Regularity of Wave-Maps in Dimension 2 + 1

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Abstract: In this article we prove a Sacks-Uhlenbeck/Struwe type global regularity result for wave-maps $\Phi : \mathbb{R}^{2+1} \to \mathcal{M}$ into general compact target manifolds $\mathcal{M}$.

Contents

1. Introduction

In this article we consider large data Wave-Maps from $\mathbb{R}^{2+1}$ into a compact Riemannian manifold $(\mathcal{M}, m)$, and we prove that regularity and dispersive bounds persist as long as a soliton-like concentration is absent. This is a companion to our concurrent article [27], where the same result is proved under a stronger energy dispersion assumption, see Theorem 1.2 below.

The set-up we consider is the same as the one in [41], using the extrinsic formulation of the Wave-Maps equation. Precisely, we consider the target manifold $(\mathcal{M}, m)$ as an isometrically embedded submanifold of $\mathbb{R}^N$. Then we can view the $\mathcal{M}$ valued functions as $\mathbb{R}^N$ valued functions whose range is contained in $\mathcal{M}$. Such an embedding always exists by Nash's theorem [20] (see also Gromov [8] and Günther [9]). In this context the Wave-Maps equation can be expressed in a form which involves the second fundamental form $\mathcal{S}$ of $\mathcal{M}$, viewed as a symmetric bilinear form:

$$\mathcal{S}: T\mathcal{M} \times T\mathcal{M} \to N\mathcal{M}, \quad \langle \mathcal{S}(X, Y), N \rangle = \langle \partial_X N, Y \rangle.$$

The Cauchy problem for the wave maps equation has the form:

$$\Box \phi^a = -\mathcal{S}^a_{bc}(\phi)\partial^\alpha \phi^b \partial_\alpha \phi^c, \quad (\phi^1, \ldots, \phi^N) := \Phi, \tag{1a}$$

$$\Phi(0, x) = \Phi_0(x), \quad \partial_t \Phi(0, x) = \Phi_1(x), \tag{1b}$$

where the initial data (Φ_0, Φ_1) is chosen to obey the constraint:

$$\Phi_0(x) \in \mathcal{M}, \quad \Phi_1(x) \in T_{\Phi_0(x)}\mathcal{M}, \quad x \in \mathbb{R}^2.$$

In the sequel, it will be convenient for us to use the notation $\Phi[t] = (\Phi(t), \partial_t \Phi(t))$.

There is a conserved energy for this problem,

$$\mathcal{E}[\Phi](t) = \frac{1}{2}\int (|\partial_t \Phi|^2 + |\nabla_x \Phi|^2)dx.$$

This is invariant with respect to the scaling that preserves the equation, $\Phi(x, t) \to \Phi(\lambda x, \lambda t)$. Because of this, we say that the problem is **energy critical**.

The present article contributes to the understanding of finite energy solutions with arbitrarily large initial data, a problem which has been the subject of intense investigations for some time now. We will not attempt to give a detailed account of the history of this subject here. Instead, we refer the reader to the surveys [40] and [15], and the references therein. The general (i.e. without symmetry assumptions) small energy problem for compact targets was initiated in the ground-breaking work of Klainerman-Machedon [13,14], and completed by the work of Tao [36] when $\mathcal{M}$ is a sphere, and Tataru [41] for general isometrically embedded manifolds; see also the work of Krieger [17] on the non-compact hyperbolic plane, which was treated from the intrinsic point of view.

At a minimum one expects the solutions to belong to the space $C(\mathbb{R}; \dot{H}^1(\mathbb{R}^2)) \cap \dot{C}^1(\mathbb{R}, L^2(\mathbb{R}^2))$. However, this information does not suffice in order to study the equation and to obtain uniqueness statements. Instead, a smaller Banach space $S \subset C(\mathbb{R}; \dot{H}^1(\mathbb{R}^2)) \cap \dot{C}^1(\mathbb{R}, L^2(\mathbb{R}^2))$ was introduced in [36], modifying an earlier structure in [39]. Beside the energy, S also contains Strichartz type information in various frequency localized contexts. The full description of S is not necessary here. However, we do use the fact that $S \cap L^\infty$ is an algebra, as well as the $X^{s,b}$ type embedding $S \subset \dot{X}_\infty^{1,\frac{1}{2}}$. See our companion paper [27] for precise definitions. The standard small data result is as follows:

Theorem 1.1 ([36, 41, 17]). *There is some $E_0 > 0$ so that for each smooth initial data (Φ_0, Φ_1) satisfying $\mathcal{E}[\Phi](0) \le E_0$ there exists a unique global smooth solution $\Phi = T(\Phi_0, \Phi_1) \in S$. In addition, the above solution operator T extends to a continuous operator from $\dot{H}^1 \times L^2$ to S with*

$$\|\Phi\|_S \lesssim \|(\Phi_0, \Phi_1)\|_{\dot{H}^1 \times L^2}.$$

We call such solutions "strong finite energy wave-maps". Furthermore, the following weak Lipschitz stability estimate holds for these solutions for $s < 1$ and close to 1:

$$\|\Phi - \Psi\|_{C(\mathbb{R}; \dot{H}^s) \cap \dot{C}^1(\mathbb{R}, \dot{H}^{s-1})} \lesssim \|\Phi[0] - \Psi[0]\|_{\dot{H}^s \times \dot{H}^{s-1}}. \tag{2}$$

Due to the finite speed of propagation, the corresponding local result is also valid, say with the initial data in a ball and the solution in the corresponding domain of uniqueness.

The aim of the companion [27] to the present work is to provide a conditional S bound for large data solutions, under a weak energy dispersion condition. For that, we have introduced the notion of the *energy dispersion* of a wave map Φ defined on an interval I,

$$ED[\Phi] = \sup_{k \in \mathbb{Z}} \|P_k \Phi\|_{L^\infty_{t,x}[I]},$$

where P_k are spatial Littlewood-Paley projectors at frequency 2^k. The main result in [27] is as follows:

Theorem 1.2 (Main Theorem in [27]). *For each $E > 0$ there exist $F(E) > 0$ and $\epsilon(E) > 0$ so that any wave-map Φ in a time interval I with energy $\mathcal{E}[\Phi] \le E$ and energy dispersion $ED[\Phi] \le \epsilon(E)$ must satisfy $\|\Phi\|_{S[I]} \le F(E)$.*

In particular, this result implies that a wave map with energy E cannot blow-up as long as its energy dispersion stays below $\epsilon(E)$.

In this article we establish unconditional analogs of the above result, in particular settling the blow-up versus global regularity and scattering question for the large data problem. We begin with some notations. We consider the forward light cone

$$C = \{0 \le t < \infty, \ r \le t\}$$

and its subsets

$$C_{[t_0, t_1]} = \{t_0 \le t \le t_1, \ r \le t\}.$$

The lateral boundary of $C_{[t_0, t_1]}$ is denoted by $\partial C_{[t_0, t_1]}$. The time sections of the cone are denoted by

$$S_{t_0} = \{t = t_0, \ |x| \le t\}.$$

We also use the translated cones

$$C^\delta = \{\delta \le t < \infty, \ r \le t - \delta\}$$

as well as the corresponding notations $C^\delta_{[t_0, t_1]}$, $\partial C^\delta_{[t_0, t_1]}$ and $S^\delta_{t_0}$ for $t_0 > \delta$.

Given a wave map Φ in C or in a subset $C_{[t_0, t_1]}$ of it we define the energy of Φ on time sections as

$$\mathcal{E}_{S_t}[\Phi] = \frac{1}{2} \int_{S_t} (|\partial_t \Phi|^2 + |\nabla_x \Phi|^2) dx.$$

It is convenient to do the computations in terms of the null frame

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r, \quad \cancel{\partial} = r^{-1}\partial_\theta.$$

We define the flux of Φ between t_0 and t_1 as

$$\mathcal{F}_{[t_0,t_1]}[\Phi] = \int_{\partial C_{[t_0,t_1]}} \left(\frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\cancel{\partial}\Phi|^2\right) dA.$$

By standard energy estimates we have the energy conservation relation

$$\mathcal{E}_{S_{t_1}}[\Phi] = \mathcal{E}_{S_{t_0}}[\Phi] + \mathcal{F}_{[t_0,t_1]}[\Phi]. \tag{3}$$

This shows that $\mathcal{E}_{S_t}[\Phi]$ is a nondecreasing function of t.

1.1. The Question of Blowup. We begin with the blow-up question. A standard argument which uses the small data result and the finite speed of propagation shows that if blow-up occurs then it must occur at the tip of a light-cone where the energy (inside the cone) concentrates. After a translation and rescaling it suffices to consider wave maps Φ in the cone $C_{[0,1]}$. If

$$\lim_{t \to 0} \mathcal{E}_{S_t}[\Phi] \le E_0,$$

then blow-up cannot occur at the origin due to the local result, and in fact it follows that

$$\lim_{t \to 0} \mathcal{E}_{S_t}[\Phi] = 0.$$

Thus the interesting case is when we are in an energy concentration scenario

$$\lim_{t \to 0} \mathcal{E}_{S_t}[\Phi] > E_0. \tag{4}$$

The main result we prove here is the following:

Theorem 1.3. *Let $\Phi : C_{(0,1]} \to \mathcal{M}$ be a C^∞ wave map. Then exactly one of the following possibilities must hold:*

A) *There exists a sequence of points $(t_n, x_n) \in C_{[0,1]}$ and scales r_n with*

$$(t_n, x_n) \to (0, 0), \quad \limsup \frac{|x_n|}{t_n} < 1, \quad \lim \frac{r_n}{t_n} = 0$$

so that the rescaled sequence of wave-maps

$$\Phi^{(n)}(t, x) = \Phi(t_n + r_n t, x_n + r_n x), \tag{5}$$

converges strongly in H^1_{loc} to a Lorentz transform of an entire Harmonic-Map of nontrivial energy:

$$\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}, \quad 0 < \|\Phi^{(\infty)}\|_{\dot{H}^1(\mathbb{R}^2)} \le \lim_{t \to 0} \mathcal{E}_{S_t}[\Phi].$$

B) *For each $\epsilon > 0$ there exists $0 < t_0 \leqslant 1$ and a wave map extension*

$$\Phi : \mathbb{R}^2 \times (0, t_0] \to \mathcal{M}$$

with bounded energy

$$\mathcal{E}[\Phi] \leq (1 + \epsilon^8) \lim_{t \to 0} \mathcal{E}_{S_t}[\Phi] \tag{6}$$

and energy dispersion,

$$\sup_{t \in (0, t_0]} \sup_{k \in \mathbb{Z}} \left(\| P_k \Phi(t) \|_{L_x^\infty} + 2^{-k} \| P_k \partial_t \Phi(t) \|_{L_x^\infty} \right) \leq \epsilon. \tag{7}$$

We remark that a nontrivial harmonic map $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}$ cannot have an arbitrarily small energy. Precisely, there are two possibilities. Either there are no such harmonic maps (for instance, in the case when $\mathcal{M}$ is negatively curved, see [19]) or there exists a lowest energy nontrivial harmonic map, which we denote by $\mathcal{E}_0(\mathcal{M}) > 0$. Furthermore, a simple computation shows that the energy of any harmonic map will increase if we apply a Lorentz transformation. Hence, combining the results of Theorem 1.3 and Theorem 1.2 we obtain the following:

Corollary 1.4. *(Finite Time Regularity for Wave-Maps). The following statements hold:*

A) *Assume that $\mathcal{M}$ is a compact Riemannian manifold so that there are no nontrivial finite energy harmonic maps $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}$. Then for any finite energy data $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{M} \times T\mathcal{M}$ for the wave map equation (1) there exists a global solution $\Phi \in S(0, T)$ for all $T > 0$. In addition, this global solution retains any additional regularity of the initial data.*
B) *Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a Riemannian covering, with $\mathcal{M}$ compact, and such that there are no nontrivial finite energy harmonic maps $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}$. If $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \widetilde{\mathcal{M}} \times T\widetilde{\mathcal{M}}$ is C^∞, then there is a global C^∞ solution to $\widetilde{\mathcal{M}}$ with this data.*
C) *Suppose that there exists a lowest energy nontrivial harmonic map into $\mathcal{M}$ with energy $\mathcal{E}_0(\mathcal{M})$. Then for any data $\Phi[0] : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{M} \times T\mathcal{M}$ for the wave map equation (1) with energy below $\mathcal{E}_0(\mathcal{M})$, there exists a global solution $\Phi \in S(0, T)$ for all $T > 0$.*

We remark that the statement in part B) is a simple consequence of A) and restricting the projection $\pi \circ \Phi$ to a sufficiently small section S_t of a cone where one expects blowup of the original map into $\widetilde{\mathcal{M}}$. In particular, since this projection is regular by part A), its image lies in a simply connected set for sufficiently small t. Thus, this projection can be inverted to yield regularity of the original map close to the suspected blowup point. Because of this trivial reduction, we work exclusively with compact $\mathcal{M}$ in the sequel. It should be remarked however, that as a (very) special case of this result one obtains global regularity for smooth Wave-Maps into all hyperbolic spaces $\mathbb{H}^n$, which has been a long-standing and important conjecture in geometric wave equations due to its relation with problems in general relativity (see Chapter 16 of [2]).

The statement of Corollary 1.4 in its full generality was known as the *Threshold Conjecture*. Similar results were previously established for the Wave-Map problem via symmetry reductions in the works [4,26,30], and [29]. General results of this type, as

well as fairly strong refinements, have been known for the Harmonic-Map heat-flow for some time (see [28] and [22]). As with the heat flow, Theorem 1.3 does not prevent the formation of multiple singularities on top of each other. To the contrary, such bubble-trees are to be expected (see [42]).

Finally, we remark that this result is sharp. In the case of $M = \mathbb{S}^2$ there exists a lowest energy nontrivial harmonic map, namely the stereographic projection Q. The results in [16] assert that blow-up with a rescaled Q profile can occur for initial data with energy arbitrarily close to $\mathcal{E}[Q]$. We also refer the reader to [23] for blow-up results near higher energy harmonic maps.

1.2. The Question of Scattering. Next we consider the scattering problem, for which we start with a finite energy wave map Φ in $\mathbb{R}^2 \times [0, \infty)$ and consider its behavior as $t \to \infty$. Here by scattering we simply mean the fact that $\Phi \in S$; if that is the case, then the structure theorem for large energy Wave-Maps in [27] shows that Φ behaves at ∞ as a linear wave after an appropriate renormalization.

We can select a ball B so that outside B the energy is small, $\mathcal{E}_{B^c}[\Phi] < \frac{1}{10} E_0$. Then outside the influence cone of B, the solution Φ behaves like a small data wave map. Hence it remains to study it within the influence cone of B. After scaling and translation, it suffices to work with wave maps Φ in the outgoing cone $C_{[1,\infty)}$ which have finite energy, i.e.

$$\lim_{t \to \infty} \mathcal{E}_{S_t}[\Phi] < \infty. \tag{8}$$

We prove the following result:

Theorem 1.5. *Let $\Phi : C_{[1,\infty)} \to \mathcal{M}$ be a C^∞ wave map which satisfies (8). Then exactly one of the following possibilities must hold:*

A) There exists a sequence of points $(t_n, x_n) \in C_{[1,\infty)}$ and scales r_n with

$$t_n \to \infty, \quad \limsup \frac{|x_n|}{t_n} < 1, \quad \lim \frac{r_n}{t_n} = 0$$

so that the rescaled sequence of wave-maps

$$\Phi^{(n)}(t, x) = \Phi(t_n + r_n t, x_n + r_n x), \tag{9}$$

converges strongly in H^1_{loc} to a Lorentz transform of an entire Harmonic-Map of nontrivial energy:

$$\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}, \quad 0 < \| \Phi^{(\infty)} \|_{\dot{H}^1(\mathbb{R}^2)} \leqslant \lim_{t \to \infty} \mathcal{E}_{S_t}[\Phi].$$

B) For each $\epsilon > 0$ there exists $t_0 > 1$ and a wave map extension

$$\Phi : \mathbb{R}^2 \times [t_0, \infty) \to \mathcal{M}$$

with bounded energy

$$\mathcal{E}[\Phi] \leq (1 + \epsilon^8) \lim_{t \to \infty} \mathcal{E}_{S_t}[\Phi] \tag{10}$$

and energy dispersion,

$$\sup_{t \in [t_0, \infty)} \sup_{k \in \mathbb{Z}} \left(\| P_k \Phi(t) \|_{L^\infty_x} + 2^{-k} \| P_k \partial_t \Phi(t) \|_{L^\infty_x} \right) \leq \epsilon. \tag{11}$$

In case B) Theorem 1.2 then implies that scattering holds as $t \to \infty$. Thus if scattering does not hold then we must be in case A). As a corollary, it follows that scattering can only fail for wave-maps Φ whose energy satisfies

$$\mathcal{E}[\Phi] \geq \mathcal{E}_0(\mathcal{M}). \tag{12}$$

Thus Corollary 1.4 can be strengthened to

Corollary 1.6 (Scattering for Large Data Wave-Maps). *The following statements hold:*

A) *Assume that there are no nontrivial finite energy harmonic maps* $\Phi^{(\infty)} : \mathbb{R}^2 \to \mathcal{M}$. *Then for any finite energy data* $\Phi[0]$ *for the wave map equation* (1) *there exists a global solution* $\Phi \in S$.

B) *Suppose that there exists a lowest energy nontrivial harmonic map, with energy* $\mathcal{E}_0(\mathcal{M})$. *Then for any data* $\Phi[0]$ *for the wave map equation* (1) *with energy below* $\mathcal{E}_0(\mathcal{M})$ *there exists a global solution* $\Phi \in S$.

Ideally one would also like to have a constructive bound of the form

$$\|\Phi\|_S \leq F(\mathcal{E}[\Phi]).$$

This does not seem to follow directly from our results. Furthermore, our results do not seem to directly imply scattering for non-compact targets in the absence of harmonic maps (only scattering of the projection). Results similar to Corollary 1.6 were previously established in spherically symmetric and equivariant cases, see [3] and [5].

Finally, we would like to remark that results similar in spirit to the ones of this paper and [27] have been recently announced. In the case where $\mathcal{M} = \mathbb{H}^n$, the hyperbolic spaces, globally regularity and scattering follows from the program of Tao [37,31–33,35] and [34]. In the case where the target $\mathcal{M}$ is a negatively curved Riemann surface, Krieger and Schlag [18] provide global regularity and scattering via a modification of the Kenig-Merle method [12], which uses as a key component suitably defined Bahouri-Gerard [1] type decompositions.

2. Overview of the Proof

The proofs of Theorem 1.3 and Theorem 1.5 are almost identical. The three main building blocks of both proofs are (i) weighted energy estimates, (ii) elimination of finite energy self-similar solutions, and (iii) a compactness result.

Our main energy estimates are established in Sect. 3. Beside the standard energy bounds involving the ∂_t vector field we also use the vector field

$$X_0 = \frac{1}{\rho}(t\partial_t + r\partial_r), \qquad \rho = \sqrt{t^2 - r^2} \tag{13}$$

as well as its time translates. This leads to a family of weighted energy estimates, see (26) below, which has appeared in various guises in the literature. The first such reference we are aware of is the work of Grillakis [7]. Our approach is closest to the work of Tao [37] and [31] (see also Chap. 6.3 of [38]). These bounds are also essentially identical to the "rigidity estimate" of Kenig-Merle [12]. It should be noted that estimates of this

type are probably the *only* generally useful time-like component concentration bounds possible for non-symmetric wave equations, and they will hold for any Lagrangian field equation on $(2 + 1)$ Minkowski space.

Next, we introduce a general argument to rule out the existence of finite energy self-similar solutions to (1). Such results are essentially standard in the literature (e.g. see the section on wave-maps in [25]), but we take some care here to develop a version which applies to the setup of our work. This crucially uses the energy estimates developed in Sect. 3, as well as a boundary regularity result of J. Qing for harmonic maps (see [21]).

The compactness result in Proposition 5.1, proved in Sect. 5, allows us to produce the strongly convergent subsequence of wave maps in case A) of Theorems 1.3, 1.5. It applies to local sequences $\Phi^{(n)}$ of small energy wave maps with the additional property that $X\Phi^{(n)} \to 0$ in L^2 for some time-like vector field X. This estimate uses only the standard small energy theory of [41], and is completely independent of the more involved regularity result in our companion paper [27].

Given these three building blocks, the proof of Theorems 1.3 and 1.5 presented in Sect. 6 proceeds as follows:

Step 1 (Extension and scaling). We assume that part B) of Theorem 1.3, respectively Theorem 1.5 does not hold for a wave map Φ and for some $\epsilon > 0$. We construct an extension of Φ as in part B) satisfying (6), respectively (10). Then the energy dispersion relation (7), respectively (11) must fail. Thus, we can find sequences t_n, x_n, and k_n so that

$$|P_{k_n}\Phi(t_n, x_n)| + 2^{-k_n}|P_{k_n}\partial_t\Phi(t_n, x_n)| > \epsilon,$$

with $t_n \to 0$ in the case of Theorem 1.3, respectively $t_n \to \infty$ in the case of Theorem 1.5. In addition, the flux-energy relation

$$\mathcal{F}_{[t_1,t_2]}[\Phi] = \mathcal{E}_{S_{t_2}}[\Phi] - \mathcal{E}_{S_{t_1}}[\Phi]$$

shows that in the case of Theorem 1.3 we have

$$\lim_{t_1,t_2\to 0} \mathcal{F}_{[t_1,t_2]}[\Phi] = 0$$

and in the case of Theorem 1.5 we have

$$\lim_{t_1,t_2\to\infty} \mathcal{F}_{[t_1,t_2]}[\Phi] = 0.$$

This allows us also to choose $\epsilon_n \to 0$ such that

$$\mathcal{F}_{[\epsilon_n t_n, t_n]}[\Phi] \leq \epsilon_n^{\frac{1}{2}}\mathcal{E}[\Phi].$$

Rescaling to $t = 1$ we produce the sequence of wave maps

$$\Phi^{(n)}(t, x) = \Phi(t_n t, t_n x)$$

in the increasing regions $C_{[\epsilon_n, 1]}$ so that

$$\mathcal{F}_{[\epsilon_n, 1]}[\Phi^{(n)}] \leq \epsilon_n^{\frac{1}{2}}\mathcal{E}[\Phi],$$

and also points $x_n \in \mathbb{R}^2$ and frequencies $k_n \in \mathbb{Z}$ so that

$$|P_{k_n}\Phi^{(n)}(1, x_n)| + 2^{-k_n}|P_{k_n}\partial_t\Phi^{(n)}(1, x_n)| > \epsilon. \tag{14}$$

From this point on, the proofs of Theorems 1.3,1.5 are identical.

Step 2 (Elimination of null concentration scenario). Using the fixed time portion of the X_0 energy bounds we eliminate the case of null concentration

$$|x_n| \to 1, \qquad k_n \to \infty$$

in estimate (14), and show that the sequence of maps $\Phi^{(n)}$ at time $t = 1$ must either have low frequency concentration in the range:

$$m(\epsilon, E) < k_n < M(\epsilon, E), \qquad |x_n| < R(\epsilon, E)$$

or high frequency concentration strictly inside the cone:

$$k_n \geq M(\epsilon, E), \qquad |x_n| < \gamma(\epsilon, E) < 1.$$

Step 3 (Time-like energy concentration). In both remaining cases above we show that a nontrivial portion of the energy of $\Phi^{(n)}$ at time 1 must be located inside a smaller cone,

$$\frac{1}{2} \int_{t=1, |x|<\gamma_1} \left(|\partial_t \Phi^{(n)}|^2 + |\nabla_x \Phi^{(n)}|^2 \right) dx \geq E_1,$$

where $E_1 = E_1(\epsilon, E)$ and $\gamma_1 = \gamma_1(\epsilon, E) < 1$.

Step 4 (Uniform propagation of non-trivial time-like energy). Using again the X_0 energy bounds we propagate the above time-like energy concentration for $\Phi^{(n)}$ from time 1 to smaller times $t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$,

$$\frac{1}{2} \int_{|x|<\gamma_2(\epsilon, E)t} \left(|\partial_t \Phi^{(n)}|^2 + |\nabla_x \Phi^{(n)}|^2 \right) dx \geq E_0(\epsilon, E), \qquad t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}].$$

At the same time, we obtain bounds for $X_0 \Phi^{(n)}$ outside smaller and smaller neighborhoods of the cone, namely

$$\int_{C^{\epsilon_n}_{[\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]}} \rho^{-1} |X_0 \Phi^{(n)}|^2 dx dt \lesssim 1.$$

Step 5 (Final rescaling). By a pigeonhole argument and rescaling we end up producing another sequence of maps, denoted still by $\Phi^{(n)}$, which are sections of the original wave map Φ and are defined in increasing regions $C_{[1,T_n]}$, $T_n = e^{|\ln \epsilon_n|^{\frac{1}{2}}}$, and satisfy the following three properties:

$$\mathcal{E}_{S_t}[\Phi^{(n)}] \approx E, \quad t \in [1, T_n] \quad \text{(Bounded Energy)},$$

$$\mathcal{E}_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}] \geq E_2, \quad t \in [1, T_n] \quad \text{(Nontrivial Time-like Energy)},$$

$$\iint_{C^{\epsilon_n^{\frac{1}{2}}}_{[1,T_n]}} \frac{1}{\rho} |X_0 \Phi^{(n)}|^2 dx dt \lesssim |\log \epsilon_n|^{-\frac{1}{2}} \qquad \text{(Decay to Self-similar Mode)}.$$

Step 6 (Isolating the concentration scales). The compactness result in Proposition 5.1 only applies to wave maps with energy below the threshold E_0 in the small data result. Thus we need to understand on which scales such concentration can occur. Using several additional pigeonholing arguments we show that one of the following two scenarios must occur:

i) Energy Concentration. On a subsequence there exist $(t_n, x_n) \to (t_0, x_0)$, with (t_0, x_0) inside $C_{[\frac{1}{2},\infty)}^{\frac{1}{2}}$, and scales $r_n \to 0$ so that we have

$$\mathcal{E}_{B(x_n, r_n)}[\Phi^{(n)}](t_n) = \frac{1}{10} E_0,$$

$$\mathcal{E}_{B(x, r_n)}[\Phi^{(n)}](t_n) \leq \frac{1}{10} E_0, \qquad x \in B(x_0, r),$$

$$r_n^{-1} \int_{t_n - r_n/2}^{t_n + r_n/2} \int_{B(x_0, r)} |X_0 \Phi^{(n)}|^2 dx dt \to 0.$$

ii) Non-concentration. For each $j \in \mathbb{N}$ there exists an $r_j > 0$ such that for every (t, x) inside $C_j = C_{[1,\infty)}^1 \cap \{2^j < t < 2^{j+1}\}$ one has

$$\mathcal{E}_{B(x, r_j)}[\Phi^{(n)}](t) \leq \frac{1}{10} E_0, \qquad \forall (t, x) \in C_j,$$

$$\mathcal{E}_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}](t) \geq E_2,$$

$$\iint_{C_j} |X_0 \Phi^{(n)}|^2 dx dt \to 0,$$

uniformly in n.

Step 7 (The compactness argument). In case i) above we consider the rescaled wave-maps

$$\Psi^{(n)}(t, x) = \Phi^{(n)}(t_n + r_n t, x_n + r_n x)$$

and show that on a subsequence they converge locally in the energy norm to a finite energy nontrivial wave map Ψ in $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$ which satisfies $X(t_0, x_0)\Psi = 0$. Thus Ψ must be a Lorentz transform of a nontrivial harmonic map.

In case ii) above we show directly that the sequence $\Phi^{(n)}$ converges locally on a subsequence in the energy norm to finite energy nontrivial wave map Ψ, defined in the interior of a translated cone $C_{[2,\infty)}^2$, which satisfies $X_0 \Psi = 0$. Consequently, in hyperbolic coordinates we may interpret Ψ as a nontrivial harmonic map

$$\Psi : \mathbb{H}^2 \to \mathcal{M}.$$

Compactifying this and using conformal invariance, we obtain a non-trivial finite energy harmonic map

$$\Psi : \mathbb{D}^2 \to \mathcal{M}$$

from the unit disk $\mathbb{D}^2$, which according to the estimates of Sect. 3 obeys the additional weighted energy bound:

$$\int_{\mathbb{D}^2} |\nabla_x \Phi|^2 \frac{dx}{1 - r} < \infty.$$

But such maps do not exist via combination of a theorem of Qing [21] and a theorem of Lemaire [19].

3. Weighted Energy Estimates for the Wave Equation

In this section we prove the main energy decay estimates. The technique we use is the standard one of contracting the energy-momentum tensor:

$$T_{\alpha\beta}[\Phi] = m_{ij}(\Phi)\left[\partial_\alpha\phi^i\,\partial_\beta\phi^j - \frac{1}{2}g_{\alpha\beta}\,\partial^\gamma\phi^i\partial_\gamma\phi^j\right], \tag{15}$$

with well chosen vector-fields. Here $\Phi = (\phi^1, \dots, \phi^n)$ is a set of local coordinates on the target manifold $(\mathcal{M}, m)$ and $(g_{\alpha\beta})$ stands for the Minkowski metric. The main two properties of $T_{\alpha\beta}[\Phi]$ are that it is divergence free $\nabla^\alpha T_{\alpha\beta} = 0$, and also that it obeys the positive energy condition $T(X, Y) \geqslant 0$ whenever both $g(X, X) \leqslant 0$ and $g(Y, Y) \leqslant 0$. This implies that contracting $T_{\alpha\beta}[\Phi]$ with timelike/null vector-fields will result in good energy estimates on characteristic and space-like hypersurfaces.

If X is some vector-field, we can form its associated momentum density (i.e. its Noether current)

$$^{(X)}P_\alpha = T_{\alpha\beta}[\Phi]X^\beta.$$

This one form obeys the divergence rule

$$\nabla^{\alpha\,(X)}P_\alpha = \frac{1}{2}T_{\alpha\beta}[\Phi]^{(X)}\pi^{\alpha\beta}, \tag{16}$$

where $^{(X)}\pi_{\alpha\beta}$ is the deformation tensor of X,

$$^{(X)}\pi_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

A simple computation shows that one can also express

$$^{(X)}\pi = \mathcal{L}_X g.$$

This latter formulation is very convenient when dealing with coordinate derivatives. Recall that in general one has:

$$(\mathcal{L}_X g)_{\alpha\beta} = X(g_{\alpha\beta}) + \partial_\alpha(X^\gamma)g_{\gamma\beta} + \partial_\beta(X^\gamma)g_{\alpha\gamma}.$$

Our energy estimates are obtained by integrating the relation (16) over cones $C^\delta_{[t_1,t_2]}$. Then from (16) we obtain, for $\delta \leq t_1 \leq t_2$:

$$\int_{S^\delta_{t_2}} {}^{(X)}P_0\,dx + \frac{1}{2}\int\!\!\int_{C^\delta_{[t_1,t_2]}} T_{\alpha\beta}[\Phi]^{(X)}\pi^{\alpha\beta}\,dxdt = \int_{S^\delta_{t_1}} {}^{(X)}P_0\,dx + \int_{\partial C^\delta_{[t_1,t_2]}} {}^{(X)}P_L\,dA, \tag{17}$$

where dA is an appropriately normalized (Euclidean) surface area element on the lateral boundary of the cone $r = t - \delta$.

The standard energy estimates come from contracting $T_{\alpha\beta}[\Phi]$ with $Y = \partial_t$. Then we have

$$^{(Y)}\pi = 0, \qquad {}^{(Y)}P_0 = \frac{1}{2}(|\partial_t\Phi|^2 + |\nabla_x\Phi|^2), \qquad {}^{(Y)}P_L = \frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\nabla\!\!\!/\,\Phi|^2.$$

Applying (17) over $C_{[t_1,t_2]}$ we obtain the energy-flux relation (3) used in the Introduction. Applying (17) over $C_{[\delta,1]}^{\delta}$ yields

$$\int_{\partial C_{[\delta,1]}^{\delta}} \frac{1}{4}|L\Phi|^2 + \frac{1}{2}|\not\partial\Phi|^2 \, dA \leq \mathcal{E}[\Phi]. \tag{18}$$

It will also be necessary for us to have a version of the usual energy estimate adapted to the hyperboloids $\rho = \sqrt{t^2 - r^2} = const$. Integrating the divergence of the $^{(Y)}P_\alpha$ momentum density over regions of the form $\mathcal{R} = \{\rho \geq \rho_0, t \leq t_0\}$ we have:

$$\int_{\{\rho=\rho_0\}\cap\{t\leq t_0\}} {}^{(Y)}P^\alpha dV_\alpha \leq \mathcal{E}[\Phi], \tag{19}$$

where the integrand on the LHS denotes the interior product of $^{(Y)}P$ with the Minkowski volume element. To express this estimate in a useful way, we use the hyperbolic coordinates (CMC foliation):

$$t = \rho\cosh(y), \quad r = \rho\sinh(y), \quad \theta = \Theta. \tag{20}$$

In this system of coordinates, the Minkowski metric becomes

$$-dt^2 + dr^2 + r^2 d\theta^2 = -d\rho^2 + \rho^2\left(dy^2 + \sinh^2(y)d\Theta^2\right). \tag{21}$$

A quick calculation shows that the contraction on line (19) becomes the one-form

$$^{(Y)}P^\alpha dV_\alpha = T(\partial_\rho, \partial_t)\rho^2 dA_{\mathbb{H}^2}, \qquad dA_{\mathbb{H}^2} = \sinh(y)dyd\Theta. \tag{22}$$

The area element $dA_{\mathbb{H}^2}$ is that of the hyperbolic plane $\mathbb{H}^2$. To continue, we note that:

$$\partial_t = \frac{t}{\rho}\partial_\rho - \frac{r}{\rho^2}\partial_y,$$

so in particular

$$T(\partial_\rho, \partial_t) = \frac{\cosh(y)}{2}|\partial_\rho\Phi|^2 - \frac{\sinh(y)}{\rho}\partial_\rho\Phi \cdot \partial_y\Phi$$
$$+ \frac{\cosh(y)}{2\rho^2}\left(|\partial_y\Phi|^2 + \frac{1}{\sinh^2(y)}|\partial_\Theta\Phi|^2\right).$$

Letting $t_0 \to \infty$ in (19) we obtain a useful consequence of this, namely a weighted hyperbolic space estimate for special solutions to the wave-map equations, which will be used in the sequel to rule out the existence of non-trivial finite energy self-similar solutions:

Lemma 3.1. *Let Φ be a finite energy smooth wave-map in the interior of the cone C. Assume also that $\partial_\rho\Phi \equiv 0$. Then one has:*

$$\frac{1}{2}\int_{\mathbb{H}^2}|\nabla_{\mathbb{H}^2}\Phi|^2\cosh(y)dA_{\mathbb{H}^2} \leq \mathcal{E}[\Phi]. \tag{23}$$

Here:

$$|\nabla_{\mathbb{H}^2}\Phi|^2 = |\partial_y\Phi|^2 + \frac{1}{\sinh^2(y)}|\partial_\Theta\Phi|^2,$$

is the covariant energy density for the hyperbolic metric.

Our next order of business is to obtain decay estimates for time-like components of the energy density. For this we use the timelike/null vector-field

$$X_\epsilon = \frac{1}{\rho_\epsilon} \left((t+\epsilon)\partial_t + r\partial_r \right), \qquad \rho_\epsilon = \sqrt{(t+\epsilon)^2 - r^2}. \tag{24}$$

In order to gain some intuition, we first consider the case of X_0. This is most readily expressed in the system of hyperbolic coordinates (20) introduced above. One easily checks that the coordinate derivatives turn out to be

$$\partial_\rho = X_0, \quad \partial_y = r\partial_t + t\partial_r.$$

In particular, X_0 is uniformly timelike with $g(X_0, X_0) = -1$, and one should expect it to generate a good energy estimate on time slices $t = const$. In the system of coordinates (20) one also has that

$$\mathcal{L}_{X_0} g = 2\rho \left(dy^2 + \sinh^2(y) d\Theta^2 \right).$$

Raising this, one then computes

$$^{(X_0)}\pi^{\alpha\beta} = \frac{2}{\rho^3} \left(\partial_y \otimes \partial_y + \sinh^{-2}(y)\partial_\Theta \otimes \partial_\Theta \right).$$

Therefore, we have the contraction identity:

$$\frac{1}{2} T_{\alpha\beta}[\Phi]^{(X_0)}\pi^{\alpha\beta} = \frac{1}{\rho} |X_0\Phi|^2.$$

To compute the components of $^{(X_0)}P_0$ and $^{(X_0)}P_L$ we use the associated optical functions

$$u = t - r, \quad v = t + r.$$

Notice that $\rho^2 = uv$. Also, simple calculations show that

$$X_0 = \frac{1}{\rho} \left(\tfrac{1}{2}vL + \tfrac{1}{2}u\underline{L} \right), \; \partial_t = \frac{1}{2}L + \frac{1}{2}\underline{L}. \tag{25}$$

Finally, we record here the components of $T_{\alpha\beta}[\Phi]$ in the null frame

$$T(L, L) = |L\Phi|^2, \quad T(\underline{L}, \underline{L}) = |\underline{L}\Phi|^2, \quad T(L, \underline{L}) = |\nabla\!\!\!/\,\Phi|^2.$$

By combining the above calculations, we see that we may compute

$$^{(X_0)}P_0 = T(\partial_t, X_0) = \frac{1}{4}\left(\frac{v}{u}\right)^{\frac{1}{2}} |L\Phi|^2 + \frac{1}{4}\left[\left(\frac{v}{u}\right)^{\frac{1}{2}} + \left(\frac{u}{v}\right)^{\frac{1}{2}}\right] |\nabla\!\!\!/\,\Phi|^2 + \frac{1}{4}\left(\frac{u}{v}\right)^{\frac{1}{2}} |\underline{L}\Phi|^2,$$

$$^{(X_0)}P_L = T(L, X_0) = \frac{1}{2}\left(\frac{v}{u}\right)^{\frac{1}{2}} |L\Phi|^2 + \frac{1}{2}\left(\frac{u}{v}\right)^{\frac{1}{2}} |\nabla\!\!\!/\,\Phi|^2.$$

These are essentially the same as the components of the usual energy currents $^{(\partial_t)}P_0$ and $^{(\partial_t)}P_L$ modulo ratios of the optical functions u and v.

One would expect to get nice space-time estimates for $X_0\Phi$ by integrating (16) over the interior cone $r \leqslant t \leqslant 1$. The only problem is that the boundary terms degenerate rather severely when $\rho \to 0$. To avoid this we simply redo everything with the shifted

version X_ϵ from line (24). The above formulas remain valid with u, v replaced by their time shifted versions

$$u_\epsilon = (t + \epsilon) - r, \quad v_\epsilon = (t + \epsilon) + r.$$

Furthermore, notice that for small t one has in the region $r \leqslant t$ the bounds

$$\left(\frac{v_\epsilon}{u_\epsilon}\right)^{\frac{1}{2}} \approx 1, \qquad \left(\frac{u_\epsilon}{v_\epsilon}\right)^{\frac{1}{2}} \approx 1, \qquad 0 < t \leq \epsilon.$$

Therefore, one has in $r \leqslant t$ that

$$^{(X_\epsilon)}P_0 \approx {}^{(\partial_t)}P_0, \qquad 0 < t \leq \epsilon.$$

In what follows we work with a wave-map Φ in $C_{[\epsilon,1]}$. We denote its total energy and flux by

$$E = \mathcal{E}_{S_1}[\Phi], \qquad F = \mathcal{F}_{[\epsilon,1]}[\Phi].$$

In the limiting case $F = 0$, $\epsilon = 0$ one could apply (17) to obtain

$$\int_{S_{t_2}^0} {}^{(X_\epsilon)}P_0 \, dx + \iint_{C_{[t_1,t_2]}^0} \frac{1}{\rho_\epsilon} |X_\epsilon \Phi|^2 \, dx dt = \int_{S_{t_1}^0} {}^{(X_\epsilon)}P_0 \, dx.$$

By (26), letting $t_1 \to 0$ followed by $\epsilon \to 0$ and taking the supremum over $0 < t_2 \leqslant 1$ we would get the model estimate

$$\sup_{t \in (0,1]} \int_{S_t^0} {}^{(X_0)}P_0 \, dx + \iint_{C_{[0,1]}^0} \frac{1}{\rho} |X_0 \Phi|^2 \, dx dt \leq E.$$

However, here we need to deal with a small nonzero flux. Observing that

$$^{(X_\epsilon)}P_L \lesssim \epsilon^{-\frac{1}{2}} {}^{(\partial_t)}P_L,$$

from (17) we obtain the weaker bound

$$\int_{S_{t_2}^0} {}^{(X_\epsilon)}P_0 \, dx + \iint_{C_{[t_1,t_2]}^0} \frac{1}{\rho_\epsilon} |X_\epsilon \Phi|^2 \, dx dt \lesssim \int_{S_{t_1}^0} {}^{(X_\epsilon)}P_0 \, dx + \epsilon^{-\frac{1}{2}} F.$$

Letting $t_1 = \epsilon$ and taking supremum over $\epsilon \leqslant t_2 \leqslant 1$ we obtain

$$\sup_{t \in (\epsilon,1]} \int_{S_t^0} {}^{(X_\epsilon)}P_0 \, dx + \iint_{C_{[0,1]}^0} \frac{1}{\rho_\epsilon} |X_\epsilon \Phi|^2 \, dx dt \lesssim E + \epsilon^{-\frac{1}{2}} F. \qquad (26)$$

A consequence of this is the following, which will be used to rule out the case of asymptotically null pockets of energy:

Lemma 3.2. *Let Φ be a smooth wave-map in the cone $C_{(\epsilon,1]}$ which satisfies the flux-energy relation $F \lesssim \epsilon^{\frac{1}{2}} E$. Then*

$$\int_{S_1^0} {}^{(X_\epsilon)}P_0 \, dx \lesssim E. \qquad (27)$$

Next, we can replace X_ϵ by X_0 in (26) if we restrict the integrals on the left to $r < t - \epsilon$. In this region we have

$$^{(X_\epsilon)}P_0 \approx {}^{(X_0)}P_0, \qquad \rho_\epsilon \approx \rho.$$

Using the second member above, a direct computation shows that in $r < t - \epsilon$,

$$\frac{1}{\rho}|X_0\Phi|^2 \lesssim \frac{1}{\rho_\epsilon}|X_\epsilon\Phi|^2 + \frac{\epsilon^2}{\rho^3}|\partial_t\Phi|^2,$$

and also

$$\int_{C^\epsilon_{(\epsilon,1]}} \frac{\epsilon^2}{\rho^3}|\partial_t\Phi|^2 dxdt \le \int_{C^\epsilon_{(\epsilon,1]}} \frac{\epsilon^{\frac{1}{2}}}{t^{\frac{3}{2}}}|\partial_t\Phi|^2 dxdt \lesssim E.$$

Thus, we have proved the following estimate which will be used to conclude that rescaling of Φ are asymptotically stationary, and also used to help trap uniformly time-like pockets of energy:

Lemma 3.3. *Let Φ be a smooth wave-map in the cone $C_{(\epsilon,1]}$ which satisfies the flux-energy relation $F \lesssim \epsilon^{\frac{1}{2}}E$. Then we have*

$$\sup_{t\in(\epsilon,1]}\int_{S^\epsilon_t} {}^{(X_0)}P_0 \, dx + \int\int_{C^\epsilon_{[\epsilon,1]}} \frac{1}{\rho}|X_0\Phi|^2 dxdt \lesssim E. \tag{28}$$

Finally, we use the last lemma to propagate pockets of energy forward away from the boundary of the cone. By (17) for X_0 we have

$$\int_{S^\delta_1} {}^{(X_0)}P_0 \, dx \le \int_{S^\delta_{t_0}} {}^{(X_0)}P_0 \, dx + \int_{\partial C^\delta_{[t_0,1]}} {}^{(X_0)}P_L \, dA, \qquad \epsilon \le \delta < t_0 < 1.$$

We consider the two components of $^{(X_0)}P_L$ separately. For the angular component, by (18) we have the bound

$$\int_{\partial C^\delta_{[t_0,1]}} \left(\frac{u}{v}\right)^{\frac{1}{2}} |\partial\!\!\!/\Phi|^2 \, dA \lesssim \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} \int_{\partial C^\delta_{[t_0,1]}} |\partial\!\!\!/\Phi|^2 \, dA \lesssim \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} E.$$

For the L component a direct computation shows that

$$|L\Phi| \lesssim \left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\Phi| + \left(\frac{u}{v}\right) |\underline{L}\Phi|.$$

Thus we obtain

$$\int_{S^\delta_1} {}^{(X_0)}P_0 \, dx \lesssim \int_{S^\delta_{t_0}} {}^{(X_0)}P_0 \, dx + \left(\frac{\delta}{t_0}\right)^{\frac{1}{2}} E + \int_{\partial C^\delta_{[t_0,1]}} \left(\left(\frac{u}{v}\right)^{\frac{1}{2}} |X_0\phi|^2 + \left(\frac{u}{v}\right)^{\frac{3}{2}} |\underline{L}\Phi|^2\right) dA.$$

For the last term we optimize with respect to $\delta \in [\delta_0, \delta_1]$ to obtain:

Lemma 3.4. *Let* Φ *be a smooth wave-map in the cone* $C_{(\epsilon,1]}$ *which satisfies the flux-energy relation* $F \lesssim \epsilon^{\frac{1}{2}} E$. *Suppose that* $\epsilon \leq \delta_0 \ll \delta_1 \leq t_0 \leq 1$. *Then*

$$\int_{S_1^{\delta_1}} {}^{(X_0)}P_0 \, dx \lesssim \int_{S_{t_0}^{\delta_0}} {}^{(X_0)}P_0 \, dx + \left(\left(\frac{\delta_1}{t_0} \right)^{\frac{1}{2}} + (\ln(\delta_1/\delta_0))^{-1} \right) E. \qquad (29)$$

To prove this lemma, it suffices to choose $\delta \in [\delta_0, \delta_1]$ so that

$$\int_{\partial C_{[t_0,1]}^{\delta}} \left[\left(\frac{u}{v} \right)^{\frac{1}{2}} |X_0 \phi|^2 + \left(\frac{u}{v} \right)^{\frac{3}{2}} |\underline{L}\Phi|^2 \right] dA \lesssim |\ln(\delta_1/\delta_0)|^{-1} E.$$

This follows by pigeonholing the estimate

$$\int_{C_{[t_0,1]}^{\delta_0} \setminus C_{[t_0,1]}^{\delta_1}} \frac{1}{u} \left[\left(\frac{u}{v} \right)^{\frac{1}{2}} |X_0 \phi|^2 + \left(\frac{u}{v} \right)^{\frac{3}{2}} |\underline{L}\Phi|^2 \right] dxdt \lesssim E.$$

The first term is estimated directly by (28). For the second we simply use energy bounds since in the domain of integration we have the relation

$$\frac{1}{u} \left(\frac{u}{v} \right)^{\frac{3}{2}} \leq \frac{\delta_1^{\frac{1}{2}}}{t^{\frac{3}{2}}}.$$

4. Finite Energy Self Similar Wave-Maps

The purpose of this section is to prove the following theorem:

Theorem 4.1 (Absence of non-trivial finite energy self similar wave-maps in 2D).
Let Φ *be a finite energy[1] solution to the wave-map equation* (1) *defined in the forward cone* C. *Suppose also that* $\partial_\rho \Phi \equiv 0$. *Then* $\Phi \equiv const$.

Remark 4.2. Theorems of this type are standard in the literature. The first such reference we are aware of is in the work of Shatah–Tahvildar-Zadeh [26] on the equivariant case. This was later extended by Shatah-Struwe [25] to disprove the existence of (initially) *smooth* self-similar profiles. However, the authors are not aware of an explicit reference in the literature ruling out the possibility of general (i.e. non-symmetric) finite energy self-similar solutions to the system (1), although this statement has by now acquired the status of a folk-lore theorem. It is this latter version in the above form that is necessary in the context of the present work.

Remark 4.3. It is important to remark that the finite energy assumption cannot be dropped, and also that this failure is not due to interior regularity. That is, there are non-trivial C^∞ self similar solutions to (1) in C but these solutions all have infinite energy. However, the energy divergence is marginal, i.e. the energy in S_t^δ only grows as $|\ln(\delta)|$ as $\delta \to 0$. Further, these solutions have finite energy when viewed as harmonic maps in $\mathbb{H}^2$.

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[1] That is to say, a weak solution of (1) (in the sense of [10]), such that ess $\sup_{0<t} \| \nabla_{t,x} \Phi(t) \|_{L^2(|x|<t)} < \infty$.

Proof of Theorem 4.1. Writing out the system (1) in the coordinates (20), and canceling the balanced factor of ρ^{-2} on both sides we have that Φ obeys:

$$\Delta_{\mathbb{H}^2}\Phi^a = -g_{\mathbb{H}^2}^{ij}\mathcal{S}_{bc}^a(\Phi)\partial_i\Phi^b\partial_j\Phi^c,$$

where in polar coordinates $g_{\mathbb{H}^2} = dy^2 + \sinh^2(y)d\Theta^2$ is the standard hyperbolic metric. Thus, Φ^a is an entire harmonic map $\Phi : \mathbb{H}^2 \to \mathcal{M}$. By elliptic regularity, such a map is smooth with uniform bounds on compact sets (see [10]). In particular, Φ is C^∞ in the interior of the cone C. Therefore, from estimate (23) of Lemma 3.1, Φ enjoys the additional weighted energy estimate:

$$\int_{\mathbb{H}^2}|\nabla_{\mathbb{H}^2}\Phi|^2\cosh(y)dA_{\mathbb{H}^2} \leq 2E(\Phi) = \int_{S_t}\left(|\partial_t\Phi|^2 + |\nabla_x\Phi|^2\right)dx, \qquad (30)$$

for any fixed $t > 0$.

To proceed further, it is convenient to rephrase all of the above in terms of the conformal compactification of $\mathbb{H}^2$. Using the pseudo-spherical stereographic projection from hyperboloids to the unit disk $\mathbb{D}^2 = \{t = 0\} \cap \{x^2 + y^2 < 1\}$ in Minkowski space (see Chap. II of [6]):

$$\pi(t, x^1, x^2) = (-1, 0, 0) - \frac{2\rho(t+\rho, x^1, x^2)}{\langle(t+\rho, x^1, x^2), (t+\rho, x^1, x^2)\rangle},$$

as well as the conformal invariance of the $2D$ harmonic map equation and its associated Dirichlet energy, we have from (30) that Φ induces a finite energy harmonic map $\Phi : \mathbb{D}^2 \to \mathcal{M}$ with the additional property that

$$\frac{1}{2}\int_{\mathbb{D}^2}|\nabla_x\Phi|^2\left(\frac{1+r^2}{1-r^2}\right)dx < \infty. \qquad (31)$$

To conclude, we only need to show that such maps are trivial. Notice that the weight $(1-r)^{-1}$ is *critical* in this respect, for there are many non-trivial[2] finite energy harmonic maps from $\mathbb{D}^2 \to \mathcal{M}$, regardless of the curvature of $\mathcal{M}$, which are also uniformly smooth up to the boundary $\partial\mathbb{D}^2$ and may therefore absorb an energy weight of the form $(1-r)^{-\alpha}$ with $\alpha < 1$.

From the bound (31), we have that there exists a sequence of radii $r_n \nearrow 1$ such that:

$$\int_{r=r_n}|\nabla\Phi|^2 dl = o_n(1).$$

From the uniform boundedness of $\Phi(r)$ in $L^2(\mathbb{S}^1)$ and the trace theorem, this implies that:

$$\|\Phi|_{\partial\mathbb{D}^2}\|_{L^2(\mathbb{S}^1)} < \infty, \qquad \|\Phi|_{\partial\mathbb{D}^2}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S}^1)} = 0,$$

so therefore $\Phi|_{\partial\mathbb{D}^2} \equiv const$. In particular, $\Phi : \mathbb{D}^2 \to \mathcal{M}$ is a finite energy harmonic map with smooth boundary values. By a theorem of J. Qing (see [21]), it follows that Φ

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[2] For example, if $\mathcal{M}$ is a complex surface with conformal metric $m = \lambda^2 d\Phi d\overline{\Phi}$, then the harmonic map equation becomes $\partial_z\partial_{\overline{z}}\Phi = -2\partial_\Phi(\ln\lambda)\partial_z\Phi\partial_{\overline{z}}\Phi$. Thus, any holomorphic function $\Phi : \mathbb{D}^2 \to \mathcal{M}$ suffices to produce an infinite energy self similar "blowup profile" for (1). See Chap. I of [24].

has uniform regularity up to $\partial \mathbb{D}^2$. Thus, Φ is C^∞ on the closure $\overline{\mathbb{D}^2}$ with constant boundary value. By Lemaire's uniqueness theorem (see Theorem 8.2.3 of [11], and originally [19]) $\Phi \equiv const$ throughout $\mathbb{D}^2$. By inspection of the coordinates (20) and the fact that $\partial_\rho \Phi \equiv 0$, this easily implies that Φ is trivial in all of C. $\square$

5. A Simple Compactness Result

The main aim of this section is the following result:

Proposition 5.1. *Let Q be the unit cube, and let $\Phi^{(n)}$ be a family of wave maps in $3Q$ which have small energy $\mathcal{E}[\Phi^{(n)}] \leq \frac{1}{10} E_0$ and so that*

$$\|X\Phi^{(n)}\|_{L^2(3Q)} \to 0 \tag{32}$$

for some smooth time-like vector field X. Then there exists a wave map $\Phi \in H^{\frac{3}{2}-\epsilon}(Q)$ with $E[\Phi] \leq \frac{1}{10} E_0$ so that on a subsequence we have the strong convergence

$$\Phi^{(n)} \to \Phi \qquad in \ H^1(Q).$$

Proof. The argument we present here is inspired by the one in Struwe's work on Harmonic-Map heat flow (see [28]). The main point there is that compactness is gained through the higher regularity afforded via integration in time. For a parabolic equation, integrating L^2 in time actually gains a whole derivative by scaling, so in the heat flow case one can control a quantity of the form $\int\int |\nabla^2 \Phi|^2 dx dt$ which leads to control of $\int |\nabla^2 \Phi|^2 dx$ for many individual points in time.

By the small data result in [41], we have a similar (uniform) space-time bound $\Phi^{(n)}$ in any compact subset K of $3Q$ which gains us $\frac{1}{2}$ a derivative over energy, namely

$$\| \chi \Phi^{(n)} \|_{X_\infty^{1,\frac{1}{2}}} \lesssim 1, \qquad \mathrm{supp}\, \chi \subset 3Q, \tag{33}$$

where $X_\infty^{1,\frac{1}{2}}$ here denotes the ℓ^∞ Besov version of the critical inhomogeneous $X^{s,b}$ space.

We obtain a strongly converging $H_{t,x}^1$ sequence of wave-maps through a simple frequency decomposition argument as follows. The vector field X is timelike, therefore its symbol is elliptic in a region of the form $\{\tau > (1 - 2\delta)|\xi|\}$ with $\delta > 0$. Hence, given a cutoff function χ supported in $3Q$ and with symbol equal to 1 in $2Q$, there exists a microlocal space-time decomposition

$$\chi = Q_{-1}(D_{x,t}, x, t)X + Q_0(D_{x,t}, x, t) + R(D_{x,t}, x, t),$$

where the symbols $q_{-1} \in S^{-1}$ and $q_0 \in S^0$ are supported in $3Q \times \{\tau > (1 - 2\delta)|\xi|\}$, respectively $3Q \times \{\tau < (1 - \delta)|\xi|\}$, while the remainder R has symbol $r \in S^{-\infty}$ with spatial support in $3Q$. This yields a decomposition for $\chi \Phi^{(n)}$,

$$\chi \Phi^{(n)} = (Q_0(D, x) + R(D, x))\Phi^{(n)} + Q_{-1}(D, x)X\Phi^{(n)} = \Phi_{bulk}^{(n)} + \mathcal{R}_n. \tag{34}$$

Due to the support properties of q_0, for the main term we have the bound

$$\|\Phi_{bulk}^{(n)}\|_{H_{t,x}^{\frac{3}{2}-\epsilon}} \lesssim \| \chi \Phi^{(n)} \|_{X_\infty^{1,\frac{1}{2}}} \lesssim 1.$$

On the other hand the remainder decays in norm,

$$\| \mathcal{R}_n \|_{H^1_{t,x}} \lesssim \| X\Phi^{(n)} \|_{L^2(3Q)} \longrightarrow_{n\to\infty} 0.$$

Hence on a subsequence we have the strong convergence

$$\chi\Phi^{(n)} \to \Phi \quad \text{in } H^1_{t,x}(3Q).$$

In addition, Φ must satisfy both

$$\Phi \in H^{\frac{3}{2}-\epsilon}, \qquad X\Phi = 0 \text{ in } 2Q.$$

It remains to show that Φ is a wave-map in Q (in fact a "strong" finite energy wave-map according to the definition of Theorem 1.1). There exists a time section $2Q_{t_0}$ close to the center of $2Q$ such that both

$$\| \Phi[t_0] \|_{(\dot{H}^1\times L^2)(2Q_{t_0})} < \infty, \qquad \| \Phi^{(n)}[t_0] - \Phi[t_0] \|_{(\dot{H}^s\times\dot{H}^{s-1})(2Q_{t_0})} \to 0,$$

for some $s < 1$. Letting $\widetilde{\Phi}$ be the solution to (1) with data $\Phi[t_0]$, from the weak stability result (2) in Theorem 1.1 we have $\Phi^{(n)} \to \widetilde{\Phi}$ in $H^s_x(Q_t)$ at fixed time for $s < 1$. Thus $\widetilde{\Phi} = \Phi$ in $H^s_{t,x}(Q)$ which suffices. □

We consider now the two cases we are interested in, namely when $X = \partial_t$ or $X = t\partial_t + x\partial_x$. If $X = \partial_t$ (as will be the case for a general time-like X vector after boosting), then Φ is a harmonic map

$$\Phi : Q \to \mathcal{M}$$

and is therefore smooth (see [10]).

If $X = t\partial_t + x\partial_x$ and $3Q$ is contained within the cone $\{t > |x|\}$ then Φ can be interpreted as a portion of a self-similar Wave-Map, and therefore it is a harmonic map from a domain

$$\Phi : \mathbb{H}^2 \supseteq \Omega \to \mathcal{M}$$

and is again smooth. Note that since the Harmonic-Map equation is conformally invariant, one could as well interpret this as a special case of the previous one. However, in the situation where we have similar convergence on a large number of such domains Ω that fill up $\mathbb{H}^2$, Φ will be globally defined as an $\mathbb{H}^2$ harmonic map to $\mathcal{M}$, and we will therefore be in a position to apply Theorem 4.1.

6. Proof of Theorems 1.3,1.5

We proceed in a series of steps:

6.1. Extension and scaling in the blowup scenario. We begin with Theorem 1.3. Let Φ be a wave map in $C_{(0,1]}$ with terminal energy

$$E = \lim_{t\to 0} \mathcal{E}_{S_t}[\Phi].$$

Suppose that the energy dispersion scenario B) does not apply. Let $\epsilon > 0$ be so that B) does not hold. We can choose ϵ arbitrarily small. We will take advantage of this to

construct an extension of Φ outside the cone which satisfies (6), therefore violating (7) on any time interval $(0, t_0]$.

For this we use energy estimates. Setting

$$\mathcal{F}_{t_0}[\Phi] = \int_{\partial C_{(0,t_0]}} \left(\frac{1}{4} |L\Phi|^2 + \frac{1}{2} |\not\partial \Phi|^2 \right) dA$$

it follows that as $t_0 \to 0$ we have

$$\mathcal{F}_{t_0}[\Phi] = \mathcal{E}_{S_{t_0}}[\Phi] - E \to 0. \tag{35}$$

Then by pigeonholing we can choose t_0 arbitrarily small so that we have the bounds

$$\mathcal{F}_{t_0}[\Phi] \ll \epsilon^8 E, \qquad \int_{\partial S_{t_0}} |\not\partial \Phi|^2 ds \ll \frac{\epsilon^8}{t_0} E. \tag{36}$$

The second bound allows us to extend the initial data for Φ at time t_0 from S_{t_0} to all of $\mathbb{R}^2$ in such a way that

$$\mathcal{E}[\Phi](t_0) - \mathcal{E}_{S_{t_0}}[\Phi] \ll \epsilon^8 E.$$

We remark that by scaling it suffices to consider the case $t_0 = 1$. The second bound in (36) shows, by integration, that the range of Φ restricted to ∂S_{t_0} is contained in a small ball of size ϵ^8 in $\mathcal{M}$. Thus the extension problem is purely local in $\mathcal{M}$, and can be carried out in a suitable local chart by a variety of methods.

We extend the solution Φ outside the cone C between times t_0 and 0 by solving the wave-map equation. By energy estimates it follows that for $t \in (0, t_0]$ we have

$$\mathcal{E}[\Phi](t) - \mathcal{E}_{S_t}[\Phi] = \mathcal{E}[\Phi](t_0) - \mathcal{E}_{S_{t_0}}[\Phi] + \mathcal{F}_{t_0}[\Phi] - \mathcal{F}_t[\Phi] \leq \frac{1}{2} \epsilon^8 E. \tag{37}$$

Hence the energy stays small outside the cone, and by the small data result in Theorem 1.1 there is no blow-up that occurs outside the cone up to time 0. The extension we have constructed is fixed for the rest of the proof.

Since our extension satisfies (6) but B) does not hold, it follows that we can find a sequence (t_n, x_n) with $t_n \to 0$ and $k_n \in \mathbb{Z}$ so that

$$|P_{k_n} \Phi(t_n, x_n)| + 2^{-k} |P_{k_n} \partial_t \Phi(t_n, x_n)| \geq \epsilon. \tag{38}$$

The relation (35) shows that $\mathcal{F}_{t_n} \to 0$. Hence we can find a sequence $\epsilon_n \to 0$ such that $\mathcal{F}_{t_n} < \epsilon_n^{\frac{1}{2}} E$. Thus

$$\mathcal{F}_{[\epsilon_n t_n, t_n]} < \epsilon_n^{\frac{1}{2}} E. \tag{39}$$

Rescaling we obtain the sequence of wave maps

$$\Phi^{(n)}(t, x) = \Phi(t_n t, t_n x)$$

in the increasing family of regions $\mathbb{R}^2 \times [\epsilon_n, 1]$ with the following properties:

a) Uniform energy size,

$$\mathcal{E}[\Phi^{(n)}] \approx E. \tag{40}$$

b) Small energy outside the cone,

$$\mathcal{E}[\Phi^{(n)}] - \mathcal{E}_{S_t}[\Phi^{(n)}] \lesssim \epsilon^8 E. \tag{41}$$

c) Decaying flux,

$$\mathcal{F}_{[\epsilon_n, 1]}[\Phi^{(n)}] < \epsilon_n^{\frac{1}{2}} E. \tag{42}$$

d) Pointwise concentration at time 1,

$$|P_{k_n} \Phi^{(n)}(1, x_n)| + 2^{-k_n} |P_{k_n} \partial_t \Phi^{(n)}(1, x_n)| > \epsilon \tag{43}$$

for some $x_n \in \mathbb{R}^2$, $k_n \in \mathbb{Z}$.

6.2. Extension and scaling in the non-scattering scenario. Next we consider the case of Theorem 1.5. Again we suppose that the energy dispersion scenario B) does not apply for a finite energy wave map Φ in $C_{[1,\infty)}$. Let $\epsilon > 0$ be so that B) does not hold. Setting

$$E = \lim_{t_0 \to \infty} \mathcal{E}_{S_{t_0}}[\Phi], \qquad \mathcal{F}_{t_0}[\Phi] = \int_{\partial C_{[t_0, \infty)}} \left(\frac{1}{4} |L\Phi|^2 + \frac{1}{2} |\partial\!\!\!/\,\Phi|^2 \right) dA$$

it follows that

$$\mathcal{F}_{t_0}[\Phi] = \mathcal{E}_{S_\infty}[\Phi] - \mathcal{E}_{S_{t_0}}[\Phi] \to 0. \tag{44}$$

We choose $t_0 > 1$ so that

$$\mathcal{F}_{t_0}[\Phi] \leq \epsilon^8 E.$$

We obtain our extension of Φ to the interval $[t_0, \infty)$ from the following lemma:

Lemma 6.1. *Let Φ be a finite energy wave-map in $C_{[1,\infty)}$ and E, t_0 as above. Then there exists a wave-map extension of Φ to $\mathbb{R}^2 \times [t_0, \infty)$ which has energy E.*

We remark that as $t \to \infty$ all the energy of the extension moves inside the cone. It is likely that an extension with this property is unique. We do not pursue this here, as it is not needed.

Proof. By pigeonholing we can choose a sequence $t_k \to \infty$ so that we have the bounds

$$\mathcal{F}_{t_k}[\Phi] \to 0, \qquad t_k \int_{\partial S_{t_k}} |\partial\!\!\!/\,\Phi|^2 dA \to 0.$$

The second bound allows us to obtain an extension $\Phi^{(k)}[t_k]$ of the initial data $\Phi[t_k]$ for Φ at time t_k from the circle S_{t_k} to all of $\mathbb{R}^2$ in such a way that

$$\mathcal{E}[\Phi^{(k)}](t_k) - \mathcal{E}_{S_{t_k}}[\Phi] \to 0.$$

By rescaling, this extension problem is equivalent to the one in the case of Theorem 1.3.

We solve the wave map equation backwards from time t_k to time t_0, with data $\Phi^{(k)}[t_k]$. We obtain a wave map $\Phi^{(k)}$ in the time interval $[t_0, t_k]$ which coincides with Φ in $C_{[t_0,t_k]}$. The above relation shows that

$$\mathcal{E}[\Phi^{(k)}] \to E. \qquad (45)$$

By energy estimates it also follows that for large k and $t \in [t_0, t_k]$ we have

$$\mathcal{E}[\Phi^{(k)}](t) - \mathcal{E}_{S_t}[\Phi^{(k)}] = \mathcal{E}[\Phi^{(k)}](t_k) - \mathcal{E}_{S_{t_k}}[\Phi] + \mathcal{F}_{[t,t_k]}[\Phi] \lesssim \epsilon^8 E. \qquad (46)$$

Hence the energy stays small outside the cone, which by the small data result in Theorem 1.1 shows that no blow-up can occur outside the cone between times t_k and t_0.

We will obtain the extension of Φ as the strong limit in the energy norm of a subsequence of the $\Phi^{(k)}$,

$$\Phi = \lim_{k \to \infty} \Phi^{(k)} \qquad \text{in } C(t_0, \infty; \dot{H}^1) \cap \dot{C}^1(t_0, \infty; L^2). \qquad (47)$$

We begin with the existence of a weak limit. By uniform boundedness and the Banach-Alaoglu theorem we have weak convergence for each fixed time

$$\Phi^{(k)}[t] \rightharpoonup \Phi[t] \quad \text{weakly in } \dot{H}^1 \times L^2.$$

Within $C_{[t_0,\infty)}$ all the $\Phi^{(k)}$'s coincide, so the above convergence is only relevant outside the cone. But by (46), outside the cone all $\Phi^{(k)}$ have small energy. This places us in the context of the results in [41]. Precisely, the weak stability bound (2) in Theorem 1.1 and an argument similar to that used in Sect. 5 shows that the limit Φ is a regular finite energy wave-map in $[t_0, \infty) \times \mathbb{R}^2$.

It remains to upgrade the convergence. On one hand, (45) and weak convergence shows that $\mathcal{E}[\Phi] \leq \mathcal{E}_{S_\infty}[\Phi]$. On the other hand $\mathcal{E}[\Phi] \geq \mathcal{E}_{S_t}[\Phi]$, and the latter converges to $\mathcal{E}_{S_\infty}[\Phi]$. Thus we obtain

$$\mathcal{E}[\Phi] = \mathcal{E}_{S_\infty}[\Phi] = \lim_{k \to \infty} \mathcal{E}[\Phi^{(k)}].$$

From weak convergence and norm convergence we obtain strong convergence

$$\Phi^{(k)}[t] \to \Phi[t] \quad \text{in } \dot{H}^1 \times L^2.$$

The uniform convergence in (47) follows by applying the energy continuity in the small data result of Theorem 1.1 outside $C_{[t_0,\infty)}$.   □

By energy estimates for Φ it follows that

$$\mathcal{E}[\Phi] - \mathcal{E}_{S_t}[\Phi] = \mathcal{F}_t[\Phi] \leq F_{t_0}[\Phi] \leq \epsilon^8 E, \qquad t \in [t_0, \infty).$$

Hence the extended Φ satisfies (10) so it cannot satisfy (11). By (35) we obtain the sequence (t_n, x_n) and $k_n \in \mathbb{Z}$ with $t_n \to \infty$ so that (38) holds. On the other hand from the energy-flux relation we have

$$\lim_{t_1,t_2 \to \infty} \mathcal{F}_{[t_1,t_2]}[\Phi] = 0.$$

This allows us to select again $\epsilon_n \to 0$ so that (39) holds; clearly in this case we must also have $\epsilon_n t_n \to \infty$. The same rescaling as in the previous subsection leads to a sequence $\Phi^{(n)}$ of wave maps which satisfy the conditions a)–d) above.

6.3. Elimination of the null concentration scenario. Due to (41), the contribution of the exterior of the cone to the pointwise bounds for $\Phi^{(n)}$ is negligible. Precisely, for low frequencies we have the pointwise bound

$$|P_k \Phi^{(n)}(1, x)| + 2^{-k}|P_k \partial_t \Phi^{(n)}(1, x)| \lesssim \left(2^k(1 + 2^k|x|)^{-N} + \epsilon^4\right) E^{\frac{1}{2}}, \qquad k \leq 0,$$

where the first term contains the contribution from the interior of the cone and the second is the outside contribution. On the other hand, for large frequencies we similarly obtain

$$|P_k \Phi^{(n)}(1, x)| + 2^k|P_k \partial_t \Phi^{(n)}(1, x)| \lesssim \left((1 + 2^k(|x| - 1)_+)^{-N} + \epsilon^4\right) E^{\frac{1}{2}}, \qquad k \geq 0. \tag{48}$$

Hence, in order for (43) to hold, k_n must be large enough,

$$2^{k_n} > m(\epsilon, E),$$

while x_n cannot be too far outside the cone,

$$|x_n| \leq 1 + 2^{-k_n} g(\epsilon, E).$$

This allows us to distinguish three cases:

(i) *Wide pockets of energy.* This is when

$$2^{k_n} < C < \infty.$$

(ii) *Sharp time-like pockets of energy.* This is when

$$2^{k_n} \to \infty, \qquad |x_n| \leq \gamma < 1.$$

(iii) *Sharp pockets of null energy.* This is when

$$2^{k_n} \to \infty, \qquad |x_n| \to 1.$$

Our goal in this subsection is to eliminate the last case. Precisely, we will show that:

Lemma 6.2. *There exists $M = M(\epsilon, E) > 0$ and $\gamma = \gamma(\epsilon, E) < 1$ so that for any wave map $\Phi^{(n)}$ as in (40)–(43) with a small enough ϵ_n we have*

$$2^{k_n} > M \implies |x_n| \leq \gamma. \tag{49}$$

Proof. We apply the energy estimate (27), with ϵ replaced by ϵ_n, to $\Phi^{(n)}$ in the time interval $[\epsilon_n, 1]$. This yields

$$\int_{S_1} (1 - |x| + \epsilon_n)^{-\frac{1}{2}} \left(|L\Phi^{(n)}|^2 + |\partial\!\!\!/\,\Phi^{(n)}|^2\right) dx \lesssim E. \tag{50}$$

The relation (49) would follow from the pointwise concentration bound in (43) if we can prove that for $k > 0$ the bound (50), together with (40) and (41) at time $t = 1$, imply the pointwise estimate

$$|P_k \Phi^{(n)}(1, x)| + 2^{-k}|P_k \partial_t \Phi^{(n)}(1, x)| \lesssim \left(\left((1 - |x|)_+ + 2^{-k} + \epsilon_n\right)^{\frac{1}{8}} + \epsilon^2\right) E^{\frac{1}{2}}. \tag{51}$$

In view of (48) it suffices to prove this bound in the case where $\frac{1}{2} < |x| < 2$. After a rotation we can assume that $x = (x_1, 0)$ with $\frac{1}{2} < x_1 < 2$.

In general we have

$$(1 - |x|)_+ \leq (1 - x_1)_+ + x_2^2,$$

and therefore from (50) we obtain

$$\int_{S_1} \frac{1}{(|1 - x_1| + x_2^2 + \epsilon_n)^{\frac{1}{2}}} \left(|L\Phi^{(n)}|^2 + |\partial\!\!\!/\,\Phi^{(n)}|^2 \right) dx \lesssim E. \tag{52}$$

At the spatial location $(1, 0)$ we have $\partial\!\!\!/\,\Phi^{(n)} = \partial_2$, so we obtain the rough bound

$$|\partial_2 \Phi^{(n)}| \lesssim |\partial\!\!\!/\,\Phi^{(n)}| + (|x_2| + |1 - x_1|)|\nabla_x \Phi^{(n)}|.$$

Similarly,

$$|\partial_t \Phi^{(n)} - \partial_1 \Phi^{(n)}| \lesssim |L\Phi^{(n)}| + (|x_2| + |1 - x_1|)|\nabla_x \Phi^{(n)}|.$$

Hence, taking into account (40) and (41), from (52) we obtain

$$\int_{t=1} \frac{1}{((1 - x_1)_+ + x_2^2 + \epsilon_n)^{\frac{1}{2}} + \epsilon^8} \left(|\partial_t \Phi^{(n)} - \partial_1 \Phi^{(n)}|^2 + |\partial_2 \Phi^{(n)}|^2 \right) dx \lesssim E. \tag{53}$$

Next, given a dyadic frequency $2^k \geq 1$ we consider an angular parameter $2^{-\frac{k}{2}} \leq \theta \leq 1$ and we rewrite the multiplier P_k in the form

$$P_k = P_{k,\theta}^1 \partial_1 + P_{k,\theta}^2 \partial_2,$$

where $P_{k,\theta}^1$ and $P_{k,\theta}^2$ are multipliers with smooth symbols supported in the sets $\{|\xi| \approx 2^k, \ |\xi_2| \lesssim 2^k \theta\}$, respectively $\{|\xi| \approx 2^k, \ |\xi_2| \gtrsim 2^k \theta\}$. The size of their symbols is given by

$$|p_{k,\theta}^1(\xi)| \lesssim 2^{-k}, \qquad |p_{k,\theta}^2(\xi)| \lesssim \frac{1}{|\xi_2|}.$$

Therefore, they satisfy the $L^2 \to L^\infty$ bounds

$$\|P_{k,\theta}^1\|_{L^2 \to L^\infty} \lesssim \theta^{\frac{1}{2}}, \qquad \|P_{k,\theta}^2\|_{L^2 \to L^\infty} \lesssim \theta^{-\frac{1}{2}}. \tag{54}$$

In addition, the kernels of both $P_{k,\theta}^1$ and $P_{k,\theta}^2$ decay rapidly on the $2^{-k} \times \theta^{-1} 2^{-k}$ scale. Thus one can add weights in (54) provided that they are slowly varying on the same scale. The weight in (52) is not necessarily slowly varying, but we can remedy this by slightly increasing the denominator to obtain the weaker bound

$$\int_{t=1} \frac{1}{((1 - x_1)_+ + x_2^2 + 2^{-k} + \epsilon_n)^{\frac{1}{2}} + \epsilon^8} \left(|\partial_t \Phi^{(n)} - \partial_1 \Phi^{(n)}|^2 + |\partial_2 \Phi^{(n)}|^2 \right) dx \lesssim E. \tag{55}$$

Using (40), (55) and the weighted version of (54) we obtain

$$
\begin{aligned}
|P_k \Phi^{(n)}(1, x)| &\leq |P_{k,\theta}^1 \partial_1 \Phi^{(n)}(1, x)| + |P_{k,\theta}^2 \partial_2 \Phi^{(n)}(1, x)| \\
&\lesssim \left(\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} \left(((1 - x_1)_+ + x_2^2 + 2^{-k} + \epsilon_n)^{\frac{1}{4}} + \epsilon^4 \right) \right) E^{\frac{1}{2}}.
\end{aligned}
$$

We set $x_2 = 0$ and optimize with respect to $\theta \in [2^{-k/2}, 1]$ to obtain the desired bound (51) for $\Phi^{(n)}$,

$$|P_k \Phi^{(n)}(1, x_1, 0)| \lesssim \left(\left((1 - x_1)_+ + 2^{-k} + \epsilon_n \right)^{\frac{1}{8}} + \epsilon^2 \right) E^{\frac{1}{2}}, \qquad \frac{1}{2} < x_1 < 2.$$

A similar argument yields

$$2^{-k} |P_k \partial_1 \Phi^{(n)}(1, x_1, 0)| \lesssim \left(\left((1 - x_1)_+ + 2^{-k} + \epsilon_n \right)^{\frac{1}{8}} + \epsilon^2 \right) E^{\frac{1}{2}}, \qquad \frac{1}{2} < x_1 < 2.$$

On the other hand, from (55) we directly obtain

$$2^{-k} |P_k (\partial_t - \partial_1) \Phi^{(n)}(1, x_1, 0)| \lesssim \left(\left((1 - x_1)_+ + 2^{-k} + \epsilon_n \right)^{\frac{1}{8}} + \epsilon^2 \right) E^{\frac{1}{2}}, \qquad \frac{1}{2} < x_1 < 2.$$

Combined with the previous inequality, this yields the bound in (51) for $\partial_t \Phi^{(n)}$. $\quad\square$

6.4. Nontrivial energy in a time-like cone. According to the previous step, the points x_n and frequencies k_n in (43) satisfy one of the following two conditions:

(i) *Wide pockets of energy.* This is when

$$c(\epsilon, E) < 2^{k_n} < C(\epsilon, E).$$

(ii) *Sharp time-like pockets of energy.* This is when

$$2^{k_n} > C(\epsilon, E), \qquad |x_n| \leq \gamma(\epsilon, E) < 1.$$

Using only the bounds (40) and (41), we will prove that there exists $\gamma_1 = \gamma_1(\epsilon, E) < 1$ and $E_1 = E_1(\epsilon, E) > 0$ so that in both cases there is some amount of uniform time-like energy concentration,

$$\frac{1}{2} \int_{t=1, |x|<\gamma_1} \left(|\partial_t \Phi^{(n)}|^2 + |\nabla_x \Phi^{(n)}|^2 \right) dx \geq E_1. \tag{56}$$

For convenience we drop the index n in the following computations. Denote

$$E(\gamma_1) = \frac{1}{2} \int_{t=1, |x|<\gamma_1} \left(|\partial_t \Phi|^2 + |\nabla_x \Phi|^2 \right) dx,$$

where $\gamma_1 \in (\frac{\gamma+1}{2}, 1)$ will be chosen later. Here γ is from line (49) of the previous subsection. Thus at time 1 the function Φ has energy $E(\gamma_1)$ in $\{|x| < \gamma_1\}$, energy $\leq E$ in $\{\gamma_1 < |x| < 1\}$, and energy $\leq \epsilon^8 E$ outside the unit disc. Then we obtain different pointwise estimates for $P_k \Phi[1]$ in two main regimes:

(a) $2^k (1 - \gamma_1) < 1$. Then we obtain

$$\|P_k \Phi(1)\|_{L_x^\infty} + 2^{-k} \|P_k \partial_t \Phi(1)\|_{L_x^\infty} \lesssim E(\gamma_1)^{\frac{1}{2}} + \left((2^k (1 - \gamma_1))^{\frac{1}{2}} + \epsilon^4 \right) E^{\frac{1}{2}},$$

with a further improvement if both

(a1) $2^k(1 - \gamma_1) < 1 < 2^k(1 - \gamma)$ and $|x| < \gamma$, namely

$$|P_k\Phi(1, x)| + 2^{-k}|P_k\partial_t\Phi(1, x)| \lesssim E(\gamma_1)^{\frac{1}{2}} + \left((2^k(1 - \gamma_1))^{\frac{1}{2}}(2^k(1 - \gamma))^{-N} + \epsilon^4\right)E^{\frac{1}{2}}.$$

(b) $2^k(1 - \gamma_1) \geq 1$. Then

$$|P_k\Phi(1, x)| + 2^{-k}|P_k\partial_t\Phi(1, x)| \lesssim E(\gamma_1)^{\frac{1}{2}} + \left((2^k(1 - \gamma))^{-N} + \epsilon^4\right)E^{\frac{1}{2}}, \qquad |x| < \gamma.$$

We use these estimates to bound $E(\gamma_1)$ from below. We first observe that if $1 - \gamma_1$ is small enough then case (i) above implies we are in regime (a), and from (43) we obtain

$$\epsilon \lesssim E(\gamma_1)^{\frac{1}{2}} + \left((C(\epsilon, E)(1 - \gamma_1))^{\frac{1}{2}} + \epsilon^4\right)E^{\frac{1}{2}},$$

which gives a bound from below for $E(\gamma_1)$ if $1 - \gamma_1$ and ϵ are small enough.

Consider now the remaining case (ii) above. If we are in regime (a) but not (a1), then the bound in (a) combined with (43) gives

$$\epsilon \lesssim E(\gamma_1)^{\frac{1}{2}} + \left(\frac{(1 - \gamma_1)^{\frac{1}{2}}}{(1 - \gamma)^{\frac{1}{2}}} + \epsilon^4\right)E^{\frac{1}{2}},$$

which suffices if $1 - \gamma_1$ is small enough. If we are in regime (a1) then we obtain exactly the same inequality directly. Finally, if we are in regime (b) then we achieve an even better bound

$$\epsilon \lesssim E(\gamma_1)^{\frac{1}{2}} + \left(\left(\frac{1 - \gamma_1}{1 - \gamma}\right)^N + \epsilon^4\right)E^{\frac{1}{2}}.$$

Thus, (56) is proved in all cases for a small enough $1 - \gamma_1$.

6.5. Propagation of time-like energy concentration. Here we use the flux relation (39) to propagate the time-like energy concentration in (56) *uniformly* to smaller times $t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$. Precisely, we show that there exists $\gamma_2 = \gamma_2(\epsilon, E) < 1$ and $E_2 = E_2(\epsilon, E) > 0$ so that

$$\frac{1}{2}\int_{|x| < \gamma_2 t}\left(|\partial_t\Phi^{(n)}|^2 + |\nabla_x\Phi^{(n)}|^2\right)dx \geq E_2, \qquad t \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]. \tag{57}$$

At the same time, we also obtain uniform weighted $L_{t,x}^2$ bounds for $X_0\Phi^{(n)}$ outside smaller and smaller neighborhoods of the cone, namely

$$\int_{C^{\epsilon_n}_{[\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]}}\rho^{-1}|X_0\Phi^{(n)}|^2 dx dt \lesssim E. \tag{58}$$

The latter bound (58) is a direct consequence of (28), so we turn our attention to (57). Given a parameter $\gamma_2 = \gamma_2(\gamma_1, E_1)$, and any $t_0 \in [\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$, we define δ_0 and δ_1 according to

$$(1 - \gamma_2)t_0 = \delta_0 \ll \delta_1 \le t_0.$$

We apply Proposition 3.4 to $\Phi^{(n)}$ with this set of small constants.

Optimizing the right-hand side in (29) with respect to the choice of δ_1 it follows that

$$\int_{S_1^{t_0}} {}^{(X_0)}P_0[\Phi^{(n)}]\, dx \lesssim \int_{S_{t_0}^{\delta_0}} {}^{(X_0)}P_0[\Phi^{(n)}]\, dx + |\ln(t_0/\delta_0)|^{-1} E.$$

Converting the X_0 momentum density into the ∂_t momentum density it follows that

$$(1 - \gamma_1)^{\frac{1}{2}} \int_{S_1^{1-\gamma_1}} {}^{(\partial_t)}P_0[\Phi^{(n)}]\, dx \lesssim (1 - \gamma_2)^{-\frac{1}{2}} \int_{S_{t_0}^{\delta_0}} {}^{(\partial_t)}P_0[\Phi^{(n)}]\, dx + |\ln(1 - \gamma_2)|^{-1} E.$$

Hence by (56) we obtain

$$(1 - \gamma_1)^{\frac{1}{2}} E_1 \lesssim (1 - \gamma_2)^{-\frac{1}{2}} \mathcal{E}_{S_{t_0}^{\delta_0}}[\Phi^{(n)}] + |\ln(1 - \gamma_2)|^{-1} E.$$

We choose γ_2 so that

$$|\ln(1 - \gamma_2)|^{-1} E \ll (1 - \gamma_1)^{\frac{1}{2}} E_1.$$

Then the second right-hand side term in the previous inequality can be neglected, and for

$$0 < E_2 \ll (1 - \gamma_1)^{\frac{1}{2}}(1 - \gamma_2)^{\frac{1}{2}} E_1$$

we obtain (57).

6.6. Final rescaling.

The one bound concerning the rescaled wave maps $\Phi^{(n)}$ which is not yet satisfactory is (57), where we would like to have decay in n instead of uniform boundedness. This can be achieved by further subdividing the time interval $[\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$.

For $2 < N < \epsilon^{-\frac{1}{4}}$ we divide the time interval $[\epsilon_n^{\frac{1}{2}}, \epsilon_n^{\frac{1}{4}}]$ into about $|\ln \epsilon_n|/\ln N$ subintervals of the form $[t, Nt]$. By pigeonholing, there exists one such subinterval which we denote by $[t_n, Nt_n]$ so that

$$\iint_{C^{\epsilon_n}_{[t_n, Nt_n]}} \frac{1}{\rho} |X_0 \Phi^{(n)}|^2 dx dt \lesssim \frac{\ln N}{|\ln \epsilon_n|} E. \tag{59}$$

We assign to $N = N_n$ the value

$$N_n = e^{\sqrt{|\ln \epsilon_n|}}.$$

Rescaling the wave maps $\Phi^{(n)}$ from the time interval $[t_n, N_n t_n]$ to the time interval $[1, N_n]$ we obtain a final sequence of rescaled wave maps, still denoted by $\Phi^{(n)}$, defined on increasing sets $C_{[1, T_n]}$, where $T_n \to \infty$, with the following properties:

a) Bounded energy,

$$\mathcal{E}_{S_t}[\Phi^{(n)}](t) \approx E, \qquad t \in [1, T_n]. \tag{60}$$

b) Uniform amount of nontrivial time-like energy,

$$\mathcal{E}_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}](t) \geq E_2, \qquad t \in [1, T_n]. \tag{61}$$

c) Decay to self-similar mode,

$$\iint_{C_{[1,T_n]}^{\frac{1}{\epsilon_n^2}}} \frac{1}{\rho} |X_0 \Phi^{(n)}|^2 dx dt \lesssim |\log \epsilon_n|^{-\frac{1}{2}} E. \tag{62}$$

6.7. Concentration scales. We partition the set $C_{[1,\infty)}^1$ into dyadic subsets

$$C_j = \{(t, x) \in C_{[1,\infty)}^1; \ 2^j < t < 2^{j+1}\}, \qquad j \in \mathbb{N}.$$

We also consider slightly larger sets

$$\widetilde{C}_j = \{(t, x) \in C_{[\frac{1}{2},\infty)}^{\frac{1}{2}}; \ 2^j < t < 2^{j+1}\}, \qquad j \in \mathbb{N}.$$

Then we prove that

Lemma 6.3. *Let $\Phi^{(n)}$ be a sequence of wave maps satisfying* (60), (61) *and* (62). *Then for each $j \in \mathbb{N}$ one of the following alternatives must hold on a subsequence:*

(i) *Concentration of non-trivial energy. There exist points $(t_n, x_n) \in \widetilde{C}_j$, a sequence of scales $r_n \to 0$, and some $r = r_j$ with $0 < r < \frac{1}{4}$ so that the following three bounds hold:*

$$\mathcal{E}_{B(x_n, r_n)}[\Phi^{(n)}](t_n) = \frac{1}{10} E_0, \tag{63}$$

$$\mathcal{E}_{B(x, r_n)}[\Phi^{(n)}](t_n) \leq \frac{1}{10} E_0, \qquad x \in B(x_n, r), \tag{64}$$

$$r_n^{-1} \int_{t_n - r_n/2}^{t_n + r_n/2} \int_{B(x_n, r)} |X_0 \Phi^{(n)}|^2 dx dt \to 0. \tag{65}$$

(ii) *Nonconcentration of uniform energy. There exists some $r = r_j$ with $0 < r < \frac{1}{4}$ so that the following three bounds hold:*

$$\mathcal{E}_{B(x, r)}[\Phi^{(n)}](t) \leq \frac{1}{10} E_0, \qquad \forall (t, x) \in C_j, \tag{66}$$

$$\mathcal{E}_{S_t^{(1-\gamma_2)t}}[\Phi^{(n)}](t) \geq E_2, \qquad \text{when } B(0, (1-\gamma_2)t) \subseteq C_{[1,\infty)}^1, \tag{67}$$

$$\iint_{C_j} |X_0 \Phi^{(n)}|^2 dx dt \to 0. \tag{68}$$

Proof. The argument boils down to some straightforward pigeonholing, and is essentially identical for all $j \in \mathbb{N}$, which is now fixed throughout the proof. Given any large parameter $N \in \mathbb{N}$ we partition the time interval $[2^j, 2^{j+1}]$ into about $N2^j$ equal intervals,

$$I_k = [2^j + (k-1)/(10N), 2^j + k/(10N)], \qquad k = 1, 10N2^j.$$

Then it suffices to show that the conclusion of the lemma holds with C_j and $\widetilde{C}_j$ replaced by

$$C_j^k = C_j \cap I_k \times \mathbb{R}^2, \qquad \widetilde{C}_j^k = \widetilde{C}_j \cap I_k \times \mathbb{R}^2.$$

We begin by constructing a low energy barrier around C_j^k. To do this we partition $\widetilde{C}_j^k \setminus C_j^k$ into N sets

$$\widetilde{C}_j^{k,l} = \left\{ (t,x) \in \widetilde{C}_j^k; \ \frac{1}{4} + \frac{l-1}{4N} < t - |x| < \frac{1}{4} + \frac{l}{4N} \right\}, \qquad l = 1, N.$$

By integrating energy estimates we have

$$\sum_{l=1}^{N} \int_{I_k} \mathcal{E}_{\widetilde{C}_j^{k,l}}[\Phi^{(n)}](t)dt \leq \int_{I_k} \mathcal{E}_{\widetilde{C}_j^k}[\Phi^{(n)}](t)dt \leq \frac{1}{10N}E.$$

Thus by pigeonholing, for each fixed n there must exist l_n so that

$$\sum_{l=l_n-1}^{l_n+1} \int_{I_k} \mathcal{E}_{\widetilde{C}_j^{k,l}}[\Phi^{(n)}](t)dt \leq \frac{3}{10N^2}E,$$

and further there must be some $t_n \in I_k$ so that

$$\sum_{j=j_n-1}^{j_n+1} \mathcal{E}_{\widetilde{C}_j^{k,l}}[\Phi^{(n)}](t_n) \leq \frac{3}{N}E.$$

For $t \in I_k$ we have $|t - t_n| < 1/(10N)$, and therefore the t section of $\widetilde{C}_j^{k,l_n}$ lies within the influence cone of the t_n section of $\widetilde{C}_j^{k,l_n-1} \cap \widetilde{C}_j^{k,l_n} \cap \widetilde{C}_j^{k,l_n+1}$. Hence it follows that one has the uniform bound

$$\mathcal{E}_{\widetilde{C}_j^{k,l_n}}[\Phi^{(n)}(t)] \leq \frac{3}{N}E, \qquad t \in I_k.$$

We choose N large enough so that we beat the perturbation energy

$$\frac{3}{N}E \leq \frac{1}{20}E_0.$$

Then the set $\widetilde{C}_j^{k,l_n}$ acts as an energy barrier for $\Phi^{(n)}$ within $\widetilde{C}_j^k$, separating the evolution inside from the evolution outside with a small data region. We denote the inner region by $\widetilde{C}_j^{k,<l_n}$ and its union with $\widetilde{C}_j^{k,l_n}$ by $\widetilde{C}_j^{k,\leq l_n}$. We fix r_0 independent of n so that

$$(t,x) \in \widetilde{C}_j^{k,<l_n} \implies \{t\} \times B(x, 4r_0) \subset \widetilde{C}_j^{k,\leq l_n}.$$

To measure the energy concentration in balls we define the functions

$$f_n : [0, r_0] \times I_k \to \mathbb{R}^+, \qquad f_n(r, t) = \sup_{\{x; \{t\} \times B(x,r) \subset \widetilde{C}_j^{k, \leq l_n}\}} \mathcal{E}_{B(x,r)}[\Phi^{(n)}](t).$$

The functions f_n are continuous in both variables and nondecreasing with respect to r. We also define the functions

$$r_n : I_k \to (0, r_0], \qquad r_n(t) = \begin{cases} \inf\{r \in [0, r_0]; \ f_n(t, r) \geq \frac{E_0}{10}\}, & \text{if } f_n(t, r_0) \geq \frac{E_0}{10}; \\ r_0, & \text{otherwise.} \end{cases}$$

which measure the lowest spatial scale on which concentration occurs at time t. Due to the finite speed of propagation it follows that the r_n are Lipschitz continuous with Lipschitz constant 1,

$$|r_n(t_1) - r_n(t_2)| \leq |t_1 - t_2|.$$

The nonconcentration estimate (66) in case ii) of the lemma corresponds to the case when all functions r_n admit a common strictly positive lower bound (note that (61) and (62) give the other conclusions).

It remains to consider the case when on a subsequence we have

$$\lim_{n \to \infty} \inf_{I_k} r_n = 0$$

and show that this yields the concentration scenario i). We denote "kinetic energy" in $\widetilde{C}_j$ by

$$\alpha_n^2 = \int_{\widetilde{C}_j} |X_0 \Phi^{(n)}|^2 dx dt.$$

By (62) we know that $\alpha_n \to 0$. Using α_n as a threshold for the concentration functions r_n, after passing to a subsequence we must be in one of the following three cases:

Case 1 (r_n Dominates). For each n we have $r_n(t) > \alpha_n$ in I_k. Then we let t_n be the minimum point for r_n in I_k and set $r_n^0 = r_n(t_n) \to 0$. By definition we have $f_n(t_n, r_n(t_n)) = E_0/10$. We choose a point x_n where the maximum of $f_n(t_n, r_n(t_n))$ is attained. This directly gives (63). For (64) we observe that, due to the existence of the energy barrier, x_n must be at least at distance $3r_0$ from the lateral boundary of $\widetilde{C}_j^{k, \leq l_n}$. Hence, if $x \in B(x_n, r_0)$ then x is at distance at least $2r_0$ from $\partial \widetilde{C}_j^{k, \leq l_n}$ and (64) follows. For (65) it suffices to know that $r_n(t_n)^{-1} \alpha_n^2 \to 0$, which is straightforward from the assumptions of this case.

Case 2 (Equality). For each n there exists $t_n \in I_k$ such that $\alpha_n = r_n(t_n)$. This argument is a repeat of the previous one, given that we define $r_n^0 = r_n(t_n)$ and set up the estimate (64) around this t_n as opposed to the minimum of $r_n(t)$.

Case 3 (α_n Dominates). For each n we have $r_n(t) < \alpha_n$ in I_k. For $t \in I_k$ set

$$g(t) = \int_{\widetilde{C}_j^{k, \leq l_n}} |X_0 \Phi^{(n)}(t)|^2 dx.$$

Then by definition

$$\int_{I_k} g(t) = \alpha_n^2.$$

Let $\widetilde{I}_k$ be the middle third of I_k, and consider the localized averages

$$\mathcal{I} = \int_{\widetilde{I}_k} \frac{1}{r_n(t)} \int_{t-r_n(t)/2}^{t+r_n(t)/2} g(s)\,ds\,dt.$$

Since $r_n(t)$ is Lipschitz with Lipschitz constant 1, if $s \in [t - r_n(t)/2, t + r_n(t)/2]$ then $\frac{1}{2}r_n(s) < r_n(t) < 2r_n(s)$ and $t \in [s - r_n(s), s + r_n(s)]$. Hence changing the order of integration in $\mathcal{I}$ we obtain

$$\mathcal{I} \leq 2 \int_{\widetilde{I}_k} \int_{t-r_n(t)/2}^{t+r_n(t)/2} \frac{1}{r_n(s)} g(s)\,ds\,dt \leq 4 \int_{I_k} g(s)\,ds = 4\alpha_n^2.$$

Hence by pigeonholing there exists some $t_n \in \widetilde{I}_k$ so that

$$\frac{1}{r_n(t_n)} \int_{t_n-r_n(t_n)/2}^{t_n+r_n(t_n)/2} g(s)\,ds \leq 4\alpha_n^2 |\widetilde{I}_k|^{-1}$$

Then let x_n be a point where the supremum in the definition of $f(t_n, r_n)$ is attained. The relations (63)-(66) follow as above. □

6.8. *The compactness argument.* To conclude the proof of Theorems 1.3,1.5 we consider separately the two cases in Lemma 6.3:

(i) *Concentration on small scales.* Suppose that the alternative (i) in Lemma 6.3 holds for some $j \in \mathbb{N}$. On a subsequence we can assume that $(t_n, x_n) \to (t_0, x_0) \in \widetilde{C}_j$. Then we define the rescaled wave maps

$$\Psi^{(n)}(t, x) = \Phi^{(n)}(t_n + r_n t, x_n + r_n x)$$

in the increasing sets $B(0, r_0/r_n) \times [-\frac{1}{2}, \frac{1}{2}]$ They have the following properties:
a) Bounded energy:

$$\mathcal{E}[\Psi^{(n)}](t) \leq \mathcal{E}[\Phi], \qquad t \in [-\frac{1}{2}, \frac{1}{2}].$$

b) Small energy in each unit ball:

$$\sup_x \mathcal{E}_{B(x,1)}[\Psi^{(n)}](t) \leq \frac{1}{10} E_0, \qquad t \in [-\frac{1}{2}, \frac{1}{2}].$$

c) Energy concentration in the unit ball centered at $t = 0$, $x = 0$:

$$\mathcal{E}_{B(0,1)}[\Psi^{(n)}](0) = \frac{1}{10} E_0.$$

d) Time-like energy decay: There exists a constant time-like vector $X_0(t_0, x_0)$ such that for each x we have

$$\iint_{[-\frac{1}{2}, \frac{1}{2}] \times B(x,1)} |X_0(t_0, x_0) \Psi^{(n)}|^2 dx dt \to 0.$$

Note in part (d) we used the fact that $(t_n, x_n) \to (t_0, x_0)$.

By the compactness result in Proposition 5.1 it follows that on a subsequence we have strong uniform convergence on compact sets,

$$\Psi^{(n)} \to \Psi \qquad \text{in } H^1_{loc}\left(B(0, r_0/(2r_n) \times [-\frac{1}{2}, \frac{1}{2}]\right),$$

where $\Psi \in H^{\frac{3}{2} - \epsilon}_{loc}$ is a wave-map. Thus, we have obtained a wave map Ψ defined on all of $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R}^2$, with the additional properties that

$$\frac{1}{10} E_0 \leq \mathcal{E}[\Psi] \leq E[\Phi]$$

and

$$X_0(t_0, x_0) \Psi = 0.$$

Then Ψ extends uniquely to a wave map in $\mathbb{R} \times \mathbb{R}^2$ with the above properties (e.g. by transporting its values along the flow of $X_0(t_0, x_0)$). After a Lorentz transform that takes $X_0(t_0, x_0)$ to ∂_t the function Ψ is turned into a nontrivial finite energy harmonic map with energy bound $\mathcal{E}[\Psi] \leq E[\Phi]$.

(ii) *Nonconcentration.* Assume now that the alternative (ii) in Lemma 6.3 holds for *every* $j \in \mathbb{N}$. There there is no need to rescale. Instead, we successively use directly the compactness result in Proposition 5.1 in the interior of each set $C_j \cap C^2_{[2,\infty)}$. We obtain strong convergence on a subsequence

$$\Phi^{(n)} \to \Psi \qquad \text{in } H^1_{loc}(C^2_{[2,\infty)})$$

with $\Psi \in H^{\frac{3}{2} - \epsilon}_{loc}(C^2_{[2,\infty)})$. From (67) and energy bounds (e.g. (61)) we obtain

$$0 < E_2 \leq \sup_{t \geqslant 2} \mathcal{E}_{B(0, t-2)}[\Psi](t) \leq \mathcal{E}[\Phi].$$

From (68) it follows also that

$$X_0 \Psi = 0.$$

By rescaling (i.e. extending Ψ via homogeneity), we may replace the interior of the translated cone $C^2_{[2,\infty)}$ with the interior of the full cone $t > r$ and retain the assumptions on Ψ, in particular that it is non-trivial with finite energy up to the boundary $t = r$. But this contradicts Theorem 4.1, and therefore shows that scenario (i) above is in fact the only alternative.

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