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REGULARITY OF SOLUTIONS TO NONLINEAR EQUATIONS OF SCHRÖDINGER TYPE

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Abstract. Regularity and local regularity of solutions to nonlinear equations of Schrödinger type are studied.

In Sjögren and Sjölin [5] we studied the local regularity of solutions to the equation $i\partial_t u = -Pu + Vu$. Here $u = u(x, t)$ where $x \in \boldsymbol{R}^n$ and $t \in \boldsymbol{R}$, P is an elliptic constant-coefficient differential operator in x, and $V = V(x)$ a suitable potential. We assume that $u(x, 0) = f(x)$ and that f belongs to some Sobolv space $H_s = H_s(\boldsymbol{R}^n)$. To formulate the results we introduce the class

$$\mathscr{A} = \{\varphi \in C^\infty(\boldsymbol{R}^n)\,;\ \text{there exists } \varepsilon > 0 \text{ such that } |D^\alpha \varphi(x)| \le C_\alpha (1 + |x|)^{-1/2-\varepsilon} \text{ for every } \alpha\}$$

and set $I = [0, T]$ where $T > 0$. In the special case when $P = \Delta^k$, $k = 1, 2, 3, \dots,$ it follows from the results in [5] that

$$\|\varphi u\|_{L^2(I; H_{s+k-1/2}(\boldsymbol{R}^n))} \le C_T \|f\|_{H_s}, \qquad s \ge 1/2 - k, \tag{1}$$

where C_T depends on φ and φu stands for $\varphi(x)u(x, t)$.

Kato [2], [3] has studied the existence and regularity of solutions to the non-linear equation

$$i\partial_t u = -\Delta u + F(u), \qquad x \in \boldsymbol{R}^n, \quad t \ge 0, \tag{2}$$

and in Sjölin [6] we obtained results about the local regularity of these solutions.

We shall study here the equation

$$i\partial_t u = -\Delta^k u + F(u), \qquad k = 1, 2, 3, \dots. \tag{3}$$

To formulate the conditions of F we introduce a parameter γ satisfying $1 < \gamma < \infty$ for $n = 1$ and 2, and $1 < \gamma < (n+2)/(n-2)$ for $n \ge 3$. We assume that $F \in C^1(\boldsymbol{R}^2) = C^1(\boldsymbol{C})$, F is complex-valued, $F(0) = 0$ and

$$|D^\alpha F(\zeta)| \le C|\zeta|^{\gamma-1} \tag{4}$$

for $|\zeta| \ge 1$ and $|\alpha| = 1$. An example is $F(\zeta) = |\zeta|^{\gamma-1}\zeta$.

We also introduce the spaces $L^{p,r} = L^r(I; L^p(\boldsymbol{R}^n))$, $1 \le p \le \infty$, $1 \le r \le \infty$, and let L^p_s

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denote Bessel potential spaces for $1 \leq p \leq \infty$ and $s \in \boldsymbol{R}$. Hence $L_s^p = J_s L^p$, where J_s is the Bessel potential operator, defined by multiplication on the Fourier transform side by $(1+|\xi|^2)^{-s/2}$. In particular $H_s = L_s^2$. We also set $L_s^{p,r} = L^r(I; L_s^p(\boldsymbol{R}^n))$ for $1 \leq p \leq \infty$, $1 \leq r \leq \infty$ and $s \in \boldsymbol{R}$. We write $u(t) = u(\cdot, t)$ and use the notation $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ and $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$.

We shall prove the following result.

THEOREM. *Assume that* $f \in H_1(\boldsymbol{R}^n)$. *Then there exists a* $T > 0$ *such that* (3) *has a solution* $u \in C(I; H_1)$ *with* $u(0) = f$. *The functions* u *and* ∂u *belong to* $L_s^{p+1,r}$, *where* $1 < p < \infty$ *for* $n = 1$ *and* 2, *and* $1 < p < (n+2)/(n-2)$ *for* $n \geq 3$, $r = 4(p+1)/n(p-1)$ *and* $s = 2(k-1)/r$. *The solution* u *is unique.*

Assume $\varphi \in \mathscr{A}$. *If* $k \geq 2$ *or if* $k = 1$, $1 \leq n \leq 6$, *then*

$$\varphi u \in L^2(I; H_{k+1/2}) = L_{k+1/2}^{2,2}. \tag{5}$$

If $k = 1$ *and* $n \geq 7$ *then* (5) *holds under the additional assumption* $\gamma < 1 + 2/(n-4)$.

In the case $k = 1$ the first part of the theorem is proved in [2] and [3], and in this case the second part about local regularity is partially contained in [6].

In the proof of the theorem we need two lemmas. We set $P = \Delta^k$ and write $P(\xi)$ for the corresponding symbol $(-1)^k |\xi|^{2k}$. Our first lemma is a consequence of estimates in Kenig, Ponce and Vega [4].

LEMMA 1. *Set* $u(t) = e^{itP} u_0$, $t \geq 0$. *For* $T > 0$ *we then have*

$$\| u \|_{L_s^{p+1,r}} \leq C_T \| u_0 \|_2, \tag{6}$$

where p, r *and* s *are as in the theorem. Also*

$$\| u(t) \|_{L_s^{2/(1-\theta)}(\boldsymbol{R}^n)} \leq C_T |t|^{-\theta n/2} \| u_0 \|_{2/(1+\theta)}, \qquad 0 \leq t \leq T, \tag{7}$$

where $0 \leq \theta \leq 1$ *and* $s = n(k-1)\theta$.

PROOF. We set

$$V_s(t) u_0(x) = \int e^{i(tP(\xi) + x \cdot \xi)} |\xi|^s \hat{u}_0(\xi)\, d\xi .$$

It is proved in [4] that

$$\| V_s(t) u_0 \|_{L^r(\boldsymbol{R}; L^{p+1}(\boldsymbol{R}^n))} \leq C \| u_0 \|_2, \tag{8}$$

where p, r and s are as above. To obtain (6) we shall estimate

$$J_{-s} u(t)(x) = c \int e^{i(tP(\xi) + x \cdot \xi)} (1 + |\xi|^2)^{s/2} \hat{u}_0(\xi)\, d\xi .$$

We choose $\psi \in C_0^\infty(\boldsymbol{R}^n)$ so that $\psi(x) = 0$ for $|x| > 2$, and $\psi(x) = 1$ for $|x| \leq 1$. One then has

$$
\begin{aligned}
J_{-s}u(t)(x) &= c\int e^{i(tP(\xi)+x\cdot\xi)}\psi(\xi)(1+|\xi|^2)^{s/2}\hat{u}_0(\xi)\,d\xi \\
&\quad + c\int e^{i(tP(\xi)+x\cdot\xi)}(1-\psi(\xi))(1+|\xi|^2)^{s/2}\hat{u}_0(\xi)\,d\xi \\
&= A(x,t)+B(x,t)\,.
\end{aligned}
$$

It is clear that

$$|A(x,t)|\leq C\int_{|\xi|\leq 2}|\hat{u}_0(\xi)|\,d\xi\leq C\|u_0\|_2$$

and from Plancherel's theorem it also follows that

$$\left(\int|A(x,t)|^2dx\right)^{1/2}\leq C\|u_0\|_2\,.$$

We conclude that

$$\|A(t)\|_{L^{p+1}(\mathbf{R}^n)}\leq C\|u_0\|_2$$

and hence

$$\|A\|_{L^r(I;L^{p+1})}\leq C_T\|u_0\|_2\,. \tag{9}$$

We have

$$B(x,t)=c\int e^{i(tP(\xi)+x\cdot\xi)}(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s}|\xi|^s\hat{u}_0(\xi)\,d\xi \tag{10}$$

and since

$$(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s}$$

is bounded, (8) shows that

$$\|B\|_{L^r(I;L^{p+1}(\mathbf{R}^n))}\leq C\|u_0\|_2\,. \tag{11}$$

The inequality (6) is then a consequence of (9) and (11).

To prove (7) we then set $s=n(k-1)\theta$, where $0\leq\theta\leq 1$. We write $J_{-s}u(t)=A(t)+B(t)$ as above and it then follows from the Hausdorff-Young theorem and Hölder's inequality that

$$\|A(t)\|_{2/(1-\theta)}\leq C\|\psi\hat{u}_0\|_{2/(1+\theta)}\leq C\|\psi\hat{u}_0\|_{2/(1-\theta)}\leq C\|\hat{u}_0\|_{2/(1-\theta)}\leq C\|u_0\|_{2/(1+\theta)}\,. \tag{12}$$

To study B we use the formula (10) again. It follows from the results in [4] that

$$\|B(t)\|_{2/(1-\theta)}\leq C|t|^{-\theta n/2}\|v_0\|_{2/(1+\theta)}\,,$$

where

$$\hat{v}_0(\xi)=(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s}\hat{u}_0(\xi).$$

We want to prove that

$$\| v_0 \|_{2/(1+\theta)} \leq C \| u_0 \|_{2/(1+\theta)}, \tag{13}$$

which follows if we can prove that

$$(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s} \in M_{2/(1+\theta)}(\boldsymbol{R}^n), \tag{14}$$

where $M_q(\boldsymbol{R}^n)$ denotes the space of Fourier multipliers for $L^q(\boldsymbol{R}^n)$. For $0 \leq \theta < 1$ (14) is a consequence of the Hörmander-Mihlin multiplier theorem, and for $\theta = 1$ one can argue as follows. We have $s = n(k-1)$ and have to prove that

$$(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s} \in M_1(\boldsymbol{R}^n). \tag{15}$$

The case $k=1$ is trivial and we may therefore assume $k \geq 2$. According to Stein [7, p. 133], one has

$$(1+|\xi|^2)^{s/2} = \hat{v}(\xi) + |\xi|^s \hat{\lambda}(\xi),$$

where v and λ denote finite Borel measures. Hence

$$(1-\psi(\xi))\frac{(1+|\xi|^2)^{s/2}}{|\xi|^s} = (1-\psi)\frac{\hat{v}(\xi)}{|\xi|^s} + (1-\psi)\hat{\lambda}(\xi).$$

Setting $g = (1-\psi)|\xi|^{-s}$ it is easy to see that g and $D^\alpha g$ belong to L^2 for every α and hence $\hat{g} \in L^1$. We conclude that (15) holds and hence (13) is proved for all θ. It follows that

$$\| B(t) \|_{2/(1-\theta)} \leq C |t|^{-\theta n/2} \| u_0 \|_{2/(1+\theta)}.$$

Hence

$$\| J_{-s}u(t) \|_{2/(1-\theta)} \leq C(1+|t|^{-\theta n/2}) \| u_0 \|_{2/(1+\theta)} \leq C_T |t|^{-\theta n/2} \| u_0 \|_{2/(1+\theta)}, \qquad 0 < t \leq T,$$

and the lemma is proved.

In the following lemma we shall use the notation

$$(G_0 f)(t) = e^{itP} f \quad \text{and} \quad (Gv)(t) = \int_0^t e^{i(t-s)P} v(s)\, ds.$$

LEMMA 2. *G_0 and G have the properties*

(16) $$\| G_0 f \|_{L^{2,\infty}} \le C_T \| f \|_2 ,$$

(17) $$\| G_0 f \|_{L_s^{p+1,r}} \le C_T \| f \|_2 ,$$

(18) $$\| Gv \|_{L^{2,\infty}} \le C_T \| v \|_{L^{2,1}} ,$$

(19) $$\| Gv \|_{L_s^{p+1,r}} \le C_T \| v \|_{L^{2,1}} ,$$

(20) $$\| Gv \|_{L^{2,\infty}} \le C_T \| v \|_{L_{-s}^{1+1/p,r'}}$$

and

(21) $$\| Gv \|_{L_s^{p+1,r}} \le C_T \| v \|_{L_{-s}^{1+1/p,r'}} ,$$

where p, r and s are as in the theorem. The constant C_T has the property that $\sup_{0<T\le A} C_T < \infty$ for every $A>0$.

PROOF. The lemma is well-known for $k=1$ (see [2] and [3]) and essentially the same proof works for $k\ge 2$ if we use the estimates in Lemma 1.

It is clear that (16) is trivial and (17) follows from (6) in Lemma 1. The estimate (18) is a consequence of (16).

To prove (19) we observe that

$$\| (Gv)(t) \|_{L_s^{p+1}(\mathbf{R}^n)} \le \int_0^T \| e^{i(t-t_1)P} v(t_1) \|_{L_s^{p+1}(\mathbf{R}^n)} dt_1 ,$$

and

$$\| Gv \|_{L_s^{p+1,r}} \le \int_0^T \| e^{itP} e^{-it_1P} v(t_1) \|_{L_s^{p+1,r}} dt_1 \le C_T \int_0^T \| e^{-it_1P} v(t_1) \|_2 dt_1 = C_T \| v \|_{L^{2,1}} ,$$

where we have used (17).

To prove (21) we observe that it follows from Lemma 1 that

$$\| u(t) \|_{L_s^{2/(1-\theta)}} \le C_T |t|^{-\theta n/2} \| u_0 \|_{L_{-s}^{2/(1+\theta)}} , \qquad 0\le t\le T, \quad 0\le\theta\le 1 ,$$

where $s=n(k-1)\theta/2$. We set $p+1=2/(1-\theta)$ so that $\theta=(p-1)/(p+1)$ where $0<\theta<1$. One then also has

$$\frac{2}{1+\theta} = 1 + \frac{1}{p}$$

and

$$s = \frac{1}{2} n(k-1) \frac{p-1}{p+1} = (k-1)\frac{2}{r} .$$

The above estimate therefore gives

$$\| (Gv)(t) \|_{L_s^{p+1}(\mathbf{R}^n)} \leq \int_0^t \| e^{i(t-t_1)P} v(t_1) \|_{L_s^{p+1}(\mathbf{R}^n)} dt_1$$

$$\leq C_T \int_0^t |t-t_1|^{-\theta n/2} \| v(t_1) \|_{L_{-s}^{1+1/p}} dt_1 , \qquad 0 \leq t \leq T .$$

We have

$$\frac{1}{r'} - \frac{1}{r} = 1 - \frac{\theta n}{2}$$

and (21) now follows if we invoke Hardy's inequality.

Finally (20) can be proved as in the proof in the case $k=1$ in [3, Lemma 3.2].

We remark that it is easy to see that in (16), (18) and (20) $L^{2,\infty}$ can be replaced by $C(I; L^2)$.

PROOF OF THE THEOREM. To prove the first part of the theorem we shall generalize the proof in the case $k=1$ in [2].

We set

$$r = r(\gamma) = \frac{4(\gamma+1)}{n(\gamma-1)}, \quad s = s(\gamma) = (k-1)\frac{2}{r}$$

and introduce the following spaces:

$$X = L^{2,\infty} \cap L_s^{\gamma+1,r}, \quad \bar{X} = C(I; L^2) \cap L_s^{\gamma+1,r}, \quad X' = L^{2,1} + L_{-s}^{1+1/\gamma, r'},$$

$$Y = \{v \in X; \partial v \in X\}, \quad \bar{Y} = \{v \in \bar{X}; \partial v \in \bar{X}\}, \quad Y' = \{v \in X'; \partial v \in X'\} .$$

It then follows from Lemma 2 that

$$\| G_0 f \|_{\bar{X}} \leq C_T \| f \|_2 , \tag{22}$$

$$\| G_0 f \|_{\bar{Y}} \leq C_T \| f \|_{H_1} , \tag{23}$$

$$\| Gv \|_{\bar{X}} \leq C_T \| v \|_{X'} \tag{24}$$

and

$$\| Gv \|_{\bar{Y}} \leq C_T \| v \|_{Y'} . \tag{25}$$

It also follows from Lemma 2.2 in [2] that F maps Y into Y' and

$$\| F(v) \|_{Y'} \leq C(T + T^{1-\alpha} \| v \|_Y^{\gamma-1}) \| v \|_Y ,$$

where $0 < \alpha < 1$. Hence there exists a number β, $0 < \beta < 1$, such that

$$\| F(v) \|_{Y'} \leq C T^{\beta} (\| v \|_Y + \| v \|_Y^{\gamma}) \tag{26}$$

for $0 < T < 1$.

We now fix $f \in H_1(\mathbf{R}^n)$ and set $\Phi(v) = G_0 f - iGF(v)$, $v \in Y$. It follows from the above estimates that

$$\| GF(v) \|_Y \leq C_T \| F(v) \|_{Y'} \leq C_T T^\beta (\| v \|_Y + \| v \|_Y^\gamma) .$$

We set $B_R(Y) = \{v \in Y \colon \| v \|_Y \leq R\}$ and choose $R > 1$ and $v \in B_R(Y)$. Then

$$\| \Phi(v) \|_Y \leq C_T \| f \|_{H_1} + C_T T^\beta R^\gamma .$$

We now choose $R > C' \| f \|_{H_1}$, where $C' = \sup_{0 < T \leq 1} C_T$, and then choose T so small that

$$C' \| f \|_{H_1} + C' T^\beta R^\gamma < R .$$

It follows that Φ maps $B_R(Y)$ into $B_R(Y)$.

If v and $w \in B_R(Y)$ it follows from [2, p. 117], that

$$\| F(v) - F(w) \|_{X'} \leq C(R) T^\beta \| v - w \|_X ,$$

where $0 < \beta < 1$. Invoking (24) we obtain

$$\| GF(v) - GF(w) \|_X \leq d \| v - w \|_X ,$$

where $0 < d < 1$, if T is small enough.

It is easy to prove that $B_R(Y)$ with the X-metric is a complete metric space and it follows that Φ is a contraction on this space. Invoking the contraction theorem we find that Φ has a fixed point $u \in Y$ and that $u = \Phi(u) \in \bar{Y}$. Hence

$$u = G_0 f - iGF(u) \tag{27}$$

and $u(0) = f$. It follows from (27) that u satisfies the equation (3). We remark that in proving the equivalence of (27) and (3) it is useful to observe that $F(u) \in C(I; H_{-1})$, which can be proved by use of the implications

$$u(t) \in H_1 \Rightarrow u(t) \in L^2 \cap L^{\gamma+1} \Rightarrow F(u(t)) \in L^2 + L^{1+1/\gamma} \subset H_{-1}$$

(see [2, Lemma 1.3 and its proof]).

To prove that u is unique assume that v is another solution of (3) with $v(0) = f$, $v \in \bar{Y}$. It follows that

$$v = G_0 f - iGF(v) \quad \text{and} \quad u - v = -i(GF(u) - GF(v)) .$$

An application of the contraction property of GF then shows that $u = v$.

We have thus found a unique solution $u \in \bar{Y}$ of (3) with $u(0) = f$. It follows that $u \in C(I; H_1)$ and that u and $\partial u \in L_{s(\gamma)}^{\gamma+1, r(\gamma)}$. We shall now prove that u and ∂u also belong to $L_s^{p+1,r}$, where p, r and s satisfy the conditions in the theorem. For $1 < p < \gamma$ this follows from the properties of the spaces $L_s^{p+1,r}$ (see Bergh and Löfström [1, pp. 107 and 153]). For $p > \gamma$ we can simply use the fact that

$$| D^\alpha F(\zeta) | \leq C | \zeta |^{\gamma-1} \quad \text{implies} \quad | D^\alpha F(\zeta) | \leq C | \zeta |^{p-1}$$

($|\zeta|\geq 1$) and we can apply the above result with γ replaced by p.

It remains to prove the local regularity (5). We first choose $\psi\in C_0^\infty(\boldsymbol{R}^2)$ so that $\psi=1$ in a neighbourhood of the origin. Set $F_1=\psi F$ and $F_2=(1-\psi)F$ so that $F=F_1+F_2$. The proof of Lemma 2.2 in [2] shows that

$$F_1(u) \quad \text{and} \quad \partial(F_1(u))\in L^{2,1} \tag{28}$$

and

$$F_2(u) \quad \text{and} \quad \partial(F_2(u))\in L^{1+1/\gamma,r(\gamma)'}\,. \tag{29}$$

We have

$$u(t)=e^{itP}f-i\int_0^t e^{i(t-\tau)P}F(u(\tau))\,d\tau$$

and choosing $\varphi\in\mathscr{A}$ we obtain

$$\|\varphi u(t)\|_{H_{k+1/2}}\leq\|\varphi e^{itP}f\|_{H_{k+1/2}}+\int_0^t\|\varphi e^{i(t-\tau)P}F(u(\tau))\|_{H_{k+1/2}}d\tau\,.$$

Hence

$$\|\varphi u\|_{L^2(I;H_{k+1/2})}\leq\|\varphi e^{itP}f\|_{L^2(I;H_{k+1/2})}+\int_0^T\left(\int_0^T\|\varphi e^{itP}e^{-i\tau P}F(u(\tau))\|^2_{H_{k+1/2}}dt\right)^{1/2}d\tau\,.$$

Invoking the estimate (1) we then get

$$\|\varphi u\|_{L^2(I;H_{k+1/2})}\leq C\|f\|_{H_1}+C\int_I\|F(u(t))\|_{H_1}dt\,.$$

To prove (5) it is therefore sufficient to prove that $F(u)\in L^1(I;H_1)$. We have $F(u)=F_1(u)+F_2(u)$ and it follows from (28) that $F_1(u)\in L^1(I;H_1)$. Furthermore

$$F_2(u)\in L_1^{1+1/\gamma,r(\gamma)'}\subset L_1^{1+1/\gamma,1}\subset L^{2,1}$$

and it remains to prove that

$$\partial(F_2(u))\in L^1(I;L^2)\,. \tag{30}$$

We shall use the estimate

$$|\partial(F_2(u))|\leq C|u|^{\gamma-1}|\partial u| \tag{31}$$

(see [6, p. 149]).

In proving (30) we first assume $k=1$. Using Hölder's inequality we obtain

$$\int_{\boldsymbol{R}^n}|\partial(F_2(u))|^2dx\leq C\int_{\boldsymbol{R}^n}|u|^{2\gamma-2}|\partial u|^2dx \tag{32}$$

$$\leq C\left(\int |u|^{(2\gamma-2)\alpha} dx\right)^{1/\alpha}\left(\int |\partial u|^{\gamma+1} dx\right)^{2/(\gamma+1)},$$

where

$$\frac{2}{\gamma+1}+\frac{1}{\alpha}=1$$

and thus $\alpha=(\gamma+1)/(\gamma-1)$.

We now first consider the case $n=1$ or 2. We have

$$\| u \|_{2\gamma+2} \leq C \| u \|_{L_1^2}$$

since

$$\frac{1}{2\gamma+2} \geq \frac{1}{2}-\frac{1}{n},$$

and it follows from (32) that

$$\| \partial(F_2(u)) \|_2 \leq C\left(\int |u|^{2\gamma+2} dx\right)^{(\gamma-1)/2(\gamma+1)} \| \partial u \|_{\gamma+1}$$
$$\leq C \| u \|_{L_1^2}^{\gamma-1} \| \partial u \|_{\gamma+1} \leq C_u \| \partial u \|_{\gamma+1},$$

where we have used the fact that $u \in C(I; H_1)$. Now (30) follows since $\partial u \in L^{\gamma+1, r(\gamma)}$.

We then consider the case $3 \leq n \leq 5$. We have $\gamma<(n+2)/(n-2)$ and $r=4(\gamma+1)/n(\gamma-1)$ and we may assume that γ is close to $(n+2)/(n-2)$. Setting

$$p=\frac{2\gamma(n-1)+n-2}{n+2+2\gamma},$$

we observe that since γ is close to $(n+2)/(n-2)$, p is close to

$$\frac{2(n+2)(n-1)/(n-2)+n-2}{n+2+2(n+2)/(n-2)}=\frac{3n-2}{n+2}.$$

We have

$$1<\frac{3n-2}{n+2}<\frac{n+2}{n-2}$$

and it follows that

$$1<p<\frac{n+2}{n-2}.$$

From the definition of p we conclude that

$$p+1=\frac{2n(\gamma+1)}{n+2+2\gamma}$$

and

$$\frac{1}{p+1}-\frac{1}{n}=\frac{n+2+2\gamma}{2n(\gamma+1)}-\frac{1}{n}=\frac{1}{2\gamma+2}.$$

We have $u\in L_1^{p+1,r_1}$, where $r_1=4(p+1)/n(p-1)$, and it follows from Sobolev's theorem that $u\in L^{2\gamma+2,r_1}$.

From (32) we conclude that

$$\|\partial(F_2(u))\|_2\leq C\|u\|_{2\gamma+2}^{\gamma-1}\|\partial u\|_{\gamma+1} \tag{33}$$

and hence

$$\|\partial(F_2(u))\|_{L^{2,1}}\leq C\int_I\|u\|_{2\gamma+2}^{\gamma-1}\|\partial u\|_{\gamma+1}dt\leq C\left(\int_I\|u\|_{2\gamma+2}^{(\gamma-1)r'}dt\right)^{1/r'}\left(\int_I\|\partial u\|_{\gamma+1}^{r}dt\right)^{1/r}.$$

Since $\partial u\in L^{\gamma+1,r}$ and $u\in L^{2\gamma+2,r_1}$ the above right hand side is finite if $(\gamma-1)r'\leq r_1$. To show this we shall prove that

$$\frac{1}{r_1}-\frac{1}{(\gamma-1)r'}\leq 0. \tag{34}$$

We have

$$\begin{aligned}\frac{1}{r_1}-\frac{1}{(\gamma-1)r'}&=\frac{n(p-1)}{4(p+1)}-\frac{1}{\gamma-1}\left(1-\frac{1}{r}\right)=\frac{n}{4}\left(1-\frac{2}{p+1}\right)-\frac{1}{\gamma-1}+\frac{n}{4(\gamma+1)}\\&=\frac{n}{4}-\frac{n+2+2\gamma}{4(\gamma+1)}-\frac{1}{\gamma-1}+\frac{n}{4(\gamma+1)}=\frac{n-2}{4}-\frac{1}{\gamma-1}\\&=\frac{(n-2)\gamma-n-2}{4(\gamma-1)}=\frac{(n-2)(\gamma-(n+2)/(n-2))}{4(\gamma-1)},\end{aligned}$$

and since the right hand side is negative we have proved (34) and (30).

We then assume $n\geq 6$. One has

$$\int|\partial(F_2(u))|^2\,dx\leq C\int|u|^{2\gamma-2}|\partial u|^2\,dx$$

and we assume $\gamma<1+2/(n-4)$ and that γ is close to $1+2/(n-4)$. We remark that $1+2/(n-4)\leq(n+2)/(n-2)$ with equality for $n=6$. We shall choose p such that $\gamma<p<(n+2)/(n-2)$ and use the fact that $u\in L_1^{p+1,r}$, where $r=4(p+1)/n(p-1)$.

Using Hölder's inequality one obtains

$$\|\partial(F_2(u))\|_2\leq C\|u\|_{2(\gamma-1)(p+1)/(p-1)}^{\gamma-1}\|\partial u\|_{p+1}. \tag{35}$$

Now assume that we can choose p so that

$$\frac{1}{p+1} \geq \frac{p-1}{2(\gamma-1)(p+1)} \geq \frac{1}{p+1} - \frac{1}{n}. \tag{36}$$

Then

$$\| u \|_{2(\gamma-1)(p+1)/(p-1)} \leq C \| u \|_{L_1^{p+1}}$$

and it follows from (35) that

$$\| \partial(F_2(u)) \|_2 \leq C \| u \|_{L_1^{p+1}}^{\gamma} \quad \text{and} \quad \| \partial(F_2(u)) \|_{L^{2,1}} \leq C \int_I \| u \|_{L_1^{p+1}}^{\gamma} dt .$$

However, the above right hand side is finite since $\gamma < 2 \leq r$.

It remains to prove that the above choice of p is possible. The right hand side inequality in (36) is equivalent to

$$\frac{p-1}{2(\gamma-1)} \geq 1 - \frac{p-1}{n}$$

and to

$$p\left(\frac{1}{2(\gamma-1)} + \frac{1}{n}\right) - \frac{1}{2(\gamma-1)} \geq 1 - \frac{1}{n}.$$

Thus we can find a suitable p by choosing p close to $(n+2)/(n-2)$ if

$$\frac{n+2}{n-2}\left(\frac{1}{2(\gamma-1)} + \frac{1}{n}\right) - \frac{1}{2(\gamma-1)} > 1 - \frac{1}{n}.$$

This inequality is equivalent to

$$\frac{1}{2(\gamma-1)}\left(\frac{n+2}{n-2} - 1\right) + \frac{n+2}{n(n-2)} > 1 - \frac{1}{n}$$

and to

$$\frac{2}{\gamma-1} > n-4,$$

which holds since $\gamma < 1 + 2/(n-4)$.

The left hand side inequality in (36) is equivalent to $2(\gamma-1) \geq p-1$, which is easily seen to be true if p is chosen close to $(n+2)/(n-2)$. Thus (30) is proved also in the case $n \geq 6$.

We shall then study the case $k \geq 2$. The above argument for $k=1$ clearly works also in the case $k \geq 2$. Thus it only remains to prove (30) in the case $k \geq 2$ and $n \geq 7$. In fact, in the following proof it is sufficient to assume $n \geq 5$.

We start from the estimate

$$\int |\partial(F_2(u))|^2 \, dx \leq C \int |u|^{2\gamma-2} |\partial u|^2 \, dx \tag{37}$$

and define q by

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{n} .$$

It then follows that $q = 2n/(n-2)$ and

$$\| u(t) \|_q \leq C \| u(t) \|_{L_1^2} . \tag{38}$$

We have

$$2\gamma - 2 < 2\frac{n+2}{n-2} - 2 = \frac{8}{n-2} < q ,$$

since $n \geq 5$, and we set $\alpha_1 = q/(2\gamma-2) = n/(n-2)(\gamma-1)$. Also define α_2 by

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1 .$$

From (37), (38) and the fact that $u \in C(I; H_1)$ we obtain

$$\int |\partial(F_2(u))|^2 \, dx \leq C\left(\int |u|^q \, dx\right)^{1/\alpha_1} \left(\int |\partial u|^{2\alpha_2} \, dx\right)^{1/\alpha_2}$$

and

$$\| \partial(F_2(u)) \|_2 \leq C_u \| \partial u \|_{2\alpha_2} . \tag{39}$$

We have $\partial u \in L_s^{\gamma+1, r}$, where $r = r(\gamma)$, $s = s(\gamma)$ and we will obtain (30) from (39) if we can prove that

$$\| \partial u \|_{2\alpha_2} \leq C \| \partial u \|_{L_s^{\gamma+1}} . \tag{40}$$

To prove (40) it is sufficient to prove the inequality

$$\frac{1}{\gamma+1} \geq \frac{1}{2\alpha_2} \geq \frac{1}{\gamma+1} - \frac{s}{n} . \tag{41}$$

The right hand side inequality in (41) is equivalent to

$$\frac{s}{n} \geq \frac{1}{\gamma+1} - \frac{1}{2}\left(1 - \frac{1}{\alpha_1}\right) = \frac{1}{\gamma+1} - \frac{1}{2} + \frac{1}{2\alpha_1} ,$$

which gives

$$\frac{(k-1)(\gamma-1)}{2(\gamma+1)} \geq \frac{1}{\gamma+1} - \frac{1}{2} + \frac{(n-2)(\gamma-1)}{2n}$$

and

$$\frac{(k-1)(\gamma-1)n-2n+n(\gamma+1)-(n-2)(\gamma-1)(\gamma+1)}{2n(\gamma+1)} \geq 0 .$$

We may assume $k=2$ and the above numerator then equals

$$(2-n)\gamma^2+2n\gamma-n-2=(2-n)\left(\gamma^2-\frac{2n}{n-2}\gamma+\frac{n+2}{n-2}\right)=(2-n)(\gamma-1)\left(\gamma-\frac{n+2}{n-2}\right),$$

which is positive since $1<\gamma<(n+2)/(n-2)$.

The left hand side inequality in (41) leads in a similar way to the inequality

$$(n-2)\gamma^2-n\gamma+2\geq 0 .$$

However,

$$(n-2)\gamma^2-n\gamma+2=(n-2)\left(\gamma^2-\frac{n}{n-2}\ \gamma+\frac{2}{n-2}\right)=(n-2)(\gamma-1)\left(\gamma-\frac{2}{n-2}\right),$$

which is positive for $1<\gamma<(n+2)/(n-2)$. Hence (41) is proved and (40) and (30) follow. The proof of the theorem is complete.

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