-solution of the Neumann problem, i.e. the nontangential maximal function of $|\nabla u|$ is in $L^p(\partial G)$ and the boundary condition is fulfilled in the sence of a nontangential limit (or equivalently, $u \in B^{p+1,2}_{1/p}(G)$ for $1 < p \leq 2$, $u \in L^{p+1}_{1/p}(G)$ for $2 \leq p < \infty$ and the boundary condition is fulfilled in the sence of a trace). It is well-known that the L^p-solution of the Dirichlet problem with the boundary condition g is an L^q-solution of the Neumann problem if and only if $g \in L^q_1(\partial G)$. We find in this paper the necessary and sufficient condition for the L^p-solution of the Neumann problem to be an L^q-solution of the Dirichlet problem. If $m = 2$ then every L^p-solution of the Neumann problem is continuous on the closure of G. So, we restrect ourselves to the case when $m > 2$.

Let $G \subset R^m$, $m > 2$, be a bounded domain with C^1 boundary ∂G. It means that for each $x \in \partial G$ there is a coordinate system centered at x and a function Φ of the class C^1 on R^{m-1} such that $\Phi(0, \dots, 0) = 0$ and in some neighbourhood of x the set G lies under the graph of Φ and $R^m \setminus \mathrm{cl}G$ lies above the graph of Φ. Here $\mathrm{cl}G$ denotes the closure of G.

If $x \in \partial G$, $\alpha > 0$, denote *the non-tangential approach region of opening α at the point x*

$$\Gamma_\alpha(x) = \{y \in G; |x - y| < (1 + \alpha)\mathrm{dist}(y, \partial G)\}.$$

If u is a function in G we denote on ∂G *the non-tangential maximal function of u*

$$N_\alpha(u)(x) = \sup\{|u(y)|; y \in \Gamma_\alpha(x)\}.$$

If

$$c = \lim_{y \to x, y \in \Gamma_\alpha(x)} u(y)$$

for each $\alpha > \alpha_0$, we say that c is the nontangential limit of u *at* x.

Since G has C^1 boundary there is $\alpha_0 > 0$ such that $x \in \mathrm{cl}\Gamma_\alpha(x)$ for each $x \in \partial G$, $\alpha > \alpha_0$.

If $g \in L^p(\partial G)$, $1 < p < \infty$, we define L^p-solution of the problem

$$\Delta u = 0 \quad \text{in } G,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial G \tag{1}$$

as follows:

Find a function u harmonic in G, such that $N_\alpha(|\nabla u|) \in L^p(\partial G)$ for each $\alpha > \alpha_0$, ∇u has the nontangential limit $\nabla u(x)$ for almost all $x \in \partial G$ and $g(x) = n(x) \cdot \nabla u(x)$ for allmost all $x \in \partial G$, where $n(x)$ is the exterior unit normal of G at x.

If $g \in L^p(\partial G)$, $1 < p < \infty$ then there is an L^p solution of the Neumann problem (1) if and only if

$$\int_{\partial G} f \; d\mathcal{H}_{m-1} = 0. \tag{2}$$

(Here $\mathcal{H}_k$ denotes the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k.) This solution is unique up to additive constant and there is $f \in L^p(\partial G)$ such that the single layer potential with density f

$$\mathcal{S}f(x) = \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G} f(y)|x-y|^{2-m} \; d\mathcal{H}_{m-1}(y)$$

is a solution of the problem (1) (see [3], Theorem 2.6). (Here $\Omega_r(x)$ denotes the open ball with the centre x and the diameter r.)

For $f \in L^p(\partial G)$ and $x \in \partial G$ define

$$K^* f(x) = \lim_{\epsilon \to 0_+} \frac{1}{\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int\limits_{\partial G \setminus \Omega_\epsilon(x)} \frac{n(x)\cdot(y-x)}{|x-y|^m} f(y) \; d\mathcal{H}_{m-1}(y). \tag{3}$$

This limit exists for allmost all $x \in \partial G$ and K^* is a bounded linear operator on $L^p(\partial G)$ (see [3], Theorem 1.9). Moreover, $\mathcal{S}f$ is an L^p-solution of the Neumann problem (1) with $g \in L^p(\partial G)$ if and only if $\frac{1}{2}f + K^*f = g$ (see [3], Theorem 1.10).

If $f \in L^p(\partial G)$, $1 < p < \infty$ then $\mathcal{S}f(x)$ has a sence for allmost all $x \in \partial G$, $\mathcal{S}f \in L^p(\partial G)$ and $\mathcal{S}f$ is an L^p-solution of the Dirichlet problem for the Laplace equation with the boundary condition $h = \mathcal{S}f$ (see [14], Lemma 1.8 and [2], Theorem 2).

If $h \in L^p(\partial G)$, $1 < p < \infty$, we define L^p-solution of the problem

$$\Delta u = 0 \quad \text{in } G,$$

$$u = h \quad \text{on } \partial G \tag{4}$$

as follows:

Find a function u harmonic in G, such that $N_\alpha(|u|) \in L^p(\partial G)$ for each $\alpha > \alpha_0$ and u has the nontangential limit $h(x)$ at allmost all $x \in \partial G$.

Remark that this solution is unique (see [2] and [14], Theorem 0.9D).

For $1 < p < \infty$ and $0 \le s < \infty$ the Sobolev space L^p_s is defined by

$$L^p_s = \{(I - \Delta)^{-s/2} g;\, g \in L^p(R^m)\}$$

with the norm

$$\|f\|_{L^p_s} = \|(I - \Delta)^{s/2} f\|_{L^p(R^m)}.$$

Remark that $f \in L^p_{1+s}$ if and only if $f \in L^p_s$ and $\nabla f \in L^p_s$.

Define $L^p_s(G)$ as the space of restrictions of functions in L^p_s to G. (If s is integer then $L^p_s(G)$ is the set of functions from $L^p(G)$ which partial derivatives in the sense of distributions up to the order s are from $L^p(G)$.)

For $0 < s < 1$, $1 < p, q < \infty$ let us introduce Besov spaces

$$B^{p,q}_s \equiv \left\{ f \in L^p(R^m);\, \int \frac{1}{|y|^{m+ps}} \left[\int |f(x) - f(x+y)|^p dx \right]^{q/p} dy < \infty \right\},$$

$$B^{p,q}_{1+s} \equiv \left\{ f \in B^{p,q}_s;\, \nabla f \in B^{p,q}_s \right\}.$$

Define $B^{p,q}_s(G)$ as the space of restrictions of functions in $B^{p,q}_s$ to G.

If u is an L^p solution of the Dirichlet problem (4) and $2 \le p < \infty$ then $u \in L^p_{1/p}(G)$. If u is an L^p solution of the Dirichlet problem (4) and $1 < p \le 2$ then $u \in B^{p,2}_{1/p}(G)$. If u is an L^p solution of the Neumann problem (1) and $2 \le p < \infty$ then $u \in L^p_{1+1/p}(G)$. If u is an L^p solution of the Neumann problem (1) and $1 < p \le 2$ then $u \in B^{p,2}_{1+1/p}(G)$. (See [4], Theorem 5.15 and [5], Theorem 2.2.22.)

We use the following result proved in [6] (Lemma 2.18):

Lemma 1. *Let ν be a real measure with a compact support in R^m. Denote*

$$\mathcal{S}\nu(x) = \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G} |x - y|^{2-m} \, d\nu(y)$$

whenever this integral has a sence. Let $k > 0$, $\lambda > m - 2$ be such constants that

$$|\nu|(\Omega_r(x) \le k r^\lambda$$

for all $x \in R^m$ and all $r > 0$. If $0 < \alpha < \min(1, \lambda - m + 2)$ then $\mathcal{S}\nu$ is an α-Hölder function in each bounded subset of R^m.

Lemma 2. *Let $f \in L^p(\partial G)$. If $1 < p < m-1$ then $\mathcal{S}f \in L^{p(m-1)/(m-1-p)}(\partial G)$. If $p = m-1$ then $\mathcal{S}f \in L^q(\partial G)$ for all $1 < q < \infty$. If $p > m-1$ then $\mathcal{S}f \in C^\alpha(\mathrm{cl}G)$ for each $\alpha \in (0, 1-(m-1)/p)$.*

Proof. Since $\mathcal{S}f \in L_1^p(\partial G)$ by [14], Lemma 1.8 we deduce from [1], Theorem 5.4 that $\mathcal{S}f \in L^{p(m-1)/(m-1-p)}(\partial G)$ for $p < m-1$ and $\mathcal{S}f \in L^q(\partial G)$ for all $1 < q < \infty$ and $p = m-1$. Let now $p > m-1$, $\alpha \in (0, 1-(m-1)/p)$. Since the boundary of G is locally a graph of a Lipschitz function there is a constant M such that $\mathcal{H}_{m-1}(\partial G \cap \Omega_r(x)) \le M r^{m-1}$ for each $x \in \partial G$ and $r > 0$. For such x, r we have

$$\int_{\partial G\cap\Omega_r(x)} |f|\, d\mathcal{H}_{m-1} \le \left[\int_{\partial G\cap\Omega_r(x)} |f|^p\right]^{1/p} \left[\mathcal{H}_{m-1}(\Omega_r(x))\right]^{(p-1)/p}$$
$$\le \|f\|_{L^p(\partial G)} M^{(p-1)/p} r^{(m-1)(p-1)/p}.$$

If $x \in R^n$, $r > 0$ then

$$\int_{\partial G\cap\Omega_r(x)} |f|\, d\mathcal{H}_{n-1} \le \|f\|_{L^p(\partial G)} M^{(p-1)/p} (2r)^{(m-1)(p-1)/p}.$$

Thus $\mathcal{S}f \in C^\alpha(\mathrm{cl}G)$ by [6], Lemma 1.

Remark 3. Let u be an L^p solution of the Neumann problem (1), $1 < p < \infty$. Then there is the nontangential limit $h(x)$ of u at allmost all $x \in \partial G$. If $p \ge m-1$ then u is an L^q-solution of the Dirichlet problem (4) for each $1 < q < \infty$. If $p < m-1$ then u is an L^q-solution of the Dirichlet problem (4) for each $1 < q \le p(m-1)/(m-1-p)$.

Proof. According to [10], Theorem 5.1, [10], Theorem 5.2 and [5], Corollary 2.1.12 there is $f \in L^p(\partial G)$ such that $u = \mathcal{S}f$. Put $h = \mathcal{S}f$ on ∂G. Then $h(x)$ is a nontangential limit of u at allmost all $x \in \partial G$. If $p \ge m-1$ then $h \in L^q(\partial G)$ for $1 < q < \infty$; if $p < m-1$ then $h \in L^q(\partial G)$ for $1 < q \le p(m-1)/(m-1-p)$ by Lemma 2. If $h = \mathcal{S}f \in L^q(\partial G)$, $1 < q < \infty$ then $u = \mathcal{S}f$ is an L^q-solution of the Dirichlet problem (4) by [2], Theorem 2.

The following result was proved in [14] (Theorem 1.12):

Lemma 4. *There is a sequence of C^∞ domains G_j with the following properties:*

1. $\mathrm{cl}G_j \subset G$.

2. *There are homeomorphisms* $\Lambda_j : \partial G \to \partial G_j$, *such that* $\sup\{|y - \Lambda_j(y)|; y \in \partial G\} \to 0$ *as* $j \to \infty$ *and there is* $\alpha > 0$ *such that* $\Lambda_j(y) \in \Gamma_\alpha(y)$ *for each* j *and each* $y \in \partial G$.

3. *There are positive functions* ω_j *on* ∂G *bounded away from zero and infinity uniformly in* j *such that for any measurable set* $E \subset \partial G$, $\int_E \omega_j \, d\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\Lambda_j(E))$, *and so that* $\omega_j \to 1$ *pointwise a.e. and in every* $L^q(\partial G)$, $1 \le q < \infty$.

4. *The normal vectors to* G_j, $n(\Lambda_j(y))$, *converge pointwise a.e. and in every* $L^q(\partial G)$, $1 \le q < \infty$, *to* $n(y)$.

Lemma 5. *Let* $1 < p < \infty$, u *be an* L^p*-solution of the problem (1). Then there is the nontangential limit* $h(x)$ *of* u *at a.a.* $x \in \partial G$ *and* $h \in L^p(\partial G)$. *For* $f \in L^1(\partial G)$ *and* $x \in G$ *denote*

$$\mathcal{D}f(x) = \frac{1}{\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G} f(y) \frac{n(y)\cdot(y-x)}{|x-y|^m} \, d\mathcal{H}_{m-1}(y)$$

the double layer potential with density f. *Then* $u = \mathcal{S}g + \mathcal{D}h$ *in* G.

Proof. According to [3], Theorem 2.6 there are $f \in L^p(\partial G)$ and a constant c such that $u = \mathcal{S}f + c$. Using the boundary properties of single layer potentials (see [14], Lemma 1.8 and [2], Theorem 2) we get that there is the nontangential limit $h(x)$ of u at a.a. $x \in \partial G$ and $N_\alpha(u) \in L^p(\partial G)$.

Let G_j be domains from Lemma 4. If $x \in G$ we get using Lebesque's Lemma and [7], p. 136.

$$\mathcal{S}g(x) + \mathcal{D}h(x) =$$

$$\lim_{j\to\infty} \left[\frac{1}{(m-2)\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G_j} \frac{\partial u(y)}{\partial n} |x-y|^{2-m} \, d\mathcal{H}_{m-1}(y) + \right.$$

$$\left. \frac{1}{\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G_j} u(y) \frac{n(y)\cdot(y-x)}{|x-y|^m} \, d\mathcal{H}_{m-1}(y) \right] = u(x).$$

Lemma 6. *Let* $1 < p, q < \infty$. *Denote* $SL^{p,q} = \{f \in L^p(\partial G); \mathcal{S}f \in L^q(\partial G)\}$. *Then* $\mathcal{S} : f \mapsto \mathcal{S}f$ *is a closed linear operator from* $SL^{p,q} \subset L^p(\partial G)$ *to* $L^q(\partial G)$. *For* $f \in L^q(\partial G)$ *and* $x \in \partial G$ *define*

(5)
$$Kf(x) = \lim_{\epsilon\to 0_+} \frac{1}{\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G\setminus\Omega_\epsilon(x)} \frac{n(y)\cdot(x-y)}{|x-y|^m} f(y) \, d\mathcal{H}_{m-1}(y).$$

This limit exists for allmost all $x \in \partial G$ *and* K *is a bounded linear operator on* $L^q(\partial G)$*. If* $f \in SL^{p,q}$ *then* $K^* f \in SL^{p,q}$ *and* $\mathcal{S}K^* f = K\mathcal{S}f$.

Proof. Let $f_n \in SL^{p,q}$, $f_n \to f$ in $L^p(\partial G)$ and $\mathcal{S}f_n \to g$ in $L^q(\partial G)$ as $n \to \infty$. Put $r = \min(p, q)$. Since $f_n \to f$ in $L^r(\partial G)$, $\mathcal{S}f_n \to g$ in $L^r(\partial G)$ as $n \to \infty$ and $\mathcal{S}$ is a continuous linear operator on $L^r(\partial G)$ (see [14], Lemma 1.8) we get $\mathcal{S}f = g$. Therefore $\mathcal{S}$ is a closed linear operator from $SL^{p,q} \subset L^p(\partial G)$ to $L^q(\partial G)$. It is well-known fact that the limit (5) exists for allmost all $x \in \partial G$ and K is a bounded linear operator on $L^q(\partial G)$ (see [5], Theorem 2.2.13).

Let $f \in SL^{p,q}$. Then $\mathcal{S}f$ is an L^p-solution of the problem (1) with $g = \frac{1}{2} f + K^* f$ (see [3], Theorem 1.10). Since $\mathcal{S}f(x)$ is the nontangential limit of $\mathcal{S}f$ at a.a. $x \in \partial G$ (see [14], Lemma 1.8) Lemma 5 gives that $\mathcal{S}f = \mathcal{S}[(1/2)f + K^* f] + \mathcal{D}\mathcal{S}f$ in G. Using boundary properties of single layer and double layer potentials (see [5], Theorem 2.2.13) we get $\mathcal{S}f = \mathcal{S}[(1/2)f + K^* f] + [(1/2) - K]\mathcal{S}f$ on ∂G and thus $\mathcal{S}K^* f = K\mathcal{S}f$. Since $\mathcal{S}f \in L^q(\partial G)$ and K is a bounded linear operator in $L^q(\partial G)$ we have $\mathcal{S}K^* f = K\mathcal{S}f \in L^q(\partial G)$ what forces $K^* f \in SL^{p,q}$.

Definition 7. Let X be a Banach space, T be a bounded linear operator in X. Denote by Ker$T$ the kernel of T, by X^* the dual space of X, by T^* the adjoint operator on X^*, by $\alpha(T)$ the dimension of Ker$T$ and by $\beta(T)$ the dimension of KerT^*. The operator T is called Fredholm if $T(X)$ is a closed subspace of X, $\alpha(T) < \infty$ and $\beta(T) < \infty$. We then denote $i(T) = \alpha(T) - \beta(T)$ the index of T. Denote by I the identity operator. If X is a complex Banach space denote by $\Phi(T)$ the set of all $\lambda \in C$ for which $\lambda I - T$ is a Fredholm operator.

We use the following result proved in [9] (Theorem 5):

Lemma 8. *Let* X *and* $\tilde{X}$ *be complex Banach spaces,* Y *be a subspace of* X*. Let* T *be a bounded linear operator in* X*,* $\tilde{T}$ *be a bounded linear operator in* $\tilde{X}$*,* $T(Y) \subset Y$*. Let* S *be a closed linear operator from* Y *to* $\tilde{X}$ *such that* $\tilde{T}Sy = STy$ *for each* $y \in Y$*. Denote by* Ω *the unbounded component of* $\Phi(T) \cap \Phi(\tilde{T})$*. Let* $\mu \in \Omega$ *be such that* $\mathrm{Ker}(T - \mu I)^n \subset Y$ *for each* $n \in N$*. If* $x, y \in X$*,* $(T - \mu I)x = y$ *then* $x \in Y$ *if and only if* $y \in Y$*.*

Lemma 9. *Let* X*,* Y *be Banach spaces,* T *be a bounded linear Fredholm operator in* X*,* $\mathcal{T}$ *be a bounded linear Fredholm operator* *in Y, $i(T) = i(\mathcal{T}) = 0$. Let $X \subset Y$, $Y^* \subset X^*$ and $Tx = \mathcal{T}x$ for each $x \in X$, $T^*z = \mathcal{T}^*z$ for each $z \in Y^*$. Then* $\mathrm{Ker}T^n = \mathrm{Ker}\mathcal{T}^n$ *for each $n \in N$.*

Proof. Clearly, $\mathrm{Ker}T^n \subset \mathrm{Ker}\mathcal{T}^n$, $\mathrm{Ker}(\mathcal{T}^*)^n \subset \mathrm{Ker}(T^*)^n$. Since T^n, $\mathcal{T}^n$ are Fredholm operators with index 0 (see [12], Theorem 5.7) we obtain

$$\alpha(T^n) \le \alpha(\mathcal{T}^n) = \beta(\mathcal{T}^n) = \alpha((\mathcal{T}^*)^n) \le \alpha((T^*)^n) = \beta(T^n) = \alpha(T^n) < \infty$$

by [12], p. 58. Therefore $\mathrm{Ker}T^n = \mathrm{Ker}\mathcal{T}^n$.

Theorem 10. *Let $1 < p, q < \infty$. Let u be an L^p-solution of the Neumann problem (1) with the boundary condition g. Denote by h the nontangential limit of u on ∂G. Then the following are equivalent*

1) *$N_\alpha(u) \in L^q(\partial G)$ for each $\alpha > \alpha_0$.*
2) *$h \in L^q(\partial G)$.*
3) *$N_\alpha(\mathcal{S}g) \in L^q(\partial G)$ for each $\alpha > \alpha_0$.*
4) *$\mathcal{S}g \in L^q(\partial G)$.*

Proof. Suppose first that $q \le p$. Then 1), 2) are fulfilled by Remark 3 and 3), 4) are fulfilled by [14], Lemma 1.8 and [2], Theorem 2.

We can suppose that $p < q$. Put $r = q/(q-1)$. Then $q = r/(r-1)$. Since $r \le p$ the function u is an L^r-solution of the Neumann problem (1). Therefore we can suppose that $q = p/(p-1)$.

1) $\Rightarrow$ 2) Since $|h(x)| \le N_\alpha u(x)$ for each $x \in \partial G$ and $N_\alpha u \in L^q(\partial G)$ we deduce $h \in L^q(\partial G)$.

2) $\Rightarrow$ 1) For $x \in G$ denote by ω^x the harmonic measure corresponding to x. Denote $Hh(x) = \int h(y)\, d\omega^x$ for $x \in G$. Then Hh is an L^q-solution of the Dirichlet problem (4) by [2], Theorem 2. Put $r = \min(p, q)$. Then Hh is an L^r-solution of the Dirichlet problem (4). Since $u = \mathcal{S}f$ for some $f \in L^p(\partial G)$ (see [3], Theorem 2.6) the function u is an L^p-solution of the Dirichlet problem (4) by [14], Lemma 1.8 and [2], Theorem 2. This forces that u is an L^r-solution of the problem (4). From the uniqueness of an L^r-solution of the Dirichlet problem (4) we get that $u = Hh$ (see [2] and [14], Theorem 0.9D). Thus $N_\alpha(u) \in L^q(\partial G)$ for each $\alpha > \alpha_0$.

3) $\Leftrightarrow$ 4) Since $\mathcal{S}f$ is an L^p solution of the Neumann problem with the boundary condition $(\frac{1}{2}f + K^*f)$ and the nontangential limit of $\mathcal{S}f$

on ∂G is $\mathcal{S}f$ this equivalence follows from 1) $\Leftrightarrow$ 2).

2) $\Rightarrow$ 4) $u = \mathcal{S}g + \mathcal{D}h$ in G by Lemma 5. Using boundary properties of single layer and double layer potentials (see [5], Theorem 2.2.13) we get $\mathcal{S}g = \frac{1}{2}h + Kh$ in ∂G. Since $h \in L^q(\partial G)$ and K is a bounded linear operator in $L^q(\partial G)$ we obtain $\mathcal{S}g \in L^q(\partial G)$.

4) $\Rightarrow$ 2) If X is a real linear space with the norm $\| \ \|$ we denote $\mathrm{compl}X = \{x + iy; x, y \in X\}$ the complexification of X with the norm $\|x + iy\| = \|x\| + \|y\|$. If T is a linear operator from $D(T) \subset X$ to the real linear space Z define $T(x + iy) = Tx + iTy \in \mathrm{compl}Z$ for $x, y \in D(T)$.

Since K is a compact linear operator in compl $L^q(\partial G)$ by [5], Corollary 2.2.14 we have $C \setminus \{0\} \subset \Phi(K)$ in compl $L^q(\partial G)$ by [12], Theorem 7.8. Since K^* is a compact linear operator in compl $L^p(\partial G)$ (see [5], Corollary 2.2.14) we obtain $C \setminus \{0\} \subset \Phi(K)$ in compl $L^p(\partial G)$ (see [12], Theorem 7.8). Lemma 6 yields that $\mathcal{S}$ is a closed linear operator from compl $SL^{p,q}$ to compl $L^q(\partial G)$, $K^*(\mathrm{compl}SL^{p,q}) \subset \mathrm{compl}SL^{p,q}$ and $\mathcal{S}K^*f = K\mathcal{S}f$ for each $f \in \mathrm{compl}SL^{p,q}$. Lemma 9 yields that Ker $(\frac{1}{2}I + K^*)^n \subset \mathrm{compl}L^q(\partial G)$. Since $\mathcal{S}$ is a bounded linear operator in compl $L^q(\partial G)$ (see [5], Theorem 2.2.20) this gives that Ker $(\frac{1}{2}I + K^*)^n \subset \mathrm{compl}SL^{p,q}$ for each $n \in N$.

According to [3], Theorem 2.6 there is $f \in L^p(\partial G)$ such that $u = \mathcal{S}f$. Since $(\frac{1}{2}I + K^*)f = g$ by [5], Theorem 2.2.13 and $g \in SL^{p,q}$, Lemma 8 yields that $f \in SL^{p,q}$. Thus $h = \mathcal{S}f \in L^q(\partial G)$.

Definition 11. Let $1 < p < \infty$. We say that u is a weak solution of the problem (1) in $L_1^p(G)$ if $u \in L_1^p(G)$ and

$$\int_G \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = \int_{\partial G} g\varphi \, d\mathcal{H}_{m-1}$$

for all $\varphi \in L_1^q(G)$ where $q = p/(p-1)$.

Remark 12. If u is an L^p-solution of the problem (1), $1 < p < \infty$, then u is a weak solution of the problem (1) in $L_1^p(G)$ by [8], Lemma 4.1.

Corollary 13. *Let $1 < p, q < \infty$. Let f be a Borel measurable function from $\partial G \times R^1$ such that $|f(x, y)| \leq h(x) + C|y|^\lambda$ for each $x \in \partial G$, $y \in R^1$, where $h \in L^q(\partial G)$, C, λ are constants, $0 < \lambda < \infty$ for $p \geq m$, $0 < \lambda < p(m-1)/(m-p)$ for $p < m$. If u is a weak solution of the* *problem*

$$\Delta u = 0 \quad \text{in } G,$$

$$\frac{\partial u}{\partial n}(x) = f(x, u(x)) \quad \text{on } \partial G$$

in $L_1^p(G)$ *then* $g(x) = f(x, u(x)) \in L^q(\partial G)$ *and* u *is an* L^q*-solution of the problem (1). Moreover, if* $q > m-1$ *then* $u \in C^\alpha(\mathrm{cl}G)$ *for each* $\alpha \in (0, 1-(m-1)/q)$.

Proof. Let $1 < \beta < \infty$ be such that $g \in L^\beta(\partial G)$. Put $r = \min(p, \beta)$. We get from the definition of a weak solution for $\varphi \equiv 1$ that

$$\int_{\partial G} g \; d\mathcal{H}_{m-1} = 0.$$

According to [3], Theorem 2.6 there is an L^β-solution v of the problem (1). The function v is a weak solution of the problem (1) in $L_1^r(G)$ by [8], Lemma 4.1. Since u is a weak solution of the problem (1) in $L_1^r(G)$ there is a constant c such that $u = v + c$ in G (see [13], Theorem 4.1). Therefore u is an L^β-solution of the problem (1).

If $p \geq m$ then $u \in L^{q\lambda}(\partial G)$ by [11], Theorem 4.6. Hence $g \in L^q(\partial G)$ and u is an L^q-solution of the problem (1).

Suppose now that $p < m$. Then $u \in L^{p(m-1)/(m-p)}(\partial G)$ by [11], Theorem 4.7. Hence $g \in L^r(\partial G)$ where $r = \min(q, p(m-1)(m-p)^{-1}\lambda^{-1}) > 1$ and u is an L^r-solution of the problem (1). We can restrict ourselves to the case when $p(m-1)/(m-p) < q\lambda$. We now show that $u \in L^{q\lambda}(\partial G)$.

Let $p(m-1)/(m-p) \leq \beta \leq q\lambda$ be such that $u \in L^\beta(\partial G)$. Then $\beta/\lambda > 1$ and $g \in L^{\beta/\lambda}(\partial G)$. Lemma 2 gives that $\mathcal{S}g \in L^{q\lambda}(\partial G)$ for $\beta/\lambda \geq (m-1)$ and $\mathcal{S}g \in L^{\beta(m-1)/[\lambda(m-1)-\beta]}(\partial G)$ for $\beta/\lambda < (m-1)$. If $\beta \geq \lambda(m-1)$ then $u \in L^{q\lambda}(\partial G)$ by Theorem 10. Suppose now that $p(m-1)/(m-p) \leq \beta < (m-1)\lambda$. Then $\lambda > p/(m-p)$ and $u \in L^{\beta(m-1)/[\lambda(m-1)-\beta]}(\partial G)$ by Theorem 10. Since $p(m-1)/(m-p) \leq \beta$ we have $u \in L^{s\beta}(\partial G)$ where $(m-1)/[\lambda(m-1)-\beta] \geq s \equiv (m-1)/[\lambda(m-1) - p(m-1)/(m-p)] > (m-1)/[(m-1)m/(m-p) - p(m-1)/(m-p)] = 1$. Using the induction we get $u \in L^{q\lambda}(\partial G)$. Since $u \in L^{q\lambda}(\partial G)$, $h \in L^q(\partial G)$ we have $g \in L^q(\partial G)$ and therefore u is an L^q-solution of the problem (1).

Let now p be arbitrary and $q > m-1$, $\alpha \in (0, 1-(m-1)/q)$.

According to [3], Theorem 2.6 there is $\varphi \in L^q(\partial G)$ and a constant c such that $u = \mathcal{S}\varphi + c$. Lemma 2 gives that $u \in C^\alpha(\mathrm{cl} G)$.

Corollary 14. *Let $1 < p, q < \infty$. Let f be a Borel measurable function on $\partial G \times R^{m+1}$ such that $|f(x, y, z_1, \dots, z_m)| \le h(x) + C_1|y|^\lambda + C_2(|z_1| + \dots |z_m|)^\alpha$ for each $x \in \partial G$, $y, z_1, \dots, z_m \in R^1$, where $h \in L^q(\partial G)$, C_1, C_2, α, λ are constants, $0 < \alpha < 1$, $0 < \lambda < \infty$ for $p \ge m$, $0 < \lambda < p(m-1)/(m-p)$ for $p < m$. If u is an L^p-solution of the problem*

$$\Delta u = 0 \quad \text{in } G,$$

$$\frac{\partial u}{\partial n}(x) = f(x, u(x), \nabla u(x)) \quad \text{on } \partial G \tag{6}$$

then u is an L^q-solution of the problem (6). If $q > m-1$ then $u \in C^\alpha(\mathrm{cl} G)$ for each $\alpha \in (0, 1-(m-1)/q)$.

Proof. We can suppose $p < q$. Let $p \le \beta \le q$ be such that u is an L^β-solution of the problem (6). Put $g(x, y) = f(x, y, \nabla u(x))$, $v(x) = h(x) + C_2 m|\nabla u|^\alpha$. Then g is a Borel measurable function on $\partial G \times R^1$ and $|g(x, y)| \le v(x) + C_1|y|^\lambda$. Since u is an L^β-solution of the problem (6) we have $|\nabla u| \in L^\beta(\partial G)$. Thus $v \in L^{\min(q,\beta/\alpha)}(\partial G)$. Corollary 13 gives that u is an $L^{\min(q,\beta/\alpha)}$-solution of the problem (6). Repeating the process we get that u is an L^q-solution of the problem (6).

If $q > m-1$ then $u \in C^\alpha(\mathrm{cl}\, G)$ for each $\alpha \in (0, 1-(m-1)/q)$ by Corollary 13.

Corollary 15. *Let $1 < p, q, r < \infty$. Let $f \in L^r(\partial G)$, g be a Borel measurable function in R^1 such that $|g(y)| \le C|y|^\lambda$ for each $y \in R^1$, where $0 < C < \infty$, $0 < \lambda \le 1$ are constants. Let u be a weak solution of the problem*

$$\Delta u = 0 \quad \text{in } G,$$

$$\frac{\partial u}{\partial n}(x) = f(x) + g(u(x)) \quad \text{on } \partial G \tag{7}$$

in $L_1^p(G)$. Then u is an L^q-solution of the problem (4) with $h = u$ if and only if $\mathcal{S}f \in L^q(\partial G)$.

Proof. u is an L^r-solution of the problem (7) by Corollary 13. If $r \ge m-1$ then $\mathcal{S}f \in L^q(\partial G)$ by Lemma 2, $u \in L^q(\partial G)$ by Corollary 13 and therefore u is an L^q-solution of the problem (4) by Theorem 10. Suppose now that $r < m-1$.

Suppose first that u is an L^q-solution of the problem (4). Then $\mathcal{S}f + \mathcal{S}g(u) \in L^q(\partial G)$ by Theorem 10. Since $g(u(x)) \in L^q(\partial G)$ Lemma 2 gives that $\mathcal{S}g(u) \in L^q(\partial G)$ and thus $\mathcal{S}f \in L^q(\partial G)$.

Suppose now that $\mathcal{S}f \in L^q(\partial G)$. Remark 3 gives that $u \in L^r(\partial G)$. Let $r \le \beta \le q$ be such that $u \in L^\beta(\partial G)$. Since $g(u(x)) \in L^\beta(\partial G)$ Lemma 2 gives that $\mathcal{S}g(u) \in L^{\beta(m-1)/(m-1-r)}(\partial G)$. Thus $u \in L^{\min(q,\beta(m-1)/(m-1-r))}(\partial G)$ by Theorem 10. Using the induction we get that $u \in L^q(\partial G)$ and thus u is an L^q-solution of the problem (4) by Theorem 10.

Example 16. Fix $\lambda \in (1/(m-1), 1)$. Take a coordinate system so that $[0, 0, \ldots, 0] \in \partial G$ and for some $\rho > 0$, $h > 0$ there is a function f of class C^1 in $M = \{[x_1, \ldots, x_{m-1}];\ |x_1|, \ldots, |x_{m-1}| < 2\rho\}$ such that $K \equiv \partial G \cap \{[x_1, \ldots, x_m];\ |x_1|, \ldots, |x_{m-1}| < 2\rho, |x_m| < h\} = \{[x_1, \ldots, x_{m-1}, f(x_1, \ldots, x_{m-1})]; [x_1, \ldots, x_{m-1}] \in M\}$. Put

$$g(x_1,\ldots,x_m) = \begin{cases} |x_1|^{-\lambda} & \text{for } [x_1,\ldots,x_m] \in K, x_1 > 0, |x_1|,\ldots,|x_{m-1}| < \rho, \\ -c|x_1|^{-\lambda} & \text{for } [x_1,\ldots,x_m] \in K, x_1 < 0, |x_1|,\ldots,|x_{m-1}| < \rho, \\ 0 & \textit{elsewhere} \end{cases}$$

where c is a such positive constant that

$$\int_{\partial G} g \, d\mathcal{H}_{m-1} = 0. \tag{8}$$

If $1 < p < \infty$ then $g \in L^p(\partial G)$ if and only if $p < 1/\lambda$ $(< m-1)$. If $1 < p < 1/\lambda < m-1$ then there is an L^p-solution u of the problem (1) (see [3], Theorem 2.6). Moreover, there is $\varphi \in L^p(\partial G)$ and a constant C such that $u = \mathcal{S}\varphi + C$ (see [3], Theorem 2.6). Therefore $u \in L^q(\partial G)$ for each $1 < q \le p(m-1)/(m-1-p)$ (see Remark 3). Since two L^p-solutions of the problem (1) differ by a constant (see [3], Theorem 2.6) the function u is an L^r-solution of the problem (1) for all $1 < r < 1/\lambda$ and thus $u \in L^q(\partial G)$ for each $1 < q < (m-1)/[\lambda(m-1)-1] < \infty$. Using Theorem 10 we can show that $u \in L^q(\partial G)$ for each $1 < q < \infty$.

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G. Put $\nu = g\mathcal{H}$. Denote

$$L = \sup_{|x_1|,\ldots,|x_{m-1}| \le \rho} \sqrt{1 + |\nabla f(x_1, \ldots, x_{m-1})|^2}.$$

If $z \in R^m$, $r > 0$ then

$$|\nu|(\Omega_r(z)) = \int_{K \cap \Omega_r(z)} |g| \, d\mathcal{H} \leq L(1+c) \int_{|x_1|,\dots,|x_{m-1}|<r} |x_1|^{-\lambda} \, dx_1 \dots dx_{m-1}$$

$$= r^{m-2+(1-\lambda)} L(1+c) 2^{m-1}/(1-\lambda).$$

Lemma 1 shows that $\mathcal{S}g \in C(\partial G)$ and thus $u \in L^q(\partial G)$ for all $1 < q < \infty$ by Theorem 10.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Accademic Press, New York 1975.

[2] B. E. J. Dahlberg, *On the Poisson integral for Lipschitz and C^1-domains*, Stud. Math., 66 (1979), pp. 13-24 .

[3] E. B. Fabes, M. Jodeit - N. M. Riviére, *Potential techniques for boundary value problems in C^1 domains*, Acta Math., 141 (1978), pp. 165-186.

[4] D. Jerison, C. E. Kenig, *The inhomogeneus Dirichlet problem in Lipschitz domains*, Journal of Functional Analysis, 130 (1995), pp. 161-219.

[5] C. E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, American Mathematical Society, Providence, Rhode Island (1994).

[6] J. Král, *Integral Operators in Potential Theory*, Lecture Notes in Mathematics, 823. Springer-Verlag, Berlin (1980).

[7] V. G. Maz'ya, *Boundary Integral Equations*, Analysis IV. Encyclopaedia of Mathematical Sciences 27, Springer-Verlag, New York (1991) pp. 127-222.

[8] D. Medková, *The third problem for the Laplace equation with a boundary condition from L_p*, Integr. Equ. Oper. Theory (in print).

[9] D. Medková, *Conditions ensuring $T^{-1}(Y) \subset Y$*, Extracta Mathematicae, 20 (2005), pp. 43-50.

[10] D. Mitrea, I. Mitrea, *On the Besov regularity of conformal maps and layer potentials on nonsmooth domains*, Journal of Functional Analysis, 201 (2003), pp. 380-429.

[11] J. Nečas, *Les méthodes directes en théorie des équations élliptiques*, Academia, Prague 1967.

[12] M. Schechter, *Principles of Functional Analysis*, American Mathematical Society, Providence, Rhode Island (2002).

[13] Ch. G. Simader, *The weak Dirichlet and Neumann problem for the Laplacian in L^q for bounded and exterior domains*, Applications Nonlinear analysis, function spaces and applications, Vol. 4, Proc. Spring Sch., Roudnice nad Labem (Czech, 1990), pp. 180-223, Teubner-Texte Math. 119, (1990).

[14] G. Verchota *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, Journal of Functional Analysis, 59 (1984), pp. 572-611.

Mathematical Institute
of the Academy of Sciences
of the Czech Republic
Žitná 25 - 115 67 Praha 1
Czech Republic
email: medkova@math.cas.cz