Regularity of solutions to the parabolic fractional obstacle problem

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1 Introduction

In recent years, there has been an increasing interest in studying constrained variational problems with a fractional diffusion. One of the motivations comes from mathematical finance: jump-diffusion processes where incorporated by Merton [14] into the theory of option evaluation to introduce discontinuous paths in the dynamics of the stock's prices, in contrast with the classical lognormal diffusion model of Black and Scholes [2]. These models allow to take into account large price changes, and they have become increasingly popular for modeling market fluctuations, both for risk management and option pricing purposes.

Let us recall that an American option gives its holder the right to buy a stock at a given price prior (but not later) than a given time $T > 0$. If $v(\tau, x)$ represents the rational price of an American option with a payoff ψ at time $T > 0$, then v will solve (in the viscosity sense) the following obstacle problem:

$$\begin{cases} \min\{\mathcal{L}v, v - \psi\} = 0, \\ v(T) = \psi. \end{cases}$$

Here $\mathcal{L}v$ is a (backward) parabolic integro-differential operator of the form

$$\mathcal{L}v = -v_\tau - rv + \sum_{i=1}^n (r - d_i)x_i v_{x_i} - \frac{1}{2}\sum_{i,j=1}^n x_i x_j \sigma_{ij} v_{x_i x_j}$$
$$- \int \Big[v\big(\tau, x_1 e^{y_1}, \dots, x_n e^{y_n}\big) - v(\tau, x) - \sum_{i=1}^n (e^{y_i} - 1) x_i v_{x_i}(\tau, x) \Big] \mu(dy),$$

where $r > 0$, $d_i \in \mathbb{R}$, $\sigma = (\sigma_{ij})$ is a non-negative definite matrix, and μ is a jump measure. (We refer to the book [9] for an explanation of these models and more references.) When the matrix σ is uniformly elliptic, after the change of variable $x_i \mapsto \log(x_i)$ the equation becomes uniformly

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parabolic (backward in time) and the diffusion part dominates. In particular, if no jump part is present (i.e., $\mu \equiv 0$), then the regularity theory is pretty well-understood (see, for instance, [12]).

Here we assume that there is no diffusion (i.e., $\sigma \equiv 0$), so all the regularity should come from the jump part. We also assume that the jump part behaves, at least at the leading order, as a fractional power of the Laplacian, so that the equation takes the form

$$\mathcal{L}v = -v_\tau - rv - b \cdot \nabla u + (-\Delta)^s v + \mathcal{K}v, \qquad s \in (0,1), \tag{1.1}$$

where $b = (d_1 - r, \ldots, d_n - r)$, and $\mathcal{K}v$ is a non-local operator of lower order with respect to $(-\Delta)^s v$.

We now observe that the choice of $s \in (0,1)$ plays a key role:

- $s > 1/2$: In this case $(-\Delta)^s v$ is the leading term, so the regularity theory for solutions to (1.1) is expected to be the same one as that for the equation

$$\begin{cases} \min\{-v_\tau + (-\Delta)^s v, v - \psi\} = 0 \quad \text{on } [0,T] \times \mathbb{R}^n, \\ v(T) = \psi \quad \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

- $s \leq 1/2$: If $s < 1/2$ then the leading term becomes $b \cdot \nabla v$, and we do not expect to have a regularity theory for (1.1). On the other hand, in the borderline case $s = 1/2$ one may expect some regularity due to the interplay between $b \cdot \nabla v$ and $-(-\Delta)^s v$ (but this becomes a very delicate issue). However, when $b \equiv 0$, even if the diffusion term is of lower order with respect to the time derivative, the equation is still parabolic and one may hope to prove some regularity for all values of s.

The goal of this paper is to investigate the regularity theory for the model equation (1.2). The reason for this is three-fold: first of all, considering this model case allows to avoid technicalities which may obscure the main ideas behind the regularity theory that we will develop. Moreover, since there is no transport term inside the equation, we are able to prove that solutions are as smooth as in the elliptic case [6] *for all* values of $s \in (0,1)$. Hence, although when $s < 1/2$ the time derivative is of higher order with respect to the elliptic part $(-\Delta)^s v$, the regularity of solutions is as good as in the stationary case. Finally, as described in Section 5, once the general regularity theory for solutions of (1.2) is established, the adaptation of these proofs to the more general case (1.1) when $s > 1/2$ should not present any major difficulty.

Let us remark that the fact that the smoothness of solutions of (1.2) is the same as in the elliptic case may look surprising. Indeed, the optimal regularity for the stationary problem $\min\{(-\Delta)^s v, v - \psi\} = 0$ is $C^{1+s}_x(\mathbb{R}^n)$ [1, 15, 6]. On the other hand, as we will show in Remark 3.7, for any $\beta \in (0,1)$ one can find a traveling wave solution to the parabolic obstacle problem $\min\{-v_\tau + (-\Delta)^{1/2} v, v - \psi\} = 0$ which is $C^{1+\beta}$ both in space and time, but not $C^{1+\gamma}$ for any $\gamma > \beta$. Hence, in order to prove that solutions to (1.2) are C^{1+s} in space, one has to exploit the crucial fact that v coincides with the obstacle at time T.

2 Description of the results and structure of the paper

In this section we introduce more in detail the problem, and describe our main result.

Let us observe that, by performing the change of variable $t = T - \tau$, all equations introduced in the previous section become forward in time. From now on, we will always work with t in place of τ, so the payoff ψ becomes the initial condition at time 0.

2.1 Preliminary definitions

The fractional Laplacian can be defined as

$$(-\Delta)^s f := \widehat{|\xi|^{2s}\hat{f}} \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n),$$

so that $\int f(-\Delta)^s g = \langle f, g\rangle_{\dot{H}^s}$. There are also two other different ways to define $(-\Delta)^s$. The first one is through an integral kernel: there exists a positive constant $C_{n,s}$ such that

$$-(-\Delta)^s f(x) = C_{n,s} \int \frac{f(x') - f(x)}{|x' - x|^{n+2s}}\, dx' \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n),$$

where the integral has to be intended in the principal value sense. (This can be proved, for instance, by computing the Fourier transform of $|\xi|^{2s}$.) The second one is through a Dirichlet-to-Neumann operator, as shown in [7]: given $a \in (-1, 1)$, for any function $f \in C_c^\infty(\mathbb{R}^n)$ denote by $F : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ the L_a-harmonic extension of f, i.e.,

$$\begin{cases} L_a F(x,y) := \operatorname{div}_{x,y}\big(y^a \nabla_{x,y} F(x,y)\big) = 0 & \text{on } \mathbb{R}^n \times \mathbb{R}^+, \\ F(x,0) = f(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Then there exists a positive constant $c_{n,s}$ such that

$$\lim_{y\to 0^+} y^a F_y(x,y) = -c_{n,s}(-\Delta)^s f(x,0), \qquad s := \frac{1-a}{2} \in (0,1).$$

In the sequel, we will make use of all of the three above characterizations of the fractional Laplacian. However, in order to simplify the notation, we will conventionally assume that $C_{n,s} = c_{n,s} = 1$, so that

$$-(-\Delta)^s f = \int \frac{f(x') - f(x)}{|x' - x|^{n+2s}}\, dx' = \lim_{y\to 0^+} y^a F_y(x,y). \tag{2.1}$$

We will also need the notion of semiconvex function: a function $w : \mathbb{R}^n \to \mathbb{R}$ is said to be C-semiconvex for some constant $C \in \mathbb{R}$ if $w + C|x|^2/2$ is convex.

Finally, to measure the regularity of the solutions we will use space-time Hölder, Lipschitz, and logLipschitz spaces: given $\alpha, \beta, \gamma, \delta \in (0,1)$, and for any interval $[a,b] \subset \mathbb{R}$, we say that:
$w \in C^{\alpha,\beta}_{t,x}([a,b] \times \mathbb{R}^n)$ if

$$\begin{aligned} \|w\|_{C^{\alpha,\beta}_{t,x}([a,b]\times\mathbb{R}^n)} &:= \|w\|_{L^\infty([a,b]\times\mathbb{R}^n)} + [w]_{C^{\alpha,\beta}_{t,x}([a,b]\times\mathbb{R}^n)} \\ &= \|w\|_{L^\infty([a,b]\times\mathbb{R}^n)} + \sup_{[a,b]\times\mathbb{R}^n} \frac{|w(t,x) - w(t',x')|}{|t-t'|^\alpha + |x-x'|^\beta} < +\infty; \end{aligned}$$

$w \in \mathrm{Lip}_t C_x^\beta([a,b] \times \mathbb{R}^n)$ if

$$\|w\|_{\mathrm{Lip}_t C_x^\beta([a,b]\times\mathbb{R}^n)} := \|w\|_{L^\infty([a,b]\times\mathbb{R}^n)} + \sup_{[a,b]\times\mathbb{R}^n} \frac{|w(t,x)-w(t',x')|}{|t-t'|+|x-x'|^\beta} < +\infty;$$

$w \in \mathrm{logLip}_t C_x^\beta([a,b] \times \mathbb{R}^n)$ if

$$\|w\|_{\mathrm{logLip}_t C_x^\beta([a,b]\times\mathbb{R}^n)} := \|w\|_{L^\infty([a,b]\times\mathbb{R}^n)} + \sup_{[a,b]\times\mathbb{R}^n} \frac{|w(t,x)-w(t',x')|}{|t-t'|\big(1+\big|\log|t-t'|\big|\big)+|x-x'|^\beta} < +\infty.$$

We will also use the notation $w \in C_{t,x}^{\alpha-0^+,\beta}([a,b]\times\mathbb{R}^n)$ if

$$w \in C_{t,x}^{\alpha-\varepsilon,\beta}([a,b]\times\mathbb{R}^n) \qquad \forall\, \varepsilon > 0,$$

and $w \in C_{t,x}^{\alpha,\beta}((a,b]\times\mathbb{R}^n)$ if

$$w \in C_{t,x}^{\alpha,\beta}([a+\varepsilon,b]\times\mathbb{R}^n) \qquad \forall\, \varepsilon > 0$$

(analogous definitions hold for the other spaces).

2.2 The main result

Let $\psi : \mathbb{R}^n \to \mathbb{R}^+$ be a globally Lipschitz function of class C^2 satisfying $\int_{\mathbb{R}^n} \frac{|\psi|}{(1+|x|)^{n+2s}} < +\infty$ and $(-\Delta)^s\psi \in L^\infty(\mathbb{R}^n)$. Fix $s \in (0,1)$, and let $u : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ be a (continuous) viscosity solution to the obstacle problem

$$\begin{cases} \min\{u_t + (-\Delta)^s u, u-\psi\} = 0 \quad \text{on } [0,T]\times\mathbb{R}^n, \\ u(0) = \psi \quad \text{on } \mathbb{R}^n. \end{cases} \tag{2.2}$$

Existence and uniqueness of such a solution follows by standard results on obstacle problems[1]. The main goal of this paper is to investigate the smoothness of solutions to the above equations, planning to address in a future work the regularity of the free boundary.

Our main result is the following:

Theorem 2.1. *Assume that $\psi \in C^2(\mathbb{R}^n)$, with*

$$\|\nabla\psi\|_{L^\infty(\mathbb{R}^n)} + \|D^2\psi\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)} < +\infty,$$

and let u be the unique continuous viscosity solution of (2.2). *Then u is globally Lipschitz in space-time on $[0,T]\times\mathbb{R}^n$, and satisfies*

$$\begin{cases} u_t \in \mathrm{logLip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n), & (-\Delta)^s u \in \mathrm{logLip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n) \quad \text{if } s \le 1/3; \\ u_t \in C_{t,x}^{\frac{1-s}{2s}-0^+,1-s}((0,T]\times\mathbb{R}^n), & (-\Delta)^s u \in C_{t,x}^{\frac{1-s}{2s},1-s}((0,T]\times\mathbb{R}^n) \qquad \text{if } s > 1/3. \end{cases} \tag{2.3}$$

[1] Here, existence of solutions is not the main issue: for instance, one can construct solutions by using probabilistic formulas involving stochastic processes and stopping times [9]. Another possibility is to approximate the equation using a penalization method (as done in the proof of Lemma 3.1) and then use the a priori bounds on the approximate solutions (see the proofs of Lemmas 3.2 and 3.3) to show existence by compactness. The fact that these two notions of solutions (the probabilistic one and the one constructed by approximation) coincide, follows from standard comparisons principle for viscosity solutions.

Let us make some comments. First of all we recall that, for the stationary version of the obstacle problem, solutions belong to $C_x^{1+s}(\mathbb{R}^n)$ (or equivalently, $(-\Delta)^s u \in C_x^{1-s}(\mathbb{R}^n)$), and such a regularity result is optimal [15, 6]. Hence, at least concerning the spatial regularity, our result is optimal, too.

Once the C_x^{1-s}-regularity of $(-\Delta)^s u$ is established, the fact that $s = 1/3$ plays a special role is not surprising: indeed, the operator $\partial_t + (-\Delta)^s$ is invariant under the scaling $(t, x) \mapsto (\lambda^{2s} t, \lambda x)$. Hence, a spatial regularity C_x^{1-s} naturally corresponds to a time regularity $C_t^{\frac{1-s}{2s}}$, provided $\frac{1-s}{2s} < 1$, that is, $s > 1/3$ (see (A.3)-(A.4) in the Appendix).

Finally, concerning the regularity in time, when $s = 1/2$ one can construct traveling wave solutions which are $C^{1+1/2}$ both in space and time, see Remark 3.7. Hence our result is almost optimal in time, at least when $s = 1/2$ (the result would be optimal if we did not have the 0^+ in the Hölder exponent). Moreover, the regularity in time is almost optimal also in the limit $s \to 1$ (since, when $s = 1$, it is well-known that solutions are C^1 in time and $C^{1,1}$ in space [3, 4, 5]). Hence, it may be expected that our result is almost optimal in time for all $s \in (0, 1)$ (or at least for $s > 1/3$).

2.3 Structure of the paper

The paper is structured as follows: first, in Section 3 we discuss some basic properties of solutions of (2.2), like the validity of a comparison principle, the Lipschitz regularity in space-time, the semiconvexity in space, and the boundedness of $(-\Delta)^s u$. Moreover, we will show that solutions are C^1 for $s \geq 1/2$, and, as explained in Remark 3.7, C^1-regularity in space is optimal when $s = 1/2$ unless one exploits the additional information that the solution coincides with the obstacle at the initial time.

In Section 4, we first use an iteration method to show that, for any $t > 0$, $(-\Delta)^s u(t)$ is C_x^α near any free boundary point (Subsection 4.1). Then, we prove a monotonicity formula which allows to show that $(-\Delta)^s u(t)$ is C_x^{1-s} near any free boundary point for all $t > 0$ (Subsection 4.2). Finally, combining the fact that $(-\Delta)^s u(t)$ is C_x^{1-s} on the contact set with equation (2.2), a bootstrap argument allows to prove Theorem 2.1 (Subsection 4.3).

In Section 5 we briefly describe what are the main modifications to perform in order to extend the regularity result in Theorem 2.1 to solutions of (1.1) when $s > 1/2$, leaving the details to some future work.

Finally, in the appendix we collect some regularity properties of the fractional heat operator $\partial_t + (-\Delta)^s$.

3 Basic properties of solutions

Here we discuss some elementary properties of solutions of (2.2). Actually, since many of them do not rely on the fact that u coincides with the obstacle at time 0, we consider solutions to

$$\begin{cases} \min\{u_t + (-\Delta)^s u, u - \psi\} = 0 & \text{on } [0,T] \times \mathbb{R}^n, \\ u(0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u_0 \geq \psi$ is a globally Lipschitz semiconvex function. Most of the properties of u will be a consequence of the following general comparison principle:

Lemma 3.1 (Comparison principle). *Let $\psi, \tilde{\psi} : \mathbb{R}^n \to \mathbb{R}$ be two continuous functions, and assume that $u, \tilde{u} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are viscosity solutions of*

$$\begin{cases} \min\{u_t + (-\Delta)^s u, u - \psi\} = 0 \quad \text{on } [0,T] \times \mathbb{R}^n, \\ u(0) = u_0 \quad \text{on } \mathbb{R}^n, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \min\{\tilde{u}_t + (-\Delta)^s \tilde{u}, \tilde{u} - \tilde{\psi}\} = 0 \quad \text{on } [0,T] \times \mathbb{R}^n, \\ \tilde{u}(0) = \tilde{u}_0 \quad \text{on } \mathbb{R}^n, \end{cases} \tag{3.3}$$

respectively. Assume that $u_0 \leq \tilde{u}_0$ and $\psi \leq \tilde{\psi}$. Then $u(t) \leq \tilde{u}(t)$ for all $t \in [0,T]$.

Proof. We use a penalization method: it is well-known that solutions of (3.1) can be constructed as a limit of u^ε as $\varepsilon \to 0$, where u^ε are smooth solutions of

$$\begin{cases} u^\varepsilon_t + (-\Delta)^s u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi_\varepsilon) \quad \text{on } [0,T] \times \mathbb{R}^n \\ u^\varepsilon(0) = u^\varepsilon_0 \geq \psi_\varepsilon \quad \text{on } \mathbb{R}^n, , \end{cases} \tag{3.4}$$

with $u^\varepsilon_0, \psi_\varepsilon \in C^\infty_c(\mathbb{R}^n)$, $\beta_\varepsilon(s) = e^{-s/\varepsilon}$, $\psi_\varepsilon \to \psi$, $(-\Delta)^s \psi_\varepsilon \to (-\Delta)^s \psi$, and $u^\varepsilon_0 \to u_0$ locally uniformly as $\varepsilon \to 0$ (see for instance [8, Chapter 3] for a proof in the classical parabolic case).

Hence, it suffices to prove the comparison principle at the level of the approximate equations, assuming $u^\varepsilon(0) \leq \tilde{u}^\varepsilon(0)$ and $\psi_\varepsilon \leq \tilde{\psi}_\varepsilon$. Let us observe that, since $\psi_\varepsilon \leq \tilde{\psi}_\varepsilon$ and $\beta'_\varepsilon \leq 0$, we have

$$\beta_\varepsilon(\cdot - \psi_\varepsilon) \leq \beta_\varepsilon(\cdot - \tilde{\psi}_\varepsilon),$$

which implies

$$u^\varepsilon_t + (-\Delta)^s u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi_\varepsilon) \qquad \text{on } [0,T] \times \mathbb{R}^n$$
$$\tilde{u}^\varepsilon_t + (-\Delta)^s \tilde{u}^\varepsilon = \beta_\varepsilon(\tilde{u}^\varepsilon - \tilde{\psi}_\varepsilon) \geq \beta_\varepsilon(\tilde{u}^\varepsilon - \psi_\varepsilon) \qquad \text{on } [0,T] \times \mathbb{R}^n.$$

Since $u^\varepsilon(0) \leq \tilde{u}^\varepsilon(0)$, by standard comparison principle for parabolic equations (see for instance the argument in the proof of Lemma 3.3 below) we get $u^\varepsilon \leq \tilde{u}^\varepsilon$, as desired. □

The following important properties are an immediate consequence of the above result:

Lemma 3.2. *Let u be a solution of (3.1), and assume that u_0 and ψ are globally Lipschitz and C_0-semiconvex. Then:*

(i) $u(t)$ is Lipschitz for all $t \in [0,T]$, with $\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \leq \max\{\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \|\nabla \psi\|_{L^\infty(\mathbb{R}^n)}\}$.

(ii) $u(t)$ is C_0-semiconvex for all $t \in [0,T]$.

Moreover, if $u_0 = \psi$ then

(iii) $[0,T] \ni t \mapsto u(t,x)$ is non-decreasing in time.

Proof. (i) Observe that, for every vector $v \in \mathbb{R}^n$ and any constant $C \in \mathbb{R}$, $u(t, x+v) + C|v|$ solves (3.2) starting from $u_0(x+v) + C|v|$ with obstacle $\psi(x+v) + C|v|$. Moreover, if $C := \max\{\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \|\nabla \psi\|_{L^\infty(\mathbb{R}^n)}\}$, then $u_0(x+v) + C|v| \geq u_0(x)$ and $\psi(x+v) + C|v| \geq \psi(x)$. Hence, by Lemma 3.1 we obtain

$$u(t, x+v) + C|v| \geq u(t, x) \qquad \forall\, x, v \in \mathbb{R}^n,\, t \geq 0.$$

The Lipschitz regularity of $u(t)$ follows.
(ii) As above, we just remark that $u(t, x+v) + u(t, x-v) + C|v|^2$ solves (3.2) for every $C \in \mathbb{R}$. Hence, by choosing $C := 2C_0$ we get $u_0(x+v) + u_0(x-v) + 2C_0|v|^2 \geq 2u_0(x)$ and $\psi(x+v) + \psi(x-v) + 2C_0|v|^2 \geq 2\psi(x)$, and we conclude as above using Lemma 3.1.
(iii) We observe that, for any $\varepsilon \geq 0$, the function $u(t+\varepsilon, x)$ solves (2.2) starting from $u(\varepsilon, \cdot)$. Hence, since $u(\varepsilon, \cdot) \geq \psi$, by the comparison principle we obtain

$$u(t+\varepsilon, x) \geq u(t, x) \qquad \forall\, t, \varepsilon \geq 0.$$

□

We now prove the following important bounds:

Lemma 3.3. *Let u be a solution of (3.1). Then*

$$0 \leq u_t + (-\Delta)^s u \leq \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)}, \tag{3.5}$$

$$\|u_t\|_{L^\infty([0,T]\times\mathbb{R}^n)} \leq \|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)}, \tag{3.6}$$

In particular $(-\Delta)^s u$ is bounded, with

$$\|(-\Delta)^s u\|_{L^\infty([0,T]\times\mathbb{R}^n)} \leq \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)}. \tag{3.7}$$

Proof. As in the proof of Lemma 3.1, we use a penalization method: we consider solutions u^ε to (3.4), and we prove a uniform (with respect to ε) L^∞-bound on both $\beta_\varepsilon(u^\varepsilon - \psi_\varepsilon)$ and u^ε_t.

• *L^∞-bound on $\beta_\varepsilon(u^\varepsilon - \psi_\varepsilon)$.* Since $\beta_\varepsilon \geq 0$, we only need an upper bound.

Assume that $\inf_{[0,T]\times\mathbb{R}^n}(u^\varepsilon - \psi_\varepsilon) < 0$ (otherwise the problem is trivial), and let φ be a smooth function which grows like $|x|^s$ at infinity. Then, since u^ε vanishes at infinity (being a solution to a smooth parabolic equation starting from a compactly supported initial datum), for any $\delta > 0$ small we can consider $(t^\delta_\varepsilon, x^\delta_\varepsilon)$ a minimum point for $u^\varepsilon - \psi_\varepsilon + \frac{\delta}{T-t} + \delta\varphi$ over $[0,T] \times \mathbb{R}^n$. Of course, $\min_{[0,T]\times\mathbb{R}^n}\left(u^\varepsilon - \psi_\varepsilon + \frac{\delta}{T-t} + \delta\varphi\right) < 0$ for δ sufficiently small, which implies that $(t^\delta_\varepsilon, x^\delta_\varepsilon)$ belongs to the interior of $(0,T) \times \mathbb{R}^n$. Hence

$$u^\varepsilon_t(t^\delta_\varepsilon, x^\delta_\varepsilon) + \frac{\delta}{(T - t^\delta_\varepsilon)^2} = 0, \qquad (-\Delta)^s u^\varepsilon(t^\delta_\varepsilon, x^\delta_\varepsilon) - (-\Delta)^s \psi_\varepsilon(x^\delta_\varepsilon) + \delta(-\Delta)^s \varphi(x^\delta_\varepsilon) \leq 0,$$

which combined with (3.4) gives

$$\beta_\varepsilon(u^\varepsilon - \psi_\varepsilon)(t^\delta_\varepsilon, x^\delta_\varepsilon) \leq (-\Delta)^s \psi_\varepsilon(x^\delta_\varepsilon) - \frac{\delta}{(T - t^\delta_\varepsilon)^2} - \delta(-\Delta)^s \varphi(x^\delta_\varepsilon) \leq \|(-\Delta)^s \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} + O(\delta).$$

Since $(u^\varepsilon - \psi_\varepsilon)(t^\delta_\varepsilon, x^\delta_\varepsilon) \to \inf_{[0,T]\times\mathbb{R}^n}(u^\varepsilon - \psi_\varepsilon)$ as $\delta \to 0$ and $\beta'_\varepsilon \le 0$ we obtain

$$\sup_{[0,T]\times\mathbb{R}^n} \beta_\varepsilon(u^\varepsilon - \psi_\varepsilon) = \lim_{\delta\to 0} \beta_\varepsilon(u^\varepsilon - \psi_\varepsilon)(t^\delta_\varepsilon, x^\delta_\varepsilon) \le \|(-\Delta)^s \psi_\varepsilon\|_{L^\infty(\mathbb{R}^n)},$$

so that (3.5) follows letting $\varepsilon \to 0$.

• L^∞*-bound on* u^ε_t. We use the same argument as in [11, Lemma 2.1]: differentiating (3.4) with respect to t we obtain that $w^\varepsilon := u^\varepsilon_t$ solves

$$\begin{cases} w^\varepsilon_t + (-\Delta)^s w^\varepsilon = \beta'_\varepsilon(u^\varepsilon - \psi_\varepsilon) w^\varepsilon & \text{on } [0,T] \times \mathbb{R}^n \\ w^\varepsilon(0) = -(-\Delta)^s u^\varepsilon(0) & \text{on } \mathbb{R}^n. \end{cases}$$

Since $\beta'_\varepsilon \le 0$ and $\|w^\varepsilon(0)\|_{L^\infty} = \|(-\Delta)^s u^\varepsilon_0\|_{L^\infty}$, using a maximum principle argument (as above), we infer that

$$\|u^\varepsilon_t\|_{L^\infty([0,T]\times\mathbb{R}^n)} = \|w^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)} \le \|w^\varepsilon(0)\|_{L^\infty(\mathbb{R}^n)} = \|(-\Delta)^s u^\varepsilon_0\|_{L^\infty(\mathbb{R}^n)}.$$

Letting $\varepsilon \to 0$ we get (3.6), as desired. $\square$

The above result together with Lemma 3.2(i) gives the following:

Corollary 3.4 (Lipschitz regularity in space-time)**.** *Let u be a solution of* (3.1)*. Then*

$$\|u_t\|_{L^\infty([0,T]\times\mathbb{R}^n)} + \|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^n)} \le \max\{\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \|\nabla \psi\|_{L^\infty(\mathbb{R}^n)}\} + \|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)}.$$

In the sequel we will also need the following result:

Lemma 3.5. *Let u be a solution of* (2.2) *with*

$$\|\nabla \psi\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)} < +\infty,$$

and fix $t_0 > 0$. Then

$$0 \le (-\Delta)^s u(t_0) \le \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)} \qquad \textit{for a.e. } x \in \{u(t_0) = \psi\}, \tag{3.8}$$

$$(-\Delta)^s u(t_0) \le 0 \qquad \textit{on } \{u(t_0) > \psi\}. \tag{3.9}$$

Proof. Let us recall that, thanks to Corollary 3.4, u is Lipschitz in time. So, Lemma 3.2(iii) gives $u_t \ge 0$ a.e.

Moreover, since $u_t = 0$ a.e. on the contact set $\{u = \psi\}$, u satisfies

$$u_t + (-\Delta)^s u = 0 \quad \text{in } \{u > \psi\}, \qquad u_t = 0 \quad \text{a.e. on } \{u = \psi\},$$

which gives

$$u_t + (-\Delta)^s u = \big((-\Delta)^s u\big)\chi_{\{u=\psi\}}$$

both in the almost everywhere sense and in the sense of distribution. (Observe that the above formula makes sense since $(-\Delta)^s u$ is a bounded function, see Lemma 3.3). Hence, u solves the smooth parabolic equation $u_t + (-\Delta)^s u = f$, with f globally bounded and vanishing inside the open set $\{u > \psi\}$. So, a simple application of Duhamel formula shows that u is smooth inside $\{u > \psi\}$. In particular, this fact combined with the non-negativity of u_t implies

$$(-\Delta)^s u(t_0) = -u_t(t_0) \leq 0 \qquad \text{on } \{u(t_0) > \psi\},$$

that is, (3.9).

We now prove (3.8). Since $u_t = 0$ a.e. on the contact set $\{u = \psi\}$, (3.5) gives

$$0 \leq (-\Delta)^s u \leq \|(-\Delta)^s \psi\|_{L^\infty} \qquad \text{for a.e. } (t,x) \in \{u = \psi\}. \tag{3.10}$$

Now, to show that the bound $0 \leq (-\Delta)^s u(t_0) \leq \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)}$ holds a.e. on $\mathbb{R}^n$ *for every* $t_0 \in [0,T]$, we observe that the map

$$t \mapsto u(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$$

is uniformly continuous (this is a consequence of the Lipschitz continuity in time, see Corollary 3.4), which together with the uniform bound (3.5) implies that the map

$$t \mapsto (-\Delta)^s u(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$$

is weakly continuous. Thanks to this fact, we easily deduce the desired estimate. Indeed, fix $\varepsilon > 0$, $A \subset \{u(t_0) = \psi\}$ a bounded Borel set, and test (3.10) against the function $\chi_{[t_0-\varepsilon,t_0]}\chi_A$. Since the sets $\{u(t) = \psi\}$ are decreasing in time (see Lemma 3.2(iii)), we have $[t_0-\varepsilon,t_0] \times A \subset \{u = \psi\}$, which together with (3.10) gives

$$0 \leq \int_{t_0-\varepsilon}^{t_0} \int_A (-\Delta)^s u \leq \|(-\Delta)^s \psi\|_{L^\infty} |A|\, \varepsilon.$$

Dividing by ε and letting $\varepsilon \to 0$, by the weak-L^2 continuity of $t \mapsto (-\Delta)^s u(t)$ we deduce

$$0 \leq \int_A (-\Delta)^s u(t_0) \leq \|(-\Delta)^s \psi\|_{L^\infty} |A| \qquad \forall\, A \subset \{u(t_0) = \psi\} \text{ Borel bounded},$$

so that the desired bound follows. $\square$

We now show that the uniform semiconvexity of $u(t)$, together with the L^∞-bound on $(-\Delta)^s u(t)$, implies that solutions are C^1 in space when $s \geq 1/2$ (actually, when $s > 1/2$, by elliptic regularity theory the boundedness of $(-\Delta)^s u(t)$ implies that $u(t) \in C^{2s-0^+}_{\mathrm{loc}}(\mathbb{R}^n)$). As we will show in Remark 3.7 below, unless the contact set shrinks in time, this regularity result is optimal for $s = 1/2$.

Proposition 3.6 (C^1-spatial regularity)**.** *Let u be a solution of* (3.1) *with $s \in [1/2, 1)$. Assume that u_0 and ψ are semiconvex, and that $\|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)} < +\infty$. Then $u(t) \in C^1(\mathbb{R}^n)$ for all $t \in [0,T]$. Moreover the modulus of continuity of ∇u depends only on s, the semiconvexity constant of u_0 and ψ, and on $\|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)}$.*

Proof. First of all, we claim that, for every fixed $t \in [0,T]$, the map $x \mapsto -(-\Delta)^s u(t,x)$ is lower semicontinuous. Indeed, recall that if C_0 denotes a semiconvexity constant for both u_0 and ψ, then $u(t)$ is C_0-semiconvex for all $t \in [0,T]$ (see Lemma 3.2(ii)). Hence $(-\Delta)^s u(t,x)$ is pointwise defined *at every* $x \in \mathbb{R}^n$, and is given by (see (2.1))

$$\begin{aligned}
-(-\Delta)^s u(t,x) &= \int_{B_1} \frac{u(t,x+h)+u(t,x-h)-2u(t,x)}{2|h|^{n+2s}}\,dh + \int_{\mathbb{R}^n\setminus B_1} \frac{u(t,x+h)-u(t,x)}{|h|^{n+2s}}\,dy \\
&= \int_{B_1} \frac{u(t,x+h)+u(t,x-h)-2u(t,x)+2C_0|h|^2}{2|h|^{n+2s}}\,dh \\
&\quad - C_0 C(n,s) + \int_{\mathbb{R}^n\setminus B_1} \frac{u(t,x+h)-u(t,x)}{|h|^{n+2s}}\,dy
\end{aligned}$$

where $C(n,s) := \int_{B_1} |h|^{2-n-2s}\,dh$. The last integral in the right hand side is continuous as a function of x (since u is continuous). Moreover, since the function inside the first integral is continuous in x and non-negative (by the C_0-semiconvexity), the first integral is lower semicontinuous as a function of x by Fatou's lemma. This proves the claim.

Now, we remark that $-(-\Delta)^s u(t,x_0) = +\infty$ whenever x_0 is a point such that the subdifferential of $u(t)$ at x_0 is not single valued. Indeed, suppose that

$$u(t,x) \geq \varphi_{x_0,p_1,p_2}(x) := \Big[u(t,x_0) + \max\{p_1\cdot(x-x_0), p_2\cdot(x-x_0)\} - \frac{C_0}{2}|x-x_0|^2\Big]\chi_{B_1(x_0)}(x),$$

for some $p_1 \neq p_2$. Then, it is easy to check by a simple explicit computation that

$$-(-\Delta)^s \varphi_{x_0,p_1,p_2}(x_0) = +\infty \qquad \forall\, s \geq 1/2.$$

Hence, since $u(t) \geq \varphi_{x_0,p_1,p_2}$ with equality at x_0, we get

$$(-\Delta)^s u(t,x_0) \geq (-\Delta)^s \varphi_{x_0,p_1,p_2}(x_0) = +\infty.$$

However, since $-(-\Delta)^s u(t)$ is bounded by $\|(-\Delta)^s u_0\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s \psi\|_{L^\infty(\mathbb{R}^n)}$ (see (3.7)) and it is a lower semicontinuous function, the above inequality is impossible. Thus the subdifferential of $u(t)$ at x is a singleton at every point, i.e., $u(t)$ is C^1. Finally, the last part of the statement follows by a simple compactness argument. □

Remark 3.7. The spatial C^1-regularity proved in the above proposition is optimal for $s = 1/2$. Indeed, consider the case $n = 1$ and $\psi \equiv 0$, and use the interpretation of the $(1/2)$-fractional Laplacian as the Dirichlet-to-Neumann operator for the harmonic extension, as explained in Subsection 2.1 (observe that $L_a = \Delta_{x,y}$ when $s = 1/2$). Then, we look for solutions to the problem

$$\begin{cases} \min\{u_t - u_y, u\} = 0 & \text{on } [0,T]\times\mathbb{R}, \\ \Delta_{x,y} u(t) = 0 & \text{on } [0,T]\times\mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{3.11}$$

Let us try to find traveling waves solutions to the above equation, i.e., solutions of the form $u(t,x,y) = w(at+x,y)$, with $a \in \mathbb{R}$. In this case $u_t = au_x$, so $w(x,y)$ has to solve

$$\begin{cases} aw_x(x,0) - w_y(x,0) = 0 & \text{when } \{w(x,0) > 0\}, \\ \Delta_{x,y} w = 0 & \text{on } \mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{3.12}$$

By using the complex variable $z = x+iy$ it is easy to construct C^1 solutions to the above equation: if we denote $\rho = |z| = \sqrt{x^2+y^2}$ and $\theta = \arg(z)$, then $w_\beta(x,y) := -\rho^{1+\beta}\sin((1+\beta)\theta) = -\mathrm{Im}(z^{1+\beta})$ is harmonic in the half-space $y>0$ and solves

$$\begin{cases} w_\beta(x,0) = 0 & \text{on } \{x \geq 0\}, \\ w_\beta(x,0) > 0,\ \frac{(w_\beta)_x}{\tan(\beta\pi)} = (w_\beta)_y & \text{on } \{x >< 0\}. \end{cases}$$

Observe that w_β is of class $C^{1+\beta}$ both in space and time (but not more), and solves (3.12) with $a = 1/\tan(\beta\pi)$. Since $\beta \in (0,1)$ is arbitrary, we cannot expect to prove any uniform $C_x^{1+\alpha}$-regularity for solutions to (3.11). Thus, the C_x^1-regularity proved in Proposition 3.6 is optimal.

On the other hand, we observe that the case $u_t \geq 0$ (i.e., the contact set shrinks in time) corresponds to $a \leq 0$, or equivalently to $\beta \geq 1/2$. Hence, in this case the solutions constructed above are at least $C_{t,x}^{1+1/2}$, which is the optimal regularity result for the stationary case [1, 6]. As we will show in the next section, solutions to (3.1) satisfying $u_t \geq 0$ are of class $C^{1+1/2}$ in space. In particular, by Lemma 3.2 this result applies to solutions of (2.2).

4 Proof of Theorem 2.1

The strategy of the proof is the following: first in Subsection 4.1 we prove a general $C^{\alpha+2s}$-regularity result in space which, roughly speaking, says the following: let $v : \mathbb{R}^n \to \mathbb{R}$ be a semiconvex function which touches from above an obstacle $\psi : \mathbb{R}^n \to \mathbb{R}$ of class C^2. Assume that $(-\Delta)^s v$ is non-positive outside the contact set and non-negative on the contact set. Then v detaches from ψ in a $C^{\alpha+2s}$ fashion, for some $\alpha = \alpha(s) > 0$ universal. In particular, as shown in Corollary 4.2, this implies that $(-\Delta)^s v \chi_{\{v=\psi\}} \in C_x^\alpha(\mathbb{R}^n)$.

Then, in Subsection 4.2 we use a monotonicity formula to prove the optimal regularity in space

$$(-\Delta)^s v \chi_{\{v=\psi\}} \in C_x^{1-s}(\mathbb{R}^n).$$

Finally, in Subsection 4.3 we apply the above estimate to any time slice $u(t)$ to prove that $(-\Delta)^s u(t)\chi_{\{u(t)=\psi\}} \in C_x^{1-s}(\mathbb{R}^n)$, uniformly in time. Then, exploiting (2.2) and a bootstrap argument, we get (2.3).

4.1 A general $C^{\alpha+2s}$-regularity result.

In order to underline what are the key elements in the proof, in this and in the next subsection we forget about equation (2.2), and we work in the following general setting: let $v, \psi : \mathbb{R}^n \to \mathbb{R}$ be two globally Lipschitz functions with $v \geq \psi$. Assume that[2]:

(A1) $\|D^2\psi\|_{L^\infty(\mathbb{R}^n)} =: C_0 < +\infty$;

[2]The smoothness assumption on v inside the open set $\{v > \psi\}$ (see (A5)) is not essential for the proof of the regularity of v at free boundary points, but it is only used to avoid some minor technical issues. Anyhow this makes no differences for our purposes, since all the following results will be applied to $v = u(t)$ with $t > 0$, and u is smooth inside the open set $\{u > \psi\}$ (see the proof of Lemma 3.5).

(A2) $\|(-\Delta)^s\psi\|_{C^{1-s}_x(\mathbb{R}^n)} < +\infty$;

(A3) $v-\psi, (-\Delta)^s v \in L^\infty(\mathbb{R}^n)$;

(A4) v is C_0-semiconvex;

(A5) v is smooth and $(-\Delta)^s v \le 0$ inside the open set $\{v>\psi\}$;

(A6) $\|(-\Delta)^s\psi\|_{L^\infty(\mathbb{R}^n)} \ge (-\Delta)^s v \ge 0$ a.e. on $\{v=\psi\}$.

Under these assumptions, we want to show that v is $C^{\alpha+2s}$ at every free boundary point, with a uniform bound. More precisely, we want to prove:

Theorem 4.1. *Let v be as above. Then there exist $\bar C > 0$ and $\alpha \in (0,1)$, depending on C_0, $\|(-\Delta)^s\psi\|_{C^{1-s}_x(\mathbb{R}^n)}$, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that*

$$\sup_{B_r(x)} |v-\psi| \le \bar C\, r^{\alpha+2s}, \qquad \sup_{B_r(x)} |(-\Delta)^s v \chi_{\{v=\psi\}}| \le \bar C\, r^{\alpha} \qquad \forall\, r\le 1 \tag{4.1}$$

for every $x \in \partial\{u=\psi\}$.

Before proving the above result, let us show how it implies the following:

Corollary 4.2. *Let v be as above. Then there exist $\bar C' > 0$ and $\alpha \in (0,1-s]$, depending on C_0, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that*

$$\|(-\Delta)^s v \chi_{\{v=\psi\}}\|_{C^\alpha_x(\mathbb{R}^n)} \le \bar C'.$$

Proof. Without loss of generality, we can assume that the exponent α provided by Theorem 4.1 is not greater than $1-s$. Moreover, since $(-\Delta)^s v$ is bounded on $\{v=\psi\}$ (see (A6)), it suffices to control $|(-\Delta)^s v(x_1) - (-\Delta)^s v(x_2)|$ when $x_1, x_2 \in \{v=\psi\}$ and $|x_1-x_2| \le 1/4$.

Let $M := \|v-\psi\|_{L^\infty(\mathbb{R}^n)}$. Moreover, given $x\in\{v=\psi\}$, let $d_F(x)$ denote its distance from the free boundary $\partial\{v=\psi\}$.

Fix $x_1, x_2 \in \{v=\psi\}$, with $|x_1-x_2|\le 1/4$. Two cases arise.

• *Case 1:* $\max_{i=1,2} d_F(x_i) \ge 4|x_1-x_2|$. Set $\tilde v := v-\psi$. Since $\alpha \le 1-s$ by assumption, thanks to (A2) it suffices to estimate $(-\Delta)^s\tilde v$ inside $\{\tilde v = 0\} = \{v=\psi\}$. Now, by Theorem 4.1 we have

$$\sup_{B_r(x_i)} |\tilde v| \le \bar C\, r^{\alpha+2s}, \qquad \forall\, r\le 1,\ i=1,2.$$

Hence, since $\tilde v = 0$ inside $B_{4|x_1-x_2|}(x_1) \cap B_{4|x_1-x_2|}(x_2)$ and $|\tilde v| \le M$ outside $B_1(x_1) \supset B_{1/2}(x_2)$, we get

$$
\begin{aligned}
&|(-\Delta)^s\tilde v(x_1)-(-\Delta)^s\tilde v(x_2)|\\
&=\left|\int_{\mathbb{R}^n}\frac{\tilde v(x')-\tilde v(x_1)}{|x'-x_1|^{n+2s}}\,dx'-\int_{\mathbb{R}^n}\frac{\tilde v(x')-\tilde v(x_2)}{|x'-x_2|^{n+2s}}\,dx'\right|\\
&\le\int_{\mathbb{R}^n\setminus[B_{4|x_1-x_2|}(x_1)\cap B_{4|x_1-x_2|}(x_2)]}|\tilde v(x')|\left|\frac{1}{|x'-x_1|^{n+2s}}-\frac{1}{|x'-x_1|^{n+2s}}\right|dx'\\
&\le\bar C\,|x_1-x_2|\int_{B_1(x_1)\setminus[B_{4|x_1-x_2|}(x_1)\cap B_{4|x_1-x_2|}(x_2)]}|x'|^{\alpha+2s}\left[\frac{1}{|x'-x_1|^{n+2s+1}}+\frac{1}{|x'-x_2|^{n+2s+1}}\right]dx'\\
&\qquad+2M\,|x_1-x_2|\int_{\mathbb{R}^n\setminus B_1(x_1)}\left[\frac{1}{|x'-x_1|^{n+2s+1}}+\frac{1}{|x'-x_2|^{n+2s+1}}\right]dx'\\
&\le C\left[\bar C\int_{|x_1-x_2|}^1 s^{\alpha-2}\,ds+M\right]|x_1-x_2|\le C\,|x_1-x_2|^\alpha,
\end{aligned}
$$

as desired.

• *Case 2:* $\max_{i=1,2} d_F(x_i)\le 4|x_1-x_2|$. For every $i=1,2$, let $\bar x_i\in\partial\{v=\psi\}$ denote a point such that $|x_i-\bar x_i|=d_F(x_i)$. Then, by Theorem 4.1 we get

$$
\begin{aligned}
|(-\Delta)^s v(x_1)-(-\Delta)^s v(x_2)|&\le\sup_{B_{4|x_1-x_2|}(\bar x_1)}|(-\Delta)^s v|+\sup_{B_{4|x_1-x_2|}(\bar x_2)}|(-\Delta)^s v|\\
&\le 8\bar C\,|x_1-x_2|^\alpha.
\end{aligned}
$$

□

4.1.1 Proof of Theorem 4.1

The strategy of the proof is analogous to the one used in [1] to study the stationary fractional obstacle problem with $s=1/2$ (also called "Signorini problem").

With no loss of generality, we can assume that 0 is a free boundary point, and we prove (4.1) at $x=0$. Moreover, by a slight abuse of notation, let us still denote by $v:\mathbb{R}^n\times\mathbb{R}^+\to\mathbb{R}$ the L_a-harmonic extension of v, i.e.,

$$
L_a v(x,y)=\operatorname{div}_{x,y}\big(y^a\nabla_{x,y}v(x,y)\big)=0\qquad\text{for } y>0,
$$

and $v(x,0)=v(x)$, with $v(x)$ as above. Then

$$
\lim_{y\to0^+}y^a v_y(x,y)=-(-\Delta)^s v(x,0),\qquad a=1-2s
$$

(see (2.1)). Let us observe that the C_0-semiconvexity of $v(x,0)$ (see (A4)) propagates in y: since

$$
v(x+h,0)+v(x-h,0)-2v(x,0)\ge-2C_0|h|^2\qquad\forall\,h\in\mathbb{R}^n,
$$

the maximum principle implies

$$
v(x+h,y)+v(x-h,y)-2v(x,y)\ge-2C_0|h|^2\qquad\forall\,h\in\mathbb{R}^n,\ y>0,
$$

that is,

(A7) $v(\cdot,y)$ is C_0-semiconvex for all $y \geq 0$.

In particular, since $L_a v = 0$ we get

(A8) $\partial_y(y^a v_y) \leq nC_0 y^a$.

In the sequel, we will informally call the above property "a-semiconcavity" in y[3]. Set now

$$\tilde{v}(x,y) := v(x,y) - \psi(x),$$

and denote by $\Lambda := \{\tilde{v}(x,0)=0\} = \{v(x,0)=\psi(x)\}$ the contact set. Observe that $\tilde{v}_y = v_y$, which thanks to (A1)-(A6) gives that the function $\tilde{v}$ enjoys the following properties:

(B1) $\tilde{v}(x,0) \geq 0$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^+ \setminus \Lambda \times \{0\}$.

(B2) $\partial_y(y^a \tilde{v}_y(x,y)) \leq 2nC_0 y^a$, $\tilde{v}(\cdot,y)$ is $(2C_0)$-semiconvex for all $y \geq 0$.

(B3) $\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \leq 0$ for a.e. $x \in \Lambda$, $\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \geq 0$ for a.e. $x \in \mathbb{R}^n \setminus \Lambda$.

(B4) $\tilde{v}(x,y) - \tilde{v}(x,0) \leq \frac{nC_0}{1+a}\, y^2$ for all $x \in \Lambda$.

(B5) if $\tilde{v}(x,y) \geq h$, then $\tilde{v}(\cdot,y) \geq h - C_0\rho^2$ in the half-ball

$$HB_\rho(x) := \{z \in B_\rho(x) \subset \mathbb{R}^n \,:\, \langle \nabla_x \tilde{v}(x,y), z-x\rangle \geq 0\}.$$

Observe that the proof of (B1)-(B5) is almost immediate, except for (B4) for which a (simple) computation is needed: using (B2) and (B3), for a.e. $x \in \Lambda$ we have

$$\tilde{v}(x,y) - \tilde{v}(x,0) = \int_0^y \tilde{v}_y(x,s)\,ds = \int_0^y \frac{s^a \tilde{v}_y(x,s)}{s^a}\,ds \leq \int_0^y \frac{\int_0^s 2nC_0\tau^a\,d\tau}{s^a}\,ds = \frac{nC_0}{1+a}\,y^2,$$

and by continuity the above inequality holds for all $x \in \Lambda$.

We use the notation $\Gamma_r := B_r \times [0, \eta_{n,a} r]$, where $\eta_{n,a} = \sqrt{\frac{1+a}{2n}}$. We first show a decay result for $y^a \tilde{v}_y$:

Proposition 4.3. *There exist two constants $K_1 > 0$, $\mu \in (0,1)$, depending on C_0, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that*

$$\inf_{\Gamma_{4^{-k}}} y^a \tilde{v}_y \geq -K_1\,\mu^k. \tag{4.2}$$

Proof. We prove the result by induction.
Case $k=1$: since v is L_a-harmonic, $y^a\tilde{v}_y = y^a v_y$ solves the "conjugate" equation $L_{-a}(y^a\tilde{v}_y) = 0$ inside $\mathbb{R}^n \times \mathbb{R}^+$ (see for instance [7, Subsection 2.3]). Hence, the boundedness of $\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) =$

[3]Even if we use the names "a-semiconcavity" and "C_0-semiconvexity" with different meanings, this should create no confusion. Observe also that, when $a=0$, (A8) reduces to the classical notion of semiconcavity.

$(-\Delta)^s v(x,0)$ (see (A6) and (2.1)) combined with the maximum principle implies the result.
Induction step: Assume the result is true for $k = k_0$, i.e.,

$$\inf_{\Gamma_{4^{-k_0}}} y^a \tilde{v}_y \geq -K_1\, \mu^{k_0}$$

for some constants $K_1 > 0$ and $\mu \in (0,1)$ which will be chosen later, and renormalize the solution inside Γ_1 by setting

$$\tilde{V}(x,y) := \frac{1}{K_1}\left(\frac{4^{2s}}{\mu}\right)^{k_0} \tilde{v}\left(\frac{x}{4^{k_0}}, \frac{y}{4^{k_0}}\right).$$

It will also be useful to consider the L_a-harmonic function

$$\bar{V}(x,y) := \frac{1}{K_1}\left(\frac{4^{2s}}{\mu}\right)^{k_0} \bar{v}\left(\frac{x}{4^{k_0}}, \frac{y}{4^{k_0}}\right),$$

where $\bar{v}$ is the L_a-harmonic function given by

$$\bar{v}(x,y) := v(x,y) - \psi(0) - \nabla\psi(0)\cdot x. \tag{4.3}$$

Then, thanks to (A1) and (B2):

(i) $|\tilde{V}(x,y) - \bar{V}(x,y)| \leq \frac{C_0}{K_1(4^{2(1-s)}\mu)^{k_0}}|x|^2$ and $\tilde{V}_y = \bar{V}_y$;

(ii) $\inf_{\Gamma_1} y^a \tilde{V}_y = \inf_{\Gamma_1} y^a \bar{V}_y \geq -1$;

(iii) $\partial_y(y^a\tilde{V}_y) = \partial_y(y^a\bar{V}_y) \leq \frac{2nC_0}{K_1(4^{2(1-s)}\mu)^{k_0}} y^a$, $\tilde{V}$ and $\bar{V}$ are $\left(\frac{2C_0}{K_1(4^{2(1-s)}\mu)^{k_0}}\right)$-semiconvex inside Γ_1.

Fix $L := \bar{C}_{n,a}C_0$, where $\bar{C}_{n,a} \gg 1$ is a large constant depending on n and a only (to be fixed later), and define

$$\bar{W}(x,y) := \bar{V}(x,y) - \frac{L}{K_1(4^{2(1-s)}\mu)^{k_0}}\left[|x|^2 - \frac{n}{1+a}\, y^2\right].$$

Thanks to (B1)-(B3), the function $\bar{W}$ satisfies the following properties:

1. it is L_a-harmonic in the interior of $\Gamma_{1/8}$;
2. $\bar{W}(x,0) < 0$ for $x \in \big(\Lambda \setminus \{0\}\big) \times \{0\}$;
3. $\lim_{(x,y)\to(0,0)} \bar{W}(x,y) = 0$;
4. $\lim_{y\to 0^+} y^a \bar{W}_y(x,y) \geq 0$ for $x \in B_{1/8} \setminus \Lambda$.

Hence, up to replacing $\bar{W}$ by $\bar{W} + \varepsilon y^{1-a}$ with $\varepsilon > 0$ (so that the inequality in 4. becomes strict) and then letting $\varepsilon \to 0$, by Hopf's Lemma $\bar{W}$ attains its non-negative maximum on $\partial\Gamma_{1/8} \setminus \{y = 0\}$.
Two cases arise:
Case 1: The maximum is attained on $\partial\Gamma_{1/8} \cap \{y = \eta_{n,a}/8\}$.
In this case, there exists $x_0 \in B_{1/8}$ such that

$$\bar{V}\left(x_0, \tfrac{\eta_{n,a}}{8}\right) \geq -c'_{n,a} \frac{L}{K_1(4^{2(1-s)}\mu)^{k_0}},$$

for some constant $c'_{n,a} > 0$ depending on n, a only. Thanks to the semiconvexity in x (see property (iii) above) and recalling that $L \gg C_0$ by assumption, there exists an n-dimensional half-ball $HB_{1/2}\left(x_0, \frac{\eta_{n,a}}{8}\right)$ such that

$$\bar{V}\left(x, \tfrac{\eta_{n,a}}{8}\right) \geq -2c'_{n,a} \frac{L}{K_1(4^{2(1-s)}\mu)^{k_0}} \qquad \forall\, x \in HB_{1/2}\left(x_0, \tfrac{\eta_{n,a}}{8}\right)$$

(see property (B5)). Recall now that $\lim_{y\to 0^+} y^a \bar{V}_y(x,y) = \lim_{y\to 0^+} y^a \tilde{V}_y(x,y) \geq 0$ when $\tilde{V}(x,0) > 0$, while $\lim_{y\to 0^+} y^a \bar{V}_y(x,y) = \lim_{y\to 0^+} y^a \tilde{V}_y(x,y) \leq 0$ when $\tilde{V}(x,0) = 0$. Hence, by the "a-semiconcavity" of $\bar{V}$ in y (property (iii) above) and by (i), it is easy to see that

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) \geq -C''_{n,a} \frac{L}{K_1(4^{2(1-s)}\mu)^{k_0}} \qquad \forall\, x \in HB_{1/2}(x_0, 0),$$

for some universal constant $C''_{n,a} > 0$.
Case 2: The maximum is attained on $\partial\Gamma_{1/8} \setminus \{y = \eta_{n,a}/8\}$.
Let (x'_0, y'_0) be a maximum point. Since such a point belongs to the lateral side of the cylinder, recalling the definition of $\eta_{n,a}$ we have $|x'_0|^2 \geq \frac{2n}{1+a}|y'_0|^2$, which implies

$$\bar{V}(x'_0, y'_0) \geq \frac{L}{K_1(4^{2(1-s)}\mu)^{k_0}}.$$

Again, we recall that $\lim_{y\to 0^+} y^a \bar{V}_y(x,y) = \lim_{y\to 0^+} y^a \tilde{V}_y(x,y) \geq 0$ when $\tilde{V}(x,0) > 0$, while $\lim_{y\to 0^+} y^a \bar{V}_y(x,y) = \lim_{y\to 0^+} y^a \tilde{V}_y(x,y) \leq 0$ when $\tilde{V}(x,0) = 0$. Thus, by the half-ball estimate (B5) applied to $\bar{V}$, by the "a-semiconcavity" of $\bar{V}$ in y (property (iii)) and by (i), we obtain

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) \geq 0 \qquad \forall\, x \in HB_{1/2}\left(x'_0, 0\right).$$

Hence, in both case we have reached the following conclusion:
There exist a constant $C_1 > 0$, depending on n, a, and C_0 only, and a point $\bar{x} \in B_{1/8} \subset \mathbb{R}^n$, such that

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) > -\frac{C_1}{K_1(4^{2(1-s)}\mu)^{k_0}} \qquad \forall\, x \in HB_{1/2}(\bar{x}, 0).$$

Thus, if we choose K_1 and μ satisfying $K_1 > 2C_1$ and $\mu \geq 1/4^{2(1-s)}$, then we obtain

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) > -\frac{1}{2}. \tag{4.4}$$

Moreover, thanks to (ii),

$$y^a \bar{V}_y \geq -1 \qquad \text{in } \Gamma_1. \tag{4.5}$$

As we already observed before, the fact that $\bar{V}$ is L_a-harmonic implies that $y^a \bar{V}_y$ solves the conjugate equation $L_{-a}(y^a \bar{V}_y) = 0$ inside $\mathbb{R}^n \times \mathbb{R}^+$. Hence, thanks to (4.4) and (4.5), the Poisson representation formula (see [7, Subsection 2.4]) implies the existence of a constant $\theta < 1$ such that

$$\left(\tfrac{\eta_{n,a}}{4}\right)^a \bar{V}_y\left(x, \tfrac{\eta_{n,a}}{4}\right) \geq -\theta \qquad \forall\, x \in B_{1/4}.$$

Therefore, by (i) and the "a-semiconcavity" of $\bar{v}$ in y (property (iii)), we obtain

$$y^a \tilde{V}_y(x,y) = y^a \bar{V}_y(x,y) \geq -\theta - \frac{2nC_0}{K_1(4^{2(1-s)}\mu)^{k_0}} \geq -\mu > -1$$

for all $x \in B_{1/4}$ and $y \leq \frac{\eta_{n,a}}{4}$, provided K_1 is sufficiently large and μ is sufficiently close to 1. Rescaling back, this proves (4.2) with $k = k_0 + 1$, which concludes the proof. $\square$

Recalling that $y^a \tilde{v}_y = y^a v_y$, thanks to (2.1) and (A6) the above proposition implies

$$\sup_{B_r(x)} |(-\Delta)^s v \chi_{\{v=\psi\}}| \leq \bar{C}\, r^\alpha \qquad \forall\, r \leq 1.$$

We now show that a control from below on $y^a \tilde{v}_y$ inside Γ_r gives a control from both sides on $\tilde{v}$ inside $\Gamma_{r/8}$. This will conclude the proof of Theorem 4.1.

Lemma 4.4. *Fix $K > 0$, $\alpha \in (0,1)$, and assume that*

$$\inf_{\Gamma_r} y^a \tilde{v}_y \geq -K r^\alpha \tag{4.6}$$

for some $r \in (0,1]$. Then there exists a constant $M = M(K, \alpha, a, C_0)$, independent of r, such that

$$\sup_{\Gamma_{r/8}} |\tilde{v}| \leq M r^{1+\alpha-a} = M r^{\alpha+2s}.$$

Proof. Since $\tilde{v}$ is globally bounded, it suffices to prove the result for r small. First of all, let us observe that, thanks to (B1) and (4.6),

$$\tilde{v}(x,y) \geq \tilde{v}(x,0) - K \int_0^y \frac{r^\alpha}{u^a}\, du \geq -K \frac{r^{1+\alpha-a}}{1-a} \qquad \forall\, (x,y) \in \Gamma_r, \tag{4.7}$$

which proves the lower bound on $\tilde{v}$.

To prove the upper-bound, assume that there exists a point $(\bar{x}, \bar{y}) \in \Gamma_{r/8}$ such that $\tilde{v}(\bar{x},\bar{y}) \geq M r^{1+\alpha-a}$ for some large constant M. Arguing as above, this implies

$$\tilde{v}\left(\bar{x}, \frac{\eta_{n,a} r}{2}\right) \geq \frac{M}{4} r^{1+\alpha-a}, \tag{4.8}$$

provided M is sufficiently large (depending only on K, a and n). Now, let $B' := B'_{\frac{\eta_{n,a} r}{2}}\left(\bar{x}, \frac{\eta_{n,a} r}{2}\right)$ denote the $(n+1)$-dimensional ball of radius $\frac{\eta_{n,a} r}{2}$ centered at $\left(\bar{x}, \frac{\eta_{n,a} r}{2}\right) \in \mathbb{R}^n \times \mathbb{R}^+$, and set $B'/2 := B'_{\eta_{n,a} r/4}\left(\bar{x}, \frac{\eta_{n,a} r}{2}\right)$. Then $B' \subset \Gamma_r$ and $\left(0, \frac{\eta_{n,a} r}{2}\right) \in B'/2$.

Let $\bar{v}$ be as in (4.3). Thanks to (A1), $|\bar{v}-\tilde{v}| \le C_0\, r^2$ inside Γ_r (observe that $\eta_{n,a} \le 1$), which together with (4.7) implies that $\bar{v} + K\frac{r^{1+\alpha-a}}{1-a} + C_0 r^2$ is non-negative inside Γ_r. Hence we can apply Harnack inequality inside B' (see [6, Proposition 2.2] and [10]) to obtain

$$\frac{M}{8}\, r^{1+\alpha-a} \le \sup_{B'/2}\left[\bar{v} + K\frac{r^{1+\alpha-a}}{1-a} + C_0 r^2\right] \le C\left[\bar{v}\left(0, \frac{\eta_{n,a} r}{2}\right) + K\frac{r^{1+\alpha-a}}{1-a} + C_0 r^2\right],$$

that is $\bar{v}(0, \eta_{n,a} r/2) \ge c_0 M r^{1+\alpha-a} - K\frac{r^{1+\alpha-a}}{1-a} - C_0 r^2$ for some universal constant $c_0 > 0$, which gives

$$\tilde{v}\left(0, \frac{\eta_{n,a} r}{2}\right) \ge c_0 M\, r^{1+\alpha-a} - K\frac{r^{1+\alpha-a}}{1-a} - 2C_0 r^2.$$

Since $0 \in \Lambda$, combining the above estimate with property (B4) we get

$$0 = \tilde{v}(0,0) \ge \tilde{v}\left(0, \frac{\eta_{n,a} r}{2}\right) - \frac{nC_0}{1+a} r^2 \ge c_0 M\, r^{1+\alpha-a} - K\frac{r^{1+\alpha-a}}{1-a} - \left[\frac{nC_0}{1+a} + 2C_0\right] r^2,$$

which shows that M is universally bounded, as desired. □

4.2 Towards optimal regularity: a monotonicity formula

We use the same notation as in the previous subsection.

We have proved that $(-\Delta)^s v \chi_{\{v=\psi\}}$ grows at most as r^α near any free boundary point, which implies that $(-\Delta)^s v \chi_{\{v=\psi\}} \in C^\alpha_x(\mathbb{R}^n)$ (see Corollary 4.2). Consider now the function $w : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ obtained by solving the Dirichlet problem

$$\begin{cases} L_{-a} w = 0 & \text{on } \mathbb{R}^n \times \mathbb{R}^+, \\ w(x,0) = (-\Delta)^s v(x) \chi_{\{v=\psi\}}(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{4.9}$$

Since $w(x,0) \ge 0$, the maximum principle implies $w \ge 0$ everywhere.

Assume that $0 \in \mathbb{R}^n$ is a free boundary point. Since $(-\Delta)^s v(x)$ is globally bounded (see (A3)), using the Poisson representation formula for w [7, Subsection 2.4] together with the uniform C^α_x-regularity of $w(x,0)$ (Corollary 4.2) we get

$$\sup_{|x|^2+y^2 \le r^2} w(x,y) \le C\, r^\alpha,$$

for some uniform constant C. The goal of this subsection is to show that

$$\sup_{|x|^2+y^2 \le r^2} w(x,y) \le \tilde{C}\, r^{1-s}, \tag{4.10}$$

for some constant $\tilde{C} > 0$, depending on C_0, $\|(-\Delta)^s\psi\|_{C^{1-s}_x(\mathbb{R}^n)}$, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only.

This estimate will imply that $(-\Delta)^s v$ grows at most as $|x|^{1-s}$ at every free boundary point, so that the same proof as in Corollary 4.2 will give that $(-\Delta)^s v \chi_{\{v=\psi\}} \in C^{1-s}_x(\mathbb{R}^n)$, with a uniform bound. Then, in the next subsection we will apply this estimate to $v = u(t)$ for every

$t \in (0,T]$, and using (2.2) we will obtain the desired regularity result for u.

As in the previous subsection, we consider the function $\tilde{v}(x,y) = v(x,y) - \psi(x)$. Thanks to Theorem 4.1 together with the $(2C_0)$-semiconvexity of $\tilde{v}$ (see (B2) in the previous subsection), we can mimic the proof of [1, Lemma 5]:

Lemma 4.5. *Let $\bar{C} > 0$ and $\alpha \in (0, 1-s]$ be as in Theorem 4.1, and set $\delta_\alpha = \delta_\alpha(s) := \frac{1}{4}\left(\frac{\alpha}{\alpha+2s} - \frac{\alpha}{2}\right)$. Then there exists $r_0 = r_0(\alpha, s, \bar{C}, C_0) > 0$ such that the convex hull of the set $\{x \in \mathbb{R}^n : w(x,0) \geq r^{\alpha+\delta_\alpha}\}$ in $B_r \subset \mathbb{R}^n$ does not contain the origin for $r \leq r_0$.*

Proof. Thanks to (B3),

$$\tilde{v}(x,0) = 0 \quad \text{and} \quad \lim_{y\to 0^+} y^a \tilde{v}_y(x,y) = -w(x,0) \leq -r^{\alpha+\delta_\alpha} \qquad \forall\, x \in \{w(x,0) \geq r^{\alpha+\delta_\alpha}\}.$$

Hence, by the "a-semiconcavity" (B2) of $\tilde{v}$ in y, for any $x \in \{w(x,0) \geq r^{\alpha+\delta_\alpha}\}$ we have

$$\begin{aligned} \tilde{v}(x,h) &\leq \int_0^h \frac{s^a \tilde{v}_y(x,s)}{s^a}\, ds \\ &\leq -\int_0^h \frac{r^{\alpha+\delta_\alpha}}{s^a}\, ds + \int_0^h \frac{1}{s^a}\left(\int_0^s 2nC_0 \tau^a \, d\tau\right) ds \\ &= -\frac{1}{1-a} r^{\alpha+\delta_\alpha} h^{1-a} + \frac{nC_0}{1+a} h^2 = -\frac{1}{2s} r^{\alpha+\delta_\alpha} h^{2s} + \frac{nC_0}{1+a} h^2. \end{aligned} \tag{4.11}$$

On the other hand, Theorem 4.1 gives

$$\tilde{v}(0,h) = \tilde{v}(0,h) - \tilde{v}(0,0) \geq -\bar{C} h^{\alpha+2s}. \tag{4.12}$$

Assume now by contradiction that the convex hull of the set $\{(x,0) : w(x,0) \geq r^{\alpha+\delta_\alpha}\} \cap B_r$ contains $(0,0)$. Then, by the $(2C_0)$-semiconvexity of $\tilde{v}(\cdot,h)$ (see (B2)) we get

$$\tilde{v}(0,h) \leq \sup_{x \in \{w(x,0) \geq r^{\alpha+\delta_\alpha}\}} \tilde{v}(x,h) + C_0 r^2,$$

which together with (4.12) and (4.11) gives

$$\bar{C} h^{\alpha+2s} \geq \frac{1}{2s} r^{\alpha+\delta_\alpha} h^{2s} - \frac{nC_0}{1+a} h^2 - C_0 r^2$$

for all $r, h \in (0,1)$. To get a contradiction from the above inequality, we want to choose $h = h(r)$ in such a way that

$$h^2 \ll r^2 \ll h^{\alpha+2s} \ll r^{\alpha+\delta_\alpha} h^{2s} \qquad \text{for } r \text{ sufficiently small.}$$

To this aim, set $h = r^{1+2\delta_\alpha/\alpha}$. Then $h^\alpha = r^{\alpha+2\delta_\alpha} = o(r^{\alpha+\delta_\alpha})$, and both the first and the third condition above hold. To ensure that also the second one is satisfied, it suffices to have

$$(\alpha + 2s)\left(1 + 2\frac{\delta_\alpha}{\alpha}\right) < 2,$$

that is

$$\delta_\alpha < \left(\frac{\alpha}{\alpha+2s} - \frac{\alpha}{2}\right).$$

Recalling that $\alpha + 2s < 2$ (so, the right hand side is positive) and the definition of δ_α, we get the desired contradiction, which concludes the proof. $\square$

We now want to use a monotonicity formula to improve the decay of $w(x, y)$ at the origin. We first need some preliminary results:

Lemma 4.6. *(i) There exists a constant C', depending on C_0, $\|(-\Delta)^s\psi\|_{C^{1-s}_x(\mathbb{R}^n)}$, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that*

$$\limsup_{y\to 0^+} \int_{B_r} (w^2)_y(x,y) \frac{y^{-a}}{(|x|^2+y^2)^{(n-1+a)/2}}\, dx \geq -C' r^{\alpha+1+a} \qquad \forall\, r \geq 0.$$

(ii) It holds

$$\lim_{y\to 0^+} \int_{B_r} w^2(x,y)\partial_y \left(\frac{1}{(|x|^2+y^2)^{(n-1-a)/2}}\right) y^{-a}\, dx = 0.$$

Proof. *(i)* To show the estimate, let us observe that:

(1) Since $w(\cdot, 0) = 0$ on $\mathbb{R}^n \setminus \Lambda$ while $w(\cdot, 0) \geq 0$ on Λ (see (A6)), by the maximum principle we get $w(x, y) \geq 0$. Hence

$$w(x,y) \geq w(x,0) \qquad \forall\, x \in \mathbb{R}^n \setminus \Lambda,\ y > 0.$$

(2) By the a-semiconcavity of v in y (see (A8)),

$$y^a v_y(x,y) \leq \lim_{s\to 0^+} s^a v_y(x,s) + \frac{nC_0}{1+a} y^{1+a}.$$

(Observe that the above limit always exists, since $(-\Delta)^s v(x, 0)$ is Hölder continuous on the contact set, while v is smooth outside, see (A5).)

(3) The function $y^a v_y$ solves

$$\begin{cases} L_{-a}(y^a v_y) = 0, \\ \lim_{y\to 0^+} y^a v_y(x,y) = -(-\Delta)^s v(x,0), \end{cases}$$

(see [7, Subsection 2.3]). Since $w(x, 0) \geq (-\Delta)^s v(x, 0)$ by (A5), the maximum principle gives $w \geq -y^a v_y$ on $\mathbb{R}^n \times \mathbb{R}^+$. Hence, since $w(x, 0) = -\lim_{y\to 0^+} y^a v_y(x, y)$ in Λ, by (2) above we get

$$w(x,y) \geq w(x,0) - \frac{nC_0}{1+a} y^{1+a} \qquad \forall\, x \in \Lambda,\ y > 0.$$

Combining (1) and (3) we obtain

$$w(x,y) \geq w(x,0) - \frac{nC_0}{1+a} y^{1+a} \qquad \forall\, x \in \mathbb{R}^n,\ y > 0. \tag{4.13}$$

This estimate, together with the C_x^α regularity of w and the fact that w is non-negative, implies that, for all $x \in B_r$ and $y > 0$,

$$w^2(x,y) - w^2(x,0) = [w(x,y) - w(x,0)][w(x,y) + w(x,0)] \geq -Ky^{1+a}(r+y)^\alpha, \tag{4.14}$$

for some uniform constant $K > 0$.

We now want to estimate from below

$$\limsup_{y\to 0^+} \int_{B_r} (w^2)_y(x,y) \frac{y^{-a}}{(|x|^2+y^2)^{(n-1-a)/2}}\, dx.$$

To this aim, consider the change of variable $s = s(y) := \left(\frac{y}{1+a}\right)^{1+a}$ and define $\tilde{w}(x, s(y)) := w(x,y)$. Then (4.14) becomes

$$\tilde{w}^2(x,s) - \tilde{w}^2(x,0) \geq -K's(r + s^{1/(1+a)})^\alpha \qquad \forall\, x \in B_r,\ s > 0, \tag{4.15}$$

for some uniform constant $K' > 0$. Moreover, since $y^{-a}(w^2)_y(x,y) = (\tilde{w}^2)_s(x,s)$, we are left with estimating

$$\limsup_{s\to 0^+} \int_{B_r} (\tilde{w}^2)_s(x,s) \frac{1}{(|x|^2 + (1+a)^2 s^{2/(1+a)})^{(n-1-a)/2}}\, dx.$$

To do this, we average the above expression with respect to $s \in [0,\varepsilon]$ and we use Fubini Theorem to get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^\varepsilon ds \int_{B_r} (\tilde{w}^2)_s(x,s) \frac{1}{(|x|^2 + (1+a)^2 s^{2/(1+a)})^{(n-1-a)/2}}\, dx \\
&\quad = \int_{B_r} \frac{1}{\varepsilon}\left[\frac{\tilde{w}^2(x,\varepsilon)}{(|x|^2 + (1+a)^2 \varepsilon^{2/(1+a)})^{(n-1-a)/2}} - \frac{\tilde{w}^2(x,0)}{|x|^{n-1-a}}\right] dx \\
&\qquad - \frac{1}{\varepsilon}\int_0^\varepsilon ds \int_{B_r} \tilde{w}^2(x,s) \frac{d}{ds}\left(\frac{1}{(|x|^2 + (1+a)^2 s^{2/(1+a)})^{(n-1-a)/2}}\right) dx.
\end{aligned}$$

Now, thanks to (4.15), the C_x^α-regularity of $w(x,0) = \tilde{w}(x,0)$, and the fact that

$$\frac{d}{ds}\left(\frac{1}{(|x|^2 + (1+a)^2 s^{2/(1+a)})^{(n-1-a)/2}}\right) \leq 0,$$

we obtain that the above expression is bounded from below by

$$\begin{aligned}
&\int_{B_r} \frac{1}{\varepsilon}\left[\frac{\tilde{w}^2(x,0) - K'\varepsilon(r + \varepsilon^{1/(1+a)})^\alpha}{(|x|^2 + (1+a)^2 \varepsilon^{2/(1+a)})^{(n-1-a)/2}} - \frac{\tilde{w}^2(x,0)}{|x|^{n-1-a}}\right] dx \\
&\quad \geq -K'(r + \varepsilon^{1/(1+a)})^\alpha \int_{B_r} \frac{1}{|x|^{n-1-a}}\, dx \\
&\qquad + C\int_{B_r} \frac{|x|^{2\alpha}}{\varepsilon}\left[\frac{1}{(|x|^2 + (1+a)^2 \varepsilon^{2/(1+a)})^{(n-1-a)/2}} - \frac{1}{|x|^{n-1-a}}\right] dx
\end{aligned}$$

Concerning the first term in the right hand side, since $a = 1 - 2s < 1$ we have

$$(r + \varepsilon^{1/(1+a)})^\alpha \int_{B_r} \frac{1}{|x|^{n-1-a}}\,dx \to C_{n,a} r^{\alpha+1+a} = C_{n,a} r^{\alpha+1+a} \qquad \text{as } \varepsilon \to 0.$$

For the second term, we want to prove that it converges to 0 as $\varepsilon \to 0$. To this aim, we split the integral into two terms: the integral over B_{ε^β}, and the one over $B_r \setminus B_{\varepsilon^\beta}$, where $\beta > 0$ has to be chosen. For the first term, we can bound it from below by

$$-\frac{C}{\varepsilon} \int_{B_{\varepsilon^\beta}} \frac{|x|^{2\alpha}}{|x|^{n-1-a}}\,dx = C\varepsilon^{\beta(2\alpha+a+1)-1}.$$

Thus, by choosing $\beta \in \left(\frac{1}{2\alpha+a+1}, \frac{1}{1+a}\right)$ we ensure that the above expression converges to 0 as $\varepsilon \to 0$. Moreover, the fact that $\beta < 1/(1+a)$ implies that

$$\varepsilon^{2/(1+a)} \ll |x|^2 \qquad \forall\, |x| \geq \varepsilon^\beta.$$

Therefore, for estimating the second part we can use polar coordinates and the fact that

$$(|x|^2 + (1+a)^2\varepsilon^{2/(1+a)})^{(n-1-a)/2} \sim |x|^{n-1-a} + C\varepsilon^{2/(1+a)}|x|^{n-3-a} \qquad \forall\, |x| \geq \varepsilon^\beta$$

to write

$$\begin{aligned}
&\int_{B_r \setminus B_{\varepsilon^\beta}} \frac{|x|^{2\alpha}}{\varepsilon} \left[\frac{1}{(|x|^2 + (1+a)^2\varepsilon^{2/(1+a)})^{(n-1-a)/2}} - \frac{1}{|x|^{n-1-a}}\right] dx \\
&\quad \sim \frac{C}{\varepsilon} \int_{\varepsilon^\beta}^r \rho^{n-1+2\alpha} \left[\frac{1}{\rho^{n-1-a} + C\varepsilon^{2/(1+a)}\rho^{n-3-a}} - \frac{1}{\rho^{n-1-a}}\right] d\rho \\
&\quad = \frac{C}{\varepsilon} \int_{\varepsilon^\beta}^r \rho^{2\alpha+a} \left[\frac{\rho^2}{\rho^2 + C\varepsilon^{2/(1+a)}} - 1\right] d\rho = -\frac{C}{\varepsilon} \int_{\varepsilon^\beta}^r \rho^{2\alpha+a} \frac{\varepsilon^{2/(1+a)}}{\rho^2 + C\varepsilon^{2/(1+a)}}\,d\rho \\
&\quad \geq -\frac{C\varepsilon^{2/(1+a)}}{\varepsilon} \int_{\varepsilon^\beta}^r \rho^{2\alpha+a-2}\,d\rho \geq -C\varepsilon^{2/(1+a)-1}\big[1 + \varepsilon^{\beta(2\alpha+a-1)}\big].
\end{aligned}$$

Let us remark that $2/(1+a) > 1$, so if $2\alpha + a - 1 \geq 0$ the above expression obviously converges to 0. On the other hand, if $2\alpha + a - 1 < 0$, since $\beta < 1/(1+a)$ we get

$$\frac{2}{1+a} - 1 + \beta(2\alpha + a - 1) > \frac{2}{1+a} - 1 + \frac{2\alpha + a - 1}{1+a} \geq \frac{2 - 1 - a + 2\alpha + a - 1}{1+a} = \frac{2\alpha}{1+a} > 0,$$

and again the above expression converges to 0. All in all, we conclude that

$$\int_{B_r} \frac{|x|^{2\alpha}}{\varepsilon} \left[\frac{1}{(|x|^2 + (1+a)^2\varepsilon^{2/(1+a)})^{(n-1-a)/2}} - \frac{1}{|x|^{n-1-a}}\right] dx \to 0 \qquad \text{as } \varepsilon \to 0,$$

so that combining all our estimates together we obtain

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon ds \int_{B_r} (\tilde{w}^2)_s(x,s) \frac{1}{(|x|^2 + (1+a)^2 s^{2/(1+a)})^{(n-1-a)/2}}\,dx \geq -K' C_{n,a} r^{\alpha+1+a}.$$

From this fact we easily deduce that

$$\limsup_{\varepsilon\to 0}\int_{B_r}(\tilde{w}^2)_s(x,\varepsilon)\frac{1}{(|x|^2+(1+a)^2\varepsilon^{2/(1+a)})^{(n-1-a)/2}}\,dx \geq -K'C_{n,a}r^{\alpha+1+a},$$

which concludes the proof of (i).
(ii) In this case, we use the C^α_x-regularity of w to control the integral by

$$y^{1-a}\int_{B_r}\frac{1}{(|x|^2+y^2)^{(n+1-a)/2-\alpha}}\,dx.$$

Using polar coordinates, the above integral is comparable to

$$y^{1-a}\int_0^r\frac{\rho^{n-1}}{(\rho^2+y^2)^{(n+1-a)/2-\alpha}}\,d\rho \sim y^{1-a}\int_0^r\frac{\rho^{n-1}}{(\rho+y)^{n+1-a-2\alpha}}\,d\rho \sim \frac{y^{1-a}}{y^{1-a-2\alpha}} = y^{2\alpha},$$

and the above expression converges to 0 as $y\to 0$. □

We will also need a result on the first eigenvalue of a weighted Laplacian on the half-sphere. We use $\nabla_\theta w$ to denote the derivative of w with respect to the angular variables, and we denote with $\mathbb{S}^n$ the n-dimensional sphere in $\mathbb{R}^{n+1}$.

Lemma 4.7. *Set $\mathbb{S}^n_+ := \mathbb{S}^n\cap\{x_{n+1}\geq 0\}$, $\mathbb{S}^n_0 := \partial\mathbb{S}^n_+ = \mathbb{S}^n\cap\{x_{n+1}=0\}$, $\mathbb{S}^n_{0,+} := \mathbb{S}^n\cap\{x_{n+1}=0\}\cap\{x_n\geq 0\}$. Then*

$$\inf_{h\in H^{1/2}(\mathbb{S}^n_0),\,h=0 \text{ on } \mathbb{S}^n_{0,+}}\frac{\int_{\mathbb{S}^n_+}|\nabla_\theta h|^2 y^{-a}\,d\sigma}{\int_{\mathbb{S}^n_+}h^2y^{-a}\,d\sigma} = (1-s)(n-1+s).$$

Proof. Let $\bar{h}(\theta)$ denote the restriction to $\mathbb{S}^n_+$ of

$$\bar{H}(x,y) := \left(\sqrt{x_n^2+y^2}-x_n\right)^{1-s},$$

that is $\bar{H} = r^{1-s}\bar{h}(\theta)$. As shown in [6, Proposition 5.4], $\bar{h}$ is the first eigenfunction corresponding to the above minimization problem. Moreover, $\bar{H}$ solves $L_{-a}\bar{H}=0$ for $y>0$.[4] Let λ_1 denote the eigenvalue corresponding to $\bar{h}$, so that

$$\inf_{h\in H^{1/2}(\mathbb{S}^n_0),\,h=0 \text{ on } \mathbb{S}^n_{0,+}}\frac{\int_{\mathbb{S}^n_+}|\nabla_\theta h|^2 y^{-a}\,d\sigma}{\int_{\mathbb{S}^n_+}h^2y^{-a}\,d\sigma} = -\lambda_1.$$

[4] A simple way to check this fact is the following: the function $G := \left(\sqrt{x_n^2+y^2}-x_n\right)^{1/2}$ is harmonic inside $y>0$, since it is equal to the imaginary part of the holomorphic function $z\mapsto z^{1/2}$, $z=x_n+iy$. Moreover, by a direct computation it is easily checked that G satisfies

$$|\nabla_x G|^2+(G_y)^2-\frac{GG_y}{y}=0 \qquad \forall\,(x,y)\in\mathbb{R}^n\times\mathbb{R}^+.$$

Thanks to this fact, since $\bar{H}=G^{2(1-s)}=G^{1+a}$, we get

$$L_{-a}\bar{H} = L_{-a}\big(G^{1+a}\big) = (1+a)G^a\Delta_{x,y}G+(1+a)a\,G^{a-1}\left[|\nabla_x G|^2+(G_y)^2-\frac{GG_y}{y}\right]=0,$$

as desired.

In order to explicitly compute λ_1, we observe that $\mathrm{div}_\theta(y^{-a}\nabla_\theta \bar h) = \lambda_1 \bar h$. In particular, evaluating the above identity at the point $(0,1) \in \mathbb{R}^n \times \mathbb{R}^+$ we obtain

$$\Delta_\theta \bar h(0,1) = \lambda_1 \bar h(0,1).$$

We now write the equation $L_{-a}\bar H = 0$ in spherical coordinates:

$$\Delta_r \bar H + \frac{n}{r}\bar H_r + \frac{1}{r^2}\Delta_\theta \bar H - \frac{a}{y}\bar H_y = 0$$

Evaluating the above expression at $(0,1) \in \mathbb{R}^n \times \mathbb{R}^+$ and recalling that $\bar H = r^{1-s}\bar h$, we get

$$\begin{aligned} 0 &= \Delta_r \bar H(0,1) + n\bar H_r(0,1) + \Delta_\theta \bar H(0,1) - a\bar H_r(0,1) \\ &= -(1-s)s\,\bar h(0,1) + (1-s)(n-a)\,\bar h(0,1) + \Delta_\theta \bar h(0,1). \end{aligned}$$

Hence

$$\Delta_\theta \bar h(0,1) = -(1-s)(n-a-s)\,\bar h(0,1) = -(1-s)(n-1+s)\,\bar h(0,1).$$

which gives $\lambda_1 = -(1-s)(n-1+s)$ as desired. □

To simplify notation, we use the variable z to denote a point $(x,y) \in \mathbb{R}^n \times \mathbb{R}^+$.

Lemma 4.8. *Let w be as above, set $B_r^+ := \{z \in \mathbb{R}^n \times \mathbb{R}^+ : |z| < r\}$, and define*

$$\varphi(r) := \frac{1}{r^{2(1-s)}}\int_{B_r^+} \frac{|\nabla w(z)|^2 y^{-a}}{|z|^{n-1-a}}\,dz \qquad \forall\, r \le 1.$$

There exists a constant C'', depending on C_0, $\|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)}$, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that

$$\varphi(r) \le C''\left[1 + r^{2\alpha+\delta_\alpha-a-1}\right] \qquad \forall\, r \le 1.$$

Proof. First of all, we show that $\varphi(1)$ is universally bounded, so that in particular $\varphi(r)$ is well-defined for all $r \in (0,1]$.

Set $\varphi_\varepsilon(r) := \frac{1}{r^{2(1-s)}}\int_{B_r^+\cap\{y>\varepsilon\}} \frac{|\nabla w(z)|^2 y^{-a}}{|z|^{n-1-a}}\,dz$. By the monotone convergence theorem, it suffices to estimate $\liminf_{\varepsilon\to 0}\varphi_\varepsilon(1)$. Let $\chi : \mathbb{R}^n \to [0,1]$ be a smooth compactly supported function such that $\chi \equiv 1$ on $B_1 \subset \mathbb{R}^n$. Then

$$\varphi_\varepsilon(r) \le \int_\varepsilon^1 \int_{\mathbb{R}^n} \frac{|\nabla w(z)|^2 y^{-a}}{|z|^{n-1-a}}\chi(x)\,dx\,dy.$$

Since $L_{-a}w = 0$ we have $L_{-a}(w^2) = 2wL_{-a}w + 2|\nabla w|^2 y^{-a} = 2|\nabla w|^2 y^{-a}$, which implies that the right hand side is equal to

$$\begin{aligned} \int_\varepsilon^1 \int_{\mathbb{R}^n} \frac{L_{-a}(w^2)}{2|z|^{n-1-a}}\chi(x)\,dx\,dy &= -\int_\varepsilon^1 \int_{\mathbb{R}^n} \nabla(w^2)\cdot\nabla\left(\frac{1}{2|z|^{n-1-a}}\right) y^{-a}\chi(x)\,dx\,dy \\ &\quad - \int_\varepsilon^1 \int_{\mathbb{R}^n} \nabla_x(w^2)\cdot\nabla_x\chi(x)\frac{y^{-a}}{2|z|^{n-1-a}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^n} (w^2)_y \frac{y^{-a}}{2|z|^{n-1-a}}\chi(x)\,dx\Big|_{y=\varepsilon}^{y=1}. \end{aligned}$$

Integrating by parts again the first two terms in the right hand side, and using that $L_{-a}\frac{1}{|z|^{n-1-a}} = C_{n,a}\delta_{(0,0)}$, we find that the above expression coincides with

$$\int_\varepsilon^1 \int_{\mathbb{R}^n} w^2 \nabla_x \chi(x) \cdot \nabla_x \left(\frac{1}{|z|^{n-1-a}}\right) y^{-a}\, dx\, dy + \int_\varepsilon^1 \int_{\mathbb{R}^n} w^2 \Delta_x \chi(x) \frac{y^{-a}}{2|z|^{n-1-a}}\, dx\, dy$$
$$- \int_{\mathbb{R}^n} w^2 \partial_y \left(\frac{1}{2|z|^{n-1-a}}\right) y^{-a} \chi(x)\, dx \Bigg|_{y=\varepsilon}^{y=1} + \int_{\mathbb{R}^n} (w^2)_y \frac{y^{-a}}{2|z|^{n-1-a}} \chi(x)\, dx \Bigg|_{y=\varepsilon}^{y=1}.$$

Now, since $\chi \equiv 1$ inside B_1, the first two terms above are immediately seen to be bounded. Concerning the last two terms, the integrals evaluated at $y = 1$ are clearly finite (and universally bounded), since w is smooth for $y > 0$. Finally, we apply Lemma 4.6 to estimate the integrals at $y = \varepsilon$, and we obtain

$$\varphi(1) = \liminf_{\varepsilon \to 0} \varphi_\varepsilon(1) \leq C_\varphi,$$

for some constant C_φ depending on C_0, $\|(-\Delta)^s \psi\|_{C_x^{1-s}(\mathbb{R}^n)}$, $\|v - \psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only. Observe that, as a consequence of the fact that $\varphi(1)$ is finite (i.e., $\frac{|\nabla w(z)|^2 y^{-a}}{|z|^{n-1-a}}$ is integrable over B_1^+), we deduce that $\varphi_\varepsilon(r) \to \varphi(r)$ locally uniformly over $(0, 1]$.

Now that we have proved that $\varphi(r)$ is well-defined, we want to estimate from below its derivative. Again, we will do our computations with φ_ε, and then we let $\varepsilon \to 0$[5].

Let us assume $r > \varepsilon$, and split $\partial\left(B_r^+ \cap \{y > \varepsilon\}\right)$ as the union of $\partial B_r^+ \cap \{y = \varepsilon\}$ and $\partial B_r^+ \cap \{y >$

[5]The proof of the monotonicity formula may look a bit tedious, since we always prove the result at the ϵ level, and then we show that one can take the limit as $\varepsilon \to 0$. Let us point out that this level of precision is actually needed: indeed, assume that we had chosen a different operator L_b ($b \in (-1, 1)$) to define w in (4.9), and we defined $\varphi(r)$ replacing $-a$ by b (changing, of course, the value of s correspondingly). Then, if one does a "formal" proof of the monotonicity formula, one would obtain (at least in the stationary case, so that $w(x, 0) = (-\Delta)^s v(x)$) that Lemma 4.8 is true with b in place of $-a$, and this would imply a false Hölder regularity for w (since we know that w should be only C^{1-s}). The fact that we have chosen the "right" operator L_{-a} to define w has played a key role in the proof of Lemma 4.6, which is now providing to us some fundamental estimates, which are needed to give a rigorous proof of the monotonicity formula.

$\varepsilon\}$. Using again that $L_{-a}\frac{1}{|z|^{n-1-a}} = C_{n,a}\delta_{(0,0)}$ and recalling that $a = 1-2s$, we easily get

$$
\begin{aligned}
\varphi_\varepsilon'(r) &= -\frac{1-s}{r^{1+2(1-s)}}\int_{B_r^+\cap\{y>\varepsilon\}} \frac{L_{-a}(w^2)}{|z|^{n-1-a}}\,dz + \frac{1}{r^n}\int_{\partial B_r^+\cap\{y>\varepsilon\}} |\nabla w(z)|^2 y^{-a}\,d\sigma \\
&= -\frac{2(1-s)}{r^{1+2(1-s)}}\int_{\partial(B_r^+\cap\{y>\varepsilon\})} w\,\nabla w\cdot\nu\,\frac{y^{-a}}{|z|^{n-1-a}}\,d\sigma \\
&\quad + \frac{1-s}{r^{1+2(1-s)}}\int_{B_r^+\cap\{y>\varepsilon\}} \nabla(w^2)\cdot\nabla\left(\frac{1}{|z|^{n-1-a}}\right) y^{-a}\,dz + \frac{1}{r^n}\int_{\partial B_r^+\cap\{y>\varepsilon\}} |\nabla w(z)|^2 y^{-a}\,d\sigma \\
&= -\frac{2(1-s)}{r^{1+2(1-s)}}\int_{\partial(B_r^+\cap\{y>\varepsilon\})} w\,\nabla w\cdot\nu\,\frac{y^{-a}}{|z|^{n-1-a}}\,d\sigma \\
&\quad + \frac{1-s}{r^{1+2(1-s)}}\int_{\partial(B_r^+\cap\{y>\varepsilon\})} w^2\nabla\left(\frac{1}{|z|^{n-1-a}}\right)\cdot\nu\,y^{-a}\,d\sigma \\
&\quad + \frac{1}{r^n}\int_{\partial B_r^+\cap\{y>\varepsilon\}} |\nabla w(z)|^2 y^{-a}\,d\sigma \\
&= -\frac{2(1-s)}{r^{n+1}}\int_{\partial B_r^+\cap\{y>\varepsilon\}} w\,w_r\,y^{-a}\,d\sigma + \frac{1-s}{r^{1+2(1-s)}}\int_{B_r^+\cap\{y=\varepsilon\}} (w^2)_y\,\frac{y^{-a}}{|z|^{n-1-a}}\,d\sigma \\
&\quad - \frac{(1-s)(n-1-a)}{r^{n+2}}\int_{\partial B_r^+\cap\{y>\varepsilon\}} w^2 y^{-a}\,d\sigma \\
&\quad - \frac{1-s}{r^{1+2(1-s)}}\int_{B_r^+\cap\{y=\varepsilon\}} w^2\partial_y\left(\frac{1}{|z|^{n-1-a}}\right) y^{-a}\,d\sigma + \frac{1}{r^n}\int_{\partial B_r^+\cap\{y>\varepsilon\}} |\nabla w(z)|^2 y^{-a}\,d\sigma.
\end{aligned}
$$

Thanks to Lemma 4.6, we can estimate from below both the second and the last but one term in the last expression. So, letting $\varepsilon\to 0$ and using that $\varphi_\varepsilon\to\varphi$ locally uniformly, we deduce that the distributional derivative $D_r\varphi$ of φ is bounded from below by

$$
\begin{aligned}
&-\frac{2(1-s)}{r^{n+1}}\int_{\partial B_{r,+}} w\,w_r\,y^{-a}\,d\sigma - C\,r^{\alpha-1} \\
&\qquad\qquad - \frac{(1-s)(n-1-a)}{r^{n+2}}\int_{\partial B_{r,+}} w^2 y^{-a}\,d\sigma + \frac{1}{r^n}\int_{\partial B_{r,+}} |\nabla w(z)|^2 y^{-a}\,d\sigma,
\end{aligned}
$$

for some universal constant C. Now, by Schwartz's inequality the first term in the above expression can be estimated from below by

$$
-\frac{1}{r^n}\int_{\partial B_{r,+}} (w_r)^2 y^{-a}\,d\sigma - \frac{(1-s)^2}{r^{n+2}}\int_{\partial B_{r,+}} w^2 y^{-a}\,d\sigma.
$$

Hence, recalling that $|\nabla w(z)|^2 = (w_r)^2 + \frac{1}{r^2}|\nabla_\theta w|^2$ and observing that $n-1-a+1-s = n-1+s$, we obtain

$$
D_r\varphi \ge \frac{1}{r^{n+2}}\int_{\partial B_{r,+}} |\nabla_\theta w(z)|^2 y^{-a}\,d\sigma - \frac{(1-s)(n-1+s)}{r^{n+2}}\int_{\partial B_{r,+}} w^2 y^{-a}\,d\sigma - C\,r^{\alpha-1}.
$$

Consider now the function $\bar W := (w - r^{\alpha+\delta_\alpha})^+$. Then $|\nabla_\theta \bar W(z)|^2 \le |\nabla_\theta w(z)|^2$. Moreover, by Lemma 4.5, $\bar W$ is admissible for the eigenvalue problem in Lemma 4.7. Hence

$$
\begin{aligned}
D_r\varphi &\ge \frac{1}{r^{n+2}}\int_{\partial B_{r,+}} |\nabla_\theta \bar W(z)|^2 y^{-a}\, d\sigma - \frac{(1-s)(n-1+s)}{r^{n+2}}\int_{\partial B_{r,+}} \bar W^2 y^{-a}\, d\sigma \\
&\quad + \frac{(1-s)(n-1+s)}{r^{n+2}}\int_{\partial B_{r,+}} \left[(w-\bar W)^2 + 2\bar W(w-\bar W)\right] y^{-a}\, d\sigma - C\, r^{\alpha-1} \\
&\ge \frac{(1-s)(n-1+s)}{r^{n+2}}\int_{\partial B_{r,+}} \left[(w-\bar W)^2 + 2\bar W(w-\bar W)\right] y^{-a}\, d\sigma - C\, r^{\alpha-1}.
\end{aligned}
$$

Since $|\bar W| \le |w| \le C\, r^{\alpha}$ and $|w - \bar W| \le r^{\alpha+\delta_\alpha}$ we obtain

$$
D_r\varphi \ge -C\, r^{2\alpha+\delta_\alpha-a-2} - C r^{\alpha-1},
$$

which integrated over $[r,1]$ gives

$$
\varphi(r) \le \varphi(1) + C\, r^{2\alpha+\delta_\alpha-a-1} + C \qquad \forall\, r \le 1.
$$

(Recall that $1+a>0$.) Since $\varphi(1)$ is universally bounded, this concludes the proof. $\square$

We are now ready to prove the optimal decay rate around free boundary points.

Proposition 4.9. *There exists a constant $\tilde C > 0$, depending on C_0, $\|(-\Delta)^s\psi\|_{C^{1-s}_x(\mathbb{R}^n)}$, $\|v - \psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that* (4.10) *holds.*

Proof. Define $w_\varepsilon = w * \rho_\varepsilon$, where $\rho_\varepsilon = \rho_\varepsilon(x)$ is a smooth convolution kernel. Since L_{-a} commutes with convolution in the x variable, w_ε is L_{-a}-harmonic on $\mathbb{R}^n \times \mathbb{R}^+$. Moreover, by (4.13), $w_\varepsilon(x,y) - w_\varepsilon(x,0) \ge -\frac{nC_0}{1+a} y^{1+a}$.

Set $\bar W_\varepsilon := (w_\varepsilon - r^{\alpha+\delta_\alpha})^+$. Then it is easily seen that $\bar W_\varepsilon$ is L_{-a}-subharmonic for $y>0$, and $\bar W_\varepsilon(x,y) - \bar W_\varepsilon(x,0) \ge -\frac{nC_0}{1+a} y^{1+a}$. Consider now the function

$$
\tilde w_\varepsilon(x,y) := \bar W_\varepsilon(x,|y|) + \left(1 + \frac{nC_0}{1+a}\right) |y|^{1+a} \qquad \text{on } \mathbb{R}^n \times \mathbb{R}.
$$

We observe that $\tilde w_\varepsilon$ is L_{-a}-subharmonic outside $\{y=0\}$. Moreover, since $\tilde w_\varepsilon(x,y) - \tilde w_\varepsilon(x,0) \ge |y|^{1+a}$ and $\tilde w_\varepsilon$ is smooth in the x variable, we deduce that it is L_{-a}-subharmonic on the whole $\mathbb{R}^n \times \mathbb{R}$. Letting $\varepsilon \to 0$ we obtain that

$$
\tilde w(x,y) := \big(w(x,|y|) - r^{\alpha+\delta_\alpha}\big)^+ + \left(1 + \frac{nC_0}{1+a}\right) |y|^{1+a}
$$

is globally L_{-a}-subharmonic. Thanks to Lemma 4.5, $\tilde w$ vanishes on more than half of the n-dimensional disc $B_r \times \{0\}$. So we can apply a weighted Poincaré inequality (see [10]) and the definition of φ (see Lemma 4.8) to get

$$\begin{aligned}
\int_{B_r^+} (\tilde w)^2 y^{-a}\,dz &\le C\,r^2 \int_{B_r^+} |\nabla \tilde w|^2 y^{-a}\,dz \\
&\le C\,r^2 \biggl[\int_{B_r^+} |\nabla w|^2 y^{-a}\,dz + \,r^{n+1+a}\biggr] \\
&\le C\,r^{n+2}\bigl[\varphi(r) + r^{1+a}\bigr] \le C\,r^{n+2}\bigl[\varphi(r)+1\bigr] \qquad \forall\, r \le 1,
\end{aligned}$$

which combined with the L_{-a}-subharmonicity of $\tilde w$ and Lemma 4.8 gives

$$\begin{aligned}
\sup_{B_{r/2}^+} (\tilde w)^2 &\le \frac{C}{r^{n+1-a}} \int_{B_r^+} (\tilde w)^2 y^{-a}\,dz \\
&\le C\,r^{1+a}\bigl[\varphi(r)+1\bigr] \le C\,\Bigl[r^{1+a} + r^{2\alpha+\delta_\alpha}\Bigr].
\end{aligned}$$

Hence, since $1+a = 2(1-s)$ we obtain

$$\sup_{B_{r/2}^+} w \le C\biggl[\sup_{B_{r/2}^+} \tilde w + r^{\alpha+\delta_\alpha} + r^{1+a}\biggr] \le C\,\Bigl[r^{1-s} + r^{\alpha+\delta_\alpha/2}\Bigr] \qquad \forall\, r \le 1.$$

Since the above bound holds at every free boundary point, by the very same argument as in the proof of Corollary 4.2 we obtain that $\|w\|_{C_x^{\beta_\alpha}(\mathbb{R}^n)} \le C$, where $\beta_\alpha = \beta_\alpha(s) := \min\{\alpha + \delta_\alpha/2, (1-s)\}$. Observe now that, by the formula for δ_α provided in Lemma 4.5, given $\alpha_0 > 0$ there exists $\delta_0 > 0$ such that $\delta_\alpha \ge \delta_0 > 0$ for $\alpha \in [\alpha_0, 1-s]$. Hence, by iterating the above argument k times we get

$$\sup_{B_{r/2}^+} \tilde w \le C\,\Bigl[r^{1-s} + r^{\alpha+k\delta_0/2}\Bigr] \qquad \forall\, r \le 1,$$

and after finitely many iterations we obtain (4.10). □

Arguing as in the proof of Corollary 4.2, (4.10) gives:

Corollary 4.10. *There exists a constant $\bar C'' > 0$, depending on C_0, $\|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)}$, $\|v-\psi\|_{L^\infty(\mathbb{R}^n)}$, and $\|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^n)}$ only, such that*

$$\|(-\Delta)^s v \chi_{\{v=\psi\}}\|_{C_x^{1-s}(\mathbb{R}^n)} \le \bar C''.$$

4.3 Almost optimal regularity of solutions to the parabolic fractional obstacle problem

Let u be a solution of (2.2), with $\psi \in C^2(\mathbb{R}^n)$ satisfying assumptions (A1)-(A2) of Subsection 4.1. Let us remark that $u(0) = \psi$, while for all $t > 0$ we can apply the results of the previous subsections with $v = u(t)$. Hence Corollary 4.10 gives:

Proposition 4.11. *Let u, ψ be as above. Then there exists a constant $\bar{C}_T > 0$, depending on T, $\|D^2\psi\|_{L^\infty(R^n)}$, and $\|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)}$ only, such that*

$$\sup_{t\in[0,T]} \|(-\Delta)^s u(t)\chi_{\{u(t)=\psi\}}\|_{C_x^{1-s}(\mathbb{R}^n)} \le \bar{C}_T.$$

Proof. As explained at the beginning of Subsection 4.1, $u(t)$ satisfies assumptions (A5)-(A6) of Subsection 4.1 for every $t > 0$ (see Lemma 3.5). Moreover, (A3)-(A4) follow from Lemma 3.3 (since $\|u(t) - \psi\|_{L^\infty(\mathbb{R}^n)} \le T\|u_t\|_{L^\infty([0,T]\times\mathbb{R}^n)} \le T\|(-\Delta)^s\psi\|_{L^\infty(\mathbb{R}^n)}$). Hence the result is an immediate consequence of Corollary 4.10 applied to $v = u(t)$ for any $t > 0$. (For $t = 0$ the result is trivial since $u(0) = \psi$.) □

Now, we want to exploit the fact that u solves the parabolic equation

$$u_t + (-\Delta)^s u = \big((-\Delta)^s u\big)\chi_{\{u=\psi\}} \qquad \text{on } (0,T] \times \mathbb{R}^n. \tag{4.16}$$

Thanks to Proposition 4.11, the right-hand side of (4.16) belongs to $L^\infty([0,T]; C_x^{1-s}(\mathbb{R}^n))$, which by parabolic regularity implies

$$u_t, (-\Delta)^s u \in L^\infty((0,T]; C_x^{1-s-0^+}(\mathbb{R}^n)), \tag{4.17}$$

see (A.7). We now want to use (4.16) and a bootstrap argument to obtain the desired Hölder regularity in time. We start with a preliminary result:

Lemma 4.12. *Let that $(-\Delta)^s u\,\chi_{\{u=\psi\}} \in L^\infty((0,T]; C_x^{1-s}(\mathbb{R}^n))$. Fix $\alpha \in \big[0, \min\big\{1, \frac{1-s}{2s}\big\}\big)$, and assume that:*

- $u_t \in L^\infty((0,T]\times\mathbb{R}^n)$, $(-\Delta)^s u \in L^\infty((0,T]; C_x^{1-s-0^+}(\mathbb{R}^n))$ *if* $\alpha = 0$*;*
- $u_t \in C_{t,x}^{\alpha,1-s}((0,T]\times\mathbb{R}^n)$, $(-\Delta)^s u \in L^\infty((0,T]; C_x^{1-s}(\mathbb{R}^n))$ *if* $\alpha > 0$*.*

Then

$$(-\Delta)^s u\,\chi_{\{u=\psi\}} \in \begin{cases} C_{t,x}^{\frac{1-s}{1+s}-0^+,1-s}((0,T]\times\mathbb{R}^n) & \text{if } \alpha = 0, \\ \mathrm{Lip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n) & \text{if } \alpha > 0,\ s < 1/3, \\ \mathrm{logLip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n) & \text{if } \alpha > 0,\ s = 1/3, \\ C_{t,x}^{(1+\alpha)\frac{1-s}{1+s},1-s}((0,T]\times\mathbb{R}^n) & \text{if } \alpha > 0,\ s > 1/3, \end{cases}$$

with a uniform bound.

Moreover, for $s > 1/3$ the function

$$\alpha \mapsto \Phi(\alpha) := (1+\alpha)\frac{1-s}{1+s}$$

is strictly increasing on $\big[0, \frac{1-s}{2s}\big)$, and $\Phi\big(\frac{1-s}{2s}\big) = \frac{1-s}{2s}$.

Proof. We need to estimate

$$|(-\Delta)^s u(t,x)\chi_{\{u(t)=\psi\}} - (-\Delta)^s u(s,x)\chi_{\{u(s)=\psi\}}|, \qquad 0 < s < t. \tag{4.18}$$

Since $\{u(t) = \psi\} \subset \{u(s) = \psi\}$ for $s < t$, we can assume that $x \in \{u(s) = \psi\}$ (otherwise the above expression vanishes and there is nothing to prove). Moreover, since $(-\Delta)^s u$ vanishes on the free boundary, if $x \in \{u(s) = \psi\} \setminus \{u(t) = \psi\}$ we can alway find a time $\tau \in (s,t)$ such that

$$(-\Delta)^s u(\tau,x)\chi_{\{u(\tau)=\psi\}} = (-\Delta)^s u(t,x)\chi_{\{u(t)=\psi\}} \quad \text{and} \quad x \in \partial\{u(\tau) = \psi\}.$$

Then, if we can estimate (4.18) with τ in place of t, then we will also get the desired bound by simply replacing τ with t. Hence, we only need to consider the case $x \in \{u(t) = \psi\}$.

We have to estimate $|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)|$. Let ϕ be a smooth non-negative cut-off function supported in B_1 such that $\int_{\mathbb{R}^n} \phi = 1$, set $\phi_r(x) := \frac{1}{r^n}\phi\left(\frac{x}{r}\right)$, and compute

$$\begin{aligned} |(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)| \le & \left| \int_{\mathbb{R}^n} [(-\Delta)^s u(t,x) - (-\Delta)^s u(t,z)]\phi_r(x-z)\,dz \right| \\ & + \left| \int_{\mathbb{R}^n} [(-\Delta)^s u(t,z) - (-\Delta)^s u(s,z)]\phi_r(x-z)\,dz \right| \\ & + \left| \int_{\mathbb{R}^n} [(-\Delta)^s u(s,x) - (-\Delta)^s u(s,z)]\phi_r(x-z)\,dz \right|, \end{aligned} \tag{4.19}$$

We now distinguish between two cases:

• $\alpha = 0$. Thanks to the $C_x^{1-s-0^+}$-regularity of $(-\Delta)^s u$ and the fact that supp $\phi_r \subset B_r$, the first and the third term in the right hand side of (4.19) are bounded by $C\,r^{1-s-0^+}$. For the second term, we integrate $(-\Delta)^s$ by parts, and using that $\|(-\Delta)^s \phi_r\|_{L^1(\mathbb{R}^n)} = \|(-\Delta)^s \phi\|_{L^1(\mathbb{R}^n)}/r^{2s} \sim 1/r^{2s}$ and that u is Lipschitz in time (Corollary 3.4), we get

$$|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)| \le C\left[r^{1-s-0^+} + \frac{(t-s)}{r^{2s}}\right].$$

Choosing $r := |t-s|^{1/(1+s)}$ we obtain

$$|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)| \le C(t-s)^{\frac{1-s}{1+s}-0^+},$$

as desired.

• $\alpha > 0$. Arguing as above, the first and the third term in the right hand side of (4.19) are bounded by $C\,r^{1-s}$. For the second one, we integrate again by parts and we estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} [(-\Delta)^s u(t,z) - (-\Delta)^s u(s,z)]\phi_r(x-z)\,dz \right| \\ & \quad = \left| \int_{\mathbb{R}^n} [u(t,z) - u(s,z)](-\Delta)^s \phi_r(x-z)\,dz \right| \\ & \quad \le \left| \int_{\mathbb{R}^n} \big[u(t,z) - u(s,z) - u_t(s,z)[t-s]\big](-\Delta)^s \phi_r(x-z)\,dz \right| \\ & \qquad + (t-s)\left| \int_{\mathbb{R}^n} |u_t(s,z)|\,|(-\Delta)^s \phi_r(x-z)|\,dz \right|. \end{aligned}$$

Since $\|(-\Delta)^s\phi_r\|_{L^1(\mathbb{R}^n)} \sim 1/r^{2s}$ and $u_t \in C_t^\alpha$, the first term in the right hand side is bounded by $C\,\frac{(t-s)^{1+\alpha}}{r^{2s}}$. For the second term, we observe that u_t vanishes at $(t,x) \in \{u = \psi\}$, so by the C_x^{1-s}-regularity of u_t we get

$$\left|\int_{\mathbb{R}^n} |u_t(s,z)|\,|(-\Delta)^s\phi_r(x-z)|\,dz\right| \le C\left|\int_{\mathbb{R}^n} \min\left\{|x-z|^{1-s},1\right\}\,|(-\Delta)^s\phi_r(x-z)|\,dz\right|.$$

We now remark that, since ϕ is compactly supported, $|(-\Delta)^s\phi(w)| \le \frac{C}{|w|^{n+2s}}$ for $|w|$ large. So, there exists a constant C_ϕ, depending on ϕ only, such that

$$|(-\Delta)^s\phi(w)| \le \frac{C_\phi}{1+|w|^{n+2s}} \qquad \forall\, w \in \mathbb{R}^n,$$

which by scaling gives

$$|(-\Delta)^s\phi_r(w)| \le \frac{C_\phi}{r^{n+2s}+|w|^{n+2s}} \qquad \forall\, w \in \mathbb{R}^n.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^n} \min\left\{|w|^{1-s},1\right\}|(-\Delta)^s\phi_r(w)|\,dw &\le C\int_{B_1} \frac{|w|^{1-s}}{r^{n+2s}+|w|^{n+2s}}\,dw + C\int_{\mathbb{R}^n\setminus B_1} \frac{1}{|w|^{n+2s}}\,dw\\ &\le \frac{C}{r^{n+2s}}\int_{B_r} |w|^{1-s}\,dw + C\int_{B_1\setminus B_r} |w|^{1-3s-n}\,dw + C,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^n} \min\left\{|w|^{1-s},1\right\}|(-\Delta)^s\phi_r(w)|\,dw \le \begin{cases} C & \text{if } s<1/3;\\ C(1+|\log(r)|) & \text{if } s=1/3;\\ C(1+r^{1-3s}) & \text{if } s>1/3.\end{cases}$$

All in all, we have obtained

$$|(-\Delta)^s u(t,x)-(-\Delta)^s u(s,x)| \le C\left[r^{1-s}+\frac{(t-s)^{1+\alpha}}{r^{2s}}+(t-s)\begin{cases} C & \text{if } s<1/3;\\ C(1+|\log(r)|) & \text{if } s=1/3;\\ C(1+r^{1-3s}) & \text{if } s>1/3.\end{cases}\right]$$

Choosing $r := (t-s)^{(1+\alpha)/(1+s)}$, the above estimates give:

- $s<1/3$: $|(-\Delta)^s u(t,x)-(-\Delta)^s u(s,x)| \le C(t-s)$.
- $s=1/3$: $|(-\Delta)^s u(t,x)-(-\Delta)^s u(s,x)| \le C(t-s)\big[1+|\log(t-s)|\big]$.
- $s>1/3$: Since $\alpha \le (1-s)/2s$ by assumption, we have $\alpha \le \frac{(1-s)(1+\alpha)}{1+s} \le 1+\frac{(1-3s)(1+\alpha)}{1+s}$, so

$$\begin{aligned}|(-\Delta)^s u(t,x)-(-\Delta)^s u(s,x)| &\le C\left[(t-s)^{\frac{(1+\alpha)(1-s)}{1+s}}+(t-s)+(t-s)^{1+\frac{(1-3s)(1+\alpha)}{1+s}}\right]\\ &\le C\,(t-s)^{\frac{(1+\alpha)(1-s)}{1+s}}.\end{aligned}$$

$\square$

Thanks to the above lemma, we can use (4.16) and a bootstrap argument to prove our main regularity result.

Proof of Theorem 2.1. The global Lipschitz regularity of u in space-time follows from Corollary 3.4.

By Proposition 4.11 and (4.17), we can apply Lemma 4.12 with $\alpha = 0$ to deduce that the right hand side of (4.16) belongs to $C_{t,x}^{\frac{1-s}{1+s}-0^+,1-s}((0,T]\times\mathbb{R}^n)$. Hence, since $2s < 1+s$, by the parabolic regularity theory for $\partial_t + (-\Delta)^s$ (see (A.1)) we get $u_t, (-\Delta)^s u \in C_{t,x}^{\frac{1-s}{1+s}-0^+,1-s}((0,T]\times\mathbb{R}^n)$. We now apply Lemma 4.12 with $\alpha > 0$, and we distinguish between two cases:

- $s \le 1/3$: In this case we get

$$(-\Delta)^s u\,\chi_{\{u=\psi\}} \in \mathrm{logLip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n),$$

  so by (A.1) and (A.6) we get $(-\Delta)^s u \in \mathrm{logLip}_t C_x^{1-s}((0,T]\times\mathbb{R}^n)$, and we conclude by using $u_t = (-\Delta)^s u\,\chi_{\{u=\psi\}} - (-\Delta)^s u$.

- $s > 1/3$: Lemma 4.12 gives

$$(-\Delta)^s u\,\chi_{\{u=\psi\}} \in C_{t,x}^{\Phi\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n),$$

  which by (A.1) implies $u_t, (-\Delta)^s u \in C_{t,x}^{\Phi\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n)$ (recall that $\Phi(\alpha) < \frac{1-s}{2s}$ if $\alpha < \frac{1-s}{2s}$). Hence, we can use iteratively Lemma 4.12 and (A.1) to get

$$u_t, (-\Delta)^s u \in C_{t,x}^{\Phi^n\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n),$$

  which together with (A.5) implies

$$(-\Delta)^s u \in C_{t,x}^{\frac{1-s}{2s},1-s}((0,T]\times\mathbb{R}^n).$$

  Finally, since $\Phi^n\left(\frac{1-s}{1+s}-0^+\right) \nearrow \frac{1-s}{2s}$ as $n\to\infty$, we obtain

$$u_t \in C_{t,x}^{\frac{1-s}{2s}-0^+,1-s}((0,T]\times\mathbb{R}^n),$$

  as desired.

$\square$

5 Extension to more general equations

In this section we give a brief informal description of the main modifications needed to extend the regularity result of Theorem 2.1 to solutions of (1.1), at least when $s > 1/2$. Our aim is only to point out the major differences with respect to the model case (2.2) treated above, and to explain how to handle them. There will be however no attempt to state a proper theorem, as this would need a careful analysis of the assumptions needed on $\psi, \mathcal{K}$ (for instance, since the operators are non-local, in all the estimates one should take care of the contribution coming from infinity). We plan to address this issue in a future work.

Assume that $\psi : \mathbb{R}^n \to \mathbb{R}^+$ is a smooth globally Lipschitz function, $b \in \mathbb{R}^n$ a vector, $r \geq 0$ is a constant and $\mathcal{K}$ a (smooth) non-local translation-invariant elliptic operator of lower order with respect to $(-\Delta)^s$, i.e., there exists $\kappa \in (0,1)$ such that

$$[\mathcal{K}\varphi]_{C^{\kappa}_{\mathrm{loc}}(\mathbb{R}^n)} \lesssim \|(-\Delta)^s \varphi\|_{L^\infty(\mathbb{R}^n)} \qquad \forall\, \varphi \in C^\infty_c(\mathbb{R}^n).$$

We consider $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ a (continuous) viscosity solution to the obstacle problem

$$\begin{cases} \min\{-u_t + ru + b \cdot \nabla u - (-\Delta)^s u - \mathcal{K}u, u - \psi\} = 0 & \text{on } (0,T] \times \mathbb{R}^n, \\ u(0) = \psi, \end{cases} \tag{5.1}$$

When $s > 1/2$, existence and uniqueness of such a solution follows again by standard results on obstacle problems.

Let us now analyze the properties of solutions, as we did before for (2.2).

• *Basic properties.* We proceed as in Section 3. First of all, as in Lemma 3.1 one can approximate solutions to (5.1) using a penalization method. In this way, all the results of Section 3 still hold true:

- $u(t, \cdot)$ is globally Lipschitz (see Lemma 3.2);
- $u(t, \cdot)$ is uniformly semiconvex (see Lemma 3.2);
- u_t is globally bounded (see Lemma 3.3);
- $(-\Delta)^s u + \mathcal{K}u - ru - b \cdot \nabla u$ is globally bounded (see Lemma 3.3).

In particular, by elliptic regularity for the fractional Laplacian, the L^∞-bound on $(-\Delta)^s u + \mathcal{K}u(t) - ru(t) - b \cdot \nabla u(t)$ gives

- $u \in L^\infty([0,T], C^{2s-0^+}_{\mathrm{loc}}(\mathbb{R}^n))$.

Hence, since $s > 1/2$ and $\mathcal{K}u$ is of order $\leq 2s - \kappa$, there exists $\gamma = \gamma(s,\kappa) > 0$ such that

$$R := -\mathcal{K}u + ru + b \cdot \nabla u \in L^\infty([0,T], C^{\gamma}_{\mathrm{loc}}(\mathbb{R}^n)). \tag{5.2}$$

• C^α_x*-decay for* $(-\Delta)^s u(t)$. In this setting, we have:

- $(-\Delta)^s u(t) - R(t) < 0$ inside the open set $\{u(t) > \psi\}$;
- $(-\Delta)^s u(t) - R(t) \geq 0$ a.e. on $\{u(t) = \psi\}$.

Now, fixed $t > 0$ and given a free boundary point $x_0 \in \partial\{u(t) = \psi\}$, we consider the L_a-harmonic function $v(x,y) := u(t,x,y) - \frac{R(t,x_0)}{1-a}\, y^{1-a}$, where $u(t,x,y)$ is the L_a-harmonic extension of $u(t)$. Moreover, as in Subsection 4.1 we consider the function

$$\tilde{v}(x,y) := v(x,y) - \psi(x).$$

Since u_t is globally bounded and $v(\cdot,0)$ is semiconvex, all estimates (B1)-(B2) and (B4)-(B5) of Subsection 4.1 still hold true, while (B3) becomes

(B3') $\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \leq |R(t,x) - R(t,x_0)| \leq C|x-x_0|^\gamma$ for a.e. $x \in \Lambda$,
$\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) > -C|x-x_0|^\gamma$ for $x \in \mathbb{R}^n \setminus \Lambda$.

While the proof of Lemma 4.4 works with no modifications under these assumptions, for the proof of Proposition 4.3 we remark that now we do not have $\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \geq 0$ for $x \in \mathbb{R}^n \setminus \Lambda$, which was used to apply Hopf's Lemma. To overcome this difficulty, in the induction step from k_0 to k_0+1 one should replace $v(x,y)$ with $v(x,y) + \frac{\|R(t)\|_{C^\gamma(B_1(x_0))}}{1-a}(4^{-k_0})^\gamma y^{1-a}$, and the rest of the proof should go through with minor modifications.

Hence, one still gets $\sup_{B_r(x_0)} |u(t) - \psi| \leq Cr^{\alpha+2s}$ and

$$[(-\Delta)^s u(t) - R(t)]\chi_{u(t)=\psi} = [(-\Delta)^s u(t) + \mathcal{K}u(t) - ru(t) - b\cdot\nabla u(t)]\chi_{u(t)=\psi} \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^n),$$

for some universal exponent $\alpha \in (0,\gamma)$.

• *Monotonicity formula and optimal spatial regularity.* As in Subsection 4.2, one would like to apply a monotonicity formula. However, first of all one has to do a preliminary step: using the equation

$$u_t + (-\Delta)^s u = [(-\Delta)^s u + R]\chi_{u=\psi} - R \in L^\infty([0,T]; C^\alpha_{\mathrm{loc}}(\mathbb{R}^n))$$

one deduces (thanks to a local variant of (A.5)-(A.6)) that $(-\Delta)^s u \in L^\infty([0,T]; C^{\alpha-0^+}_{\mathrm{loc}}(\mathbb{R}^n))$. So, by elliptic regularity for the fractional Laplacian, $u \in C^{\alpha+2s-0^+}_{\mathrm{loc}}(\mathbb{R}^n)$, which gives

$$R \in L^\infty([0,T], C^{\alpha+\gamma/2}_{\mathrm{loc}}(\mathbb{R}^n)).$$

This allows considering R as a lower order perturbation when applying the monotonicity formula.

Now, to apply the monotonicity formula around a free boundary point $x_0 \in \partial\{u(t) = \psi\}$, one should consider the function $w : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}$ obtained by solving the Dirichlet problem

$$\begin{cases} L_{-a} w = 0, \\ w(x,0) = [(-\Delta)^s u(t,x,0) - R(t,x_0)]\chi_{\{u(t)=\psi\}}(x). \end{cases}$$

Since $R(t) \in C^{\alpha+\gamma/2}_{\mathrm{loc}}(\mathbb{R}^n)$, we have $w \geq -Cr^{\alpha+\gamma/2}$ on $B_r(x_0)$. So, by the monotonicity formula one gets $[(-\Delta)^s u + R(t,x)]\chi_{u(t)=\psi} \in C^{\beta'_\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, $\beta'_\alpha := \min\{\alpha + \delta'_\alpha, 1-s\}$. Then, one can iterate the above strategy, first using the parabolic regularity of $\partial_t+(-\Delta)^s$ and the the elliptic regularity of $(-\Delta)^s$ to show that $R \in L^\infty([0,T]; C^{\beta'_\alpha+\gamma}_{\rm loc}(\mathbb{R}^n)$, and then applying again the monotonicity formula. In this way, after finitely many iterations we get

$$[(-\Delta)^s u + \mathcal{K}u - ru - b\cdot\nabla u]\chi_{u=\psi} \in L^\infty([0,T], C^{1-s}_{\rm loc}(\mathbb{R}^n)), \qquad R \in L^\infty([0,T], C^{1-s+\gamma}_{\rm loc}(\mathbb{R}^n)).$$

- *Parabolic regularity and conclusion.* Using Lemma 4.12, the argument in Subsection 4.3 applied to

$$\partial_t u + (-\Delta)^s u = [(-\Delta)^s u + R]\chi_{u=\psi} - R$$

allows to extend the regularity result in Theorem 2.1 (at least locally in space-time) to solutions of (5.1).

A Regularity properties of the operator $\partial_t + (-\Delta)^s$

In this appendix we describe some important properties of the parabolic operator $\partial_t + (-\Delta)^s$.

Let us first recall that fractional Laplacian works nicely in Hölder spaces: in $f \in C^\alpha(\mathbb{R}^n)$ then $(-\Delta)^{-s} f \in C^{\alpha+2s}(\mathbb{R}^n)$, see for instance [15, Subsection 2.1].

Analogously, the operator $\partial_t + (-\Delta)^s$ works nicely in space-time Hölder spaces: if $v_t + (-\Delta)^s v = f$ on $[0,T]\times\mathbb{R}^n$ and $v(0)$ is smooth (for our purposes, we can assume $v(0) \in C^2(\mathbb{R}^n)$, globally Lipschitz, and $\|D^2 v(0)\|_{L^\infty(\mathbb{R}^n)} + \|(-\Delta)^s v(0)\|_{C^{1-s}_x(\mathbb{R}^n)} < +\infty$), by classical results on multipliers on Hölder spaces (see for instance [13, Theorem 2.3] and the proof of [13, Theorem 3.1]) we get

$$\|v_t\|_{C^{\alpha,\beta}_{t,x}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s v\|_{C^{\alpha,\beta}_{t,x}((0,T]\times\mathbb{R}^n)} \lesssim 1 + \|f\|_{C^{\alpha,\beta}_{t,x}((0,T]\times\mathbb{R}^n)} \qquad \forall\, \alpha,\beta \in (0,1) \tag{A.1}$$

However, for our purposes, we also need to have some regularity estimates when f is only bounded in time (but Hölder in space).

Let us observe that we can write the solution in terms of the fundamental solution $\Gamma_s(t,x)$ of the fractional heat equation. More precisely, using Duhamel formula, we have

$$\begin{aligned} -(-\Delta)^s v(t,x) &= v_t(t,x) - f(t,x) \\ &= -\Gamma_s(t) * (-\Delta)^s v(0) + \int_0^t \int_{\mathbb{R}^n} \partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)]\,dy\,d\tau. \end{aligned} \tag{A.2}$$

We now claim that the following estimates hold (the proof of them is postponed to Subsection A.1 below):

$$\left\| \int_0^t \int_{\mathbb{R}^n} \partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)]\,dy\,d\tau \right\|_{C^{\beta/(2s),\beta-0^+}_{t,x}((0,T]\times\mathbb{R}^n)} \lesssim \|f\|_{L^\infty((0,T];C^\beta_x(\mathbb{R}^n))} \tag{A.3}$$

if $\beta < 2s$,

$$\left\| \int_0^t \int_{\mathbb{R}^n} \partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)]\, dy\, d\tau \right\|_{\mathrm{logLip}_t C_x^{\beta-0^+}((0,T]\times\mathbb{R}^n)} \lesssim \|f\|_{L^\infty((0,T];C_x^\beta(\mathbb{R}^n))} \tag{A.4}$$

if $\beta \geq 2s$.

Combining the above estimates with (A.2), and using that $-\Gamma_s(t) * (-\Delta)^s v(0)$ is smooth in space-time for $t > 0$, we get

$$\|(-\Delta)^s v\|_{C_{t,x}^{\beta/(2s),\beta-0^+}((0,T]\times\mathbb{R}^n)} \lesssim 1 + \|f\|_{L^\infty((0,T];C_x^\beta(\mathbb{R}^n))} \qquad \forall\, \beta \in (0, 2s) \tag{A.5}$$

$$\|(-\Delta)^s v\|_{\mathrm{logLip}_t C_x^{\beta-0^+}((0,T]\times\mathbb{R}^n)} \lesssim 1 + \|f\|_{L^\infty((0,T];C_x^\beta(\mathbb{R}^n))} \qquad \forall\, \beta \in [2s, 1) \tag{A.6}$$

(At $t = 0$ the time regularity may degenerate, due to the presence of the term $\Gamma_s(t) * (-\Delta)^s v(0)$.) In particular, using that $v_t = f - (-\Delta)^s v$, we obtain

$$\|v_t\|_{L^\infty((0,T];C_x^{\beta-0^+}(\mathbb{R}^n))} + \|(-\Delta)^s v\|_{L^\infty((0,T];C_x^{\beta-0^+}(\mathbb{R}^n))} \lesssim 1 + \|f\|_{L^\infty((0,T];C_x^\beta(\mathbb{R}^n))} \qquad \forall\, \beta \in (0, 1) \tag{A.7}$$

A.1 Proof of (A.3) and (A.4)

Let us recall that $t \in [0, T]$, with $T < +\infty$.

To prove (A.3) and (A.4), we use that the fundamental solution $\Gamma_s(1, y)$ behaves like $\frac{1}{1+|y|^{n+2s}}$, which by scaling implies

$$\Gamma_s(t,y) \sim \frac{t}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}}, \qquad |\partial_t \Gamma_s(t,y)| \lesssim \frac{1}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}}, \tag{A.8}$$

$$|\partial_{tt} \Gamma_s(t,y)| \lesssim \frac{1}{t} \frac{1}{(t^{\frac{n+2s}{2s}} + |y|^{n+2s})}, \quad |\nabla_y \partial_t \Gamma_s(t,y)| \lesssim \frac{1}{|y|} \frac{1}{(t^{\frac{n+2s}{2s}} + |y|^{n+2s})} \tag{A.9}$$

We will also make use of the following two basics estimates:
- There exists a constant $C > 0$ such that, for all $h \in (0, 1]$,

$$\int_{\mathbb{R}^n} \frac{\min\{|z|^\beta, 1\}}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}}\, dz \leq C\Big(1 + h^{\beta/(2s)-1}\Big). \tag{A.10}$$

- There exists a constant $C > 0$ such that, for all $h > 0$,

$$\int_0^t \frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + h^{n+2s}}\, d\tau \leq C \min\left\{\frac{1}{h^n}, \frac{1}{h^{n+2s}}\right\}. \tag{A.11}$$

The proof of both is pretty simple. For instance, to show (A.10), one splits the integral into three parts:

$$\int_{B_{h^{1/(2s)}}} \frac{|z|^\beta}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}}\, dz \leq \frac{1}{h^{\frac{n+2s}{2s}}} \int_{B_{h^{1/(2s)}}} |z|^\beta\, dz \lesssim h^{\beta/(2s)-1},$$

$$\int_{B_1\setminus B_{h^{1/(2s)}}} \frac{|z|^\beta}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}}\,dz \le \int_{B_1\setminus B_{h^{1/(2s)}}} |z|^{\beta-n-2s}\,dz \lesssim 1 + h^{\beta/(2s)-1},$$

$$\int_{\mathbb{R}^n\setminus B_1} \frac{1}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}}\,dz \le \int_{\mathbb{R}^n\setminus B_1} |z|^{-n-2s}\,dz \lesssim 1.$$

To prove (A.11), we observe that the bound is trivial if $h \ge 1$, since $\frac{1}{(t-\tau)^{\frac{n+2s}{2s}}+h^{n+2s}} \lesssim \frac{1}{h^{n+2s}}$ (recall that $|t-\tau| \le T \lesssim 1$). On the other hand, if $h \in (0,1]$, it suffices to split the integral over $[0, t-h^{2s}]$ and $[t-h^{2s}, t]$, and argue as above.

- *Proof of* (A.3). Let us observe that, for $u < t$,

$$\begin{aligned}
&\big|(-\Delta)^s v(t,x) - (-\Delta)^s v(\tau,x)\big| \\
&\quad = \bigg|\int_0^t \int_{\mathbb{R}^n} \partial_t\Gamma_s(t-\tau, x-y)[f(\tau,y)-f(\tau,x)]\,dy\,d\tau \\
&\qquad - \int_0^u \int_{\mathbb{R}^n} \partial_t\Gamma_s(u-\tau, x-y)[f(\tau,y)-f(\tau,x)]\,dy\,d\tau\bigg| \\
&\quad \lesssim \int_u^t \int_{\mathbb{R}^n} \big|\partial_t\Gamma_s(t-\tau,x-y)\big||f(\tau,y)-f(\tau,x)|\,dy\,d\tau \\
&\qquad + \int_0^u \int_{\mathbb{R}^n} \Big|\partial_t\Gamma_s(u-\tau,x-y) - \partial_t\Gamma_s(t-\tau,x-y)\Big| \min\{|x-y|^\beta, 1\}\,dy\,d\tau \\
&\quad = (T1) + (T2).
\end{aligned}$$

Now, using (A.8), by (A.10) applied with $h = (t-\tau)$ we get that (T1) is bounded by

$$\begin{aligned}
\int_u^t \int_{\mathbb{R}^n} \frac{\min\{|x-y|^\beta, 1\}}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}\,dy\,d\tau &\le \int_u^t \Big(1 + (t-\tau)^{\beta/(2s)-1}\Big)\,d\tau \\
&\lesssim (t-u) + (t-u)^{\beta/(2s)}.
\end{aligned}$$

Concerning (T2), thanks to (A.8), (A.9), and (A.10) with $h = (t-\tau)$, we can control it by

$$\begin{aligned}
&\int_0^u \int_{\mathbb{R}^n} \min\left\{\frac{t-u}{u-\tau}, 1\right\} \frac{1}{((u-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s})} \min\{|x-y|^\beta, 1\}\,dy\,d\tau \\
&\lesssim \int_0^u \min\left\{\frac{t-u}{u-\tau}, 1\right\} \int_{\mathbb{R}^n} \frac{\min\{|x-y|^\beta, 1\}}{(u-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}\,dy\,d\tau \\
&\lesssim \int_0^{u-(t-u)} \frac{t-u}{u-\tau}\Big(1 + (u-\tau)^{\beta/(2s)-1}\Big)\,d\tau + \int_{u-(t-u)}^u \Big(1 + (u-\tau)^{\beta/(2s)-1}\Big)\,d\tau \\
&\lesssim (t-u) + (t-u)|\log(t-u)| + (t-u)^{\beta/(2s)},
\end{aligned}$$

which proves the time regularity of $(-\Delta)^s v$.

• *Proof of* (A.4). The proof of the spatial regularity is analogous: we write

$$
\begin{aligned}
&(-\Delta)^s v(t,x) - (-\Delta)^s v(t,z) \\
&= \int_0^t \int_{\mathbb{R}^n} \Big(\partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)] - \partial_t \Gamma_s(t-\tau, z-y)[f(\tau,y) - f(\tau,z)] \Big)\, dy\, d\tau.
\end{aligned}
$$

Then, we split the spatial integral over two sets: the region where $\big\{|x-z| \le |x-y|/2\big\}$ and the region where $\big\{|x-z| \ge |x-y|/2\big\}$.

On the first set, since $\frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}$ and $\frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |z-y|^{n+2s}}$ are comparable, we can estimate the integrand by

$$
|f(\tau,y) - f(\tau,x)| \Big| \partial_t \Gamma_s(t-\tau, x-y) - \partial_t \Gamma_s(t-\tau, z-y) \Big| + |f(\tau,x) - f(\tau,z)| \Big| \partial_t \Gamma_s(t-\tau, x-y) \Big|,
$$

which thanks to (A.8) and (A.9) can be bounded by

$$
|y-x|^\beta \frac{|x-z|}{|x-y|} \frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}} + |x-z|^\beta \frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}.
$$

So, using (A.11) with $h = |x-y|$ we get

$$
\begin{aligned}
&\int_0^t \int_{\{|x-z| \le |x-y|/2\}} \Big| \partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)] - \partial_t \Gamma_s(t-\tau, z-y)[f(\tau,y) - f(\tau,z)] \Big|\, dy\, d\tau \\
&\lesssim \int_{\{|x-z| \le |x-y|/2\}} \left(\frac{|x-z|}{|x-y|^{1-\beta}} + |x-z|^\beta \right) \int_0^t \frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}\, d\tau\, dy \\
&\lesssim \int_{\{|x-z| \le |x-y|/2\}} |x-z|^\beta \min\left\{ \frac{1}{|x-y|^n}, \frac{1}{|x-y|^{n+2s}} \right\} dy \\
&\le |x-z|^\beta \int_{\{|x-z| \le |x-y|/2 \le 1\}} |x-y|^{-n} + |x-z|^\beta \int_{\{|x-y|/2 \ge 1\}} |x-y|^{-n-2s} \\
&\lesssim |x-z|^\beta \big| \log |x-z| \big| \lesssim |x-z|^{\beta-\varepsilon} \qquad \forall \varepsilon > 0.
\end{aligned}
$$

Concerning the integral over the second set, we simply use (A.8) to bound the integrand by

$$
\frac{|x-y|^\beta}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}} + \frac{|x-z|^\beta}{(t-\tau)^{\frac{n+2s}{2s}} + |x-z|^{n+2s}}
$$

and observing that $\{|x-z| \ge |x-y|/2\} \subset B_{3|x-z|}(x) \cap B_{3|x-z|}(z)$ we get

$$
\begin{aligned}
&\int_0^t \int_{\{|x-z| \ge |x-y|/2\}} \Big| \partial_t \Gamma_s(t-\tau, x-y)[f(\tau,y) - f(\tau,x)] - \partial_t \Gamma_s(t-\tau, z-y)[f(\tau,y) - f(\tau,z)] \Big|\, dy\, d\tau \\
&\lesssim \int_0^t \int_{B_{3|x-z|}(x)} \frac{|x-y|^\beta}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}\, dy\, d\tau \\
&= \int_{B_{3|x-z|}(x)} |x-y|^\beta \int_0^t \frac{1}{(t-\tau)^{\frac{n+2s}{2s}} + |x-y|^{n+2s}}\, d\tau\, dy \\
&\lesssim \int_{B_{3|x-z|}(x)} |x-y|^{\beta-n}\, dy \lesssim |x-z|^\beta,
\end{aligned}
$$

where for the last but one inequality we used again (A.11) with $h = |x - y|$. This concludes the proof of (A.4).

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