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REGULARITY RESULTS FOR AN ELLIPTIC-PARABOLIC FREE BOUNDARY PROBLEM

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ABSTRACT. We study an elliptic-parabolic free boundary problem in one space dimension. We give several regularity results for both the weak solution and the free boundary. In particular conditions are given which ensure that the free boundary is a C^1-curve.

1. Introduction. Consider the initial-boundary value Problem I:

$$\begin{cases} (c(u))_t = u_{xx} & \text{in } Q_T = (0,1) \times (0,T], & (1.1) \\ u_x(0,t) = 0 \text{ and } u_x(1,t) = f(t), & 0 < t \leq T, & (1.2) \\ c(u(x,0)) = v_0(x), & 0 \leq x \leq 1. & (1.3) \end{cases}$$

Here $T > 0$ and the functions c, f and v_0 satisfy the following hypotheses.

H1. $c \in C(\mathbf{R}) \cap C^{2,\alpha}(\mathbf{R}^-)$ *for some* $\alpha \in (0,1)$, $c \equiv 1$ *on* $\mathbf{R}^+$, *and* c' *is strictly positive and uniformly bounded on* $\mathbf{R}^-$.

H2. $f: [0,T] \to \mathbf{R}$ *is Lipschitz continuous and strictly positive.*

H3. *There exists a Lipschitz continuous function* $u_0: [0,1] \to \mathbf{R}$ *such that* $v_0 = c(u_0)$.

H4. $\int_0^1 v_0(x)\,dx + \int_0^T f(t)\,dt < 1$.

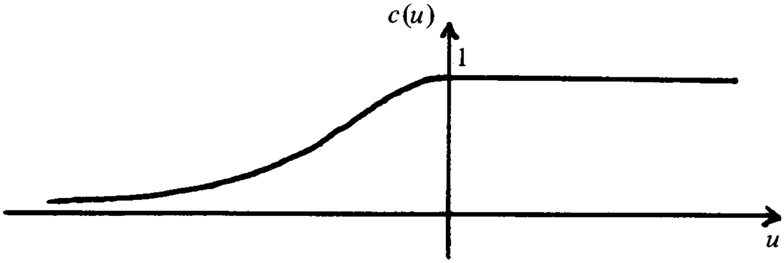

FIGURE 1. The function $c(u)$

Problem I describes the one dimensional fluid flow in a partially saturated porous medium: u denotes the hydrostatic potential due to capillary suction, c the moisture content, and we assume that c depends on u as described in Figure 1. The saturation value of c is taken to be one:

$$c(u(x,t)) = 1 \Leftrightarrow u(x,t) \geq 0 \Leftrightarrow \text{saturation at the point } (x,t).$$

From a mathematical point of view equation (1.1) is quite interesting. It is of parabolic type at points (x,t) where the medium is unsaturated, i.e., where

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$u(x,t) < 0$ ($\Leftrightarrow c'(u) > 0$), and of elliptic type in the saturated region where $u(x,t) \geq 0$ ($\Leftrightarrow c'(u) = 0$). In particular the answer to the question whether Problem I has a classical solution is not a priori clear.

Problem I does have a unique weak solution [**6**, **8**]. By this statement we mean that there exists a unique function $u \in L^2(0,T;H^1(0,1))$ which satisfies

(i) $c(u) \in C(\overline{Q}_T)$;

(ii) for every test function $\phi \in C^1(\overline{Q}_T)$ vanishing at $t = T$

$$\iint_{Q_T} \{\phi_x u_x - \phi_t c(u)\}\,dx\,dt = \int_0^1 \phi(x,0)v_0(x)\,dx + \int_0^T \phi(1,t)f(t)\,dt$$

(for more general existence results we refer to [**1**]). We observe here that condition H4 is natural in view of the conservation law

$$\int_0^1 c(u(x,t))\,dx = \int_0^1 v_0(x)\,dx + \int_0^t f(s)\,ds, \qquad 0 \leq t \leq T,$$

and it expresses the fact that the medium is not completely saturated at time T.

Of special interest is the free boundary between the regions where the medium is saturated, respectively, unsaturated. We define the set

$$P_t = \{0 \leq x \leq 1 : c(u(\cdot,t)) \equiv 1 \text{ on } [x,1]\}$$

and the function $\varsigma : [0,T] \to [0,1]$ by

$$\varsigma(t) = \begin{cases} \inf P_t & \text{if } P_t \neq \emptyset, \\ 1 & \text{if } P_t = \emptyset. \end{cases} \tag{1.4}$$

Hulshof [**6**] has shown that

$$\varsigma \text{ is continuous on } [0,T] \tag{1.5}$$

and, if $0 < \varsigma(t) < 1$, the point $x = \varsigma(t)$ is the interface between the saturated and the unsaturated region:

$$\begin{aligned} c(u(x,t)) &< 1 \quad \text{if } 0 \leq x < \varsigma(t), \quad 0 < t \leq T, \\ c(u(x,t)) &= 1 \quad \text{if } \varsigma(t) < x \leq 1, \quad 0 \leq t \leq T. \end{aligned} \tag{1.6}$$

Since $c(u(\cdot,t)) \not\equiv 1$ on $[0,1]$ it follows that $\varsigma > 0$ on $[0,T]$. It may happen however that, for some $t \in (0,T]$, $\varsigma(t) = 1$. In that case it is possible that $c(u(\varsigma(t),t)) < 1$, i.e. the medium is completely unsaturated at that time.

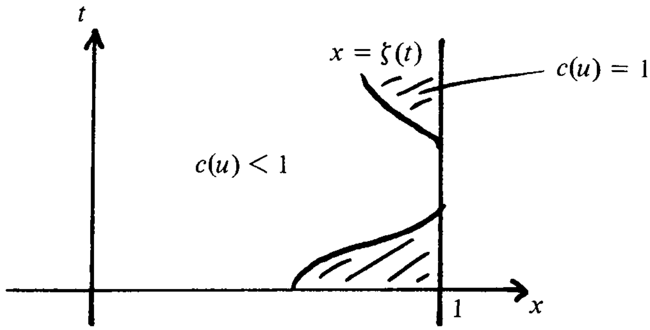

FIGURE 2

In this paper we present some regularity results for the function ς which improve (1.5). We shall give conditions which guarantee that ς is Lipschitz continuous, respectively continuously differentiable. We also prove regularity results for the weak solution u of Problem I. In particular we shall give conditions such that u is a classical solution.

Before we state the precise results, we give a list of several additional assumptions on the data.

A1. *$c(s)$ is concave near $s = 0$.*

A2. *$u_0 \in C^{2,\alpha}([a, \varsigma(0))) \cap C^1([a, \varsigma(0)])$, where $\alpha \in (0,1)$, $\varsigma(0)$ is defined by* (1.4) *and $a \in [0, \varsigma(0))$;*

$$|u_0''| \leq Mc'(u_0) \quad \text{on } [a, \varsigma(0)) \quad (M > 0),$$

u_0 is piecewise monotone, and $u_0'(\varsigma(0)) = f(0)$.

A2*. *u_0 satisfies* A2, *$u_0 \in C^{3,\alpha}([a, \varsigma(0)))$ and $u_0''/c'(u_0)$ is uniformly Lipschitz continuous on $[a, \varsigma(0))$.*

Without loss of generality we may assume that $u_0 < 0$ on $[a, \varsigma(0))$.

THEOREM 1.1. *Let the hypotheses* H1–H4 *and* A2 *be satisfied and let u be the weak solution of Problem* I. *If either $c(u(1, \cdot)) \equiv 1$ on $[0, T]$ or c satisfies assumption* A1, *then*

(i) *ς is Lipschitz continuous on $[0, T]$;*

(ii) *$u_x \in C([0,1] \times (0, T])$ and*

$$u_x(x,t) = f(t) \quad \text{if } \varsigma(t) \leq x \leq 1, \ 0 \leq t \leq T;$$

(iii) *$u_t \in L^\infty_{\mathrm{loc}}([0,1] \times (0, T])$;*

(iv) *if $c'(0^-) = 0$ then u is a classical solution of Problem* I, *i.e. $(c(u))_t \in C(Q_T)$, $u_{xx} \in C(Q_T)$ and u satisfies* (1.1)–(1.3) *pointwise.*

If we replace condition A2 by the stronger condition A2* we can improve Theorem 1.1.

THEOREM 1.2. *Let the hypotheses* H1–H4 *and* A2* *be satisfied and let u be the weak solution of Problem* I. *If either $c(u(1, \cdot)) \equiv 1$ on $[0, T]$, or c satisfies assumption* A1, *then*

(i) *ς is continuously differentiable in every $t \in [0, T]$ where $\varsigma(t) < 1$;*

(ii) *$u_{xt} \in L^\infty_{\mathrm{loc}}([0,1] \times (0, T])$;*

(iii) *$u_t \in C(\overline{D} \cap Q_T)$, where*

$$D = \{(x,t) \in Q_T : 0 < x < \varsigma(t)\}, \tag{1.7}$$

and if $\varsigma(t) < 1$

$$\varsigma'(t) = -u_t(\varsigma(t)^-, t)/f(t); \tag{1.8}$$

(iv) *if $f \in C^1([0, T])$ then $u_t \in C(Q_T)$.*

REMARK 1. In Theorems 1.1 and 1.2 the set $[0,1] \times (0, T]$ may be replaced by any compact subset K of $\overline{Q}_T$ which does not contain the set $\{(x,t): 0 \leq x \leq a, \ t = 0\}$.

REMARK 2. If $c'(0^-) > 0$ the weak solution is in general not classical in the neighborhood of a point $(\varsigma(t_0), t_0)$ with $\varsigma(t_0) < 1$. Otherwise (1.8) and the fact that $(c(u))_t = u_{xx} = 0$ in $\{(x,t) \in Q_T : \varsigma(t) < x < 1\}$ would imply

$$\varsigma'(t_0) = \frac{-u_t(\varsigma(t_0)^-, t_0)}{f(t_0)} = \frac{-(c(u(\varsigma(t_0), t_0)))_t}{c'(u(\varsigma(t_0)^-, t_0)) f(t_0)} = 0.$$

REMARK 3. The condition $\varsigma(t) < 1$ in Theorem 1.2(i), (iii) cannot be omitted. It can be shown that in general ς arrives at, respectively departs from the lateral boundary $\{x = 1\}$ with nonzero speed.

REMARK 4. The condition that $f \in C^1([0,T])$ in Theorem 1.2(iv) cannot be omitted. This follows at once from the observation that, since $u_{xx} = 0$ in $Q_T \backslash \mathcal{D}$, u is given by

$$u(x,t) = f(t)\{x - \varsigma(t)\} \quad \text{if } \varsigma(t) < x < 1.$$

REMARK 5. We believe that the conditions A2 resp. A2* are not necessary for the results of Theorem 1.1 (resp. Theorem 1.2). The proofs of these theorems are based on bounds for the functions u_t and u_{xt} near the interface $x = \varsigma(t)$. To obtain these bounds, we need that u_t (resp. u_{xt}) are bounded at $t = 0$ near $x = \varsigma(0)$. Since formally, from equation (1.1), $u_t = u_{xx}/c'(u)$, we arrive at the conditions on $u_0''/c'(u_0)$ in the assumptions A2 and A2*. We expect however that equation (1.1) has a regularizing effect for $t > 0$, i.e. also if A2 and A2* do not hold, the function $u_{xx}/c'(u)$ and its spatial derivative are bounded for every $t > 0$. In that case the theorems would still be valid (except for the smoothness of ς at $t = 0$).

REMARK 6. Our method of proof is based on the maximum principle, i.e. we use global techniques which depend heavily on the boundary conditions. Not only the method, but also the results depend on the boundary condition at $x = 1$. It follows for example from results in [**7**] that, if we drop our condition $f > 0$ on $[0,T]$, it is possible to construct a solution with a discontinuous function ς. So we cannot expect that all our results can be proved with local techniques.

In this connection we mention that DiBenedetto and Gariepy [**3**] use local techniques to prove that, in arbitrary space dimensions, for bounded weak solutions, $c(u)$ is continuous.

The proofs of Theorem 1.1 and 1.2 are given in §4 resp. §5 of this paper. §3 is devoted to the study of the level lines of u near the interface curve $x = \varsigma(t)$ (observe that, at least formally, $x = \varsigma(t)$ itself is a level line, namely $\{u = 0\}$, provided $\varsigma(t) < 1$). In §6 we comment on generalizations of our results to the case of a Dirichlet boundary condition at $x = 1$.

2. Preliminaries. First we collect some known results about the solution of Problem I [**6**, **8**].

PROPOSITION 2.1. *Let* H1–H4 *be satisfied. Then Problem* I *has a unique weak solution u, which has the following properties.*

(i) $u \in L^2(0,T; H^2(0,1)) \cap L^\infty(0,T; W^{1,\infty}(0,1))$.

(ii) $c(u) \in C^{0+1}(\overline{Q}_T) \cap H^1(Q_T)$.

(iii) *u is a classical solution in the region*

$$\mathcal{D} = \{(x,t) \in Q_T : c(u(x,t)) < 1\}.$$

(iv) *The equalities in Problem* I *all hold in the a.e. sense.*

(v) *The function ς, defined by* (1.4) *satisfies* (1.6) *and* $\varsigma \in C([0,T])$.

By Proposition 2.1(iv) $u_{xx} = 0$ a.e. in $Q_T \backslash \mathcal{D}$ and, by Proposition 2.1(i), $u(\cdot,t) \in C^1([0,1])$ for a.e. $t \in [0,T]$. Since $u_x(1,t) = f(t)$ a.e. on $[0,T]$, this implies that we may assume, after redefining u on a set of measure zero, that

$$u(x,t) = f(t)\{x - \varsigma(t)\} \quad \text{on } \overline{Q_T \backslash \mathcal{D}}. \tag{2.1}$$

The existence of a weak solution was proved in [8] by means of parabolic regularization. In this method the functions c, u_0 and f are replaced by smooth functions c_n, u_{0n} and f_n with the properties

(i) $c_n \to c$, $c_n(u_{0n}) \to v_0$, and $f_n \to f$ as $n \to \infty$;

(ii) $|u'_{0n}| \leq L$, $u'_{0n}(0) = 0$, and $u'_{0n}(1) = f_n(0)$;

(iii) $1/n \leq c'_n \leq K < \infty$.

Let $u_n(x,t)$ denote the unique solution of Problem I_n:

$$\begin{cases} (c_n(u))_t = u_{xx} & \text{in } Q_T, \\ u_x(0,t) = 0 \text{ and } u_x(1,t) = f_n(t), & 0 < t \leq T, \\ u(x,0) = u_{0n}(x), & 0 \leq x \leq 1. \end{cases}$$

Then $u_n \to u$ as $n \to \infty$.

The proofs of Theorems 1.1 and 1.2 are based on estimates of u_n which are uniform with respect to n. For that reason we have to reconstruct the initial functions u_{0n} in such a way that they satisfy the conditions A2 and A2*, uniformly with respect to n.

LEMMA 2.2. *Let u_0 satisfy assumption* A2. *Then there exists a sequence of smooth functions $u_{0n}\colon [0,1] \to \mathbf{R}$ such that*

$$\begin{aligned} &c_n(u_{0n}) \to v_0 && \textit{in } C([0,1]) \textit{ as } n \to \infty, \\ &u_{0n} \to u_0 && \textit{in } C^1([a, \varsigma(0)]) \textit{ as } n \to \infty, && (2.2) \\ &u'_{0n} \to f(0) && \textit{in } C([\varsigma(0), 1]) \textit{ as } n \to \infty, && (2.3) \\ &u'_{0n}(0) = 0, \end{aligned}$$

and

$$|u''_{0n}| \leq M c'_n(u_{0n}) \quad \textit{on } [a,1]. \tag{2.4}$$

If u_0 satisfies A2*, *we have in addition that*

$$u''_{0n}/c'_n(u_{0n}) \quad \textit{is uniformly bounded in } C^1([a,1]). \tag{2.5}$$

REMARK. Once we have constructed the functions u_{0n}, we can choose the functions f_n in Problem I_n such that they satisfy the compatibility condition $u'_{0n}(1) = f_n(0)$.

PROOF. There exists a sequence of functions $q_n \in C^\infty([a,1])$ such that

$$|q_n| \leq M \quad \text{on } [a,1] \tag{2.6}$$

and

$$q_n \to u''_0/c'(u_0) \quad \text{in } C([a, \varsigma(0) - \varepsilon)) \text{ as } n \to \infty \tag{2.7}$$

for any small $\varepsilon > 0$.

Let $u_{0n}\colon [a,1] \to \mathbf{R}$ be the unique solution of the initial-value problem

$$\begin{cases} w'' = q_n c'_n(w), & a < x \leq 1, \\ w(a) = u_0(a) \quad \text{and} & w'(a) = u'_0(a). \end{cases}$$

Here c_n is a sequence of smooth functions such that

$1/n \leq c'_n \leq K$ on $\mathbf{R}$,

$c'_n \to 0$ uniformly on $[\delta, \infty)$ for all $\delta > 0$,

$c_n \to c$ in $C^2(I)$ for every compact set $I \subset \mathbf{R}^-$.
We extend u_{0n} to a smooth function on $[0,1]$ in such a way that $u'_{0n}(0) = 0$, $|u'_{0n}| \leq L$, and $c_n(u_{0n}) \to v_0$ on $[0, a]$.

We claim that u_{0n} satisfies (2.2)–(2.4).

To prove (2.2), it is sufficient to show that $u_{0n} \to u_0$ in $C^1([a, \varsigma(0)])$ as $n \to \infty$. Since u_{0n} is uniformly bounded in $C^2([a, \varsigma(0)])$, there exists a subsequence (which we denote by $\{u_{0n}\}$ again) such that $u_{0n} \to w$ in $C^1([a, \varsigma(0)])$ as $n \to \infty$ for some $w \in C^1([a, \varsigma(0)])$. We have to show that

$$w = u_0 \quad \text{on } [a, \varsigma(0)]. \tag{2.8}$$

From the construction of q_n, c_n, and u_{0n}, it can be easily derived that the limit function w is the unique solution of the initial value problem

$$\begin{cases} w'' = u_0'' c'(w)/c'(u_0), & a < x \leq \varsigma(0) - \varepsilon, \\ w(a) = u_0(a), & w'(a) = u_0'(a), \end{cases}$$

where $\varepsilon > 0$ is an arbitrarily small number. Thus, by uniqueness, $w = u_0$ on $[a, \varsigma(0) - \varepsilon]$ and (2.8) follows.

From (2.6) and the construction of u_{0n} we find at once that (2.4) is satisfied.

To prove (2.3) we observe that, by (2.2),

$$u'_{0n}(\varsigma(0)) \to f(0) \quad \text{as } n \to \infty,$$

since $u_0'(\varsigma(0)) = f(0)$. We know that $f(0) > 0$, $u_{0n}(\varsigma(0)) \to 0$ as $n \to \infty$, u_0'' is uniformly bounded, and that $c_n' \to 0$ uniformly on $[\delta, \infty)$ for all $\delta > 0$. Combining these facts we obtain (2.3).

Finally, if u_0 satisfies A2*, we can choose the functions q_n such that

$$q_n' \text{ is uniformly bounded on } [a, 1]$$

and

$$q_n \to q_\infty \quad \text{in } C([a, 1]),$$

where $q_\infty \in C([a,1])$ is defined by

$$q_\infty(x) = \begin{cases} \dfrac{u_0''(x)}{c'(u_0(x))}, & a \leq x < \varsigma(0), \\ \lim\limits_{x \uparrow \varsigma(0)} \dfrac{u_0''(x)}{c'(u_0(x))}, & \varsigma(0) \leq x < 1, \end{cases}$$

and (2.5) is clearly satisfied.

3. The level curves. In the Introduction we mentioned already that, as long as $\varsigma(t) < 1$, the set $x = \varsigma(t)$ is, at least formally, the level curve $\{(x,t) \colon u(x,t) = 0\}$. Since our main objective is to study the regularity of ς, it seems a natural approach to study the level curves of u in a neighborhood of $x = \varsigma(t)$ (see also [**2**]).

Our plan is as follows. First, to eliminate the formal aspect of calculations, we shall study the level curves of the approximating smooth solutions u_n of Problem I_n which we introduced in §2. Given a level curve $x = X(t)$ of u_n, it follows from $(d/dt)\{u_n(X(t), t)\} = 0$ that

$$X_t = -u_{nt}/u_{nx} \quad \text{if } u_{nx} \neq 0. \tag{3.1}$$

Defining

$$\eta_n(x,t) = -u_{nt}(x,t)/u_{nx}(x,t) \quad \text{if } u_{nx}(x,t) \neq 0, \tag{3.2}$$

we derive a parabolic partial differential equation for η_n. In the following sections this equation will be the main tool to obtain uniform bounds for η_n, which, by (3.1), are bounds for the time derivatives of level curves.

Secondly, to control the condition $u_{nx} \neq 0$ in (3.1), we shall prove that the condition $f(t) > 0$ in (1.2) implies that $u_{nx} \geq \delta > 0$ in a neighborhood of $x = \varsigma(t)$ for n large enough and for some $\delta > 0$.

LEMMA 3.1. *Let u_n be the solution of Problem* I_n. *If $u_{nx} \neq 0$ at $(x_0, t_0) \in Q_T$ then the function η_n, defined by* (3.2), *satisfies*

$$\eta_t = \eta_{xx}/c'_n(u_n) - (\eta^2)_x \quad \text{at } (x_0, t_0). \tag{3.3}$$

PROOF. Although it is possible to obtain (3.3) by direct calculation, we choose a proof which uses the level curves of u_n as a new coordinate system, following an idea by Gurtin, MacCamy, and Socolovsky [**5**].

For convenience we drop the subscript n.

Since $u_x \neq 0$ in a neighborhood $\mathcal{V}$ of (x_0, t_0), we can define in $\mathcal{V}$ a coordinate transformation $(x,t) \to (p,\tau)$ defined by $x = X(p,\tau)$, $t = \tau$, where X is defined by $u(X(p,\tau),\tau) = p$. Hence $u_t + u_x X_\tau = 0$ and $u_x X_p = 1$. Using this we derive that

$$\frac{1}{X_p} X_\tau = u_x X_\tau = -u_t = -\frac{u_{xx}}{c'(u)} = -\frac{1}{c'(u)} \left(\frac{1}{X_p}\right)_p \frac{1}{X_p} = \frac{1}{c'(u)} \frac{X_{pp}}{X_p^3},$$

or

$$X_\tau = X_{pp}/c'(p)X_p^2. \tag{3.4}$$

We set $Y(p,\tau) = X_\tau(p,\tau)$ and differentiate (3.4) with respect to τ. This yields

$$Y_\tau = \frac{1}{c'(p)} \frac{Y_{pp}}{X_p^2} - \frac{2X_{pp}}{c'(p)X_p^3} Y_p = \frac{1}{c'(p)} \frac{Y_{pp}}{X_p^2} - \frac{2YY_p}{X_p},$$

where we used (3.4) another time. Defining

$$\eta(x,t) = Y(p,\tau)$$

we find that

$$\eta_t = Y_\tau - \eta_x \eta = \frac{1}{c'(u)} \eta_{xx} + \frac{1}{c'(u)} \frac{X_{pp}}{X_p^2} \eta_x - 2\eta\eta_x - \eta\eta_x = \frac{1}{c'(u)} \eta_{xx} - 2\eta\eta_x,$$

which completes the proof.

In the following lemmas we establish lower bounds for u_x and u_{nx} near $x = \varsigma(t)$.

LEMMA 3.2. *Let hypotheses* H1–H4 *be satisfied and let u_0 be piecewise monotone. Let u be the solution of Problem* I. *For every $t_0 \in [0,T]$ there exists a neighborhood $N(t_0)$ of $(\varsigma(t_0), t_0)$ in $\overline{Q_T}$ such that $u_x \geq 0$ a.e. in $N(t_0)$.*

PROOF. If $\varsigma(t_0) = 1$ and $c(u(\varsigma(t_0), t_0)) < 1$, u is a classical solution of Problem I in a neighborhood of $(\varsigma(t_0), t_0)$. Since $u_x(\varsigma(t_0), t_0) = f(t_0) > 0$, the existence of $N(t_0)$ is obvious.

So let $c(u(\varsigma(t_0), t_0)) = 1$. Since u_0 is piecewise monotone, it follows from the so-called lap-number arguments in [**6**, Appendix] that the function $u(\cdot, t)$ is piecewise monotone on $[0, \varsigma(t)]$ for all $t \in [0,T]$. Thus there exists for every t a number $\phi(t) \in [0, \varsigma(t))$ such that $u_x(\cdot,t) \geq 0$ on $[\phi(t), \varsigma(t))$. Here we recall that u is a classical solution in $\mathcal{D}$ and hence u_x is a smooth function. We may choose $\phi(t)$ as small as possible. Observe that $u_x(\cdot,t) \not\equiv 0$ on $[\phi(t), \varsigma(t)]$.

We now argue by contradiction. Since by (2.1) $u_x = f > 0$ if $x > \varsigma(t)$, we suppose that there exists a sequence $(x_n, t_n) \in \mathcal{D}$ with $(x_n,t_n) \to (\varsigma(t_0), t_0)$ as $n \to \infty$ and $u_x(x_n,t_n) < 0$.

First we consider the case that $t_n \nearrow t_0$ as $n \to \infty$. For all n there exists a maximal interval (y_n, z_n) with $0 \leq y_n < x_n < z_n < \varsigma(t_n)$, such that $u_x(\cdot, t_n) \leq 0$ on (y_n, z_n). Hence $c(u(\cdot,t_n))$ attains a local maximum in y_n. Since $u(y_n,t_n) \geq u(x_n,t_n)$ and since, by the continuity of ς, $c(u(x_n,t_n)) \to 1$ as $n \to \infty$, we have that

$$c(u(y_n,t_n)) \to 1 \quad \text{as } n \to \infty. \tag{3.5}$$

Since $u(\cdot,t)$ is piecewise monotone, the largest local maximum of $c(u(\cdot,t))$ less than one, which we denote by $S(t)$, is well defined. In [**6**] it was proved that $S(t)$ is decreasing with respect to t. Hence

$$S(t_n) \not\to 1 \quad \text{as } n \to \infty,$$

which is a contradiction to (3.5).

Finally we consider the case that $t_n \searrow t_0$ as $n \to \infty$. We choose $\bar{x} \in (\phi(t_0), \varsigma(t_0))$ and $\tau > t_0$ such that $u_x(\bar{x}, \cdot) > 0$ on $[t_0, \tau]$. Using that, by (2.1), $u_x = f$ in $Q_T \backslash \overline{\mathcal{D}}$, it follows that $u_x \geq 0$ on the parabolic boundary of the set $(\bar{x}, 1) \times (t_0, \tau]$. By the maximum principle for u_x (which can be easily proved using the approximating sequence $\{u_n\}$) it follows that $u_x \geq 0$ a.e. in this set, and we obtain a contradiction.

LEMMA 3.3. *Let the assumptions of Lemma* 3.2 *be satisfied and let* $u_0 \in C^1([a,1])$ $(0 \leq a < \varsigma(0))$ *with* $u_0'(\varsigma(0)) = f(0)$. *Then there exist a function* $\xi \in C^\infty([0,T])$ *with* $0 \leq \xi < \varsigma$ *on* $[0,T]$ *and* $\xi(0) > a$ *and a number* $\delta > 0$ *such that* $u_x \geq 2\delta$ *a.e. in* $\mathcal{B}_T$, *where*

$$\mathcal{B}_T = \{(x,t) \colon \xi(t) \leq x \leq 1,\ 0 \leq t \leq T\}. \tag{3.6}$$

PROOF. By Lemma 3.2 and a straightforward compactness argument there exist a function $\xi \in C^\infty([0,T])$ and an $\varepsilon > 0$ with $\varepsilon \leq \xi < \varsigma$ on $[0,T]$ such that $u_x \geq 0$ a.e. in the set $\{(x,t) \colon \xi(t) - \varepsilon \leq x \leq 1,\ 0 \leq t \leq T\}$. Since $u_0' > 0$ near $\varsigma(0)$ we may assume that $u_0' > 0$ on $[\xi(0), 1]$. By the strong maximum principle, applied to u_x in a neighborhood of the curve $x = \xi(t)$, there exists a number δ, with $0 < 2\delta \leq f$ on $[0,T]$, such that $u_x \geq 2\delta$ on the parabolic boundary of $\mathcal{B}_T$. Then Lemma 3.3 follows from the maximum principle for u_x.

LEMMA 3.4. *Let the hypotheses* H1–H4 *and assumption* A2 *be satisfied and let* u_n *be the solution of Problem* I_n. *Then there exists an* $N \in \mathbf{Z}^+$ *such that* $u_{nx} \geq \delta > 0$ *on* $\mathcal{B}_T$ *for all* $n \geq N$, *where* δ *and* $\mathcal{B}_T$ *are given by Lemma* 3.3.

PROOF. The approximating functions u_{0n} and f_n in Problem I_n can be chosen in such a way that $u_{0n}' \geq \delta$ on $[\xi(0), 1]$ and $f_n \geq \delta$ on $[0,T]$.

Since the curve $x = \xi(t)$ lies entirely in the region where u is a classical solution, it follows from standard a priori estimates [**9**] that $u_n \to u$ in $C^{2,1}(\overline{\mathcal{N}})$, where

$\mathcal{N}$ is a neighborhood of $x = \xi(t)$ in Q_T. Hence it follows from Lemma 3.3 that $u_{nx}(x,t) \geq \delta$ if $x = \xi(t)$ for n large enough.

Thus $u_{nx} \geq \delta$ on the parabolic boundary of $\mathcal{B}_T$ for n large, and Lemma 3.4 follows form the maximum principle.

4. Theorem 1.1. In this section we shall prove Theorem 1.1. The main step in the proof is to show the uniform boundedness of u_{nt} near $x = \varsigma(t)$. At this point we distinguish two cases. In Lemma 4.1 we give a proof in the case that

$$c(u(1,t)) = 1 \quad \text{for all } t \in [0,T]. \tag{4.1}$$

In Lemma 4.2 we consider the general case, but there we shall need concavity of $c(s)$ near $s = 0$.

Below C denotes a generic constant.

LEMMA 4.1. *Let hypotheses* H1–H4 *and assumption* A2 *be satisfied, and let u resp. u_n be the solution of Problem* I *resp.* I_n, *with u_{0n} defined by Lemma* 2.2. *If u satisfies* (4.1), *then there exists a constant C such that, for n large enough, $|u_{nt}| \leq C$ in $\mathcal{B}_T$, where $\mathcal{B}_T$ is defined by* (3.6).

PROOF. We extend the solution u by

$$u(x,t) = f(t)(x - \varsigma(t)), \qquad x \geq 1,\ 0 \leq t \leq T.$$

Since $c(u(1,\cdot)) = 1$ on $[0,T]$, it follows that u can be considered as the solution of Problem I on $[0,A] \times [0,T]$ for any $A \geq 1$, where the boundary condition $u_x = f$ holds at $x = A$ instead of at $x = 1$. This argument implies that we may assume without loss of generality that

$$\varsigma(t) < 1 \quad \text{on } [0,T]. \tag{4.2}$$

By Lemma 3.4, uniform boundedness on $\mathcal{B}_T$ of u_{nt} and of η_n are equivalent for n large enough. Below we shall estimate η_n.

We observe that η_n is the classical solution of the initial-boundary value Problem II_n:

$$\begin{cases} \eta_t = \eta_{xx}/c_n'(u_n) - (\eta^2)_x & \text{in } \mathcal{B}_T, \qquad (4.3)\\ \eta_x = -f_n'/f_n + c_n'(u_n)\eta^2, & x = 1,\ 0 \leq t \leq T, \qquad (4.4)\\ \eta = -u_{nt}/u_{nx}, & x = \xi(t),\ 0 \leq t \leq T, \qquad (4.5)\\ \eta(\cdot,0) = -u_{0n}''/c'(u_{0n})u_{0n}', & \xi(0) \leq x \leq 1. \qquad (4.6)\end{cases}$$

Here (4.3) follows from Lemma 3.1, and (4.4) is derived from

$$\eta_{nx} = -\left(\frac{u_{nt}}{u_{nx}}\right)_x = -\frac{u_{nxt}}{u_{nx}} + \frac{u_{nt}u_{nxx}}{u_{nx}^2} = -\frac{f_n'}{f_n} + c_n'(u_n)\left(\frac{u_{nt}}{u_{nx}}\right)^2 \quad \text{for } x = 1,\ 0 \leq t \leq T.$$

To prove Lemma 4.1, we construct comparison functions which do not depend on n.

Define, for $A > 0$,

$$\underline{\eta}(x,t) = -A(x-1) - e^{2A(t+1)} \quad \text{on } \mathcal{B}_T.$$

We claim that, for A and n large, $\underline{\eta}$ satisfies (4.3)–(4.6) with equality replaced by $\leq$. Concerning (4.3) and (4.4), this follows from a simple calculation. Concerning (4.5) and (4.6) we use the uniform boundedness of η_n on $x = \xi(t)$, resp. $\mathcal{B}_T \cap \{t = 0\}$, which, at $x = \xi(t)$, follows from the proof of Lemma 3.4, and, at $t = 0$, from Lemma 2.2.

Hence, for A and n large, $\underline{\eta}$ is a subsolution of Problem II_n and thus, by the maximum principle [**4**],

$$\eta_n \geq \underline{\eta} \quad \text{in } \mathcal{B}_T. \tag{4.7}$$

Next we define

$$\overline{\eta}(x,t) = B(x+1), \qquad (x,t) \in \mathcal{B}_T.$$

As above, it follows that, for B large enough $\overline{\eta}$ satisfies (4.3), (4.5) and (4.6) with equality replaced by $\geq$. To prove the same for (4.4), we have to show that

$$c_n'(u_n(1,t)) \to 0 \quad \text{as } n \to \infty \tag{4.8}$$

which enables us to neglect the term $c_n'(u_n)\eta^2$ in (4.4). Accepting (4.8) for the moment, it follows that $\overline{\eta}$ is a supersolution of Problem II_n for B and n large, and thus

$$\eta_n \leq \overline{\eta} \quad \text{in } \mathcal{B}_T. \tag{4.9}$$

Lemma 4.1 follows from (4.7) and (4.9).

So it remains to prove (4.8). For every $\phi \in C([0,T])$ with $0 \leq \phi < \varsigma$ on $[0,T]$ we known that $u_n(\phi(t),t) \to u(\phi(t),t)$ in $C([0,T])$ as $n \to \infty$. Since $u(\varsigma(t),t) = 0$ and $u_{nx}(\varsigma(t),t) \geq \delta > 0$ on $\mathcal{B}_T$, it follows from (4.2) that there exists an $\varepsilon > 0$ such that $u_n(1,\cdot) \geq \varepsilon$ on $[0,T]$ for n large enough. Since $c_n' \to 0$ uniformly on $[\varepsilon,\infty)$ as $n \to \infty$, we obtain (4.8).

LEMMA 4.2. *Let hypotheses* H1–H4 *and assumptions* A1, A2 *be satisfied. Let* u_n *be the solution of Problem* I_n *with* u_{0n} *defined by Lemma* 2.2. *Then there exists a constant* C *such that for* n *large enough* $|u_{nt}| \leq C$ *in* $\mathcal{B}_T$.

PROOF. The upper bound for u_{nt} follows, as in the proof of Lemma 4.1, by constructing a lower bound for η_n.

The construction of a supersolution of Problem II_n fails, because we can no longer neglect the term $c_n'(u_n)\eta^2$ in (4.4). For that reason we give a different proof, where we need the concavity of $c(s)$ near $s = 0$.

We choose $\varepsilon > 0$ and c_n such that for all n $c_n(s)$ is concave on $(-4\varepsilon, \infty)$. It follows easily from the uniform continuity of $c(u(1,\cdot))$ on $[0,T]$ that there exists a finite partition

$$0 = t_0 < t_1 < \cdots < t_m = T \tag{4.10}$$

of the interval $[0,T]$ such that for all $i = 1,\ldots,m$ either $u(1,t) > -2\varepsilon$ or $u(1,t) < -\varepsilon$ on (t_{i-1}, t_i).

If $u(1,t) < -\varepsilon$ on (t_{i-1},t_i), u is a classical solution on $[0,1] \times [t_{i-1},t_i]$. Thus if $t_{i-1} \neq 0$, u_t and u_{nt} are uniformly bounded on these subsets and, if $t_{i-1} = 0$, on compact subsets of $\{[0,1] \times [0,t_1]\} \backslash \{[0,a] \times \{0\}\}$. In particular, if m in (4.10) is chosen minimal, then

$$\begin{cases} u_{nt} \geq -C & \text{on } [0,1] \times \{t_i\},\ i = 1,\ldots,m, \\ u_{nt} \geq -C & \text{on } [a,1] \times \{0\}. \end{cases} \tag{4.11}$$

It remains to prove that, for some $\phi \in C^\infty([0,T])$ with $\phi(0) > a$ and $\phi < \varsigma$ on $[0,T]$, and for n large

$$u_{nt} \geq -C \text{ in } \mathcal{K}_i = \{(x,t): \phi(t) \leq x \leq 1,\ t_{i-1} \leq t \leq t_i\} \tag{4.12}$$

for all i with $u(1,t) > -2\varepsilon$ on (t_{i-1}, t_i). We fix such an i. We choose ϕ such that $u > -3\varepsilon$ on $\mathcal{K}_i$. Then, for n large, $u_n > -4\varepsilon$ on K_i. Defining $q_n = u_{nt}$ it follows from the concavity of c_n that

$$c_n'(u_n)q_{nt} = -c_n''(u_n)u_{nt}^2 + q_{nxx} \geq q_{nxx} \quad \text{on } \mathcal{K}_i.$$

In addition, $u_{nt} \geq -C$ on the lateral boundary $x = \phi(t)$ of $\mathcal{K}_i$ (where u is a classical solution) and on $\mathcal{K}_i \cap \{t = t_{i-1}\}$ (by (4.11)). Finally, $(u_{nt})_x = f_n'$ is uniformly bounded on the lateral boundary $x = 1$. Hence (4.12) follows from the maximum principle.

PROOF OF THEOREM 1.1. (iii) By Lemmas 4.1 and 4.2 and from the fact that u is a classical solution in $Q_T \backslash \mathcal{B}_T$, it follows that u_n is uniformly Lipschitz continuous with respect to t (and also, by Proposition 2.1(i), with respect to x) on $[0,1] \times [\tau, T]$ for any $\tau > 0$. Hence the limit function u is Lipschitz continuous on $[0,1] \times [\tau, T]$, and $u_t \in L^\infty_{\text{loc}}([0,1] \times (0,T])$.

(iv) Since, by (2.1), $(c(u))_t = u_{xx} = 0$ if $x > \varsigma(t)$, and since $(c(u))_t = u_{xx} \in C(\mathcal{D})$, it is enough to prove that

$$\lim_{\substack{(x,t)\to(\varsigma(t_0),t_0)\\(x,t)\in\mathcal{D}}} (c(u(x,t)))_t = 0 \quad \text{if } u(\varsigma(t_0), t_0) = 0.$$

This follows at once from (iii) and the fact that $c'(0^-) = 0$.

(i) It is sufficient to prove that $\varsigma(t)$ is uniformly Lipschitz continuous on $\{t \in [0,T]: \varsigma(t) < 1\}$.

We fix $\tau \in [0,T]$ with $\varsigma(\tau) < 1$. Then $\varsigma(t) < 1$ in a neighborhood $\mathcal{O}_\tau$ of τ. By Lemma 3.3 the level curves $x = \varsigma_\varepsilon(t)$ with $u(\varsigma_\varepsilon(t), t) = -\varepsilon$ are well defined smooth curves for ε small enough and $t \in \mathcal{O}_\tau$. Since $\varsigma_\varepsilon \nearrow \varsigma$ on $\mathcal{O}_\tau$ as $\varepsilon \searrow 0$ it is sufficient to prove that

$$|\varsigma_\varepsilon'| \leq C \quad \text{on } \mathcal{O}_\tau \tag{4.13}$$

where C does not depend on ε and τ.

We derive from the fact that $(d/dt)u(\varsigma_\varepsilon(t), t) = 0$ on $\mathcal{O}_\tau$, that

$$\varsigma_\varepsilon'(t) = -u_t(\varsigma_\varepsilon(t), t)/u_x(\varsigma_\varepsilon(t), t), \qquad t \in \mathcal{O}_\tau.$$

Since, for ε small, u_x is bounded away from zero (by Lemma 3.3), (4.13) follows from the fact that u_t is bounded in a neighborhood in $\overline{Q_T}$ of $x = \varsigma(t)$.

(ii) Because, by (2.1), $u_x = f$ if $x > \varsigma(t)$ and since $u_x \in C(\mathcal{D})$, it is enough to show that

$$\lim_{\substack{(x,t)\to(\varsigma(t_0),t_0)\\(x,t)\in\mathcal{D}}} u_x(x,t) = f(t_0), \qquad 0 \leq t_0 \leq T.$$

Since $\varsigma \in C([0,T])$, we only have to show that

$$\lim_{\varepsilon\searrow 0} u_x(\varsigma(t) - \varepsilon, t) = f(t) \quad \text{uniformly on } [0,T].$$

Because u_{xx} is bounded near $x = \varsigma(t)$, $u_x(\varsigma(t) - \varepsilon, t)$ converges uniformly on $[0,T]$ to a continuous function $g(t)$ on $[0,T]$. By Proposition 2.1(i), $u(\cdot, t) \in C^1([0,1])$ for a.e. $t \in [0,T]$, and hence $f(t) = g(t)$ a.e. on $[0,T]$. Since both f and g are continuous, $f \equiv g$ on $[0,T]$.

5. Theorem 1.2. We proceed as in the proof of Theorem 1.1: first we use Problem II_n to obtain estimates for η_n.

LEMMA 5.1. *Let the assumptions of Theorem 1.2 be satisfied, let u_{0n} be defined by Lemma 3.2, and let f_n satisfy the compatibility condition*

$$(u_{0n}''(1)/c_n'(u_{0n}(1)))' = f_n'(0).$$

Let η_n be defined by (3.2). Then $|\eta_{nx}| \le C$ on $\mathcal{B}_T$ for n large enough.

PROOF. First we observe that η_{nx} is uniformly bounded on the parabolic boundary of $\mathcal{B}_T$. On the lateral boundary $x = \xi(t)$ this follows from the fact that u is a classical solution of Problem I and $u_{nx} \to u_x$ in $C^{2,1}$ near $x = \xi(t)$. On $\mathcal{B}_T \cap \{t = 0\}$ it follows from (2.5), where we use that

$$\eta_{nx} = -\frac{u_{nxt}}{u_{nx}} + \frac{u_{nt}u_{nxx}}{(u_{nx})^2} = -\frac{1}{u_{nx}}\left(\frac{u_{nxx}}{c_n'(u_n)}\right)_x + c_n'(u_n)\eta_n. \tag{5.1}$$

Finally, (4.4) implies the uniform boundedness of η_{nx} on the lateral boundary $x = 1$. Here we use the uniform bound on η_n, established in §4.

Differentiating equation (4.3) with respect to x, we find that $w = \eta_{nx}$ satisfies

$$w_t = \{w_x/c_n'(u_n) - 2\eta_n w\}_x.$$

By [**9**, p. 181, Theorem 7.1], this implies that η_{nx} is uniformly bounded on $\mathcal{B}_T$.

PROOF OF THEOREM 1.2. (ii) By Lemma 5.1 and (5.1), u_{nxt} is uniformly bounded in $\mathcal{B}_T$, and thus also in $[0,1] \times [\tau, T]$ for all $\tau \in (0,T)$. In view of Theorem 1.1(iii), u_{nxx} is uniformly bounded in $[0,1] \times [\tau, T]$ and hence u_{nx} is Lipschitz continuous in this set. Since the Lipschitz constant does not depend on n, the same is true for u_x, which shows that $u_{xt} \in L^\infty_{\mathrm{loc}}([0,1] \times (0,T])$.

(i)+(iii) Using that u_{xt} is bounded, it follows that $u_t(\varsigma(t) - \varepsilon, t)$ converges uniformly on $[0,T]$ as $\varepsilon \searrow 0$. Hence $u_t \in C(\overline{\mathcal{D}} \cap Q_T)$.

Let $\mathcal{O}_\tau$ and $\varsigma_\varepsilon(t)$ be defined as in the proof of Theorem 1.1(i). Then, for ε small,

$$\varsigma_\varepsilon'(t) = -u_t(\varsigma_\varepsilon(t), t)/u_x(\varsigma_\varepsilon(t), t), \qquad t \in \mathcal{O}_\tau.$$

Since $(u_t/u_x)_x$ is bounded, and $\varsigma \in C([0,T])$, it follows that $\varsigma_\varepsilon \to \varsigma$ in $C^1(\overline{\mathcal{O}}_\tau)$ and ς' is given by (1.8).

(iv) By (2.1)

$$u_t = f'(x - \varsigma) - f\varsigma', \qquad \varsigma(t) < x < 1. \tag{5.2}$$

Hence $u_t \in C(Q_T \backslash \mathcal{D})$. In view of (ii), $u_t \in C(Q_T \cap \overline{\mathcal{D}})$. Finally, by (1.8) and (5.2), u_t is continuous across the interface $x = \varsigma(t)$.

6. A Dirichlet boundary condition. In this section we briefly indicate how the main results of this paper can be generalized to the case that the Neumann condition at $x = 1$ is replaced by a Dirichlet condition:

$$u(1,t) = g(t) > 0 \quad \text{on } [0,T], \tag{6.1}$$

where g is a Lipschitz continuous and positive function on $[0,T]$. We shall, for convenience, refer to this problem as the Dirichlet problem, although we still have $u_x = 0$ at $x = 0$. Again (see [**8**]) there exists a unique weak solution u and, as long as the medium is not completely saturated, i.e.,

$$c(u(\cdot, t)) \not\equiv 1 \quad \text{on } [0,1] \text{ for all } 0 \le t \le T,$$

the interface $x = \varsigma(t)$ is continuous and satisfies

(6.2) $$0 < \varsigma < 1 \quad \text{on } [0, T].$$

We claim that in the situation described above, Theorems 1.1 and 1.2 still hold, if in assumption A2 the condition $u_0'(\varsigma(0)) = f(0)$ is replaced by

$$u_0'(\varsigma(0)) = g(0)/(1 - \varsigma(0)).$$

Observe that, in view of (6.2), $c(u(1, \cdot)) \equiv 1$ on $[0, T]$ and hence assumption A1 can be omitted. Of course we do not need the assumptions concerning $f(t)$ anymore.

To prove our claim, we notice that in the saturated region $u_{xx} = 0$ a.e. and thus we may assume that (cf. (2.1))

$$u(x, t) = \frac{g(t)}{1 - \varsigma(t)}\{x - \varsigma(t)\}, \qquad x \leq \varsigma(t) \leq 1,\ 0 \leq t \leq T.$$

In particular u can be considered as the solution of Problem I with the Neumann condition $u_x(1, t) = f(t)$, where

(6.3) $$f(t) = g(t)/(1 - \varsigma(t)).$$

Since $\varsigma \in C([0, T])$ and $\varsigma < 1$ on $[0, T]$, f is a positive and continuous function, but not, as was assumed in the previous sections, a Lipschitz continuous function. In the first step of the proof of Theorem 1.1, namely Lemma 3.3, we did not use the Lipschitz continuity of f, and so Lemma 3.3 still holds.

In the rest of the proof of Theorem 1.1 we consider u as the solution of the Dirichlet problem. In particular, u can be obtained as the limit of smooth solutions u_n of an approximating Dirichlet problem, with g replaced by g_n. The main line of the proof of Theorem 1.1 remains the same. Only at two parts, which we discuss below, the proof has to be adapted.

First, to prove Lemma 3.4, we need to know that $u_{nx}(1, \cdot)$ is positive and bounded away from 0 on $[0, T]$ for n large. To do this, a supersolution $\bar{u}_n(x, t)$ with the properties $\bar{u}_{nx}(1, \cdot) \geq \delta > 0$ and $\bar{u}_n(1, \cdot) = g_n$ on $[0, T]$, $\bar{u}_n \geq 0$ in Q_T, can be used as a barrier function. The construction of $\bar{u}_n$ is left to the reader.

Secondly, the proof of Lemma 4.1 is simplified since the boundary condition (4.4) for η_n at $x = 1$ is replaced by

$$\eta_n(1, t) = -g_n'(t)/u_{nx}(1, t), \qquad 0 \leq t \leq T.$$

Hence η_n is uniformly bounded on the boundary $x = 1$ and the construction of a subsolution and a supersolution is straightforward.

Finally we observe that, once we have proved Theorem 1.1, we know that ς is Lipschitz continuous, and so is f, given by (6.3). Hence Theorem 1.2 follows at once if we consider u as the solution of the Neumann problem.

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